FREGE'S PARADISE AND THE PARADOXES

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1. Introduction

Hilbert spoke of Cantor's universe of sets as a paradise for mathematicians: "No one shall drive us out of the paradise that Cantor has created for us"[1] – a paradise that seemed to have room for all the entities that a mathematician would ever need. However, this paradise was – as we all know – threatened by paradoxes. Similarly, what one might call *Frege's paradise* – Frege's intensional ontology – seems large enough to accommodate many, if not all, of the abstract entities that a logical-semanticist may use when interpreting our language and thought: propositions (Gedanken), senses (Sinne), functions, relations, classes, and the two truth-values, the True and the False.

Frege's paradise, however, is also threatened by paradoxes. First, of course, there is Russell's paradox, that proved Frege's theory of extensions (or classes) to be inconsistent. There are also the *semantic paradoxes* that threaten his intensional ontology and his theory of sense (Sinn) and denotation (Bedeutung).[2]

The purpose of this paper is logical rather than historical. The main objective is to investigate how the semantical paradoxes threaten theories of propositions and their constituents that are in a broad sense Fregean in character, rather than to discuss whether, or to what extent, these theories were actually held by Frege. I will, of course, also discuss ways of modifying the Fregean framework in order to avoid the paradoxes.

Fregean theories about propositions and senses have recently received renewed significance, in connection with deflationary approaches to the concept of truth that take abstract propositions as the primary truth bearers. The most influential theory of this kind is Paul Horwich's *minimalist theory of truth* (or *minimalism* as it is also called).[3] Horwich's minimalism has three ingredients:

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[1] Hilbert (1967).
[2] I follow Church (1951), Kaplan (1964), and Anderson (1980, 1987) in translating Frege's term 'Bedeutung' by 'denotation' rather than 'reference'. My reason is that I want to emphasize the technical character of Frege's notion of 'Bedeutung'.
[3] Cf. Horwich (1998). See also Lindström (2001) for a discussion of the logical aspects of Horwich's minimalism about truth.

(i) *An account of the concept of truth:* Horwich claims that the word 'true' picks out an indefinable property of propositions the content of which is exhausted by (or is "implicitly defined by") a certain theory which he calls *the minimal theory of truth*, or MT for short. Roughly speaking, the axioms of MT are all the propositions that are expressed by (non-paradoxical) instances of the schema:

(E) The proposition that p is true iff p.

(ii) *An account of the utility of the truth predicate:* If the truth predicate only occurred in what we may call *primary contexts*:

(a) *The proposition that snow is white is true* (or its sentential counterparts: *'Snow is white' is true*),

then, it could be eliminated by means of the schema (E) and would thus be *redundant* (at least in extensional contexts). However, we also want to use the truth predicate to say things like:

(b) The continuum hypothesis is true.
(c) There are true propositions that are not supported by the available evidence.
(d) Every sentence is such that either it or its negation is true.
(e) Most statements that Clinton made in his deposition were true.

In the latter sentences, however, the truth predicate cannot be eliminated by means of the (E) schema. According to Horwich, *the sole purpose of having a truth predicate at all* is to be able to express claims of this latter kind.

(iii) *An account of the nature of truth:* The property truth does not have any underlying nature and the explanatory basic facts about truth are instances of the (E) schema.

Horwich's minimal theory of truth is noncommittal with respect to the nature of propositions:

> As far as the minimal theory of truth is concerned, propositions could be composed of abstract Fregean senses, or of concrete objects and properties; they could be identical to a class of sentences in some specific language, or to the meanings of sentences, or to some new and irreducible type of entity that is correlated with the meanings of certain sentences.[4]

Recently, however, Christopher Hill (2002) has developed a kind of minimalism along broadly Fregean lines. Hill's discussion of truth does not take as its starting point the truth and falsity of linguistic items. Instead, he is concerned

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[4] Horwich (1998), p. 17.

with the semantic properties of *thoughts*, i.e., the (alleged) propositional objects of such psychological states as beliefs, desires and intentions. Hill's goal is to explain what it means to say that a thought is true. Secondarily, he wants to explain what it means to say of the constituents of thoughts that they *refer to* or are *satisfied by* things. The constituents of thoughts he refers to as *concepts*.

Hill makes the following fundamental assumptions about thoughts:

- (i) Thoughts have logical structures.
- (ii) Thoughts have concepts as their constituents.
- (iii) Thoughts are individuated by their logical structures and their constituent structures.

Thoughts are themselves a particular kind of concepts. Hence, thoughts can have other thoughts as constituents.

I take these assumptions to imply the following principles:

- (C) The principle of *compositionality* for thoughts. Thoughts that are built up in the same way from the same concepts are identical (assumptions (i) together with (ii))
- (MD) The *principle of maximum distinction* for thoughts: Two thoughts are identical *only if* they are built up in the same way from the same concepts. (Assumption (iii)).

I am not going to discuss Hill's original and interesting account of truth and reference here, except to say that I regard it as a version of what I would like to call *Fregean minimalism*, a deflationary approach that takes Fregean propositions ('thoughts') and their constituents ('senses') as the primary bearers of truth and reference. Fregean minimalism is an attractive approach to truth. It is threatened, however, by the kind of logical difficulties ("paradoxes") that I am going to discuss in this paper. Unless we can come to terms with these difficulties, there is little hope of developing a satisfactory theory of truth and reference along Fregean lines.

2. The Fregean ontology and the theory of sense and denotation

Let us briefly recapitulate Frege's theory of sense and denotation and its accompanying ontology.

2.1. *The distinction between objects and functions*. Frege makes a fundamental ontological distinction between *objects* and *functions*. We may think of objects and functions as constituting two separate mutually exclusive domains. The basic notion when distinguishing functions from objects is functional *application*.

A function f is the kind of entity that can be *applied* to one or several entities $a_1,..., a_n$ (within its *domain* or *range of definition*) to yield an object $f(a_1,..., a_n)$, called the *value* of f for the *arguments* $a_1,..., a_n$. An object, on the other hand, cannot be *applied* to anything. Frege indicates this distinction by saying that functions are incomplete (or, unsaturated), while objects are complete (or, saturated). Functions are divided into levels, roughly, as follows. First-level functions only take objects as arguments. Second-level functions take first-level functions as arguments, and so on.[5] The result of applying a function to something is for Frege always an object. Hence, there is in Frege's ontology no room for functions that yield functions as values.

In addition to functions, we may expect Frege's ontology also to contain *properties* and *relations*. Intuitively, we speak of items as *having properties* and *standing in relations* to each other. Frege, however, identifies properties and relations with functions of a special kind that he calls *concepts*. Among the objects, there are the two truth-values *the True* (**t**) and *the False* (**f**). These are abstract logical objects. A *Fregean concept* F is a function, which for any argument (or sequence of arguments) within its domain, yields one of the truth-values, **t** or **f**, as value. So, instead of saying that an item a has the property F, Frege says that F(a) = **t**. And instead of saying that, $a_1,..., a_n$ (in that order) stand in the relation F to each other, Frege says that $F(a_1,..., a_n) = \mathbf{t}$. Intuitively, $F(a_1,..., a_n) = \mathbf{f}$ means that F is defined for the arguments $a_1,..., a_n$, although $a_1,..., a_n$ *do not* stand in the relation F to each other.

In the following, we shall use the term 'attribute' with the same meaning as Frege's 'concept'.[6] Thus, an n-ary *attribute* is an n-ary function taking a sequence of n items as arguments and yielding one of the truth-values **t** or **f** as values. Usually, we are only going to considering *first-level attributes*, i.e., attributes taking objects as arguments and yielding truth-values as values. A *property* is a unary attribute and, for $n \geq 2$, an n-ary *relation* is an n-ary attribute. If f is an n-ary (first-level) attribute, and $a_1..., a_n$ are objects, then $f(a_1,..., a_n)$ is either **t** (the True) or **f** (the False).

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[5] To be more exact, there is a *type hierarchy* of items of different types. First, there is the type **i** of all *objects*; Then, for any $n \geq 1$ and types $\alpha_1,..., \alpha_n$, there is a type $[\alpha_1,..., \alpha_n]$ of all functions f taking n arguments $a_1,..., a_n$ from respective types $\alpha_1,..., \alpha_n$ and yielding an object as value. We can assign *levels* to types as follows. The type **i** has level 0. If the maximal level among the types $\alpha_1,..., \alpha_n$ is n, then the type $[\alpha_1,..., \alpha_n]$ has level n+1. Items of a type of level n are called items of level n. Hence, objects are entities of level 0. Functions from objects to objects, are first-level functions, etc.

[6] By adopting this terminology, I can follow Alonzo Church in using the term 'concept' for those items that can serve as appropriate senses of linguistic expressions.

2.2. *Sense and denotation*. Any well-formed expression E of a logically proper language has a *denotation* (*Bedeutung*), den('E'), and a *sense* (*Sinn*), sense('E'). The *sense* of a well-formed expression E is said to be the *mode of presentation* of its denotation. We shall say that an expression *denotes* its denotation and *expresses* its sense. den('E') is the object designated by (or presented by) sense('E').

Singular terms ("names") and general terms ("predicates") have, respectively, objects and attributes (i.e., Frege's "concepts") as their denotations. A sentence has as its denotation one of the truth-values, **t** or **f**. With the possible exception of so-called oblique contexts, the denotation of a complex expression is functionally determined by the denotations of its parts. For instance, if $Pt_1...t_n$ is a sentence, where P is an n-ary predicate, and $t_1,..., t_n$ are singular terms, then

(i) den('P') is an n-ary relation, den('t_1'),..., den('t_n') are objects, and
(ii) den('$Pt_1...t_n$') = den('P')(den('t_1'),..., den('t_n')); which is either **t** or **f**.

We shall assume that the *principle of compositionality* also applies to senses, i.e., that the sense of a complex expression is a function of the senses of its constituents and the way in which it is built up from these constituents. [7]

Following Church (1951), and deviating from Frege, we speak of the entities that can serve as appropriate senses of expressions as *concepts*. Thus, a concept is anything that is capable of being the Fregean sense of a linguistic expression. For any concept x, x *is a concept of* y iff x is capable of serving as the sense of an expression denoting y. We also say that x *designates* y if x is a concept of y.

We follow Church in using the symbol Δ for the *concept relation*, which holds between a concept and the entity that it is a concept of. Every concept is a concept of at most one entity, i.e.,

$$\text{If } \Delta(x, y) \text{ and } \Delta(x, z), \text{ then } y = z.$$

The denotation of an expression A is uniquely determined by its sense and by the concept relation in the following way:

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[7] Although clearly part of a Fregean perspective on language, the principle of compositionality was never explicitly formulated by Frege. It appears that the first attribution of the principle to Frege is in Carnap (1947, p. 121), where Carnap formulates versions of the principle of compositionality for denotations as well as for senses. See also Church (1956), pp. 8, 9 and Kaplan (1964) for discussions of compositionality within the context of Fregean semantics. Whether, or in what form, Frege himself was committed to the principle is a difficult and much debated historical question (cf. Pelletier (2001), Janssen (2001)).

$$x = \text{den}(A) \text{ iff for some } y, y = \text{sense}(A) \text{ and } \Delta(y, x).$$

In other words, the denotation relation is the relative product of the relation between an expression and its sense and the concept relation. A meaningful expression A will lack a denotation, if sense(A) is an *empty concept*, i.e., if there is no y such that Δ(sense(A), y).

There are various kinds of concepts corresponding to the different categories of entities in the Fregean ontology and to the different kinds of meaningful expressions in a logically well-constructed language. A *singular concept* is a concept that can serve as an appropriate sense of a singular term in some (actual or merely possible) language, i.e., a singular concept is a concept of an object. *Function concepts*, *property concepts*, and *relation concepts* are appropriate senses of function terms, concept terms, and relation terms, respectively. A (*Fregean*) *proposition* is the appropriate sense of a (non-indexical) declarative sentence, i.e., it is a concept of a truth-value. A proposition P is *true* if it is a concept of the truth-value **t**; and it is false if it is a concept of the truth-value **f**. We also sometimes speak of Fregean propositions as *thoughts* (*Gedanken*). Thoughts, or propositions, are abstract objects that are true or false and can serve as the contents of propositional attitudes, like belief, desires, and intentions.

If we allow for meaningful sentences that lack truth-values, then the senses of such sentences must be propositions (thoughts) that are neither true nor false. Such propositions, that we might call *empty propositions*, would so to speak aspire to a truth-value without actually having one. Admitting empty propositions and empty concepts could, of course, be useful in the treatment of the Liar paradox and other semantic paradoxes.

The question whether a Fregean theory of sense and denotation can allow for propositions and other senses being empty, is a controversial one. However, at least on some interpretations of the sense-denotation distinction, it makes good sense to speak of senses that do not determine any denotation. For instance, Carnapian intensions, i.e., functions from possible worlds to appropriate extensions, may very well be *partial functions* that are undefined at certain worlds. A Carnapian proposition, in particular, i.e., a proposition from possible worlds to truth-values, may be undefined at the actual world, and thereby lack a truth-value at the actual world. Such a Carnapian proposition would be neither true nor false. So, if we were to identify senses with Carnapian intensions, then it would also make sense to assume the existence of senses that do not determine denotations and Fregean propositions that lack truth-values. Even if we interpreted Fregean senses as structured entities that are built up from Carnapian intensions, it would still make sense to let such senses not to determine denotations at some worlds. Frege, of course, did not allow for the senses of the expressions of a well-constructed formal language to be empty.

The distinction between objects and functions also applies to concepts (senses). Thus, singular concepts and propositions are themselves objects, while function concepts, property concepts, and relation concepts are functional ("unsaturated") in character.

We say that an object a *satisfies* a property concept **F** if and only if a has the property F determined by **F**, i.e., if and only if F(a) = **t**. In general, a sequence $<a_1,.., a_n>$ of objects *satisfies* an n-ary relation concept **R** if and only $R(a_1,..., a_n)$ = **t**. Finally, a proposition P is *true* iff P designates the truth-value **t**. P is false iff P designates **f**.

It is important, on the Fregean view, to distinguish between propositions (thoughts) and *judgments*. A proposition is an abstract object that is either true or false. A judgment, on the other hand, is a type of *mental act*: the judgment that φ is the mental act of affirming the truth of the proposition that φ. This mental act has the proposition that φ as its *propositional content*. We can say that the judgment is (objectively) *correct* just in case its propositional content is true (designates the true). According to Frege, we perform the same act when we judge that φ and when we judge that it is true that φ. Thus, the judgment *that* φ and the judgment *that it is true that* φ have the same propositional content. It follows, that the proposition that φ is the very same proposition as the proposition that it is true that φ. That is, according to Frege, we have:

sense('it is true that φ') = sense('φ').

Finally, we should note that it is also important to distinguish between *thinking* that φ in the sense of *judging* that φ, and *thinking* that φ in the sense of entertaining (considering, grasping) the thought that φ.

3. A Fregean intensional language

We consider a formal language L (or, rather a group of languages), constructed with the purpose of representing various theories of sense and denotation.[8] L is a second-order predicate language equipped with special terms denoting propositions and concepts (Sinne); and with a symbol Δ (Church's concept relation) for the relation of a concept *being a concept of* an object, a function, or an attribute.

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[8] The language L is an extension – with special sense terms and the concept relation Δ – of the modernized version of Frege's *Begriffschrift* described in Zalta (2003).

The well-formed expressions of L are divided into the following categories: (a) *singular terms*, denoting objects; (b) n-ary *predicate terms* (for n ≥ 1), denoting n-ary attributes; (c) n-ary *function terms* (for n ≥ 1), denoting n-ary functions from objects to objects; and (d) *formulas*, denoting truth-values.

For every closed well-formed expression X, L contains another well-formed expression <X> that functions as a name of the sense of X. Syntactically, but not semantically, X is a part of <X>. This means that the ordinary substitution rules for co-denoting expressions do not apply inside contexts of the form <...>. We say that <X> is the *sense-name* corresponding to X, and we speak of the symbols '<' and '>' as *sense-quotes*.

We deviate from Frege's original approach by not counting the truth-values *true* and *false* among the domain of ordinary objects. Thus, L is interpreted informally relative to the following domains of entities: (i) the domain **U** of objects; (ii) the domain {**t**, **f**} of the truth-values *true* and *false*; (iii) for each n ≥ 1, the domain $\mathbf{U}^n \to \mathbf{U}$ of all n-ary functions from objects to objects; and, finally, (iv) for each n ≥ 1, the domain $\mathbf{U}^n \to \{\mathbf{t}, \mathbf{f}\}$ of all n-ary attributes, i.e., n-ary functions from objects to truth-values. Since concepts (senses) of objects, so-called *singular concepts*, are themselves objects, they belong to **U**. Likewise, concepts of truth-values, i.e., *propositions*, also belong to **U**. *Predicate concepts* and *function concepts* are identified with appropriate functions that map singular concepts to propositions and singular concepts, respectively. Hence, they are (identified with) appropriate n-ary functions.

If σ is a sentence or a singular term, then the sense of σ is an object and <σ> is a singular term denoting the sense of σ. If X is an n-ary predicate term, then we let <X> denote a function from objects to objects representing the sense of X. We define this function in such a way that for all closed singular terms $\tau_1,..., \tau_n$, $<X\tau_1...\tau_n>$ denotes the result of applying the function denoted by <X> to the singular concepts denoted by $<\tau_1>,..., <\tau_n>$.

To be more precise: If X is a closed n-ary predicate term with sense **F** and denotation F, then <X> is an n-ary function term that denotes an n-ary first-level function $G_{\mathbf{F}}$ satisfying the following condition: if $c_1,..., c_n$ are singular concepts of the objects $a_1,..., a_n$, respectively, then $G_{\mathbf{F}}(c_1,..., c_n)$ = the proposition $[\mathbf{F}, c_1,..., c_n]$, i.e., the Fregean proposition which is the result of applying the predicate concept **F** to the singular concepts $c_1,..., c_n$. Intuitively speaking, $G_{\mathbf{F}}$ is a function from singular concepts to propositions that represents the predicate concept **F**. We call $G_{\mathbf{F}}$ the *concept function* representing the predicate concept **F**.[9] Hence,

[9] Strictly speaking, $G_{\mathbf{F}}$ is not uniquely defined since we have not indicated how the function is defined for arguments that are not concepts. This can easily be remedied by assigning the

for every closed n-ary predicate term X, <X> denotes the concept function representing the sense of X. In the following, however, we will always identify the concept F with the concept function G_F. Thus, for any singular terms $\tau_1,\ldots, \tau_n$,

(1) $\text{sense}(`X\tau_1\ldots,\tau_n') = \text{den}(`<X>')(\text{sense}(`\tau_1'),\ldots, \text{sense}(`\tau_n'))$.

Thus, we also have,

(2) $\text{den}(`<X\tau_1\ldots\tau_n>') = \text{den}(`<X>')(\ \text{den}(`<\tau_1>'),\ldots, \text{den}(`<\tau_n>') = \text{den}(`<X>(<\tau_1>,\ldots, <\tau_n>)')$.

Informally, we may think of <X> as a name of the sense of X.

If X is an n-ary function term, then <X> is an n-ary function term that denotes the sense of X. We define the denotation of <X>, in analogy with the case for predicate terms, as the concept function representing the sense of X. Thus, (1) and (2) above also hold for sense-names <X> corresponding to function terms X.

The language L contains the following symbols:

(i) There are object variables x, y, z,... ranging over the domain of all objects.

(ii) For each $n \geq 1$, there are predicate variables F^n, G^n,... for first level n-ary attributes; i.e., n-ary functions from objects to truth-values.

(iii) For each $n \geq 1$, there are function variables f^n, g^n,... for first level n-ary functions; i.e., n-ary functions from objects to objects;

(iv) L may contain constants of the various categories of terms. Thus, there may be object constants, predicate constants, and function constants.

(v) L contains the logical symbols $\bot$, $\rightarrow$, $\forall$, $=$, with their standard classical interpretations. We define: $\neg\varphi =_{df} (\varphi \rightarrow \bot)$, $T =_{df.} \neg\bot$, etc.

(vi) L contains the symbol Δ (the concept relation symbol) as well as sense-quotes and parentheses.

The *well-formed expressions* of L are built up from the symbols of L in accordance with the following formation rules:

(R1) Object variables, n-ary attribute variables, and n-ary function variables are, respectively, singular terms, n-ary predicate terms, and n-ary function terms.

(R2) If Φ is an n-ary predicate term [n-ary function term] and $\tau_1,\ldots, \tau_n$ are singular terms, then $\Phi\tau_1\ldots\tau_n$ is a formula [singular term].

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function a fixed and arbitrary value for such arguments.

(R3) $\perp$ is a formula, denoting the truth-value false.

(R4) If σ and τ are singular terms, then $\sigma = \tau$ is a formula.

(R5) If φ and ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula.

(R6) If φ is a formula and v is an object variable, predicate variable, or function variable, then $\forall v\varphi$ is a formula.

(R7) If φ is a formula, then $[\lambda x_1...x_n\varphi]$ is an n-ary predicate term which denotes the n-ary attribute of being objects $x_1,..., x_n$ such that $\varphi(x_1,..., x_n)$.

(R8) If τ is a singular term, then $[\lambda x_1...x_n\tau]$ is an n-ary function term which denotes the n-ary function which for any objects $x_1,..., x_n$ yields the value denoted by $\tau(x_1,..., x_n)$.

(R9) If X is a closed singular term or a sentence (closed formula), then <X> is a singular term denoting the sense of X. Thus, if X is a sentence, then <X> denotes the proposition expressed by X.

(R10) If Φ is a closed n-ary function term or n-ary predicate term, then $<\Phi>$ is an n-ary function term, denoting the sense function representing the sense of Φ.

(R11) If σ is a singular term and X is either a singular term or a sentence, then $\Delta(\sigma, X)$ is a formula that, intuitively, says that the object denoted by σ is a concept of (or designates) the object denoted by X. Hence, $\Delta(\sigma, X)$ is true iff the object denoted by σ is a concept of (or designates) the object denoted by X.

(R12) If ϕ is an n-ary function term and Φ is an n-ary predicate or function term, then $\Delta(\phi, \Phi)$ is a formula that, intuitively, says that the function denoted by ϕ is a concept of the function or attribute denoted by Φ.

Frege regarded coextensional functions as being identical. Hence, we can define:

(i) If Φ, Γ are n-ary predicate terms, then
$(\Phi = \Gamma) =_{df.} \forall x_1...\forall x_n(\Phi x_1...x_n \leftrightarrow \Gamma x_1...x_n)$.

(ii) If ϕ, γ are n-ary function terms, then
$(\phi = \gamma) =_{df.} \forall x_1...\forall x_n(\phi x_1...x_n = \gamma x_1...x_n)$.

4. Basic Fregean intensional logic

We now describe, in brief outline, a deductive system BFIL, (*Basic Fregean Intensional Logic*) for Frege's intensional logic.[10]

4.1 The extensional part. The extensional part of BFIL is an ordinary second-order predicate logic calculus, with standard axioms and rules governing the sentential connectives, the first- and second-order quantifiers, and the λ-terms. Hence, the ordinary introduction and elimination rules for the existential and universal quantifiers apply both to the first- and to the second-order quantifiers. For the identity sign, we have the axiom:

(Id) $\forall x \forall y(x = y \leftrightarrow \forall F(Fx \leftrightarrow Gx))$.

Since, for every formula φ and variables $x_1,..., x_n$, the language contains a λ-term $[\lambda x_1,...,x_n\varphi]$ that denotes the n-ary attribute defined by the condition $\varphi(x_1,..., x_n)$, it is assumed that every formula with free variables designates an attribute. Thus, the basic logic contains the following theorem schemata:

λ-conversion for λ–predicate terms:

$$\forall y_1...\forall y_n([\lambda x_1...x_n\varphi] \leftrightarrow \varphi(y_1/x_1,..., y_n/x_n))$$

Comprehension schema for attributes:

$$\exists F\forall x_1...\forall x_n(Fx_1...x_n \leftrightarrow \varphi),$$

the attribute F that the comprehension schema says exist is, of course, the same attribute that is denoted by the term $[\lambda x_1...x_n\varphi]$.

We have the following rule of proof:

Rule of substitution

The λ-term $[\lambda x_1...x_n\varphi]$ may be uniformly substituted for the occurrence of an n-ary predicate variable F^n in any theorem of the basic logic containing F^n free.

We also have corresponding principles for functions and λ–function terms, which we do not state here.

4.2 The intensional part.

BFIL contains the following axioms for the concept relation and for the sense terms.

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[10] The present system is simpler than Church's (1951) logic of sense and denotation. For one thing, it is based on standard second-order predicate logic rather than on the simple theory of types. Moreover, the method of referring to senses by means of sense-quotes (se below) is simpler than Church's so-called method of direct discourse.

First, there are axioms for the concept relation:

(A1) $\forall x \forall y \forall z(\Delta(x, y) \wedge \Delta(x, z) \rightarrow y = z)$,

i.e., every (singular) concept designates at most one object.

(A2) $\forall x \exists y \Delta(y, x)$,

i.e., for every object, there is a concept of that object.

The next two axioms say that for any n-ary property or function, there is an n-ary function that is a concept of it.

(A3) $\forall F^n \exists f^n \Delta(f^n, F^n)$.

(A4) $\forall f^n \exists g^n \Delta(g^n, f^n)$.

Next, we have:

(A5) $\forall F^n \forall f^n \forall x_1 ... \forall x_n \forall y_1 ... \forall y_n(\Delta(x_1, y_1) \wedge ... \wedge \Delta(x_n, y_n) \wedge \Delta(f^n, F^n) \rightarrow \Delta(f^n x_1 ..., x_n, F^n y_1 ... y_n))$,

i.e., if f^n is a concept of F^n and $x_1,..., x_n$ are concepts of $y_1,..., y_n$, respectively, then $f^n x_1 ... x_n$ is a proposition which is true iff $F^n y_1 ... y_n$; and

(A6) $\forall f^n \forall g^n \forall x_1 ... \forall x_n \forall y_1 ... \forall y_n(\Delta(x_1, y_1) \wedge ... \wedge \Delta(x_n, y_n) \wedge \Delta(g^n, f^n) \rightarrow \Delta(g^n x_1 ... x_n, f^n y_1 ... y_n)$,

i.e., if g^n is a concept of f^n and $x_1,..., x_n$ are concepts of $y_1,..., y_n$, respectively, then $g^n x_1,... x_n$ is a concept of the object $f^n y_1 ... y_n$.

Next, comes the axiom-schemata for the sense-terms:

(A7) For any closed singular term τ,

$$\Delta(<\tau>, \tau),$$

i.e., $<\tau>$ is a concept of τ.

(A8) For any sentence φ,

$$\Delta(<\varphi>, \psi) \leftrightarrow (\psi \leftrightarrow \varphi).$$

i.e., the proposition expressed by φ is a concept of (the truth-value of) φ.

(A9) $\forall F^n[\Delta(<[\lambda x_1 ... x_n \varphi]>, F^n) \leftrightarrow F^n = [\lambda x_1 ... x_n \varphi]]$.

(A10) $\forall f^n[\Delta(<[\lambda x_1 ... x_n \tau]>, f^n) \leftrightarrow f^n = [\lambda x_1 ... x_n \tau]]$.

We also have the following *compositionality principles* for concept terms:

(A11) For any closed n-ary predicate term Φ, and closed singular terms $\tau_1,..., \tau_n$,

$$<\Phi \tau_1 ... \tau_n> = <\Phi><\tau_n>...<\tau_n>.$$

(A12) For any closed n-ary function term ϕ, and closed singular terms $\tau_1,..., \tau_n$,

$<\phi\tau_1...\tau_n> = <\phi><\tau_n>...<\tau_n>$.

It is well known that the extensional part of BFIL is consistent. To see that the full BFIL is consistent, assume that for any sentence or closed term X, <X> and X have the same denotation and that Δ is interpreted as the identity relation. Intuitively, this means that we interpret two expressions as having the same sense if and only if they are coextensional. All of the axioms (A1)-(A12) are true under this trivialization of the sense-denotation distinction. Thus, given that the extensional part of the system is consistent, the full system BFIL is also consistent.

5. Fregean truth

The notions of truth and falsity can be applied to assertions, propositions, as well as to sentences. For Frege, these notions apply primarily to propositions (Gedanken). The basic notions are here the two truth-values **t** (the True) and **f** (the False). These are presumably introduced by *abstraction* from the notion of a proposition (Gedanke). The general procedure that Frege followed when introducing a kind of abstract entity was by means of a *principle of abstraction*. Examples of such principles are:

> The direction of l_1 = the direction of l_2 iff l_1 and l_2 are parallel to each other.
>
> The number of F's = number of G's iff there is a one-to-one correspondence between the F's and the G's.
>
> $\{x: F(x)\} = \{x: G(x)\} \leftrightarrow \forall x(Fx \leftrightarrow Gx)$.

A principle of abstraction provides a *criterion of identity* for a kind of entity. In the days before the discovery of Russell's paradox, Frege apparently thought that providing an appropriate principle of abstraction for a kind of entity was sufficient to assure the existence of the entities in question. The notion of a truth-value comes with the following principle of abstraction:

> For any two propositions p, q, the truth-value of p = the truth-value of q iff p and q are materially equivalent, i.e., if φ and ψ are sentences that express p and q, respectively, then $(\varphi \leftrightarrow \psi)$.

One can then define:

The True = the truth-value of the proposition that $\forall x(x = x)$.
The False = the truth-value of the proposition that $\forall x(x \neq x)$.

Once the two abstract objects **t** (the True) and **f** (the False) are given, Frege can define truth and falsity for propositions in the following way:

(i) $\mathrm{True}(x) =_{\mathrm{df.}} \Delta(x, T)$,

i.e., x is *true* iff x designates the truth-value **t**.

(ii) $\mathrm{False}(x) =_{\mathrm{df.}} \Delta(x, \bot)$,

i.e., x is *false* iff x designates the truth-value **f**.

By axiom schema (A8) of BFIL, we get:

$$\mathrm{True}(\langle\varphi\rangle) \leftrightarrow \Delta(\langle\varphi\rangle, T) \leftrightarrow (T \leftrightarrow \varphi) \leftrightarrow \varphi.$$

Hence, the equivalence schema for propositional truth is a theorem of BFIL:

(E) $\mathrm{True}(\langle\varphi\rangle) \leftrightarrow \varphi$.

That is, as a matter of logic, the proposition that φ is true iff φ.

Frege also maintained that $\mathrm{True}(\langle\varphi\rangle)$ and φ have the same sense, i.e., that the *redundancy thesis* holds:

(RT) $\langle\mathrm{True}(\langle\varphi\rangle)\rangle = \langle\varphi\rangle$.[11]

In the article ‘Thoughts’, he writes:

> ...we cannot recognize a property of a thing without at the same time finding the thought *this thing has this property* to be true. So with every property of a thing is tied up a property of a thought, namely truth. It also worth noticing that the sentence ‘ I smell the scent of violets’ has just the same content as the sentence ‘It is true that I smell the scent of violets.’ So it seems, then, that nothing is added to the thought by my ascribing to it the property of truth. And yet is it not a great result when the scientist after much hesitation and laborious research can finally say ‘My conjecture is true’? The meaning of the word ‘true’ seems to be altogether *sui generis*. May we not be dealing here with something which cannot be called a property in the ordinary sense at all? In spite of this doubt I shall begin by expressing myself in accordance with ordinary usage, as if truth were a property, until some more appropriate way of speaking is found.[12]

In spite of what Frege says here, the schema (RT) is not self-evident. Consider the proposition expressed by the sentence ‘The Earth is round’. This proposition, let us call it P, concerns the Earth and ascribes a property to it. Consider, on the other hand, the proposition expressed by ‘The proposition that the Earth is round is true’. The latter proposition is about the proposition P and describes it as being true. Since these two propositions have different subject matters, it is reasonable to conclude that they are distinct.

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[11] Frege’s arguments for the redundancy thesis are discussed in Pagin (2001).

[12] Frege (1984), pp. 354-355.

6. The theory of sense and denotation and the Russell-Myhill antinomy

As is well known, Frege's logic in the *Grundgesetze* incorporated a theory of extensions, or classes, that Russell proved to be inconsistent. The theory in question was based on the following assumptions:

(F1) The comprehension schema for properties:

$$\exists F\forall x(Fx \leftrightarrow \varphi(x)),$$

i.e., every formula $\varphi(x)$ of the language of Frege's *Grundgesetze* defines a property of objects.

(F2) For every property F of objects, there exists an object ext(F), called the extension of F.

(F3) *Frege's Basic Law V*:

$$\forall F\forall G(\text{ext}(F) = \text{ext}(G) \leftrightarrow \forall x(Fx \leftrightarrow Gx)),$$

i.e., two properties have the same extension if and only if they are extensionally equivalent.

Russell used the following Cantorian diagonal argument to show that these assumptions are inconsistent.

By (F1), there is a property **R**, which is defined by the formula:

$$\exists F(x = \text{ext}(F) \wedge \neg F(x)),$$

i.e., **R** is the property of x being the extension of a property that does not apply to x. Let us call this property the *Russell property*. By (F2), there is an object **r** such that **r** = ext(**R**). We call **r** the *Russell class*. Using (F3), i.e., Frege's Basic Law V, it was easy for Russell to prove:

$$\mathbf{R}(\mathbf{r}) \leftrightarrow \neg\mathbf{R}(\mathbf{r}),$$

i.e., the Russell class has the Russell property if and only if it does not have the Russell property. Thus, Frege's theory of extensions, or classes, is inconsistent.

So much for Frege's theory of extensions, or classes. But what about Frege's intensional ontology – his theory of propositions (thoughts) and concepts (senses)? Is this theory consistent? One might suspect that the theory of concepts and their objects is susceptible to paradoxes that are analogous to those of naive set theory. Whether this is actually the case is, however, not obvious, since Frege never axiomatized his theory of sense and denotation in a rigorous form.

It is clear, however, that theories of propositions and concepts along Fregean lines are threatened by antinomies, in particular, variants of the *Epimenides paradox* for propositions and the so-called *Russell-Myhill antinomy*, which is analogous to Russell's paradox for extensions. That the latter paradox is a threat has been pointed out, first by Russell in letters to Frege;[13] then apparently independently by Myhill (1958), in connection with Church's logic of sense and denotation (1951); and by Anthony C. Anderson (1980, 1987), also in connection with various versions of Church's logic of sense and denotation. Recently Kevin Klement, in a careful study (2002), has shown that Frege's logical system in the *Grundgesetze* becomes inconsistent when it is extended with a theory of sense and denotation.[14] Klement's point is that Frege would have had to face the threat of paradox, had he decided to formalize the theory of sense and denotation.

The Russell-Myhill Antinomy. The Russell-Myhill antinomy comes in several versions. Russell's original version of the paradox formulated in Russell (1903, Appendix B, p. 527), can be formulated informally as follows:

(i) propositions and classes are objects.
(ii) By Cantor's theorem, there are more classes of propositions than there are propositions. Hence, there are more classes of objects than there are propositions.
(iii) For any class X of objects, there is the proposition, $\forall x(x \in X)$, that every object belongs to X.
(iv) If X and Y are distinct classes, then $\forall x(x \in X)$ and $\forall x(x \in Y)$ are distinct propositions.
(v) From (iii) and (iv) it follows that there is a one-to-one mapping from classes of objects to propositions. Hence, there are at least as many propositions as there are classes of proposition.
(vi) (ii) and (v) yield a contradiction.

However, this version of the antinomy presupposes a theory of classes. We want to prove a contradiction in the Fregean theory of sense and denotation, without assuming a theory of classes. Instead of the above paradox, we are looking for a paradox that is stated in terms of the Fregean notions of objects, senses and propositions. Roughly, we want to prove something of the following kind:

(i) There are more concepts (senses), than there are objects.
(ii) There are at least as many objects as there are senses.

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[13] See Klement (2002) for a detailed discussion of Frege's reaction to Russell's attempt to derive a contradiction in the theory of sense and denotation. Frege did not see Russell's new paradox (the Russell-Myhill paradox) as a threat to his logic and philosophy of language, since (i) it can not be formalized in the language of the *Grundgesetze*; and (ii) Russell's way of formulating the paradox, in a logical notation of his own, made Frege think that Russell's derivation of a contradiction was based on equivocations and misunderstandings.

[14] Klement (2002), see also Klement (2001).

Next, we are going to show that a contradiction of the desired kind, a version of the Russell-Myhill antinomy, can be derived in a extension $BFIL^+$ of BFIL. The system $BFIL^+$ appears to be sound from the point of view of Frege's theory of sense and denotation. We are going to analyze which of the assumptions behind $BFIL^+$ that are responsible for the contradiction.

The intuitive reasoning behind the paradox is perhaps best presented in the form of an informal cardinality argument:

(i) From Russell's paradox (which is, in essence, Cantor's diagonal proof that card($\wp$(A)) > card(A)), we know that the assumption that there exists a one-to-one mapping from properties of objects to objects leads to a contradiction. So, intuitively speaking, there are more properties of objects than there are objects.

(ii) For each property of objects F, there is a corresponding predicate concept C_F that is true of an object x iff x has the property F. Hence, there are at least as many predicate concepts (i.e., concepts of properties of objects) as there are properties of objects.

(iii) From (i) and (ii) it follows that there are more predicate concepts than there are objects.

(iv) On the other hand, for each predicate concept f, there is the proposition that the concept f is satisfied by all objects. Let Π(f) be this proposition.

(v) However, if f and g are distinct concepts, then it seems intuitively obvious that the proposition that *f is true of every object* and the proposition that *g is true of every object* are distinct propositions. So, we assume that:

$$\text{for all predicate concepts } f, g,\ \Pi(f) = \Pi(g) \rightarrow f = g.$$

Intuitively, this means that the mapping Π from predicate concepts to propositions is one-to-one.

(vi) Propositions are objects, so by (v) there is a one-to-one mapping from predicate concepts to objects.

(vii) Hence, there are at least as many objects as there are predicate concepts.

(viii) From (iii) and (vii) we obtain a contradiction.

Let us now replace this informal cardinality argument with an argument that can be formalized in an extension of the language L of our Basic Fregean Intensional Logic (BFIL). First, we add a new symbol Π to the language of BFIL together with the new formation rule:

(R13) for any unary function term Φ, $\Pi(\Phi)$ is a singular term.

In this way, we obtain a formal language L^+ with the formation rules (R1)-(R13).

Our intuitive interpretation of Π is as follows: For any (first-level) predicate concept f, $\Pi(f)$ is the proposition that the concept f is true of all objects. Thus, if f is a concept of the property F, then $\Pi(f)$ is true iff $F(x)$ is true for every object x. Intuitively, Π is the concept expressed by the universal quantifier, so we can safely assume that there is such a concept.

Next, we add a new axiom to the formal system BFIL:

$$\text{(A12)}\quad \forall f \forall g \forall F \forall G[\Delta(f, F) \wedge \Delta(g, G) \rightarrow [\Pi(f) = \Pi(g) \rightarrow \forall x(F(x) \leftrightarrow G(x))]].$$

(A12) says that if f and g are concepts of the properties F and G, respectively, and the propositions $\Pi(f)$ and $\Pi(g)$ are identical, then the properties F and G are extensionally equivalent.

The intuitive motivation of (A12) is the following. Suppose that f is a concept of F, g is a concept of G, and F and G are not extensionally equivalent. Clearly, then, f and g must be distinct concepts. Consider now the two propositions:

$\Pi(f)$: the proposition that the concept f is true of every object.
$\Pi(g)$: the proposition that the concept g is true of every object.

If $f \neq g$, then it seems possible to have one epistemic attitude (for instance, belief) towards $\Pi(f)$, without having the same attitude towards $\Pi(g)$. But, for this to be possible, $\Pi(f)$ and $\Pi(g)$ have to be distinct propositions.

We let $BFIL^+$ be the system that is obtained from BFIL by extending the language from L to L^+ and by letting the axioms of $BFIL^+$ be the axioms of BFIL, together with all instances in L^+ of axiom schemata in BFIL, together with the new axiom (A12).

Next, we proceed to give an informal proof of a contradiction from the axioms of $BFIL^+$. It should be clear that our informal proof can actually be formalized in the formal system $BFIL^+$.

Let us say that an object x is *irreflexive* iff there is a predicate concept f such that $x = \Pi(f)$ and x does not satisfy f. That is, x is irreflexive iff, for some concept f, (i) x is the proposition that f is true of every object; and (ii) f is not true of x. In symbols, this becomes:

$$\text{(Def. Ir)}\qquad Ir(x) \leftrightarrow \exists f \exists F(\Delta(f, F) \wedge x = \Pi(f) \wedge \neg F(x)),$$

Here, 'Ir' means irreflexive and F is the property designated by f. It follows from the comprehension schema for properties that the property Ir exists. Thus,

$$\mathrm{Ir} = [\lambda x \exists f \exists F(\Delta(f, F) \wedge x = \Pi(f) \wedge \neg F(x))].$$

Let, **Ir** be the sense of the predicate term defining Ir:

$$\mathbf{Ir} = <[\lambda x \exists f \exists F(\Delta(f, F) \wedge x = \Pi(f) \wedge \neg F(x))]>,$$

i.e., **Ir** is *the concept of an object being irreflexive*.

By (A9), we have:

(3) $\forall F(\Delta(\mathbf{Ir}, F) \leftrightarrow F = \mathrm{Ir})$.

We let

$$\mathbf{r} = \Pi(\mathbf{Ir}),$$

i.e., **r** is the proposition that the concept of being irreflexive is true of all object.

Next, we want to prove that:

(*) $\mathrm{Ir}(\mathbf{r}) \leftrightarrow \neg\mathrm{Ir}(\mathbf{r})$.

Suppose that Ir(**r**), i.e., Ir(Π(**Ir**)). It follows, by (Def. Ir) that,

(4) $\exists f \exists F[\Delta(f, F) \wedge \Pi(\mathbf{Ir}) = \Pi(f) \wedge \neg F(\Pi(\mathbf{Ir}))]$,

Using existential instantiation and the fact that Δ(**Ir**, Ir) (which we derive from (3)), we get:

(5) $\Delta(f, F) \wedge \Delta(\mathbf{Ir}, \mathrm{Ir}) \wedge \Pi(\mathbf{Ir}) = \Pi(f) \wedge \neg F(\Pi(\mathbf{Ir}))$.

From (5) using (A12) we get:

(6) $\forall x(F(x) \leftrightarrow \mathrm{Ir}(x))$.

(5) and (6) yield:

(7) $\neg\mathrm{Ir}(\Pi(\mathbf{Ir}))$, (i.e., $\neg\mathrm{Ir}(\mathbf{r})$).

Thus, we have proved:

(8) $\mathrm{Ir}(\mathbf{r}) \rightarrow \neg\mathrm{Ir}(\mathbf{r})$.

To prove the other direction of (*), we assume ¬Ir(**r**), i.e., ¬Ir(Π(**Ir**)). (Def. Ir) yields:

(9) $\forall f \forall F[\Delta(f, F) \wedge \Pi(\mathbf{Ir}) = \Pi(f) \rightarrow F(\Pi(\mathbf{Ir}))]$.

From (9), we get by predicate logic:

(10) $\Delta(\mathbf{Ir}, \mathrm{Ir}) \wedge \Pi(\mathbf{Ir}) = \Pi(\mathbf{Ir}) \rightarrow \mathrm{Ir}(\Pi(\mathbf{Ir}))$.

Since we have Δ(**Ir**, Ir), we finally get:

(11) $\mathrm{Ir}(\Pi(\mathbf{Ir}))$.

Thus, we have also proved:

(12) $\neg \mathrm{Ir}(\mathbf{r}) \to \mathrm{Ir}(\mathbf{r})$.

(8) and (12) yield the contradictory conclusion (*).

Since this derivation can easily be formalized in BFIL^{+}, it follows that that system is inconsistent.

7. The Epimenides paradox

It is now time to turn to the Epimenides paradox, a version of the Liar paradox applied to propositions rather than sentences.[15] Suppose that one and only one sentence is engraved on Epimenides' tomb, namely:

(Λ) No proposition expressed by a sentence engraved on Epimenides' tomb is true.

Let λ be the proposition expressed by the sentence Λ and let 'Q' be a predicate constant denoting the property (Fregean concept) of being a proposition expressed by a sentence engraved on Epimenides' tomb. We then have:

(1) $\lambda = <\forall x(Q(x) \to \neg \mathrm{True}(x))>$

(2) $Q(\lambda)$

(3) $\forall x(Q(x) \to x = \lambda)$

i.e., (1) λ is the proposition that no object having the property Q is true; (2) λ itself has the property Q; and (3) λ is the only object having the property Q. As the above story shows, these assumptions are intuitively consistent.

However, we can derive a contradiction from (1) - (3):

(4)	$\mathrm{True}(\lambda)$	assumption
(5)	$\mathrm{True}(<\forall x(Q(x) \to \neg \mathrm{True}(x))>)$	1, 4, logic of identity
(6)	$\forall x(Q(x) \to \neg \mathrm{True}(x))$	5, by the (E) schema
(7)	$\neg \mathrm{True}(\lambda)$	2, 6, predicate logic
(8)	$\bot$	4, 7, ⊥-introduction
(9)	$\neg \mathrm{True}(\lambda)$	4–8, ¬-introduction, 4 is cancelled
(10)	$\neg \mathrm{True}(<\forall x(Q(x) \to \neg \mathrm{True}(x))>)$	1, 9, logic of identity
(11)	$\neg \forall x(Q(x) \to \neg \mathrm{True}(x))$	10, the (E) schema.
(12)	$\exists x(Q(x) \wedge \mathrm{True}(x))$	11, predicate logic
(13)	$\mathrm{True}(\lambda)$	3, 12, predicate logic
(14)	$\bot$	9, 13, ⊥ introduction.

[15] There are, of course, countless variants of the Epimenides paradox for propositions. For instance, C. Anthony Anderson (1987) uses the following Liar sentence: 'Church's favorite proposition is not true'. The assumption is then made that the proposition expressed by this sentence happens to be Church's favorite proposition.

This derivation of a contradiction from the intuitively consistent assumptions (1)-(3) can easily be turned into a formal derivation in BFIL.

The Epimenides paradox is simpler than the Russell-Myhill antinomy, since it can be stated in the Basic Fregean Intensional Logic, BFIL. Moreover, it does not depend on any assumptions about the individuation of Fregean senses. On the other hand, it is based on the assumption that situations like the one described in the paradox do occur, or at least are possible.

8. A way out?

Let us return to the Russell-Myhill antinomy. Intuitively, the following assumptions built into the system $BFIL^+$ are responsible for the contradiction:

(i) There is a domain of absolutely all objects over which the individual variables of L^+ range.
(ii) Every formula $\varphi(x)$ of L^+, with x as its only free variable, defines a property of objects.
(iii) For every property P of objects, there is a predicate concept f, which is a concept of that property.
(iv) For every predicate concept f, there is the proposition $\Pi(f)$ that f is true of every object.
(v) Propositions are objects.
(vi) Properties (of objects) are defined for all objects.
(vii) If f and g are distinct concepts, then $\Pi(f)$ and $\Pi(g)$ are distinct propositions.

By denying any of these assumptions, we can avoid the contradiction. One way to avoid the Russell-Myhill antinomy is to give up the assumption (vii). This assumption would not hold if we identified concepts (Fregean senses) with Carnapian intensions, i.e. functions from possible worlds to extensions. If we let f be the intension of ‘odd natural number’ and g the intension of ‘even natural number’, then ‘everything is an odd natural number’ and ‘everything is an even natural number’ are both necessarily false. Hence, they have the same Carnapian intension. This only shows, however, that Fregean senses cannot be identified with Carnapian intensions. Carnapian intensions are too coarse grained to play the role of Fregean senses. (vii) is a consequence of the principle of maximum distinction,[16] and it is needed for explaining failures of substitutivity in propositional attitude contexts. So (vii) is not an assumption that a Fregean philosopher would like to give up.

[16] Cf. Section 1 above.

The assumptions (ii)-(iii) also seem indispensable from a Fregean perspective. These assumptions are needed in order for every well-formed expression of L^+, or any extension of L^+ with additional constant symbols for objects, properties, relations, or functions, – to have both a sense and a denotation. Assumption (v) seems hard to deny once we admit propositions as entities and understand what Frege means by an object.

So let us look at the remaining assumptions (i) and (vi). One interesting alternative would be to give up (vi), but keep the assumption (i). Presumably, that would mean abandoning classical logic in favor of a logic with partially defined predicates. There are such approaches in the literature, but we are not going to consider them here. Instead, we want to see what can be accomplished within the confines of classical logic.

8.1. The paradoxes as diagonal arguments. Now, we want to focus on the remaining assumption, i.e., the assumption (i) that there is a domain of absolutely all objects for the individual quantifiers to range over. That there is a logical difficulty involved in this assumption was first stated by Bertrand Russell:

> If m be a class of propositions, the proposition "every m is true" may or may not be itself an m. But there is a one-one relation of this proposition to m: if n be different from m, "every n is true" is not the same proposition as "every m is true." Consider now the whole class of propositions of the form "every m is true," and having the property of not being members of their respective m's. Let this class be w, and let p be the proposition "every w is true". If p is a w, it must posses the defining property of w; but this property demands that p should not be a w. On the other hand, if p be not a w, then p does possess the defining property of w, and therefore is a w. Thus, the contradiction appears unavoidable.
> ...The totality of all logical objects, or of all propositions, involves, it would seem, a fundamental logical difficulty. What the complete solution of the difficulty may be, I have not succeeded in discovering; but as it affects the very foundations of reasoning, I earnestly commend the study of it to the attention of all students of logic. [17]

In his later work, within the framework of the ramified theory, Russell resolves the paradox by denying the existence of a totality of all propositions that one can quantify over:

> Whatever we suppose to be the totality of propositions, statements about this totality generate new propositions which, on pain of contradiction, must lie outside this totality. It is useless to enlarge the totality, for that equally enlarges the scope of statements about the totality. Hence, there must be no totality of propositions, and "all propositions" must be a meaningless phrase.[18]

[17] Russell (1903), Appendix B, p. 527-528.
[18] Russell (1908), (p. 154 in van Heijenoort (1967)).

Russell's idea is to look upon the paradoxes as *diagonal arguments*. Adapting the idea to Frege's ontologi, the idea is that every domain of entities of some kind (in particular, any domain of objects) is extendible, in the sense that there will always be entities of the given kind that cannot, on pain of contradiction, belong to the given domain. The paradoxes that we have considered so far can all be viewed in this way as diagonal arguments. Consider, first, Russell's paradox for classes. Suppose that U is any domain of objects and that every property F of objects in U, has an extension $\{x \in U: F(x)\}$ such that:

$$(\forall x \in U)[x \in \{x \in U: F(x)\} \leftrightarrow F(x)].$$

Consider, now, the Russell class for U:

$$r_U = \{x \in U: F(x)\}.$$

By Russell's argument, r_U cannot, on pain of contradiction, be a member of U. Of course, if we assume that U is a domain containing absolutely all objects, including all classes, we get a contradiction. But we can also view the argument as a proof that there is no such domain.

Next, let us consider the Russell-Myhill paradox. Suppose that U is any domain of objects. Let Ir_U be the property of an object in U being non-reflexive. $\mathbf{Ir}_U$ is the corresponding concept, and $\mathbf{r}_U = \Pi(\mathbf{Ir}_U)$ is the proposition that every object in U is irreflexive. Now, for every $x \in U$,

$$Ir(x) \leftrightarrow \exists f \exists F(\Delta(f, F) \wedge x = \Pi(f) \wedge \neg F(x)).$$

So, if $\mathbf{r}_U$ were in U, we would have:

$$Ir(\mathbf{r}_U) \leftrightarrow \neg\, Ir(\mathbf{r}_U).$$

Hence, we conclude that $\mathbf{r}_U$ does not belong to U. Thus, U cannot contain every object. So, this paradox too is viewed as a diagonal argument for the conclusion that there is no domain of absolutely all objects.

Finally, let us consider the Epimenides paradox. Consider again the inscription on Epimenides´ tomb:

(Λ) No proposition expressed by a sentence engraved on Epimenides' tomb is true.

Now, we should ask ourselves what domain of objects U it is that is presupposed in Λ. Thus, Λ will express different propositions for different domains U. So let us write λ_U for the proposition that Λ expresses relative to the domain of quantification U. On the assumption that λ_U belongs to U, we can prove:

$$True(\lambda_U) \leftrightarrow \neg True(\lambda_U).$$

Thus, λ_U cannot belong to U. Thus, we have still another proof that there is no domain of absolutely all objects. A paradoxically sounding corollary is that there cannot be any absolutely general propositions.

Our conclusion from the three paradoxes is the same: No domain of objects is universal.[19] For any domain, there are objects that cannot, on pain of contradiction, belong to that domain.[20]

8.2. A hierarchical approach. Now, one might ask whether Frege's ontology and theory of sense and denotation can be modified in such a way that the assumption of a universal domain of objects is given up. It may seem natural then to look for some kind of hierarchical approach. Here, I am going give a rough sketch of what *one* such approach might look like and what its philosophical motivation might be.

Frege's views about intensional entities, like propositions (thoughts) and concepts (senses), seem to be the expression of an extreme form of Platonic realism. Propositions, concepts, together with classes and other abstract entities, are thought of as existing in a "third realm", quite independently of human cognitive or linguistic activities. From this point of view, it seems natural to assume the existence of a fixed *universal domain* U containing absolutely all objects including concrete objects in the physical world, abstract extensional objects like sets, numbers and truth-values, as well as abstract intensional objects like propositions and singular concepts. A Platonic realist of this kind is also committed to the universal applicability of classical two-valued logic. Frege, of course, endorsed what we might call the *Fregean conception of sets* as extensions of properties governed by the above axioms (F1)-(F3). As we have learned from Russell's paradox, the Fregean conception of sets is inconsistent. A Platonic realist is not willing to give up classical logic, so a Platonic realist about sets who is also committed to the Fregean conception of sets is in deep trouble. Fortunately, there is a competing conception of sets that is apparently consistent, namely

[19] See Glanzberg (to appear) for a defense of the claim it is impossible to quantify over absolutely everything.

[20] There is the fairly obvious objection that these statements are self-defeating, since they appear to presuppose quantification over absolutely all domains and all objects. I believe that the objection can be met though, perhaps, by considering statements like 'any domain can be expanded', not as ordinary quantifications realistically construed but rather, along intuitionistic lines, as expressing *rules*: for any domain D, one can construct a more comprehensive domain. However, I am not going to discuss this issue further here. See also Glanzberg's (to appear) discussion of this objection.

Zermelo's *iterative conception of sets*, that yields a conceptual foundation for Zermelo-Fraenkel set theory.

The question now arises whether there is an iterative conception of propositions and their constituents that is comparable to Zermelo's iterative conception of sets. Perhaps, we could define a cumulative hierarchy of intensional entities comparable to the cumulative hierarchy of sets, or perhaps, more like the hierarchy of constructible sets. These two hierarchies usually referred to as V and L, respectively, differ with respect to their respective modes of defining, at a given stage, the sets at the next stage from the old ones. In the cumulative hierarchy, one uses the full power set operation to define the new sets, thereby referring to all sets belonging to the full universe of sets, V, that are subsets of the given stage. This procedure is, obviously, impredicative. When defining the various stages of the constructible hierarchy, on the other hand, one is only allowed to quantify over and refer to such sets that are obtained previously in the construction of the hierarchy. In this respect, the constructible hierarchy of sets is analogous to Russell's (1908) ramified theory of types, where (i) the propositions are divided into orders; and (ii) a propositional quantifier never ranges over the totality of all propositions, but only over the propositions of some given order; and (iii) a proposition that involves quantification over the propositions of a given order will itself be of higher order than the propositions in the domain of quantification.

Now let us see how we can construct a hierarchy of Fregean entities in a way that is analogous to the construction of the constructible sets. First, we define a hierarchy of objects of different order α, where α is an ordinal. We let:

(1) D_0 = the domain of all extensional objects.
(2) $D_{n+1} = D_n \cup$ {x: x is a mode of presentation (sense) of an object of domain D_n that is defined only by reference to items of level α}.
(3) $D_\delta = \cup_{\alpha<\delta} D_\alpha$ for δ a limit ordinal.

The definition of $D_{\alpha+1}$ should be understood in such a way that a proposition or concept of order α can only refer to or quantify over objects of order $< \alpha$. In other words, no impredicative propositions or concepts are allowed.

This definition of a hierarchy of propositions and concepts is analogous to the definition of the sets in Gödel's hierarchy of constructible sets:[21] When defining a new set at a certain level $\alpha+1$ of the constructible hierarchy one allows in the definiens only quantifiers ranging over the domain D_α and parameters (constants) belonging to D_α. Similarly, a sense or "a mode of presentation" at order

[21] See, for instance Devlin (1977).

α+1 should only quantify over or refer to objects that are given previously, i.e., that belong to D_α.

If D_α is the domain of all objects of a certain order α, then n-ary functions of order α are functions from $(D_\alpha)^n$ to D_α and n-ary attributes of order α are functions from $(D_\alpha)^n$ to $\{\mathbf{t}, \mathbf{f}\}$. That is, functions and attributes of order α are not defined for objects outside of D_α. The order of an item is the smallest ordinal α such that the item occurs at order α.

It is important to notice that, according to this approach, there is no domain (totality) of absolutely all objects. Consider now the concept relation Δ between objects. This relation does not exist as one single relation in the hierarchy. Instead, we have for each $\alpha > 1$, a concept relation Δ^α of order α that holds between two objects x, y in D_α iff x is a mode of presentation in D_a of y. For this to hold, the order of x must be greater than the order of y.

There is also no single notion of truth that is applicable to all propositions. Instead, we have the following notions of truth and falsity for each order $\alpha > 1$,

$$\mathrm{True}_\alpha(x) =_{\mathrm{df.}} \Delta^\alpha(x, T)$$
$$\mathrm{False}_\alpha(x) =_{\mathrm{df.}} \Delta^\alpha(x, \perp).$$

If φ is a sentence expressing a proposition of order α, then:

$$\mathrm{True}_\alpha(<\alpha>) \leftrightarrow \alpha$$

The Russell-Myhill paradox is no longer a threat to the theory of sense and denotation. There is not one property of an object x being irreflexive, instead we have for each $\alpha > 1$, a property Ir_α such that for all $x \in D_\alpha$,

$$\mathrm{Ir}_\alpha(x) \leftrightarrow \exists f \exists F(\Delta^\alpha(f, F) \wedge x = \Pi(f) \wedge \neg F(x)),$$

where f is a function variable of order α and F a predicate variable of order α.

Consider now the concept $\mathbf{Ir}_{\alpha+1}$ of Ir_a such that $\Delta^{\alpha+1}(\mathbf{Ir}_{\alpha+1}, \mathrm{Ir}_a)$ and the proposition $\Pi(\mathbf{Ir}_{\alpha+1})$ that says that $\mathbf{Ir}_{\alpha+1}$ is true of all objects of order α. However, $\mathbf{r}_{\alpha+1}$ is a proposition of level α+1, so the property Ir_a is not defined for $\mathbf{r}_{\alpha+1}$. In other words, the Russell-Myhill Paradox does not arise. The Russell Paradox and the Liar Paradox are treated analogously.

Of course, this is only a very rough sketch of a theory of well-founded propositions and concepts. Maybe, this idea is worth pursuing. On the other hand, one might feel the restriction to well-founded propositions to be overly confining. However, the discussion of self-referential, or circular, senses and propositions will have to await another occasion.*

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