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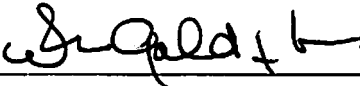## Graduate School of Arts and Sciences

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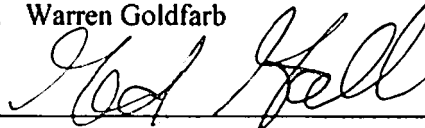
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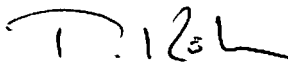
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Typed name: Prof. Warren Goldfarb

Signature _____

Typed name: Prof. Ned Hall

Signature _____

Typed name: Prof. Peter Koellner

Signature _____

Typed name: Prof.

Date: March 21, 2012

Topics in Philosophical Logic

a dissertation presented

by

Jon Erling Litland

to

The Department of Philosophy

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

in the subject of

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Warren Goldfarb   Jon Erling Litland

Topics in Philosophical Logic

Abstract

In "Proof-Theoretic Justification of Logic", building on work by Dummett and Prawitz, I show how to construct use-based meaning-theories for the logical constants. The assertability-conditional meaning-theory takes the meaning of the logical constants to be given by their introduction rules; the consequence-conditional meaning-theory takes the meaning of the logical constants to be given by their elimination rules. I then consider the question: given a set of introduction (elimination) rules $\mathcal{R}$, what are the strongest elimination (introduction) rules that are validated by an assertability (consequence) conditional meaning-theory based on $\mathcal{R}$? I prove that the intuitionistic introduction (elimination) rules are the strongest rules that are validated by the intuitionistic elimination (introduction) rules. I then prove that intuitionistic logic is the strongest logic that can be given either an assertability-conditional or consequence-conditional meaning-theory.

In "Grounding Grounding" I discuss the notion of grounding. My discussion revolves around the problem of *iterated grounding-claims*. Suppose that Δ grounds ϕ; what grounds that Δ grounds that ϕ? I argue that unless we can get a satisfactory answer to this question the notion of grounding will be useless. I discuss and reject some proposed accounts of iterated grounding claims. I then develop a new way of expressing grounding, propose an account of iterated grounding-claims and show how we can develop logics for grounding.

In "Is the Vagueness Argument Valid?" I argue that the Vagueness Ar-

gument in favor of unrestricted composition isn't valid. However, if the premisses of the argument are true and the conclusion false, mereological facts fail to supervene on non-mereological facts. I argue that this failure of supervenience is an artifact of the interplay between the necessity and determinacy operators and that it does not mean that mereological facts fail to depend on non-mereological facts. I sketch a deflationary view of ontology to establish this.

Contents

For my father

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Acknowledgments

While writing this thesis, the Harvard Philosophy Department was my intellectual home; it was here that I came of age as a philosopher.

I owe a great debt to Warren Goldfarb for suggesting that I look into Dummett's proof-theoretic justifications of logical laws. Untold false starts later, I've finally settled all the technical questions I started out with in my second year paper—and more. Many thanks also to Peter Koellner for forcing me to think hard about what the philosophical significance of this work is. Although none of the technical work in this dissertation directly relates to his own work, my debt to Koellner's seminars in the philosophy of mathematics, mathematical logic and set theory is enormous.

For the more metaphysical papers, my largest debt is to Ned Hall. The paper on the vagueness argument started out as a two-page note. Many patient discussions later the result is the chapter in the dissertation. For discussions about this paper I'm also very grateful to Bryan Pickel.

I've been interested in grounding, and the logic of grounding, for many years, but my attempts were not successful. In spring 2011 I came up with the idea behind the logics of ground which I sketch in this paper. Apart from my committee, I here owe special thanks to Susanna Siegel and the other members of the job-market seminar at Harvard for reading and commenting on numerous drafts. For their patient and insightful comments on earlier versions of this paper I'm very grateful to Shamik Dasgupta and Michael Raven. Most of all, thanks to Louis deRosset for lengthy and insightful exchanges about the logic of ground.

Needless to say, any errors remain my own.

On a personal note, this dissertation would not have existed without my family—Na'ama, Omri and Ingunn.

Part I

Proof-theoretic Justification of Logic

1

Intuitionistic Rules

1.1 INTRODUCTION

According to Dummett (see e.g., his 1973, 1976, 1978, 1991), there are two
points where our linguistic practice makes contact with the extralinguistic
world. First, we *verify* statements on the basis of observation; second, we
draw ultimate conclusions from our statements—conclusions which have
consequences for action. For Dummett our practice of verifying statements
and our practice of drawing ultimate conclusions from statements are fea-

tures which are *manifest* in our use of language. Dummett suggests that each of these aspects of use can be the central notion of a meaning-theory; the former leading to 'verificationist' meaning-theory, the latter to a 'pragmatist' meaning theory. Such meaning-theories would not be subject to the manifestation argument.

In a verificationist meaning theory the meaning of a statement ϕ is specified by laying down what counts as a *canonical* verification of ϕ. In a pragmatist meaning theory the meaning of a statement ϕ is specified by specifying which ultimate consequences can be canonically derived from ϕ.[1]

Central to Dummett's philosophy is the concern that these two aspects of use have to be in *harmony*. Our actual usage can be criticized if they're not. For instance, our actual usage can be criticized if we draw conclusions from our statements which aren't licensed by the meaning conferred on the statements by their verification conditions. Depending on whether we adopt a verificationist or a pragmatist meaning theory we can give the following definitions of harmony:

VERIFICATIONIST-HARMONY Our usage is *harmonious in the verificationist sense iff whenever we take ourselves in a position to conclude ϕ from Γ, any class C of canonical verifications of Γ can be transformed into a canonical verification $f(C)$ of ϕ;

---

[1]What are these ultimate consequences? Formally, we'll model them as atomic sentences, but how are they supposed to be interpreted? One way of interpreting such an atomic sentence is by specifying which actions it justifies us in performing. Prawitz (2007) suggests that we rather treat the meaning of an atomic sentence p as being given by the class C of observations such that if one accepts p one would not be surprised to make any observation in C. As far as which logic turns out to be validated this difference doesn't make a difference.

Pragmatist-Harmony Our usage is *harmonious in the pragmatist sense* iff whenever we take Γ to verify ϕ, then any ultimate conclusions which can be canonically derived from ϕ already can be canonically derived from Γ.

It is an enormous task to construct verificationist (pragmatist) meaning-theories for a sizable fragment of natural language; it is an even larger task to verify whether the two aspects of our language are in harmony. Discussions of harmony have therefore often focused on the simple case of logic. A main reason for this is that if we formulate logic in natural deduction style we may take the introduction rules for the connectives to determine what counts as canonical verifications; similarly we may take what counts as canonical consequences of a formula ϕ to be determined by the elimination rules for the connectives in question. The apparatus of (structural) proof-theory can then be used to give precise accounts of harmony.

For verificationist meaning-theory the leading idea goes back to Gentzen:

> the introductions represent, as it were, the "definitions" of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. (Gentzen, 1935, p. 80)

Since the meaning of a logical constant is taken to be conferred by the introduction rules for that constant, we'll dub the account of harmony which results from this *introduction (i)-harmony*.[2]

What does it mean to say that the introduction represent the "definitions" of the connectives? On the verificationist approach the meaning of a sentence is given by specifying the class of canonical proofs of that sentence; the basic

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[2]Dummett calls introduction-harmony the 'upwards justification procedure'.

idea—refined and made rigorous in § 1.3—is that a canonical proof of a sentence with a logical connective λ as the dominant operator ends with an application of an introduction rule for λ.[3]

A pragmatist meaning theory, on the other hand, would take the meaning of the logical constants to be given by their elimination rules, with the introduction rules as "consequences". Since we now take the elimination rules for granted we'll dub this approach *elimination (e) harmony*. Dummett puts the guiding idea as follows:

> The underlying principle of the inverse procedure will be that an argument is valid if any ultimate consequence that can be drawn in a canonical manner from the conclusion can already be drawn in a canonical manner from the premisses. (Dummett, 1991, p. 281)

The bulk of this paper consists in the rigorous development of both verificationist and pragmatist meaning-theories. Whereas broadly verificationist meaning theories have been studied a fair bit, the present approach is more general and we obtain various completeness results for intuitionistic logic. Pragmatist meaning-theories have not been rigorously developed before (though see Queiroz, 2008); Dummett's sketch (ch. 13 Dummett, 1991) does not work (see § 1.4.6). The investigation culminates in the result that, in a precise sense, intuitionistic logic is the strongest logic that can be validated by either a verificationist or a pragmatist meaning-theory. In particular, classical logic isn't harmonious. This goes significantly beyond previous work where one, by and large, have been content to note that this or that formulation of classical logic is inharmonious.

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[3]For a lengthy discussion of the philosophical underpinnings of this, the discussion of the "fundamental assumption" in (Dummett, 1991, ch. 12) is invaluable.

Before I embark on the technical details, some discussion of the philosophical importance of the technical results are in order.[4]

1.2 PHILOSOPHICAL REMARKS

What's the philosophical importance of the technical results to come?

First, we can construct pragmatist meaning-theories, that is, meaning-theories based on elimination rules! This is of some importance for the project of developing "anti-realist" semantics. Verificationist meaning-theories run into a lot of trouble with empirical discourse where such verification as there is always is defeasible. Maybe pragmatist meaning-theories can be made to work better. The technical results here at least show that there is no problem in developing pragmatist meaning-theories for logic.[5]

Secondly, for some connectives it's quite plausible that their introduction rules are meaning-constitutive; this is quite plausible for conjunction and disjunction. This is not very plausible for the conditional and negation. The introduction-rule for the conditional is conditional proof. If we can derive ψ from ϕ, then we can, discharging the assumption ϕ, conclude that if ϕ then ψ. Certainly, we are often willing to assert sentences of the form "if ϕ then ψ" when we're not in position of a proof of ψ conditional on ϕ. A verificationist meaning-theory has to show that we can account for this use in terms of the meaning-constitutive rule of conditional proof. This is not straightforward.[6]

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[4]I will limit myself to some brief remarks. I plan to deal with some of these points at greater length elsewhere.

[5]See also (Prawitz, 2007).

[6]For a thorough discussion of these problems see the discussion of the "fundamental

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Thirdly, if one thought that any connective which did deserve the honorific "logical" had to be governed by what I call general introduction rule or elimination rules.[7] then the results of part 2 show that intuitionistic logic is the strongest possible logic.

Fourthly, even if one doesn't think that a logical connective *has* to be governed by general introduction rules, the proof-theoretic meaning-theories can play an important indirect role. A point Dummett returns to often (see e.g., Dummett, 1978, 1991) is that which logic is validated by a meaning-theory can be very sensitive to the logic taken to govern the meta-language. What you put in, is what you get out.[8] The proof-theoretic meaning-theories sketched here do not have this feature: even if we take the meta-language to be governed by classical logic we cannot use the proof-theoretic meaning-theories to validate anything stronger than intuitionistic logic.

Fifthly, this allows the proof-theoretic meaning theories to play a particular rôle in settling the dispute between and adherent of classical logic and the and one of his opponents . Since we can agree about the logic induced by a proof-theoretic meaning-theory while disagreeing about which logic is correct *tout court*, we can agree to rely only on the logic induced by a proof-theoretic meaning-theory when trying to settle the dispute between an adherent of classical logic and one of his opponents.[9]

[7](Tennant, 1987, 1997) holds a position quite close to this.

[8]This is obvious is if the meaning-theory is a Davidsonian truth-theory; it's also the case if we give a meaning-theory for propositional logic in terms of truth-tables.

[9]As Dummett puts it "We took notice of the problem what meta-language is to be used in giving a semantic explanation of a logic to one whose logic is different. A meta-language whose underlying logic is intuitionistic now appears a good candidate for the rôle, since its logical constants can be understood and its logical laws acknowledged, without appeal to any semantic theory and with only a very general meaning-theoretical background. If that is not *the* right logic, at least it may serve as a medium by means which to discuss other logics

Sixthly, the project gives yet another perspective on what the intuitionistic connectives are. What is intutionistic logic the logic of? A good formulation *C* of classical (propositional) logic is complete in the sense that any truth-function can be expressed by means of the connectives used in *C*; we can say that classical logic is the logic of truth-functions. Is there a similar sense in which intuitionistic logic is complete? Well, one idea would be that intuitionistic logic is complete in the sense that any connective which can be given a proof-theoretic meaning-theory is definable in terms of the intuitionistic connectives. In part 2 I make this precise and show that this is, indeed, the case.

Let me end with a caveat. The entire investigation takes place in *single-conclusion*, *unilateral* natural deduction. I have made no attempt to extend the framework to "multiple conclusion" natural deduction (as in Read, 2000).[10] Nor have I made an attempt at extending the framework to a *bilateralist* framework (as in Rumfitt, 2000).[11] It is well known that these frameworks are more friendly to classical logic than the one used here; one might worry that by working in the present framework I've stacked the deck against classical logic. If so, one can take the results of this paper to be *proof* that one has to consider bilateral or multiple conclusion logics if one wants to account for classical logic.

." (Dummett, 1991, p. 300)

[10]For the record, I'm very skeptical that our regular assertoric practice contains anything analogous to multiple conclusions (see e.g., Steinberger, 2011).

[11]For the record, I'm somewhat skeptical of *bilateralism* for the reasons given in (Humberstone, 2000).

1.3 Verificationist Meaning-Theory

The task for a verificationist meaning-theory is to define canonical proofs of complex sentences in terms of canonical proofs of simpler sentences. (Cf. The task for a truth-conditional theory of meaning is to define the truth-conditions of complex sentences in terms of the truth-conditions for simples sentences). In order to get the account going we have to take as given the canonical proofs of atomic sentences. Philosophically, a lot turns on what counts as a canonical argument for an atomic conclusion. The verificationist picture—certainly Dummett's own use of it—suggests that the meaning of an atomic sentence be given by specifying which *observations* would conclusively verify the sentence. A lot is going to turn on *how* those observations can be specified, but we need not be concerned with this here. For our purposes we only need the following formal features:

1. atomic sentences have canonical proofs;

2. any given atomic sentence can have several distinct canonical proofs

3. if one is in possession of a canonical proof π of an atomic sentence p, one might *thereby* be in possession of a canonical proof π' of another atomic sentence q.

Conditions 2 and 3 are there to model the following feature. Consider a transparent goblet and consider the claim that it is transparent. Observing the goblet from angle a_0 (in good viewing conditions, while sober, etc.) is, let's assume, a canonical verification of the claim; observing the goblet from angle a_1 is a different canonical verification of the claim; and so on.

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Furthermore, by making any of these observations I could also be making an observation verifying that the goblet is wider at the top than the bottom.[12]

In order to model these features we'll need the notions of a *verifier* and a *boundary rule*.

Definition 1.3.1. Let p be an atomic sentence. Let V_p be a countably infinite set $\{p^i | i \in \mathbb{N}\}$. Such a p^i is a *verifier* of p; the set V_p is the *set of verifiers of p*. The collection $\{V_p | p$ is an atomic sentence$\}$ satisfies the following condition: if $p \neq q$, we have $V_p \cap V_q = \emptyset$.

For now we'll develop the theory without boundary rules; their need will become apparent later.

Definition 1.3.2.   1. If v is verifier, then v is a formula.

2. p is a formula for each atomic sentence letter p.

3. $\perp$ is a formula.

4. If ϕ and ψ are formulæ, and neither is a verifier, then $\phi \wedge \psi, \phi \vee \psi$ and $\phi \rightarrow \psi$ are formulæ.

In what follows $v_0, v_1, \ldots$ will be variables for verifiers; $p, q, r, \ldots$, will be variables for atomic sentences; $\phi, \psi, \sigma, \ldots$ will be variables for formulæ; $\Gamma, \Delta, \Sigma, \ldots$ will be variables for sets of formulæ.

Definition 1.3.3. A *sequent* is an expression of the form $\Gamma : \phi$ where Γ is a finite multiset of formulæ, and ϕ is a formula which is not a verifier.

---

[12]Could an observation (canonically) verifying that the goblet is transparent also be an observation canonically verifying that the goblet is wider at the top than the bottom? It is formally more convenient to assume that this never happens; rather what happens is that making the first type of observation is correlated with making an observation of the second type. Nothing substantial hinges on this.

Definition 1.3.4.   1. An *argument* is a tree Π of finite height where the nodes are labeled with sequents, each node being assigned a sequent $\Gamma : \phi$. Intuitively, Γ are the formulæ upon which ϕ depends.

2. Let Π be an argument; let $\Gamma : \phi$ be the sequent labelling the *root* of the tree. Then $\Gamma : \phi$ is the *conclusion* of the argument Π.

3. If Π is an argument and $\Pi' \subseteq \Pi$ is a tree, we call Π' a *subargument* of Π.

4. Let Π be an argument. Let $\Gamma : \phi$ be the conclusion of Π. Let $u_0, u_1, \ldots, u_n$ be the top nodes of Π; let $\Gamma_0 : \phi_0, \Gamma_1 : \phi_1, \ldots, \Gamma_n : \phi_n$ label these nodes. We say that

    (a) $\{\psi | \psi \in \Gamma_i, \text{ for some } i \leq n\} \cup \Gamma$ is the set of *premisses* of the argument Π;

    (b) $\Gamma : \phi$ is the *conclusion* of the argument Π

    (c) $\{\psi | \psi \in \Gamma_i, \text{ for some } i \leq n\} \setminus \Gamma$ is the set of *discharged* premisses of Π. Γ is the set of *undischarged premisses* of the argument Π.

Definition 1.3.5. The degree of a formula $\deg(\phi)$ is defined inductively.

1. $\deg(v)$ is 0 for all verifiers v.

2. $\deg(p)$ is 0 for all atomic sentences p;

3. $\deg(\bot)$ is 1.

4. $\deg(\phi \circ \psi) = \max\{\deg(\phi), \deg(\psi)\} + 1$, for $\circ \in \{\wedge, \vee, \rightarrow\}$

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$$\wedge\text{-INTRODUCTION} \quad \frac{\Gamma : \phi \qquad \Gamma' : \psi}{\Gamma,\Gamma' : \phi \wedge \psi}$$

$$\vee\text{-INTRODUCTION} \quad \frac{\Gamma : \phi}{\Gamma : \phi \vee \psi} \qquad \frac{\Gamma : \psi}{\Gamma : \phi \vee \psi}$$

$$\rightarrow\text{-INTRODUCTION} \quad \frac{\Gamma,\phi : \psi}{\Gamma : \phi \rightarrow \psi}$$

$$\perp\text{-INTRODUCTION} \quad \frac{\Gamma_0 : p_0 \qquad \Gamma_1 : p_1 \ldots \Gamma_i : p_i \ldots}{\bigcup \Gamma_i : \perp} \text{, where } p_0, p_1, \ldots \text{ enumerate all}$$

the atomic formulæ.

Figure 1.3.1: Intuitionistic Introduction Rules

The degree of a sequent $\Gamma : \phi$, $(\deg(\Gamma : \phi))$, is the highest degree of a formula occurring in the sequent. The degree of the argument Π is the maximum degree of sequents occurring in Π.

Definition 1.3.6. Let Π be an argument; let u be a node in Π and $u_0, u_1, \ldots, u_n$ be the nodes immediately above u; let $\Gamma : \phi$ and $\Gamma_0 : \phi_0, \Gamma_1 : \phi_1, \ldots, \Gamma_n : \phi_n$ label these nodes. Let $\mathcal{R}$ be a rule of inference. If $\Gamma : \phi$ follows from $\Gamma_0 : \phi_0, \Gamma_1 : \phi_1, \ldots, \Gamma_n : \phi_n$ by the rule $\mathcal{R}$, then we say that the subargument from $u_0, u_1, \ldots, u_n$ to u is an *instance* of rule $\mathcal{R}$.

We'll be working with the intuitionistic introduction rules shown in figure 1.3.1. In the $\rightarrow$-introduction rules we allow *vacuous* and *multiple* discharge. $\frac{\phi : \phi}{\phi : \psi \rightarrow \phi}$ and $\frac{\phi,\phi : \phi}{\phi : \phi \rightarrow \phi}$ are thus instances of $\rightarrow$-introduction.

We can now define *valid canonical argument* and *valid sequent* simultaneously by recursion.

Definition 1.3.7. Let α be a set of verifiers; let Π be an argument with

conclusion $\alpha : \phi$. Π is a *valid canonical argument* iff either

1. Π is the one-place argument $\alpha^-, p^i : p$ for p atomic and $p^i \in V_p$.

2. ϕ is $\perp$; the nodes immediately above α are $\alpha_0 : p_0, \alpha_1 : p_1, \ldots$ where the p_i enumerate all the atomic formulæ and the subarguments Π_i to each $a_i : p_i$ is canonical.

3. ϕ is $\psi_0 \wedge \psi_1$; the nodes immediately above $\alpha : \phi$ are $\alpha_0 : \psi_0$ and $\alpha_1 : \psi_1$ ($\alpha = \alpha_0 \cup \alpha_1$) and the subarguments Π_i to $\alpha_i : \psi_i$ are canonical

4. ϕ is $\psi_0 \vee \psi_1$; the node immediately above $\alpha : \phi$ is $\alpha : \psi_i$, for $i = 0$ or $i = 1$ and the argument Π_i to $\alpha : \psi_i$ is canonical. Or:

5. ϕ is $\psi_0 \rightarrow \psi_1$. In this case the sequent immediately above $\alpha : \phi$ is $\alpha, \psi_0 : \psi_1$ and this sequent is valid.

Definition 1.3.8. A sequent $\Gamma : \phi$ is *valid* iff either

1. it is $\Gamma^-, p^i : p$ for p atomic and $p^i \in V_p$; or

2. for all α, if for all $\psi_i \in \Gamma$ there is a valid canonical argument Π_i with conclusion $\alpha : \psi_i$, then there is a valid canonical argument Π with conclusion $\alpha : \phi$.

Remark 1.3.9. In Dummett's own definition of valid sequent (he calls it "valid argument") there is always the demand that we can "effetely transform" canonical proofs of the premisses into canonical proofs of the conclusion. Prawitz makes the same demand. In the above definitions no such demand is made; I only demand that for all canonical proofs of the premisses *there is* a canonical proof of the conclusion. Does this mean that I don't engage

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with Dummett and Prawitz at all? Not so. Since we're working only with propositional logic it's always decidable whether there is a canonical proof of ϕ from verifiers α. There is no need to specify that the transformation has to be effective.[13]

Observation 1.3.10. *Let Π be a canonical argument; let $\Gamma \colon \phi$ be a sequent occurring in Π. Then ϕ is not a verifier.*

In order to make this more legible, I'm adopting an observation of (Goldfarb, 1999) and will introduce a notion of a set of verifiers α 'forcing a conclusion' ϕ. We'll write $\Vdash$ for this relation.

Definition 1.3.11. Let α be a finite set of verifiers; let ϕ be a sentence. α forces ϕ ('$\alpha \Vdash \phi$') iff there is a canonical argument, Π with conclusion $\alpha \colon \phi$.

Remark 1.3.12. We can now express the definition of valid sequent more compactly as follows.

1. Let ϕ and ψ be formulæ. $\phi \Vdash \psi$ iff for all sets of verifiers α, if $\alpha \Vdash \phi$ then $\alpha \Vdash \psi$.

2. Let Γ be a set of formulæ and ψ be a formula. $\Gamma \Vdash \psi$ iff, for all sets of verifiers α, if $\alpha \Vdash \phi$ for all $\phi \in \Gamma$ then, $\alpha \Vdash \psi$.

Theorem 1.3.13. *Let α be a finite set of verifiers. The following holds.*

1. $\alpha \Vdash p$ iff $p^i \in \alpha$, for some $p^i \in V_p$.

2. $\alpha \Vdash \bot$ for no α.

3. $\alpha \Vdash \phi \wedge \psi$ iff $\alpha \Vdash \phi$ and $\alpha \Vdash \psi$

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[13]Thanks to Dag Prawitz for forcing me to get clear on this.

4. $\alpha \Vdash \phi \lor \psi$ iff $\alpha \Vdash \phi$ or $\alpha \Vdash \psi$

5. $\alpha \Vdash \phi \to \psi$ iff for all $\beta \supseteq \alpha$, if $\beta \Vdash \phi$ then $\beta \Vdash \psi$

Proof. By inspection of the definitions. I'll prove the clauses for $\bot$ and $\to$ as an illustration.

$\bot$: Let α be a finite set of verifiers. Since α is finite, there is an atomic sentence q such that $\alpha \nVdash q$. (We here use that if $p \neq q$ then $V_p \cap V_q = \emptyset$). But then there can be no canonical proof of $\bot$ from α, for if there was we would have $\alpha \Vdash p$ for all atomic p.

$\to$: Suppose that $\alpha \Vdash \phi \to \psi$. Let $\beta \supseteq \alpha$ be given and suppose that $\beta \Vdash \phi$. Let Π be a canonical argument witnessing $\alpha \Vdash \phi \to \psi$, then Π is of the form:

$$\frac{\alpha, \phi \; : \; \psi}{\alpha \; : \; \phi \to \psi}$$

Here $\alpha, \phi \colon \psi$ is a valid sequent. By the definition of valid sequent that means that if $\beta \supseteq \alpha$ is such that $\beta \Vdash \phi$ then $\beta \Vdash \psi$; which was to be shown. Conversely, if for all $\beta \supseteq \alpha$, such that $\beta \Vdash \phi$ we have $\beta \Vdash \psi$ this means that the sequent $\alpha, \phi \colon \psi$ is valid, and hence that we have a canonical argument from α to $\phi \to \psi$. $\qquad\qquad \square$

As the perceptive reader no doubt has noticed, the conditions on $\alpha \Vdash \phi$ in theorem 1.3.13 are exactly parallel to the conditions for a node w to force a sentence ϕ in a Kripke model for intuitionistic logic.

Corollary 1.3.14. *Every instance of an intuitionistic elimination rule is forced; hence, the intuitionistic elimination rules are justified.*

Proof. Immediate from theorem 1.3.13 above. $\qquad\qquad \square$

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Unfortunately, as shown by Goldfarb (1999), the procedure has unacceptable results.

Observation 1.3.15.   *1. If p,q are atomic and p is not $\bot$ then the sequent*

$$p \rightarrow q \ : \ q \ \ \text{is valid.}$$

2. Let p be atomic. Then the argument

$$: \ (p \rightarrow \bot) \rightarrow \bot \ \ \text{is valid}$$

Proof. We prove 1. Suppose $\alpha \Vdash p \rightarrow q$. Then for every $\beta \supseteq \alpha$, if $\beta \Vdash p$ then $\beta \Vdash q$. Suppose that $\alpha \nVdash q$. Then by theorem 1.3.13 above, $q^i \notin \alpha$ for all i. Put $\beta_0 := \alpha \cup \{p^1\}$; then $\beta_0 \supseteq \alpha$ with $\beta_0 \Vdash p$ and $\beta \nVdash q$, contradicting $\alpha \Vdash p \rightarrow q$.

The proof of 2 is left to the reader. □

1.3.1   Boundary Rules

The problem in these counterexamples is that atomic sentences are *logically independent*; if $p^i \notin \alpha$ for all i then $\alpha \nVdash p$ and so the only way for α to force $p \rightarrow q$ is for α to include some verifier of q. As noted in remark 1.3 above, $\Vdash$ mirrors forcing in Kripke models. The problem is that $\Vdash$ corresponds to truth in the particular Kripke model $\mathcal{K}$ where for all atomic sentences p, $\forall w \in \mathcal{K}, \exists w' \in \mathcal{K}, w' \models p$; validity in Kripke semantics corresponds to truth in *all* models. Following Goldfarb, we obtain the effect of quantification over all Kripke models by utilising Dummett's notion of a *boundary rule.* The notion of a "production rule" plays a similar role in (Prawitz, 1971) and he considers similar counterexamples at (Prawitz, 1971, pp. 275–279,285). (See also Schroeder-Heister, 2006, p. 544).

Definition 1.3.16. Let $v_0, v_1, \ldots, v_n, v$ be verifiers. A boundary rule takes the

form $v_0, \ldots, v_n \Rightarrow v$. Let $\mathcal{S}$ be a collection of boundary rules and let α be a collection of verifiers. The closure of α under $\mathcal{S}$, $\mathrm{cl}_{\mathcal{S}}(\alpha)$ is the smallest set of verifiers β such that $\alpha \subseteq \beta$ and if $\{v_0, v_1, \ldots, v_n\} \subseteq \beta$ and $v_0, v_1, \ldots v_n \Rightarrow v$ is a boundary rule in $\mathcal{S}$ then $v \in \beta$.

We now emend the definition of canonical argument and valid sequent to take boundary rules into account. This is most easily expressed using the relation $\Vdash$.

Definition 1.3.17. Let $\mathcal{S}$ be a set of boundary rules and let α be a set of verifiers; let $\alpha_{\mathcal{S}}$ be the closure of α under $\mathcal{S}$.

1. $\alpha \Vdash_{\mathcal{S}} \phi$ iff $\alpha_{\mathcal{S}} \Vdash \phi$

2. $\Gamma \Vdash_{\mathcal{S}} \phi$ iff for all $\mathcal{S}_0 \supseteq S$ and α: if $\alpha \Vdash_{\mathcal{S}} \Gamma$ then $\alpha \Vdash_{\mathcal{S},} \phi$.

Let $\mathcal{S}$ be a set of boundary rules. Let $\mathcal{V}_{\mathcal{S}}$ be the collection of sets of verifiers closed under $\mathcal{S}$. Take $\mathcal{V}_{\mathcal{S}}$ to be partially ordered under inclusion in the obvious way. $(\mathcal{V}_{\mathcal{S}}, \subseteq)$ gives rise to a Kripke-model $\mathcal{K}$ in a natural way: let the nodes of $\mathcal{K}$ be $\{V | V \in \mathcal{V}_{\mathcal{S}}$ such that there is p such that $p^i \notin V$, for all $i\}$; let the ordering $\prec$ of $\mathcal{K}$ be given by $\subseteq$. We define forcing for atomic sentences by $V \models p$ iff $p^i \in V$, for some i. The idea behind the soundness-proof for $\Vdash$ below is to do the converse, i.e., for each Kripke-model $\mathcal{K}$ to construct a set of boundary rules $\mathcal{S}$ such that $(\mathcal{V}_{\mathcal{S}}, \subseteq)$ is isomorphic to $\mathcal{K}$.

To prove soundness for $\Vdash$, we prove that if $\Gamma : \phi$ has a countermodel then $\Gamma \nVdash \phi$.

Definition 1.3.18. Call a Kripke model *pruned* iff there are no superfluous nodes in it, i.e., if a node w verifies all and only the atomic sentences in a

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set X, then there is no node $w' < w$ such that w' also verifies all and only the atomic sentences in X.

Lemma 1.3.19. *Let $\mathcal{K}$ be a countable (finite) Kripke-model: there is a countable (finite) pruned Kripke-model $\mathcal{K}'$ and a map $f : \mathcal{K} \mapsto \mathcal{K}'$ which is onto and order preserving, such that for all $w \in \mathcal{K}$, $w \models p$ iff $f(w) \models p$.*

Proof. Left to the reader. □

Lemma 1.3.20. *Let $\mathcal{K}$ be a countable (finite) Kripke-model; and let w_0 be a node in $\mathcal{K}$. Let $\mathcal{K} \restriction w_0$ be the restriction of $\mathcal{K}$ to the nodes above w_0. Then there is a tree-like Kripke model $\mathcal{K}'$ and an order-preserving map $f : \mathcal{K} \restriction w_0 \mapsto \mathcal{K}'$, such that $w \models \phi$ iff $f(w) \models \phi$.*

Proof. Standard. (By unraveling the model $\mathcal{K} \restriction w_0$.) □

Lemma 1.3.21. *For all pruned, tree-like Kripke-models $\mathcal{K}$ there is a set of boundary rules $\mathcal{S}$ such that*

1. *there is a one-to-one map $f : \mathcal{K} \mapsto \mathcal{V_S}$ such that $w_0 \leq w_1$ iff $f(w_0) \subseteq f(w_1)$; and*

2. *$w \models p$ iff $f(w) \Vdash p$, for all atomic sentences p.*

3. *if $V \in \mathcal{V_S} \neq f(w)$ for all w then $V \Vdash p$, for all atomic p.*

From the lemma we easily prove:

Theorem 1.3.22 (Intuitionistic soundness of $\Vdash$)**.** *If $\Gamma \Vdash \phi$, then $\Gamma \models \phi$.*

Proof. Let $\mathcal{K}$ be such that $\mathcal{K} \models \Gamma$, but $\mathcal{K} \not\models \phi$. We can assume that $\mathcal{K}$ is pruned and tree-like with root w. We find a set of boundary rules $\mathcal{S}$ and a map f

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satisfying the conditions of lemma 1.3.21. The proof is by induction. The only problematic clause is that for $\rightarrow$. We will show that $w \models \psi_0 \rightarrow \psi_1$ iff $f(w) \Vdash \psi_0 \rightarrow \psi_1$.

Right-to-left: suppose that $w_0 \models \psi_0 \rightarrow \psi_1$. Then for all $w \geq w_0$ such that $w \models \psi_0$, $w \models \psi_1$. Now, let $\beta \supseteq f(w_0)$ be such that $\beta \Vdash \psi_0$. By choice of $\mathcal{S}$ and f we know that either $\beta \Vdash p$ for all p, in which case $\beta \Vdash \psi_1$ or else β is $f(w_1)$ for some $w_1 \geq w_0$ and thus it follows by the induction hypothesis that $w_1 \models \psi_0$. But then $w_1 \models \psi_1$ and, again by the induction hypothesis, $f(w_1) \Vdash \psi_1$, i.e., $\beta \Vdash \psi_1$.

Left-to-right: Suppose that $w_0 \not\models \psi_0 \rightarrow \psi_1$; then there is $w_1 \geq w_0$ such that $w_1 \models \psi_0$ but $w_1 \not\models \psi_1$. But then by the induction hypothesis $f(w_1) \Vdash \psi_0$ but $f(w_1) \nVdash \psi_1$. This shows that $f(w_0) \nVdash \psi_0 \rightarrow \psi_1$. $\qquad\qquad\square$

Proof of lemma 1.3.19. Let $\mathcal{K}$ be a pruned tree-like Kripke-model; let $\leq$ be the partial ordering of the nodes of $\mathcal{K}$. We are going to pick verifiers and boundary rules so as to mimic $\mathcal{K}$. Let $w_0, w_1, \dots$ be an enumeration of the nodes of $\mathcal{K}$ such that if $w_i \prec w_j$ then $i < j$. For each node $w_i \in \mathcal{K}$ we define the set $V_i = \{p^k | (k = i \wedge w_i \models p \wedge \neg \exists w \prec w_i, w \models p) \vee (w_i \models p \wedge w_k \prec w_i \wedge p^k \in V_k)\}$. We construct the set $\mathcal{S}$ of boundary rules in the following steps:

1. For each node w_i and each $p^k \in V_i$ such that $w_i \models p$ and for no $w \prec w_i$ do we have $w \models p$: for each $v \in V_i$ add the boundary rule $p^k \Rightarrow v$.

2. For each node w_i and each p, if for *no* $w \geq w_i$ do we have $w \models p$: add the boundary rules $v_0, v_1, \dots, v_n, p^j \Rightarrow q^1$, for each j and for each atomic sentence q (where $v_0, v_1, \dots, v_n$ is a list of the verifiers in V_i). This ensures that $V_i \Vdash p \rightarrow \bot$.

3. For all w_i, w_j if neither $w_i \le w_j$ nor $w_j \le w_i$ then for all verifiers v and v', such that $v \in V_i \setminus V_j$ and $v' \in V_i \setminus V_j$: add the boundary rules $v, v' \Rightarrow q^1$ for each atomic sentence q.

4. Once we have selected boundary rules in this way for each w_i add the following boundary rules. If v is a verifier which is not in any V_i: add the boundary rules $v \Rightarrow q^1$, for all q. (This ensures that we do not have to consider verifiers which are in no V_i.)

5. Suppose that $V_0 \subset V_1 \subset \ldots$ is a chain such that $\bigcup_{i < \omega} V_i$ is not a V_k, then add the boundary rules $\bigcup_{i < \omega} V_i \Rightarrow q^1$ for each atomic q.

Let $\mathcal{S}$ be the collection of all boundary rules so chosen. Let $\mathcal{V}_{\mathcal{S}}^-$ be that subset of $\mathcal{V}_{\mathcal{S}}$ such that $V \in \mathcal{V}_{\mathcal{S}}^-$ such that there is p such that $p^i \notin V$, for all i};

Claim: $\mathcal{V}_{\mathcal{S}}^- = \{V_i | i \in \mathbb{N}\}$. *Proof of Claim.* $\{V_i | i \in \mathbb{N}\} \subseteq \mathcal{V}_{\mathcal{S}}^-$: pick a V_i, it suffices to show that $\text{cl}_{\mathcal{S}}(V_i) = V_i$. We consider each type of boundary rule in turn. If the boundary rule is of the first type, it will allow us to infer $q^k \in V_i$ from $p^i \in V_i$. That does not take us out of V_i. If the rule is of the second type, it allows us to infer any v from p, if p is in no $V_j \supseteq V_i$, but that is not going to take us out of V_i. If the rule is of the fourth form, it will allow us to infer any v from a w which is not in any V_j, but that is not going to take us out of V_i.

$\{V_i | i \in \mathbb{N}\} \supseteq \mathcal{V}_{\mathcal{S}}^-$: let $W = \{w_0, w_1, \ldots\}$ be a set of verifiers. We will show that if $\text{cl}_{\mathcal{S}}(W) \in \mathcal{V}_{\mathcal{S}}^-$ then $\text{cl}_{\mathcal{S}}(W) = V_i$ for some i. Suppose that w_0 does not occur in any V_i. Then the penultimate type of boundary rule applies and we'll have $q^1 \in \text{cl}_{\mathcal{S}}(W)$, for all q; thus $\text{cl}_{\mathcal{S}}(W) \notin \mathcal{V}_{\mathcal{S}}^-$. Similarly for each $v_i, i \ge 0$. For each v_i let V_i be the first V_j in which the verifier v_i occurs. We show that the V_i are pairwise comparable. Suppose otherwise and let V_i and V_j

be such that neither $V_i \subseteq V_j$ nor $V_j \subseteq V_i$. This can only happen if V_i, V_j are associated with two nodes w_i, w_j which are incomparable. But then by the boundary rules of type 3 there is $v_i' \in V_i$ and $v_j' \in V_j$ such that $v_i', v_j' \Rightarrow q^1$ for all atomic q. Since V_i is the first collection of verifiers in which v_i occurs, the boundary rules $v_i \Rightarrow v_i'$ is in $\mathcal{S}$; similarly $v_j \Rightarrow v_j'$ is in $\mathcal{S}$. It then follows that $q^1 \in \mathrm{cl}_\mathcal{S}(v_i, v_j)$ for each atomic q. This shows that $W \notin \mathcal{V}_\mathcal{S}^-$.

It then follows that the closure of W must be a V_k. For if the closure of the union is not identical to the union of the closure this can only be because the last type of boundary rule applies, but then $\mathrm{cl}_\mathcal{S}(W)$ is not in $\mathcal{V}_\mathcal{S}^-$.

We now define $f : \mathcal{K} \mapsto V_\mathcal{S}$ by $f(w_i) = V_i$; by the claim this operation is well-defined and onto. It is one-to-one since $\mathcal{K}$ is pruned, and we have $w_i \leq w_j$ iff $f(w_i) \leq f(w_j)$ by construction. Clearly, $w \models p$ iff $f(w) \Vdash p$. And we have shown that if $\mathrm{cl}_\mathcal{S}(W) \in \mathcal{V}_\mathcal{S}$ is not equal to a V_i then $\mathrm{cl}_\mathcal{S}(W) \Vdash p$, for every atomic p. $\qquad\square$

We have now shown that the only elimination rules which can be justified on the basis of the intuitionistic introduction rules are the intuitionistic elimination rules; more exactly, any elimination rule which can be validated by a verificationist meaning theory based on the intuitionistic introduction rules can be derived from the standard intuitionistic introduction and elimination rules. This was conjectured in (Prawitz, 1973, p. 246) and reiterated in (Prawitz, 2007, p. 476).[14]

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[14]This shows that the so-called "inversion principle" (see e.g., Read, 2000, 2010), when applied to the intuitionistic introduction rules, gives the strongest elimination rules which can be justified by a verificationist meaning theory based on the intuitionistic introduction rules.

1.3.2 ANOTHER APPROACH

What happens if we instead of using verifiers let an atomic sentence be a canonical proof of itself and we let boundary rules govern atomic sentences and not verifiers of atomic sentences? In this case sequents like $p \to q \lor r: (p \to q) \lor (p \to r)$ become valid, as can easily be checked. There is no problem, as such, in getting a different (and stronger) logic than intuitionistic logic being i-harmonious. The more serious problem is that the relation $\Vdash$ now is not closed under substitution. In order to fix this one could simply demand that the relation $\Vdash$ be closed under substitution.[15]

Definition 1.3.23. Let $p_0, p_1, \ldots, p_n$ be all the atomic sentences which occur in Γ, ϕ. Then we say that $\Gamma \Vdash'_S \phi$ iff for all sentences $\sigma_0, \sigma_1, \ldots, \sigma_n$, $\Gamma[\bar{\sigma}/\bar{p}] \Vdash_S \phi[\bar{\sigma}/\bar{p}]$.

Theorem 1.3.24. *If $\Gamma \Vdash'' \phi$, then $\Gamma \models \phi$.*

Proof. Essentially the same proof as the proof of 1.3.22. Consider a countermodel to $\Gamma: \phi$. For each atomic sentence p which occurs in Γ, ϕ we do as follows. Let $v_0, \ldots, v_n$ be all the nodes where the atomic sentence p is forced such that for all i there is no node $u \le v_i$ which also forces p. Add $r_0, \ldots, r_n$ fresh atomic formulæ and substitute $\bigvee r_i$ for p in Γ and ϕ. Do the same for each atomic sentence. The boundary rules now connect the atomic sentences $r_0, r_1, \ldots$ rather than verifiers. □

If the only thing we wanted was to obtain a definition of logical consequence this would be unproblematic; formally, of course, the two approaches are essentially identical. However, this approach is no good if we want to

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[15]This is the approach of (Goldfarb, 1999).

produce a verificationist meaning-theory for the logical constants. Since we now have to consider arbitrary substitutions, what counts as a canonical verification of a sentence ϕ now depends on what counts as canonical verifications of sentences of arbitrarily high complexity. Furthermore, we get the awkward result that even though $p \rightarrow q \vee r : (p \rightarrow q) \vee (p \rightarrow r)$ isn't logically valid, by knowing just that p, q, r are atomic we're entitled to make the inference from $p \rightarrow (q \vee r)$ to $(p \rightarrow q) \vee (p \rightarrow r)$.

1.4 Pragmatist Meaning-Theories

Just as with verificationist meaning-theories, the key is to define the notion of a valid canonical argument. Since the idea behind the pragmatist meaning-theories is that the elimination rules are "self-justifying", an argument which uses only elimination rules will be canonical. But we cannot live by elimination rules alone. Consider an argument like:

$$\frac{\Gamma : (\phi \rightarrow \phi) \rightarrow \psi \qquad : \phi \rightarrow \phi}{\Gamma : \psi}$$

There is no argument to the minor premiss $\phi \rightarrow \phi$ which uses only elimination rules. Just as in the case of verificationist meaning-theories, our definition of 'canonical argument' has to allow for valid sequents to occur in them. That being so, we have to define the notions of canonical argument and valid sequent simultaneously by recursion.

We'll be working in a natural deduction setting, but since we'll need to keep track of where premises are discharged it'll be convenient to do natural deduction in sequent style. In particular, it will be important to keep in mind that our sequents have multisets on the left. We use the

same definitions of *formula*, *sequent*, *argument* and *degree* as in section 1.3. However, we also need a notion of the *order* of a sequent.

Definition 1.4.1. Let $\Gamma : \phi$ be a sequent. The *order* of $\Gamma : \phi$, $o(\Gamma : \phi)$ is the Hessenberg (commutative) sum of the multiset $\left\{ \omega^{\deg(\psi)} | \psi \in \Gamma \text{ or } \psi = \phi \right\}$.

Obviously the order of a sequent is bounded below ω^ω.

Definition 1.4.2. Let Π be an argument with final conclusion $\Gamma : \phi$. The order of Π, $o(\Pi)$, is the order of $\Gamma : \phi$.

1.4.1 Examples

$$\frac{p \wedge p \; : \; p}{p \; : \; p}$$

has degree 1 and order 2.

The argument

$$\frac{p \wedge p \; : \; p \vee q \qquad p \; : \; r \qquad q \; : \; r}{p \wedge p \; : \; r}$$

has degree 1 and order $\omega + 1$. Obviously the order of any argument is bounded below ω^ω.

Remark 1.4.3. The definitions of order and degree for arguments only consider the final sequent. Is this the right definition? For 'degree' one might worry that in the course of a proof Π a sequent will occur which has higher degree than the final sequent (as happens in the first example) or as higher order than the final sequent (as happens in the second example); if so, if taking the degree of Π to be the degree of its final sequent seems to leave out some information about the complexity of Π. However, given the form of the elimination rules we will consider and the definition of canonical

argument to be given below, it is clear that this never can happen: canonical proofs have the sub-formula property.

As far as order is concerned one might worry that some premisses in an argument have as high an order as the conclusion of the argument. Consider, e.g., this sequent proof:

$$\frac{(p \to q) \to p, p \to q \ : \ p \qquad q \ : \ q}{(p \to q) \to p, p \to q \ : \ q}$$

Here the subsequent $(p \to q) \to p, p \to q\colon p$ has the same order as the final sequent. In order to deal with this problem, the elimination rules will be written "multiplicatively", that is, without tacit contraction. If we write the rules additively we'll have to complicate the definition of the order of an argument.

1.4.2 Rules of inference

We will work with the elimination rules given in figure 1.4.1. These elimination rules for intuitionistic propositional logic are in a somewhat unfamiliar form; the benefit is that it makes the theory run smoothly.[16] Note that all rules have atomic conclusions.[17]

Remark 1.4.4. We will again allow both *vacuous* and *multiple* discharge. Thus the following are instances of $\wedge$-elimination:

$$\frac{p \wedge q \ : \ p \wedge q \qquad r \ : \ r}{p \wedge q, r \ : \ r} \qquad \frac{p \wedge q \ : \ p \wedge q \qquad p, p, p \to (p \to r) \ : \ r}{p \wedge q, p \to (p \to r) \ : \ r}$$

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[16]For this type of elimination rule see, e.g., (von Plato, 2003, p. 197).

[17]Dummett's approach is different. He distinguishes between validity in the "narrow" and the "wide" sense. (Dummett, 1991, p. 284) An argument is valid in the broad sense if it can be transformed into an argument (with the same premises and conclusion) which is valid in the narrow sense.

$$\bot\text{-ELIMINATION} \quad \frac{\Gamma : \bot \qquad \Gamma',q : p}{\Gamma,\Gamma' : p} \text{ for all } p \text{ and } q$$

$$\wedge\text{-ELIMINATION} \quad \frac{\Gamma : \phi \wedge \psi \qquad \Delta,\phi,\psi : p}{\Gamma,\Delta : p}$$

$$\vee\text{-ELIMINATION} \quad \frac{\Gamma : \phi_0 \vee \phi_1 \qquad \Delta_0,\phi_0, : p \qquad \Delta_1,\phi_1 : p}{\Gamma,\Delta_0,\Delta_1 : p}$$

$$\rightarrow\text{-ELIMINATION} \quad \frac{\Gamma : \phi \rightarrow \psi \qquad \Delta : \phi \qquad \Gamma',\psi : p}{\Gamma,\Delta,\Gamma' : p}$$

Figure 1.4.1: Intuitionistic Elimination Rules
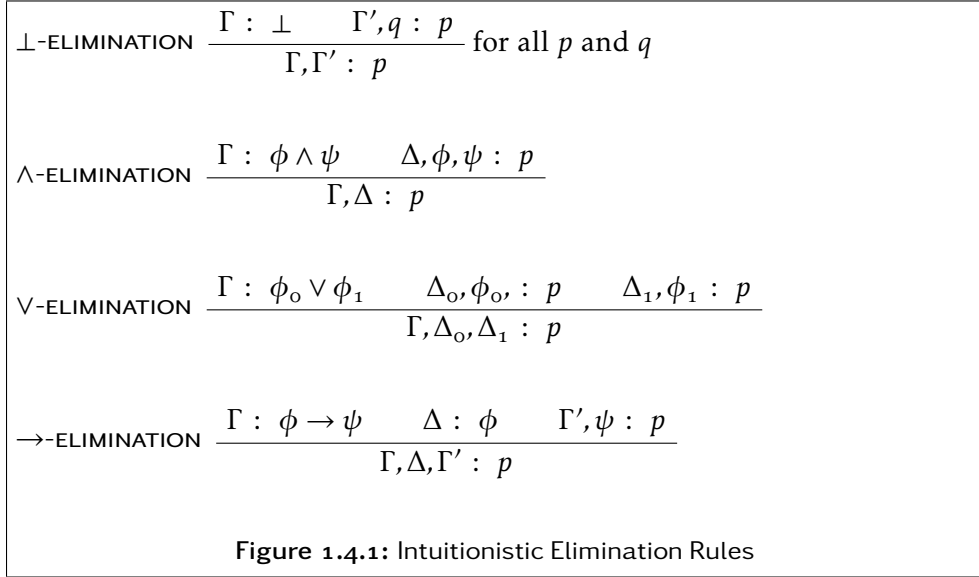
Definition 1.4.5. In an elimination rule the left-most sequent is called the *major* premiss. The other premisses are the *minor* premisses.

Definition 1.4.6. Let $\dfrac{\Gamma : \phi \rightarrow \psi \qquad \Delta : \phi \qquad \Delta',\psi : p}{\Gamma,\Delta,\Delta' : p}$ be an instance of $\rightarrow$ elimination; the occurrence of the sequent $\Delta : \phi$ is called a critical minor premiss. (We'll abuse notation and speak of a critical sequent instead of a critical occurrence of a sequent.) All other minor premisses are called non-critical minor premisses.

Definition 1.4.7. Let Π be an argument. Let $\Gamma : \phi \circ \psi$, $\circ \in \{\wedge, \vee, \rightarrow, \bot\}$ be a sequent occurring in Π. $\Gamma : \phi \circ \psi$ is the *principal major premiss* iff the conclusion of Π follows by $\circ$-elimination from $\Gamma : \phi \circ \psi$.

1.4.3 Canonical Proofs

The definition of *e*-canonical arguments is somewhat more delicate than the definition of an *i*-canonical argument. Let Δ be a multiset. Then Δ^m is the

$$\overbrace{\text{multiset } \Delta, \Delta, \dots, \Delta.}^{m \text{ times}}$$

Definition 1.4.8. **Base-clause:** 1. The one-step argument $\Gamma \colon p$, where $p \in \Gamma$, is a valid canonical argument for all p and Γ.

2. The sequent $\Gamma \colon p$ is valid for all Γ and p such that $p \in \Gamma$.

Recursion-clause The argument Π is valid canonical if Π has a principal premiss $\Gamma \colon \phi$, such that $\phi \in \Gamma$. Furthermore,

$\bot$-**condition:** if $\Gamma \colon \bot$ is the principal major premiss of Π, and $\Gamma', q \colon p$ is the minor premiss to the application of $\bot$-elimination, the argument to the conclusion $\Gamma, \Gamma' \colon p$ is also valid canonical.

$\wedge$-**condition:** For all Δ, ϕ, ψ, p, if $\Gamma \colon \phi \wedge \psi$ is the principal premiss and the minor premiss to the application of $\wedge$-elimination is $\Delta, \phi, \psi \colon p$, then the subargument to conclusion $\Delta, \phi, \psi \colon p$ is also valid canonical.

$\vee$-**condition:** For all Δ, ϕ, ψ, p, if $\Gamma \colon \phi \vee \psi$ is the principal major premiss and the minor premisses of this application of $\vee$-elimination are $\Delta, \phi \colon p$ and $\Delta, \psi \colon p$ then the subarguments Π_ϕ, Π_ψ with conclusions $\Delta, \phi \colon p$ and $\Delta, \psi \colon p$ respectively are also valid canonical.

$\rightarrow$-**condition:** Suppose that $\Gamma \colon \phi \rightarrow \psi$ is the principal major premiss. Suppose further that $\Delta_0 \colon \phi$ is the critical minor premiss and $\Delta_1, \psi \colon p$ is the non-critical minor premiss. Then we demand that $o(\Delta_0 \colon \phi) < o(\Pi)$ and $\Delta_0 \colon \phi$ is a valid sequent. Moreover, we demand that the subargument to $\Delta_1, \psi \colon p$ be canonical.

Valid Sequent A sequent $\Gamma \colon \phi$ is *valid* iff, for every Δ and p and every valid canonical argument Π (of degree at most $\deg(\phi)$) such that

1. the conclusion of Π is $\Delta, \phi \colon p$;

2. the principal major premiss of Π is $\Delta', \phi : \phi$, for some $\Delta' \subseteq \Delta$.

then there is a valid canonical argument Π' such that:

(3) the conclusion of Π' is $\Gamma, \Delta* : p$, where $\Delta*$ is a subset of Δ^m for some m.

Remark 1.4.9. To see that this definition makes sense observe the following. Suppose that Π is an argument ending with an application of $\rightarrow$-elimination. Then Π has the following form:

$$\frac{\Gamma, \phi \rightarrow \psi : \phi \rightarrow \psi \qquad \Delta_0 : \phi \qquad \Delta_1, \psi : p}{\phi \rightarrow \psi, \Gamma, \Delta_0, \Delta_1, : p}$$

Let Δ_2 be such that $\deg(\Delta_2) \leq \deg(\phi)$, then it's clear that $o(\Delta_0, \Delta_2^m) < o(\Delta_0)\#\omega^{\deg(\phi)+1} \leq o(\Delta_0, \phi \rightarrow \psi)$, for all m. But then in order to check whether $\Delta_0 : \phi$ is valid we only have to consider canonical proofs of order less than $o(\Delta_0, \phi \rightarrow \psi) \leq o(\Pi)$. It is here critical that we are not allowed to consider copies of Δ_0 itself.[18]

EXAMPLES OF CANONICAL ARGUMENTS.

$$\frac{p \wedge q : p \wedge q \qquad \dfrac{p \rightarrow r : p \rightarrow r \qquad p : p \qquad r : r}{p \rightarrow r, p : r}}{p \wedge q, p \rightarrow r : r}$$

This argument shows that r is canonically provable from $p \wedge q$ and $p \rightarrow r$.

$$\frac{p \rightarrow q : p \rightarrow q \qquad p : p \qquad \dfrac{q \rightarrow r : q \rightarrow r \qquad q : q \qquad r : r}{q \rightarrow r, q : r}}{p \rightarrow q, q \rightarrow r, p : r}$$

This argument shows that r is canonically provable from $p \rightarrow q, q \rightarrow r$ and p.

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[18]Dummett (1991, pp. 285–6), in effect, defines a sequent $\Delta : \phi$ to be valid iff for every Δ' of degree at most $\deg(\Delta, \phi)$ and p such that there is a canonical argument with conclusion $\Delta, \phi : p$ and principal major premiss ϕ there is a canonical argument with conclusion $\Delta, \Delta' : p$. If we used this definition of valid sequent the recursion would not work. For suppose Π is an argument. Since it's possible that $\deg(\Delta) = \deg(T)$ in order to check whether $\Delta : \phi$ we might have to check canonical arguments of order higher that Π.

1.4.4 Typographical conventions

Canonical proofs get very wide very fast. It will occasionally be convenient to adopt a more compact formulation. Instead of writing down

$$\frac{\phi \to \psi \colon \phi \to \psi \qquad \Delta \colon \phi \qquad \begin{array}{c} \Pi \\ \Gamma, \psi \colon p \end{array}}{\phi \to \phi, \Delta, \Gamma \colon p}$$

we may write

$$\frac{\phi \to \psi \qquad \Delta \colon \phi \qquad \begin{array}{c} \Gamma, [\psi]^x \\ \Pi \\ p \end{array}}{p} \ \text{x}$$

The notation $\Gamma, [\psi]^x$ is to indicate that there are some occurrences of ψ which are discharged by the application of $\to$-elimination.

1.4.5 Basic facts about canonical arguments

In what follows I'll abuse notation and say that p is principal in a canonical argument Π if Π is the one-step argument $\Gamma, p \colon p$.

Definition 1.4.10. We write $\Pi(\Gamma \vdash_\phi p)$ to mean that Π is a canonical argument with principal major premiss ϕ and conclusion $\Gamma, \phi \colon p$. We write $\Gamma \vdash_\phi p$ for the claim that there is some Π such that $\Pi(\Gamma \vdash_\phi p)$. We write $\Gamma \vdash p$ for the claim that there is a $\phi \in \Gamma$ such that $\Gamma \setminus \{\phi\} \vdash_\phi p$.

Definition 1.4.11. The relation 'ϕ is a subformula of ψ' is defined as the transitive closure of the immediate subformula relation, which is defined as follows.

1. p is an immediate subformula of $\bot$, for all p.

2. ϕ, ψ are immediate subformulæ of $\phi \wedge \psi, \phi \vee \psi$ and $\phi \rightarrow \psi$.

Observation 1.4.12. *Let Π be a canonical argument. Let $\Gamma: \phi$ be the principal major premiss of Π and let $\Gamma': p$ be the conclusion of Π. Then $\Gamma' \supseteq \Gamma$ and $\phi \in \Gamma$.*

Proof. By inspection of the elimination rules. □

Definition 1.4.13. An argument Π has the *subformula property* iff for all sequents $\Gamma: \phi$ occurring in Π, each formulæ $\psi \in \Gamma, \phi$ is a subformula of some formula occurring in a sequent higher up in the proof.

Proposition 1.4.14. *Let Π be a canonical argument; then Π has the subformula property.*

Proof. Obvious. (By induction on the order of the canonical argument.) □

In fact, the subformula property holds in the stronger form.

Definition 1.4.15. Let Π be a canonical proof. A *track* in Π is a maximal sequence of sequents $\Gamma_0: \phi_0, \ldots, \Gamma_n: \phi_n$ such that $\Gamma_n: \phi_n$ is the final sequent of Π and such that each $\Gamma_i: \phi_i$ is a non-critical minor premiss to an argument with conclusion $\Gamma_{i+1}: \phi_{i+1}$.

Observation 1.4.16. *Let $\Gamma_0: \phi_0, \ldots, \Gamma_n: \phi_n$ be a track, then there is an atomic q such that all ϕ_i are identical to q.*

Observation 1.4.17. *Let $\Gamma_0: \phi_0, \ldots, \Gamma_n: \phi_n$ be a track. Let $\phi \in \Gamma_i$. Then ϕ is a subformula of some formula $\psi \in \Gamma_{i+1}$.*

1.4.6 Justifying introduction rules

Dummett assumes that if we start with the intuitionistic elimination rules we'll be able to justify (at least) the intuitionistic introduction rules. The way I have set this up, this is not obvious.

Let's consider the case of $\to$-introduction. Suppose we know that the sequent $\Gamma, \phi : \psi$ is valid. Then we need to be able to conclude that the sequent $\Gamma : \phi \to \psi$ is valid. So let Π be a canonical argument such that $\Pi(\Delta \vdash_{\phi \to \psi} p)$ (with $\deg(\Pi) \leq \deg(\phi \to \psi)$). We may assume that the argument looks like this:

$$\frac{\phi \to \psi : \phi \to \psi \qquad \Delta_0 : \phi \qquad \Delta_1, \psi : p}{\phi \to \psi, \Delta_0, \Delta_1 : p}$$

And now we want to say that since $\Gamma, \phi : \psi$ is valid there must be a canonical argument from Γ, Δ_1, ϕ to p. But we're not entitled to assume that: the validity of $\Gamma, \phi : \psi$ means that for all canonical arguments Π' such that $\Pi'(\Delta \vdash_\psi p)$ where $\deg(\Pi') \leq \deg(\psi)$, there's a corresponding canonical argument $\Pi''(\Delta, \Gamma, \phi \Vdash p)$. And, first, we can't assume that ψ occurs as a principal major premiss in the canonical argument witnessing $\Delta_1, \psi : p$. Second, even if ψ so occurs, since $\phi \to \psi$ is of higher degree than ψ, it's possible that a proof witnessing $\Delta_1, \psi : p$ is of higher degree than ψ. If that's the case, the assumption that $\Gamma, \phi : \psi$ is valid doesn't help.

Furthermore, even if there is a canonical argument witnessing $\Gamma, \Delta_1, \phi : p$, we would need to get rid of ϕ to get that $\Gamma, \Gamma, \Delta_0, \Delta_1, : p$ is valid. This *should* be true—after all $\Delta_0 : \phi$ is valid; but we run into the same problems. It turns out that there is no problem here, but this is surprisingly complicated to prove. We have to prove a "cut-elimination" theorem.

Danger lurks very close by: Dummett's own account of pragmatist meaning-theories breaks down at this point.

Dummett's original definition is, apart from being a recursion on *degree*, the same as my definition. For Dummett, that is, a critical sequent is required to be of lower *degree* than the argument as a whole. That difference, however, makes all the difference: cut-elimination fails. For consider that (in Dummett's sense of 'canonical argument')

1. there is no canonical argument witnessing the validity of $p \rightarrow (p \rightarrow q), (p \rightarrow q) \rightarrow r : r$;

2. there is no canonical argument witnessing the validity of $p \rightarrow q \wedge q, (p \rightarrow q) \rightarrow r : r$;

3. there is no canonical argument witnessing the validity of $p \rightarrow q \vee q, (p \rightarrow q) \rightarrow r : r$.

Consider now, e.g., that the sequent $p \rightarrow (p \rightarrow q) : p \rightarrow q$ is valid and there is trivially a canonical argument witnessing that $p \rightarrow q, (p \rightarrow q) \rightarrow r : r$ is valid. This means that the consequence relation isn't transitive![19]

It is worth pointing out that there is no analogue of this problem when we instead start with introduction rules. When we try to justify elimination rules on the basis of introduction rules we assume that we have canonical proofs of the premisses and show that we can construct canonical proofs of the conclusions. In that case the complexity drops in the passage from premiss(es) to conclusion; when we try to justify introduction rules on the basis of elimination rules the complexity goes up in the passage from premisses to conclusion. Our trouble flows from that.

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[19]Prawitz (2007, p. 479n15) also notices this problem.

1.5 Properties of Valid Sequents

We'll now establish that the valid sequents behave just like a sequent calculus for intuitionistic logic.

1.5.1 Structural properties

Proposition 1.5.1. Reflexivity $\phi\colon \phi$ *is valid for all* ϕ.

Weakening *If* $\Gamma\colon \theta$ *is valid then* $\Gamma, \phi\colon \theta$ *is valid.*

Cut *If* $\Gamma, \phi\colon \psi$ *is valid and* $\Delta\colon \phi$ *is valid, then* $\Gamma, \Delta\colon \psi$ *is valid.*

Proof. Reflexivity is obvious. In order to prove that weakening holds, suppose that $\Gamma\colon \theta$ is valid. . Suppose now that Δ is such that $\deg(\Delta) \le \theta$ and let Π be such that $\Pi(\Delta \vdash_\theta p$. Then find Π' such that Π' witnesses that $\Gamma, \Delta \Vdash p$. Now, just add ϕ to the left-hand-side of the critical major premiss of Π; this preserves canonicity. □

On the other hand, contraction is not admissible. While there is a canonical proof witnessing that $p \to (p \to q), p, p\colon q$, there is no canonical proof witnessing that $p \to (p \to q), p\colon q$.[20]

The proof of cut is rather involved; surprisingly, the difficult case turns out to be disjunction.

1.5.2 Cut-elimination

Theorem 1.5.2. *Suppose* $\Gamma, \phi\colon \psi$ *is valid and* $\Delta\colon \phi$ *is valid, then* $\Gamma, \Delta\colon \psi$ *is valid.*

Theorem 1.5.3. *Suppose* $\Gamma, \phi\colon p$ *is valid and* $\Delta\colon \phi$ *is valid then* $\Gamma, \phi\colon p$ *is valid.*

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[20]This is not to say that the sequent $p \to (p \to q)\colon p \to q$ isn't valid.

Theorem 1.5.2 follows from theorem 1.5.3. Suppose that $\Gamma, \phi \colon \psi$ and $\Delta \colon \phi$ are both valid. Now suppose that Δ' and p are such there is a canonical argument Π such that $\Pi(\Delta' \vdash_\psi p)$, and $\deg(\Delta') \le \deg(\psi)$. Then we have that $\Gamma, \phi \colon p$ is valid. It then follows from theorem 1.5.3 that $\Gamma, \Delta, \Delta' \colon p$ is valid. So it suffices to prove 1.5.3. (Note, however, that in the proof of 1.5.3 we'll in effect use instances of 1.5.2 which are of sufficiently low order.)

In order to establish the result we'll need the following two lemmas. In order to state the results, we'll adopt the following conventions. $\phi[\frac{\bar{q}}{\bar{p}}]$ is to mean the result of uniformly substituting the $\bar{q}$ for the $\bar{p}$ in ϕ. If we have to consider several substitutions we substitute from right to left; thus when we write $\phi[\frac{\bar{r}}{\bar{s}}][\frac{\bar{q}}{\bar{p}}]$ we mean the result of first substituting the $\bar{q}$ for the $\bar{p}$ and then substituting the $\bar{r}$ for the $\bar{s}$.

Lemma 1.5.4. *Let Π be a canonical argument with principal premiss ϕ. Let $p_0, p_1, \ldots, p_n$ be $n+1$ distinct atomic formulæ that occur in Π. Let $q_0, \ldots, q_n$ be $n+1$ distinct atomic formulæ such that no q_j occurs in Π. Let Π' be obtained from Π by uniformly substituting the atomic formulæ $\bar{q}$ for the atomic formulæ $\bar{p}$. Then Π' is a canonical argument with principal premiss $\phi[\frac{\bar{q}}{\bar{p}}]$.*

Lemma 1.5.5. *Suppose $\Gamma, [\phi_i \to q_i]_{i \in I}, [q_i]_{i \in I} \Vdash \psi$, where each q_i is not $\bot$ and does not occur in either ϕ, ψ or Γ. Then $\Gamma \Vdash \psi$.*

Proof of lemma 1.5.4. By induction on the order of the canonical argument Π. Let $\bar{p}$ and $\bar{q}$ be given. The base case is straightforward. The only problematic case is when the principal major premiss in Π is $\phi_0 \to \phi_1$. Suppose that the non-critical minor premiss is $\Gamma, \phi_1 \colon p$ and the critical minor premiss is $\Delta \colon \phi_0$. The canonical argument Π_0 witnessing that $\Gamma, \phi_1 \Vdash p$ is of lower

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order than Π. The induction hypothesis thus applies and $\Gamma[\frac{\bar{q}}{\bar{p}}], \psi[\frac{\bar{q}}{\bar{p}}] \Vdash p[\frac{\bar{q}}{\bar{p}}]$. It thus suffices to show that $\Delta[\frac{\bar{q}}{\bar{p}}] \colon \phi_0[\frac{\bar{q}}{\bar{p}}]$ also is a valid sequent.

So suppose that Δ_0, q are such that $\Delta_0 \Vdash_{\phi_0[\frac{\bar{q}}{\bar{p}}]} s$. Now let $\bar{r}$ be the atomic formulæ occurring in Δ_0, s which are not in $\bar{q}$ or not in $\phi_0[\frac{\bar{q}}{\bar{p}}]$. Let $\bar{r}'$ be some fresh atomic constants; in particular no formulæ in $\bar{r}'$ occurs in Δ. Since the canonical argument Π_1 witnessing that $\Delta_0 \Vdash_{\phi_0[\frac{\bar{q}}{\bar{p}}]} s$ has lower order than the canonical argument Π, the induction applies and we get that $\Delta_0[\frac{\bar{p}}{\bar{q}}][\frac{\bar{r}'}{\bar{r}}] \Vdash_{\phi_0[\frac{\bar{p}}{\bar{q}}][\frac{\bar{q}}{\bar{p}}]} s[\frac{\bar{p}}{\bar{q}}][\frac{\bar{r}'}{\bar{r}}]$. That is we get that

$$\Delta_0[\tfrac{\bar{p}}{\bar{q}}][\tfrac{\bar{r}'}{\bar{r}}] \Vdash_{\phi_0} s[\tfrac{\bar{p}}{\bar{q}}][\tfrac{\bar{r}'}{\bar{r}}]$$

Then since $\Delta \Vdash \phi_0$ we get that $\Delta, \Delta_0[\frac{\bar{p}}{\bar{q}}][\frac{\bar{r}'}{\bar{r}}] \Vdash s[\frac{\bar{p}}{\bar{q}}][\frac{\bar{r}'}{\bar{r}}]$.

The canonical argument witnessing this has lower order than Π. Moreover, no formula in $\bar{q}$ occurs in $\Delta, \Delta_0[\frac{\bar{p}}{\bar{q}}][\frac{\bar{r}'}{\bar{r}}]$ or $s[\frac{\bar{p}}{\bar{q}}][\frac{\bar{r}'}{\bar{r}}]$. The induction hypothesis applies and we can substitute $\bar{q}$ for $\bar{p}$ to get $\Delta[\frac{\bar{q}}{\bar{p}}], \Delta_0[\frac{\bar{r}'}{\bar{r}}] \Vdash s[\frac{\bar{r}'}{\bar{r}}]$. The proof witnessing this also has lower order than Π and no formula in $\bar{r}$ occurs in either $\Delta[\frac{\bar{q}}{\bar{p}}], \Delta_0[\frac{\bar{r}'}{\bar{r}}]$ or $s[\frac{\bar{r}'}{\bar{r}}]$. The induction hypothesis thus applies and we can substitute $\bar{r}$ for $\bar{r}'$, to obtain $\Delta[\frac{\bar{q}}{\bar{p}}], \Delta_0 \Vdash s$.

$\square$

Proof of lemma 1.5.5. The result follows from the special case where ψ is atomic. For suppose that $\Gamma, [\phi_i \to q_i]_{i \in I}, [q_i]_{i \in I} \Vdash \psi$, where q_i is not $\bot$ and does not occur in either ϕ, ψ or Γ. Let Δ be such that $\deg(\Delta) \leq \deg(\psi)$. And suppose that $\Delta \vdash_\psi p$ for some p.

Pick some $\bar{r}$ which occur neither in $\Delta, \Gamma, \psi, \phi_i, q_i$ nor p. By lemma 1.5.4 and the fact that no q_i occurs in ψ, we get that $\Delta[\frac{\bar{r}}{\bar{q}_i}] \vdash_\psi p[\frac{\bar{r}}{\bar{q}_i}]$. We then get

that $\Gamma, \Delta[\frac{\bar{r}}{\bar{q}_i}], [\phi_i \to q]_{i \in I}, [q_i]_{i \in I} \Vdash p[\frac{\bar{r}}{\bar{q}}]$. The special case now applies, and we get a proof witnessing that $\Gamma, \Delta[\frac{\bar{r}}{\bar{q}_i}] : p[\frac{\bar{r}}{\bar{q}_i}]$ is valid. By lemma 1.5.4 and the fact that $\bar{q}_i$ does not occur in $\Gamma, \Delta[\frac{\bar{r}}{\bar{q}_i}], p[\frac{\bar{r}}{\bar{q}_i}]$, we get that $\Gamma[\frac{\bar{q}_i}{\bar{r}}], \Delta[\frac{\bar{q}_i}{\bar{r}}][\frac{\bar{r}}{\bar{q}_i}] \Vdash p[\frac{\bar{q}_i}{\bar{r}}][\frac{\bar{r}}{\bar{q}_i}]$, i.e., we get that $\Gamma, \Delta \Vdash p$.

So let's prove the special case. Let Π be a canonical proof witnessing that $\Gamma, [\phi_i \to q_i]_{i \in I}, [q_i]_{i \in I} : p$, where q_i is not $\perp$ and does not occur in either ϕ_i or Γ or p. We prove the result by induction on the order of Π.

If the principal major premiss is an axiom or is by $\perp$-elimination the result follows immediately. If the principal premiss is a disjunction or a conjunction the result follows immediately from the induction hypothesis. So suppose that the principal premiss is a conditional. There are two cases.

The conditional in question is $\phi_i \to q_i$, for some i. We may assume that the argument looks like this:

$$\frac{\phi_i \to q_i \,:\, \phi_i \to q_i \qquad \Gamma_0, [\phi_j \to q_j]_{j \in I \setminus \{i\}}, [q_j]_{j \in I \setminus \{i\}} \,:\, \phi \qquad \Gamma_1, [\phi_j \to q_j]_{j \in I' \setminus \{i\}}, [q_j]_{j \in J'}, q_i \,:\, p}{\Gamma_0, \Gamma_1, \phi \to q : p}$$

By the induction hypothesis it follows that $\Gamma_1 : p$ is valid. The result follows by weakening.

Suppose, then, that the conditional is $\psi_0 \to \psi_1$. Then we may assume that the canonical argument Π looks like this.

$$\frac{\psi_0 \to \psi_1 \,:\, \psi_0 \to \psi_1 \qquad \Gamma_0, [\phi_i \to q_i]_{i \in I}, [q_i]_{i \in I} \,:\, \psi_0 \qquad \Gamma_1, \psi_1 \,:\, p}{\Gamma_0, \Gamma_1, \psi_0 \to \psi_1, \phi \to q \,:\, p}$$

(For if $\phi_i \to q_i, q_i$ occurs in the noncritical minor premiss, the induction hypothesis immediately applies.) But it also follows from the induction hypothesis that $\Gamma_0 : \psi_0$ is valid. For suppose that $\Delta \vdash_{\psi_0} p$ where $\deg(\Delta) \leq \deg(\psi_0)$. We now repeat the reasoning proving the general case from the special case,

paying special attention to the order of the arguments in question. We again pick some distinct $\bar{r}$ which are not in $\Gamma_o, \Delta, \phi_i, q_i$ or ψ_o. We then have a canonical proof witnessing $\Delta[\frac{\bar{r}}{\bar{q}_i}] \vdash_{\psi_o} p[\frac{\bar{r}}{\bar{q}_i}]$. Since $\Delta[\frac{\bar{r}}{\bar{q}_i}]$ is of the same degree as Δ there is a canonical proof Π' witnessing that $\Gamma_o, \Delta[\frac{\bar{r}}{\bar{q}_i}], [\phi_i \to q]_{i \in I}, [q_i]_{i \in I} : p[\frac{\bar{r}}{\bar{q}_i}]$ is valid. This canonical proof is of order less than Π. The induction hypothesis therefore applies and $\Gamma_o, \Delta[\frac{\bar{r}}{\bar{q}_i}] : p[\frac{\bar{r}}{\bar{q}_i}]$ is valid. Since q_i does not occur in Γ_o we can substitute $\bar{q}_i$ for $\bar{r}$ again we get that $\Gamma_o, \Delta_o \Vdash p$. $\qquad\qquad \square$

Proof of theorem 1.5.3. We prove this by induction on the degree of the cut-formula ϕ and a subsidiary induction on the order of the canonical argument Π witnessing that $\Gamma, \phi : p$.

Let Π be a canonical argument witnessing that $\Gamma, \phi : p$. There are two cases to consider, each with five subcases.

1. ϕ is principal in Π. There are five sub-cases.

(a) ϕ is p. In this case we have a canonical argument Π' witnessing that $\Delta : p$ is valid; the result follows by weakening.

(b) ϕ is $\bot$. Since $\Delta : \bot$ is valid, $\Delta : q$ is valid for all q; in particular we have a canonical argument Π' witnessing that $\Delta \Vdash q$. The result follows by weakening.

(c) ϕ is $\phi_o \wedge \phi_1$. We may assume that Π looks like this:

$$\frac{\phi_o \wedge \phi_1 : \phi_o \wedge \phi_1 \qquad \Gamma, \phi_o, \phi_1 : p}{\Gamma, \phi_o \wedge \phi_1 : p}$$

It is easily seen that if $\Delta : \phi_o \wedge \phi_1$ is valid then $\Delta : \phi_o$ and $\Delta : \phi_1$ are both valid. (It is essential here that $\wedge$-elimination allows (partially) vacuous discharge.) $\Gamma, \Delta : p$ now follows by cutting on ϕ_o and ϕ_1.

(d) ϕ is $\phi_o \to \phi_1$. We may assume that the argument Π looks like this.

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$$\frac{\phi_0 \to \phi_1 \ : \ \phi_0 \to \phi_1 \qquad \Gamma_0 \ : \ \phi_0 \qquad \Gamma_1, \phi_1 \ : \ p}{\Gamma_0, \Gamma_1, \phi_0 \to \phi_1 \ : \ p}$$

Since $\Delta \colon \phi_0 \to \phi_1$ is valid, $\Delta, \phi_0 \colon \phi_1$ is also valid. But it then follows from the induction hypothesis that $\Gamma_1, \Delta, \phi_0 \ : \ p$ is valid. But then by the validity of $\Gamma_0 \colon \phi_0$ it again follows by the induction hypothesis that $\Gamma_0, \Delta, \Gamma_1 \colon p$ is valid. Graphically this is what we are doing:

$$\frac{\Gamma_0 \ : \ \phi_0 \qquad \dfrac{\Gamma_1, \phi_1 \ : \ p \qquad \Delta, \phi_0 \ : \ \phi_1}{\Gamma_1, \Delta \ : \ p} \ \text{Cut}}{\Gamma_0, \Gamma_1, \Delta \ : \ p} \ \text{Cut}$$

(e) ϕ is $\phi_0 \vee \phi_1$. We may then assume that the argument has the following form.

$$\frac{\phi_0 \vee \phi_1 \ : \ \phi_0 \vee \phi_1 \qquad \Gamma_0, \phi_0 \ : \ p \qquad \Gamma_1, \phi_1 \ : \ p}{\phi_0 \vee \phi_1, \Gamma_0, \Gamma_1 \ : \ p}$$

If either $\Delta \colon \phi_0$, $\Delta \colon \phi_1$ or $\Delta \colon \bot$ is valid, the result follows easily. If, e.g., $\Delta \colon \phi_0$ is valid the result follows by a cut on ϕ_0.

If $\deg(\Pi) \leq \deg(\phi_0 \vee \phi_1)$ the result follows immediately from $\Delta \Vdash \phi_0 \vee \phi_1$. So suppose that $\deg(\Pi) > \deg(\phi_0 \vee \phi_1)$. Now let q be atomic and such that q does not occur in any formula which occurs in Δ or $\phi_0 \vee \psi$. Since $\Delta \colon \phi_0 \vee \phi_1$ there is a valid canonical argument Π_0 witnessing that $\Delta, \phi_0 \to q, \phi_1 \to q \Vdash q$. Let σ_0 be the principal premiss in this argument. Since q does not occur in Δ the principal major premiss cannot be an axiom. Since $\Delta \colon \bot$ is not valid, σ_0 is either a conjunction, disjunction or a conditional. We'll construct a canonical argument Π_2 witnessing $\Gamma, \Delta \Vdash p$ by modifying the proof Π'.

We first transform the proof Π_0 into the proof Π_1 as follows. Suppose $\Gamma, (\phi_0 \to q)^m, (\phi_1 \to q)^n, \psi \colon \psi$ is a principal major premiss in a sub-proof of Π'. We replace this premiss with $\Gamma, \psi \colon \psi$. Suppose that $\Delta', (\phi_0 \to q)^m, (\phi_1 \to q)^n, q^l \colon \psi$ is a critical minor premiss occurring in a subproof of Π_0. ψ does not contain q. For if ψ did contain q, q would occur in the antecedent of

a conditional $\theta_0 \to \theta_1 \in \Delta$. But q does not occur in Δ. The conditions of lemma 1.5.5 thus obtain and we can replace this sequent with $\Delta': \psi$.

Making these replacements we obtain the proof Π_1. Π_1 has the following feature. Every instance of $\phi_i \to q$ occurs first as a principal major premiss to an application of $\to$-elimination.

We now consider each track in Π_1 such that q occurs on the right-hand-side of the sequents in the track. Let T be a track $\Sigma_0^T: q, \ldots, \Sigma_{n_T}^T, : q$; here $\Sigma_{n_T}: q$ is the conclusion—$\Delta_0, (\phi_0 \to q)^{m_0}, (\phi_1 \to q)^{n_0}: q$—of Π_1. Since q does not occur in $\Delta, (\phi_0 \to q)^{m_0}, (\phi_1 \to q)^{n_0}: q$ except in $\phi_i \to q$, any occurrence of q on the left side of a sequent in T has to have been discharged by either $\perp$-elimination or $\to$-elimination. (If q had been discharged by either $\wedge$ or $\vee$ elimination q would occur in a formula in Δ_0.) Now Let $\Sigma_j^T: q$ be the greatest j such that in passing to $\Sigma_{j+1}: q$ an occurrence of q is discharged by $\perp$ or $\to$ elimination.

If the occurrence of q is discharged by $\perp$-elimination the proof looks like this.

$$\frac{\Sigma_j', \perp: \perp \qquad \Sigma_j'', q: q}{\Sigma_{j+1}, \perp: q}$$

Replace this proof with

$$\frac{\Sigma_j', \perp: \perp \qquad \Sigma_j'' \setminus \{\phi_i \to q\}_{i=0,1}, p: p}{\Sigma_{j+1} \setminus \{\phi_i \to q\}_{i=0,1}, \perp: p}$$

If the occurrence of q is discharged by an application of $\to$ elimination the proof looks like this.

$$\frac{\phi_i \to q: \phi_i \to q \qquad \Sigma_j': \phi_i \qquad \Sigma_j'', q: q}{\phi_i \to q, \Sigma_{j+1}: q}$$

But we have that $\Gamma_i, \phi_i : p$ is valid and is witnessed to be valid by a canonical argument of order less than Π. The induction hypothesis applies and by cut we get that $\Gamma, \Sigma'_j : p$ is valid. By weakening we then get that $\Gamma, \Sigma_{j+1} \setminus \{\phi_i \to q\}_{i=0,1} : p$ is valid. Let Π^T_j be a canonical argument witnessing this. Note that since j was the greatest j such that an instance of q is discharged in passing to $j+1$, any formulæ which are discharged along the track T after $j+1$ are in $\Sigma_{i+1} \setminus \{\phi_i \to q\}_{i=0,1}$. For each track T in Π_0 we make the same transformations and obtain a proof Π^T. Now let Π'' be the result of making the following replacements in Π_0.

For each track $T = \Sigma^T_0 : q, \ldots, \Sigma^T_n : q$ in Π_1 find the largest j such that in passing from $\Sigma^T_j : q$ to $\Gamma^T_{j+1} : q$ an occurrence of q on the left in Σ^T_j is discharged. Replace the sequent $\Sigma^T_{j+1} : q$ with the conclusion of Π^T; replace all occurrences of q on the right below $\Sigma^T_{j+1} : q$ with p. In this way we obtain a canonical argument witnessing that $\Gamma, \Delta \Vdash p$.

2. The cut-formula ϕ is not principal in Π. There are again five subcases depending on the principal premiss in Π.

(a) Π is an axiom, $\Gamma', \phi, p : p$, say. $\Gamma, p : p$ is likewise an axiom and the result follows by weakening.

(b) The principal premiss in Π is $\bot$. In that case the argument has the form.
$$\frac{\bot : \bot \qquad \Gamma, p, \phi : q}{\Gamma, \phi, \bot : q}.$$ By the induction hypothesis we get that $\Gamma, \Delta, p : q$.
And hence that $\Gamma, \Delta, \bot : q$.

(c) The principal premiss is a conjunction $\psi_0 \wedge \psi_1$. Then we may assume that Π looks like this:

$$\frac{\psi_0 \wedge \psi_1 : \psi_0 \wedge \psi_1 \qquad \Gamma, \phi, \psi_0, \psi_1 : p}{\Gamma, \phi, \psi_0 \wedge \psi_1 : p}$$

Since the canonical argument to the minor premiss has lower order than Π

we can apply the induction hypothesis to that argument to get.

$$\dfrac{\psi_0 \wedge \psi_1 \,:\, \psi_0 \wedge \psi_1 \qquad \dfrac{\Gamma, \phi, \psi_0, \psi_1 \,:\, p \qquad \Delta \,:\, \phi}{\Gamma, \Delta, \psi_0, \psi_1 \,:\, p}\ \text{Cut}}{\Gamma, \phi, \psi_0 \wedge \psi_1 \,:\, p}$$

(d) The case where the principal major premiss of Π is a disjunction $\psi_0 \vee \psi_1$ is proved similarly.

(e) The major premiss of Π is a conditional $\psi_0 \to \psi_1$. In this case we may assume that the proof looks like this.

$$\dfrac{\psi_0 \to \psi_1 \,:\, \psi_0 \to \psi_1 \qquad \Gamma_0, \phi \,:\, \psi_0 \qquad \Gamma_1, \psi_1 \,:\, p}{\Gamma_0, \Gamma_1, \phi, \psi_0 \to \psi_1 \,:\, p}$$

(If we have $\Gamma_1, \phi, \psi_1 : p$, we can use the induction hypothesis in the same way as we do for disjunction and conjunction.) It suffices to show that $\Gamma_0, \Delta : \psi_0$ is valid. So suppose that Δ' is such that $\Delta' \Vdash_{\psi_0} q$ for some q with $\deg(\Delta') \leq \deg(\psi_0)$. Since $\Gamma_0, \phi : \psi_0$ is valid it follows that $\Gamma_0, \Delta', \phi : q$ is valid. But since $\deg(\Delta') \leq \deg(\psi_0)$ we must have $o(\Delta', \Gamma_0, \phi : q) < o(\Pi)$. Any proof witnessing that $\Delta', \Gamma_0, \phi \Vdash q$ thus has lower order than Π. The induction hypothesis applies and we get that $\Gamma_0, \Delta', \Delta : q$ is valid. Since q and Δ' were arbitrary this shows that $\Gamma_0, \Delta : \psi_0$ is valid which was to be shown.

This concludes the proof. □

Having proved cut, we can now prove that the class of canonical arguments is closed under arbitrary substitutions.

Theorem 1.5.6 (Substitution). *Let Π be a canonical argument with principal premiss ϕ. Let Π' be obtained from Π by uniformly substituting the atomic formulæ $\bar{q}$ for the atomic formulæ $\bar{p}$. Then Π' is a canonical argument with principal premiss $\phi[\frac{\bar{q}}{\bar{p}}]$.*

Proof of theorem 1.5.6. The proof is by induction on the order of canonical arguments. The case of $\rightarrow$-elimination is the only difficulty.

Let Π be a canonical argument and suppose that $\phi \rightarrow \psi$ is the principal major premiss, and suppose that $\Delta\colon \phi$ is the critical sequent, while $\Gamma, \psi\colon p$ is the non-critical sequent. The canonical argument Π_0 witnessing that $\Gamma, \psi\colon p$ is valid is of lower order than Π. The induction hypothesis thus applies and $\Gamma[\frac{\bar{q}}{\bar{p}}], \psi[\frac{\bar{q}}{\bar{p}}]\colon p[\frac{\bar{q}}{\bar{p}}]$ is valid. It thus suffices to show that $\Delta[\frac{\bar{q}}{\bar{p}}]\colon \phi[\frac{\bar{q}}{\bar{p}}]$ also is a valid sequent.

This is by induction on ϕ. If ϕ is atomic there is a canonical argument Π_0 witnessing that $\Delta\colon \phi$ is valid. Π_0 is of lower order Π so the induction hypothesis applies and we get that $\Delta[\frac{\bar{q}}{\bar{p}}]\colon \phi[\frac{\bar{q}}{\bar{p}}]$ is valid.

Suppose then that ϕ is either $\bot, \phi_0 \wedge \phi_1, \phi_0 \vee \phi_1$ or $\phi_0 \rightarrow \phi_1$. I leave the cases of $\bot$ and $\wedge$ to the reader. Suppose first that ϕ is $\phi_0 \vee \phi_1$. Let Δ_0 be such that $\Delta_0 \vdash_{(\phi_0 \vee \phi_1)[\frac{\bar{q}}{\bar{p}}]} r$; let Π_0 be a canonical argument witnessing this. We have to show that $\Delta[\frac{\bar{q}}{\bar{p}}], \Delta_0 \Vdash r$. Since $(\phi_0 \vee \phi_1)[\frac{\bar{q}}{\bar{p}}]$ is principal in Π_0, Π_0 looks like this.

$$\frac{(\phi_0 \vee \phi_1)[\frac{\bar{q}}{\bar{p}}]\colon (\phi_0 \vee \phi_1)[\frac{\bar{q}}{\bar{p}}] \qquad \Delta_0', \phi_0[\frac{\bar{q}}{\bar{p}}]\colon r \qquad \Delta_0'', \phi_1[\frac{\bar{q}}{\bar{p}}]\colon r}{(\phi_0 \vee \phi_1)[\frac{\bar{q}}{\bar{p}}], \Delta_0', \Delta_0''\colon r}$$

There are here canonical arguments Π_0' and Π_0'' witnessing the validity of the respective minor premisses. Now let $r_0, \ldots, r_m$ be all the atomic formula which occur in $\Delta_0, (\phi_0 \vee \phi_1)[\frac{\bar{q}}{\bar{p}}], r$. Let $r_0', \ldots, r_m'$ be distinct atomic constants which are disjoint from all of the constants which occur in Δ and $(\phi_0 \vee \phi_1)[\frac{\bar{q}}{\bar{p}}]$. The induction hypothesis applies so we get that $\Delta_0[\frac{\bar{r}'}{\bar{r}}] \vdash_{(\phi_0 \vee \phi_1)[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]} r[\frac{\bar{r}'}{\bar{r}}]$

Consider now the premisses $\phi_0 \rightarrow \phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \phi_1 \rightarrow \phi_1[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]$. The following argument

$$\cfrac{\cfrac{\phi_0 \vee \phi_1 \quad \cfrac{\phi_0 \to \phi_0[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}] \quad \phi_0 : \phi_0 \quad r[\tfrac{\vec{r}'}{\vec{r}}]}{r[\tfrac{\vec{r}'}{\vec{r}}]}\;\Delta_0'[\tfrac{\vec{r}'}{\vec{r}}],\phi_0[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}] \quad\quad \cfrac{\phi_1 \to \phi_1[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}] \quad \phi_1 : \phi_1 \quad r[\tfrac{\vec{r}'}{\vec{r}}]}{r[\tfrac{\vec{r}'}{\vec{r}}]}\;\Delta_0''[\tfrac{\vec{r}'}{\vec{r}}],\phi_1[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}]}{}}{r[\tfrac{\vec{r}'}{\vec{r}}]}$$

witnesses that

$$\Delta_0[\tfrac{\vec{r}'}{\vec{r}}], \phi_0 \to \phi_0[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}], \phi_1 \to \phi_1[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}] \vdash_{(\phi_0 \vee \phi_1)} r[\tfrac{\vec{r}'}{\vec{r}}]$$

It follows from the validity of $\Delta : \phi_0 \vee \phi_1$ that

$$\Delta, \Delta_0[\tfrac{\vec{r}'}{\vec{r}}], \phi_i \to \phi_i[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}] \Vdash r[\tfrac{\vec{r}'}{\vec{r}}]$$

Since a canonical argument witnessing this has lower order than the canonical argument Π the induction hypothesis applies and we can substitute $\bar{q}$ for $\bar{p}$ and $\vec{r}'$ for $\vec{r}$ to get.

$$\Delta[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}], \Delta_0[\tfrac{\vec{r}'}{\vec{r}}], \phi_i[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}] \to \phi_i[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}] \Vdash r[\tfrac{\vec{r}'}{\vec{r}}]$$

Since $: \phi_i[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}] \to \phi_i[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}]$ is valid we can use cut (theorem 1.5.3) to get that $\Delta[\tfrac{\vec{r}'}{\vec{r}}][\tfrac{\bar{q}}{\bar{p}}], \Delta_0[\tfrac{\vec{r}'}{\vec{r}}] \Vdash r[\tfrac{\vec{r}'}{\vec{r}}]$. We then substitute $\bar{r}$ for $\bar{r}'$ to get that $\Delta[\tfrac{\bar{q}}{\bar{p}}], \Delta_0 \Vdash r$.

We reason similarly if ϕ is $\phi_0 \to \phi_1$. The proof then looks like this.

$$\cfrac{(\phi_0 \to \phi_1)[\tfrac{\bar{q}}{\bar{p}}] : (\phi_0 \to \phi_1)[\tfrac{\bar{q}}{\bar{p}}] \quad \Delta_0' : \phi_0[\tfrac{\bar{q}}{\bar{p}}] \quad \Delta_0', \phi_1[\tfrac{\bar{q}}{\bar{p}}] : r}{(\phi_0 \to \phi_1)[\tfrac{\bar{q}}{\bar{p}}], \Delta_0', \Delta_1' : r}$$

Again let $r_0, \ldots, r_m$ be all the atomic formula which occur in $\Delta_0, (\phi_0 \to \phi_1)[\tfrac{\bar{q}}{\bar{p}}], r$. Let $r_0', \ldots, r_m'$ be distinct atomic constants which are disjoint from

all of the constants which occur in $\Delta, \phi_i[\frac{\bar{q}}{\bar{p}}]$. The induction hypothesis applies so we get that $\Delta_0[\frac{\bar{r}'}{\bar{r}}] \vdash_{(\phi_0 \rightarrow \phi_1)[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]} r[\frac{\bar{r}'}{\bar{r}}]$

Consider now the premisses $\phi_0[\frac{\bar{q}}{\bar{p}}] \rightarrow \phi_0$ and $\phi_1 \rightarrow \phi_1[\frac{\bar{q}}{\bar{p}}]$. We then get that $\Delta_0[\frac{\bar{r}'}{\bar{r}}], \phi_0[\frac{\bar{q}}{\bar{p}}] \rightarrow \phi_0, \phi_1 \rightarrow \phi_1[\frac{\bar{q}}{\bar{p}}] \vdash_{\phi_0 \rightarrow \phi_1} r$. This is witnessed by the following proof.

$$
\frac{\phi_0 \rightarrow \phi_1 \qquad \Delta_0[\frac{\bar{r}'}{\bar{r}}]', \phi_0[\frac{\bar{q}}{\bar{p}}] \rightarrow \phi_0 : \phi_0 \qquad \dfrac{\phi_1 \rightarrow \phi_1[\frac{\bar{q}}{\bar{p}}] \qquad \phi_1 : \phi_1 \qquad \dfrac{\phi_1[\frac{\bar{q}}{\bar{p}}], \Delta_0''[\frac{\bar{r}'}{\bar{r}}]}{r[\frac{\bar{r}'}{\bar{r}}]}}{r[\frac{\bar{r}'}{\bar{r}}]}}{r[\frac{\bar{r}'}{\bar{r}}]}
$$

Since $\Delta \Vdash \phi_0 \rightarrow \phi_1$ it then follows that

$$\Delta, \Delta_0[\frac{\bar{r}'}{\bar{r}}], \phi_0[\frac{\bar{q}}{\bar{p}}] \rightarrow \phi_0, \phi_1 \rightarrow \phi_1[\frac{\bar{q}}{\bar{p}}] \Vdash r[\frac{\bar{r}'}{\bar{r}}]$$

A canonical proof witnessing this is of lower order than Π so the induction hypothesis applies and we can conclude that

$$\Delta[\frac{\bar{q}}{\bar{p}}], \Delta_0[\frac{\bar{r}'}{\bar{r}}], \phi_0[\frac{\bar{q}}{\bar{p}}] \rightarrow \phi_0[\frac{\bar{q}}{\bar{p}}], \phi_1[\frac{\bar{q}}{\bar{p}}] \rightarrow \phi_1[\frac{\bar{q}}{\bar{p}}] \Vdash r[\frac{\bar{r}'}{\bar{r}}]$$

Since the $\bar{r}'$ are disjoint from any constants in $\Delta[\frac{\bar{q}}{\bar{p}}], \Delta$, we can again substitute $\bar{r}$ for $\bar{r}'$ to get that

$$\Delta[\frac{\bar{q}}{\bar{p}}], \Delta_0, \phi_0[\frac{\bar{q}}{\bar{p}}] \rightarrow \phi_0[\frac{\bar{q}}{\bar{p}}], \phi_1[\frac{\bar{q}}{\bar{p}}] \rightarrow \phi_1[\frac{\bar{q}}{\bar{p}}] \Vdash r$$

We use theorem 1.5.3 twice to get that

$$\Delta[\frac{\bar{q}}{\bar{p}}], \Delta_0, \Vdash r$$

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$\square$

1.5.3 Introduction rules

Having proved cut, it's now straightforward to prove that the introduction rules for intuitionistic logic are valid.

Theorem 1.5.7 (Intuitionistic introduction rules). *1. If $\Gamma \Vdash \phi$ and $\Delta \Vdash \psi$, then $\Gamma, \Delta \Vdash \phi \wedge \psi$*

2. If $\Gamma \Vdash \phi$, then $\Gamma \Vdash \phi \vee \psi$; if $\Gamma \Vdash \psi$, then $\Gamma \Vdash \phi \vee \psi$.

3. If $\Gamma, \phi \Vdash \psi$, then $\Gamma \Vdash \phi \rightarrow \psi$.

Proof. 1: Suppose $\Gamma \Vdash \phi$ and $\Delta \Vdash \psi$. Let Δ' and p be such that there is a canonical argument Π from premisses $\Delta', \phi \wedge \psi$ with conclusion p and principal major premiss $\phi \wedge \psi$. Then we must have that $\Delta', \phi, \psi \Vdash p$. Now apply cut twice to conclude that $\Gamma, \Delta, \Delta' \Vdash p$. 2 Assume that $\Gamma \Vdash \phi$; let Δ' and p be such that there is a valid canonical argument from premisses $\Delta', \phi \vee \psi$ to conclusion p with principal major premiss $\phi \vee \psi$. Then there is a canonical argument witnessing that $\Delta', \phi \Vdash p$. Apply cut to conclude that $\Gamma, \Delta' \Vdash p$. 3: Suppose that $\Gamma, \phi \Vdash \psi$. Let Δ and p be such that there is valid canonical argument Π with premisses $\Delta, \phi \rightarrow \psi$, conclusion p and major principal premiss $\phi \rightarrow \psi$. Then this argument looks like this:

$$\frac{\phi \rightarrow \psi : \phi \rightarrow \psi \qquad \Delta_0 : \phi \qquad \Delta_1, \psi : p}{\phi \rightarrow \psi, \Delta_0, \Delta_1 : p}$$

We first cut on ψ to get that $\Gamma, \phi, \Delta_1 \Vdash p$ and we then cut on ϕ to get that $\Gamma, \Delta_0, \Delta_1 \Vdash p$.

$\square$

We now know that the introduction rules for intuitionistic logic can be justified; but maybe more can be justified? It is easy to see that we cannot justify classical logic—$p \vee (p \to \bot)$, e.g., is clearly not valid; that, however, leaves it open that some intermediate logic is downwards justified. This is not the case.

In order to prove this we'll make use of the following characterization of the valid sequents, which, since it coincides with a standard sequent calculus for intuitionistic logic is of some independent interest.

Remark 1.5.8. Sequents have multisets on the left and we have seen that contraction does not hold. In what follows we'll disregard this and treat the left-hand side of a sequent as being a set. When we say that a sequent $\Gamma : \phi$ is valid, we'll mean that there is some m such that the sequent $\Gamma^m : \phi$ is valid.

Theorem 1.5.9 (Left-conditions). *1. $\Gamma, \phi \wedge \psi \Vdash \theta$ iff $\Gamma, \phi, \psi \Vdash \theta$;*

   2. $\Gamma, \phi \vee \psi \Vdash \theta$ iff both $\Gamma, \phi \Vdash \theta$ and $\Gamma, \psi \Vdash \theta$;

   3. if $\Gamma, \phi \to \psi \Vdash \theta$, then $\Gamma, \psi \Vdash \theta$; if both $\Delta \Vdash \phi$ and $\Gamma, \psi \Vdash \theta$, then $\Gamma, \Delta, \phi \to \psi \Vdash \theta$;

   4. $\Gamma, \bot \Vdash \theta$ for all θ.

Theorem 1.5.10 (Right-conditions). *1. $\Gamma \Vdash \phi \wedge \psi$ iff for all $\Gamma' \supseteq \Gamma$ and all p, (if $\Gamma', \phi \Vdash p$, then $\Gamma' \Vdash p$ and if $\Gamma', \psi \Vdash p$ then $\Gamma' \Vdash p$)*

   2. $\Gamma \Vdash \phi \vee \psi$ iff for all $\Gamma' \supseteq \Gamma$, if $\Gamma', \phi \Vdash p$ and $\Gamma', \psi \Vdash p$ then $\Gamma' \Vdash p$;

   3. $\Gamma \Vdash \phi \to \psi$ iff for all $\Gamma' \supseteq \Gamma$, if $\Delta \Vdash \phi$, then if $\Gamma', \psi \Vdash p$, then $\Gamma', \Delta \Vdash p$.

4. $\Gamma \Vdash \bot$ *iff* $\Gamma \Vdash p$ *for all p.*

Proof of theorem 1.5.9. We'll prove the left-to-right direction first. Consider the case of conjunction. Notice that it suffices to show that we can transform any canonical argument Π with conclusion $\Gamma, \phi \wedge \psi \colon p$ into a canonical argument with conclusion $\Gamma, \phi, \psi \colon p$. (For if we can do this for *any* canonical argument, we can certainly do it for canonical argument ending in $\Gamma, \Delta, \phi \wedge \psi \colon p$, where Δ is bounded by $\deg(\theta)$.)

We do this by induction on the order of the canonical argument Π such that $\Pi(\Gamma, \phi \wedge \psi) \vdash p$. So let Π be such an argument and suppose the result holds for all Π' of lower order. The base case is that $p \in \Gamma$ in which case the result obviously holds. (Use weakening.) If the principal premiss is $\bot$ the result is likewise immediate.

For the induction step there are two cases.

Case I $\phi \wedge \psi$ is the principal major premiss of Π. In this case there is a canonical subargument with conclusion $\Gamma', [\phi, \psi] \colon p$. The result follows (using weakening if necessary.)

Case II Suppose that $\phi \wedge \psi$ is not the principal major premiss; without loss of generality we may assume that $\phi \wedge \psi$ does not occur on the left in the principal major premiss. If the principal major premiss is a disjunction, $\theta_0 \vee \theta_1$, $\phi \wedge \psi$ is introduced in one of the minor premisses; the proof, e.g., is:

$$\frac{\Gamma_0, \theta_0 \vee \theta_1 \; \colon \; \theta_0 \vee \theta_1 \qquad \Gamma_1, \theta_0, \; \colon \; p \qquad \Gamma_2, \theta_1, \phi \wedge \psi \; \colon \; p}{\Gamma_0, \Gamma_1, \Gamma_2, \phi \wedge \psi \; \colon \; p}$$

In this case we apply the induction hypothesis to the canonical argument for the minor premiss. We then obtain a canonical argument:

$$\frac{\Gamma_0, \theta_0 \vee \theta_1 \; \colon \; \theta_0 \vee \theta_1 \qquad \Gamma_1, \theta_1 \; \colon \; p \qquad \Gamma_2, \theta_1, \phi, \psi \; \colon \; p}{\Gamma_0, \Gamma_1, \Gamma_2, \phi, \psi \; \colon \; p}$$

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The case where the principal major premiss in Π is a conjunction is treated in the same way. Suppose that the principal major premiss in Π is a conditional $\theta_0 \to \theta_1$. So we might assume that the proof looks like this:

$$\frac{\Gamma, \theta_0 \to \theta_1 : \theta_0 \to \theta_1 \qquad \Delta, \phi \wedge \psi : \theta_0 \qquad \Gamma', \theta_1 : p}{\Gamma, \Delta, \Gamma', \theta_0 \to \theta_1, \phi \wedge \psi : p}$$

In this case, we can use the induction hypothesis to conclude that $\Delta, \phi, \psi : \theta_0$ and that $\Gamma', \Delta, \phi, \psi, \theta_1 : p$. This application of the induction hypothesis is acceptable since the order of $\Delta, \phi \wedge \psi : \theta$ is strictly less than the order of $\Gamma, \Delta, \Gamma', \phi \wedge \psi, \theta_0 \to \theta_1$. Thus in order to check whether $\Delta, \phi, \psi : \theta$ is valid we only have to inspect arguments of order less than Π.

We thus obtain a canonical proof:

$$\frac{\Gamma, \theta_0 \to \theta_1 : \theta_0 \to \theta_1 \qquad \Delta, \phi, \psi : \theta_0 \qquad \Gamma', \theta_1 : p}{\Gamma, \Delta, \Gamma', \theta_0 \to \theta_1, \phi, \psi : p}$$

The right-to-left direction is easier. Suppose that $\Gamma, \phi, \psi : \theta$ is valid. Let Δ be such that $\deg(\Delta) \le \deg(\theta)$ and such that $\Delta \Vdash_\theta p$. Since, *ex hypothesi*, $\Gamma, \Delta, \phi, \psi : p$ is valid, let Π' be a canonical argument witnessing that. The following is a canonical argument:

$$\frac{\phi \wedge \psi : \phi \wedge \psi \qquad \overset{\Pi}{\Gamma, \Delta, \phi, \psi : p}}{\Gamma, \Delta, \phi \wedge \psi : p}$$

This shows that $\Gamma, \Delta, \phi \wedge \psi \Vdash p$; since p was arbitrary, $\Gamma, \phi \wedge \psi \Vdash \theta$ follows.

Disjunction we first consider the left-to-right direction. So suppose that $\Gamma, \phi \vee \psi : p$ is valid. Let Π be a canonical argument witnessing this fact.

Case I $\phi \vee \psi$ is the principal major premiss in Π. In this case the argument looks like this:

$$\frac{\Gamma', \phi \vee \psi : \phi \vee \psi \qquad \Gamma'', \phi : p \qquad \Gamma''', \psi : p}{\Gamma', \Gamma'', \Gamma''', \phi \vee \psi : p}$$

This means that we have that $\Gamma'', \phi \colon p$ and $\Gamma''', \psi \colon p$ are valid; using weakening (if necessary) we get that $\Gamma, \phi \colon p$ and $\Gamma, \psi \colon p$ are valid.

Case II $\phi \lor \psi$ is not principal in Π. This case is treated in the same way as Case II for conjunction.

Right-to-left. Suppose that $\Gamma, \phi \colon p$ and $\Gamma, \psi \colon p$ are valid. The following canonical argument then shows that $\Gamma, \phi \lor \psi \colon p$ is valid.

$$\frac{\phi \lor \psi \colon \ \phi \lor \psi \qquad \Gamma, \phi \colon \ p \qquad \Gamma, \psi \colon \ p}{\Gamma, \phi \lor \psi \colon \ p}$$

The conditional. Suppose, first, that $\Gamma, \phi \to \psi \colon p$ is valid. Let Π witness this. Again there are two cases.

Case I Suppose that $\phi \to \psi$ is principal in Π. Then the proof looks like this:

$$\frac{\Gamma', \phi \to \psi \colon \ \phi \to \psi \qquad \Delta, \colon \ \phi \qquad \Gamma'', \psi \colon \ p}{\Gamma', \Delta, \Gamma'', \phi \to \psi \colon \ p}$$

The result now follows by weakening from $\Gamma'', \psi \colon p$.

Case II This case is parallel to the second case for conjunction and disjunction and does not raise further problems.

Right-to-left. Suppose then that $\Delta \colon \phi$ is valid. It suffices to show that if $\Gamma, \psi \colon p$ is valid then $\Gamma, \Delta, \phi \to \psi \colon p$ is valid. But this is straightforward.

$$\frac{\phi \to \psi \colon \ \phi \to \psi \qquad \Delta \colon \ \phi \qquad \Gamma, \psi \colon \ p}{\Gamma, \Delta, \phi \to \psi \colon \ p}$$

$\square$

Proof of theorem 1.5.10. **Conjunction. Left-to-right:** Suppose that $\Gamma \colon \phi \land \psi$ is valid; we will show that if $\Gamma' \supseteq \Gamma$ is such that $\Gamma', \phi \colon p$ is valid then $\Gamma' \colon p$ is valid. So let Π be such that $\Pi(\Gamma', \phi) \vdash p$. The following witnesses that $\Gamma \Vdash_{\phi \land \psi} p$:

$$\cfrac{\phi \wedge \psi \,:\; \phi \wedge \psi \qquad \cfrac{\Pi}{\Gamma', \phi \,:\; p}}{\Gamma', \phi \wedge \psi \,:\; p}$$

The problem here is that $\Gamma', \phi\colon p$ might be of higher degree than $\phi \wedge \psi$ in which case we cannot use the validity of $\Gamma\colon \phi \wedge \psi$ to conclude that $\Gamma'\colon p$ is valid. However, the result follows by cut.

Right-to-left: assume that for all $\Gamma' \supseteq \Gamma$, if $\Gamma', \phi\colon p$ is valid then $\Gamma\colon p$ is valid (and similarly for ψ). Suppose now that $\Delta \Vdash_{\phi \wedge \psi} p$; let Π witness this. We have to show that $\Gamma, \Delta\colon p$ is valid. Since $\phi \wedge \psi$ occurs principally in Π, we have to have used $\wedge$-elimination; thus we can conclude that $\Delta, \phi, \psi\colon p$ is valid (here we might need weakening) and in particular $\Gamma, \Delta, \phi, \psi\colon p$ is valid. By (two applications of) our assumption it follows that $\Gamma, \Delta\colon p$ is valid.

Disjunction. Left-to-Right: Suppose that $\Gamma\colon \phi \vee \psi$ is valid; let $\Gamma' \supseteq \Gamma$ be given such that both $\Gamma', \phi \Vdash p$ and $\Gamma', \psi \Vdash p$. We must show that $\Gamma' \Vdash p$. We have that $\Gamma' \Vdash \phi \vee \psi$. So let Π and Π' witness that that $\Gamma, \phi \Vdash p$ and $\Gamma', \psi \Vdash p$. The following witnesses that $\Gamma' \Vdash_{\phi \vee \psi} p$.

$$\cfrac{\phi \vee \psi \,:\; \phi \vee \psi \qquad \cfrac{\Pi_\phi}{\Gamma', \phi \,:\; p} \qquad \cfrac{\Pi_\psi}{\Gamma', \psi \,:\; p}}{\Gamma', \phi \vee \psi \,:\; p}$$

Since $\Gamma\colon \phi \vee \psi$ is valid the result follows by cut.

Right-to-Left: assume that for all $\Gamma' \supseteq \Gamma$, if $\Gamma', \phi\colon p$ and $\Gamma', \psi\colon p$ are both valid, then $\Gamma'\colon p$ is valid. We must show that for all Δ and p with $\deg(\Delta) \leq \phi \vee \psi$ if $\Delta \Vdash_{\phi \vee \psi} p$, then $\Gamma, \Delta \Vdash p$. Suppose then that $\Delta \Vdash_{\phi \vee \psi} p$. Then an argument witnessing this has the following form:

$$\cfrac{\Delta_0, \phi \vee \psi \,:\; \phi \vee \psi \qquad \Delta_1, \phi \,:\; p \qquad \Delta_2, \psi \,:\; p}{\Delta_0, \Delta_1, \Delta_2, \phi \vee \psi \,:\; p}$$

It now follows, by weakening, that $\Gamma, \Delta_0, \Delta_1, \Delta_2, \phi \Vdash p$ and $\Gamma, \Delta_0, \Delta_1, \Delta_2, \psi \Vdash p$; by our assumption it then follows that $\Gamma, \Delta_0, \Delta_1, \Delta_2 \Vdash p$, which is what was

to be shown.

Conditional. Left-to-Right: Suppose that $\Gamma: \phi \to \psi$ is valid. Let Δ be such that $\Delta: \phi$ is valid. Let $\Gamma' \supseteq \Gamma$. We need to show that if $\Gamma', \psi: p$ is valid, then $\Gamma', \Delta: p$ is likewise valid. Suppose then that Π witnesses that $\Gamma', \psi \Vdash p$. Then the following argument Π' is also canonical:

$$
\frac{\phi \to \psi : \ \phi \to \psi \qquad \Delta : \ \phi \qquad \overset{\textstyle \Pi}{\Gamma', \psi : \ p}}{\Gamma', \Delta, \phi \to \psi : \ p}
$$

Since $\Gamma \Vdash \phi \to \psi$, the result now follows by cut.

Right-to-Left: Suppose that for all Δ such that if $\Delta: \phi$ is valid then if $\Gamma', \psi: p$ is valid, for $\Gamma \supseteq \Gamma$ then $\Gamma', \Delta: p$ is valid. We have to show that if Δ' and p are such that $\Delta' \Vdash_{\phi \to \psi} p$, $(\deg(\Delta) \leq \deg(\phi \to \psi))$ then $\Gamma, \Delta' \Vdash p$. So suppose that $\Delta' \Vdash_{\phi \to \psi} p$ and let Π be a valid canonical argument witnessing this. Then we may assume that Π has the form.

$$
\frac{\phi \to \psi : \ \phi \to \psi \qquad \Delta_0, : \ \phi \qquad \Delta_1, \psi : \ p}{\Delta_0, \Delta_1, \phi \to \psi : \ p}
$$

But then we have that $\Delta_0 \Vdash \phi$ and $\Delta_1, \Gamma, \psi \Vdash p$. By the assumption we then get that $\Delta_0, \Delta_1, \Gamma \Vdash p$.

Absurdity: Obvious.

$\square$

It's now rather straightforward to show completeness with respect to Kripke-semantics for intuitionistic logic. The idea behind the proof is to show that there is no least counterexample to the claim that $\Vdash$ is sound with respect to validity in Kripke-models; we show this by showing how to reduce the degree of any counterexample. First we need a lemma.

Lemma 1.5.11. *If $\Gamma \not\models \phi \vee \psi$ then there is a $\Gamma' \supseteq \Gamma$ and a p such that $\Gamma', \phi \models p$ and $\Gamma', \psi \models p$ but $\Gamma' \not\models p$ Furthermore, we can make sure that this Γ' has degree $\max\{\deg(\Gamma), \deg(\phi \vee \psi)\}$.*

Proof. Assume that $\Gamma \not\models \phi$ and $\Gamma \not\models \psi$. Pick p not occurring in Γ or in ϕ or ψ. Let Γ' be $\Gamma \cup \{\phi \to p\} \cup \{\psi \to p\}$. Clearly, $\Gamma', \phi \models p$ (similarly for ψ), but we can construct a Kripke model $\mathcal{K}$ with $\mathcal{K} \models \Gamma'$ and $K \not\models p$. $\qquad\square$

Theorem 1.5.12 (Reduction of counterexamples)**.** *If $\Gamma \not\models \phi$ but $\Gamma \colon \phi$ is valid, then there is a $\Gamma' \supseteq \Gamma$ and a ψ such that $\Gamma' \not\models \psi$ and $\Gamma' \models \psi$ is valid. Furthermore, we can pick ψ to be of lower degree than ϕ and Γ' such that $o(\Gamma' \colon \psi) < o(\Gamma \colon \phi) \# w^{\deg(\phi)+1}$.*

Proof. We call $[\Gamma, \phi]$ a *counterexample* if $\Gamma \colon \phi$ is valid but $\Gamma \not\models \phi$. Let $[\Gamma, \phi]$ be a counterexample; we show how to produce ϕ with minimal degree.

1. If ϕ is p (atomic) there is nothing to reduce.

2. Suppose $[\Gamma, \phi \wedge \psi]$ is a counterexample. We know that $\Gamma \colon \phi \wedge \psi$ is valid iff $\Gamma \colon \phi$ and $\Gamma \colon \psi$ are both valid. Since $\Gamma \not\models \phi \wedge \psi$ we must have $\Gamma \not\models \phi$ or $\Gamma \not\models \psi$ (by the definition of validity in Kripke semantics). This shows that either $[\Gamma, \phi]$ or $[\Gamma, \psi]$ is a counterexample.

3. Suppose $[\Gamma, \phi \vee \psi]$ is a counterexample. By lemma 1.5.11 above we can then find a $\Gamma' \supseteq \Gamma$ with $\deg(\Gamma') \leq \max\{\deg(\Gamma), \deg(\phi \vee \psi)\}$ such that $\Gamma' \cup \{\phi\} \models p$, $\Gamma' \cup \{\psi\} \models p$ but $\Gamma' \not\models p$. Since $\Gamma \Vdash \phi \vee \psi$, it follows from theorem 1.5.9 above that $\Gamma' \colon p$ is valid. Note that the order of Γ' is below $\omega^{\deg \phi + 1}$.

4. Suppose $[\Gamma, \phi \to \psi]$ is a counterexample. We then have that $\Gamma \not\models \phi \to \psi$ and that $\Gamma : \phi \to \psi$ is valid. By our theorem 1.5.7 above we then have that for all Δ if $\Delta : \phi$ is valid, then $\Gamma', \Delta : \psi$ is valid. Put $\Delta := \phi$. Then $\Gamma', \phi : \psi$ is valid. . It is clear that $\Gamma, \phi \not\models \psi$. For take a countermodel $\mathcal{K}$ to the claim that $\Gamma \models \phi \to \psi$. Then in $\mathcal{K}$ there is a node w_0 such that $w_0 \models \phi$ but $w_0 \not\models \psi$ (by definition of '$\models$' for Kripke-models). Then the model $\mathcal{K}' = \langle \{w | w \geq w_0\}, \geq' \rangle$, where $\geq'$ is the restriction of $\geq$ to $\{w | w \geq w_0\}$, witnesses that $[\Gamma' \cup \{\phi\}, \psi]$ is a counterexample

5. Suppose $[\Gamma, \bot]$ is a counterexample. Let $\mathcal{K}$ be a Kripke-model witnessing this. Find a node $w_0 \in \mathcal{K}$ such that $w_0 \models \Gamma$. (There is such a node, or otherwise no node in $\mathcal{K}$ forces all of Γ contradicting that $\mathcal{K}$ witnesses $\Gamma \not\models \bot$.) It follows that there is a p such that $w_0 \not\models p$. Then $[\Gamma, p]$ is our required counterexample.

$\square$

Theorem 1.5.13 (Soundness of canonical arguments)**.** *If $\Gamma : p$ is valid, then $\Gamma \models p$.*

Proof. We prove this by proving the soundness of canonical arguments by induction on their order. Let Π be a canonical argument. If Π has finite order, then Π is obviously valid: Π has the form $\Gamma : p$ where, Γ is all atomic sentences and $p \in \Gamma$.

So suppose Π is a canonical argument of degree α and that for all canonical arguments Π' of degree $\beta < \alpha$ that Π' is sound. First consider the principal premiss of Π. This has the form: $\Gamma_0, \phi : \phi$ for some ϕ. This is clearly sound. The conclusion of a canonical argument follows from its

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premisses by a clearly valid rules, so it remains to check whether the minor premisses are sound. Let $\Gamma_1 : p$ be a non-critical minor premiss, and let Π' be the canonical argument leading to $\Gamma_1 : p$. Now Π' is of lower order than Π, so by the induction hypothesis Π' is sound so $\Gamma_1 : p$ is sound.

Consider now a critical minor premiss $\Delta : \psi$. Suppose that $\Delta : \psi$ is not sound. We now use 1.5.13 to get $\Delta' \supseteq \Delta$ and p such that $[\Delta', p]$ is a counterexample of lower order than $(o(\Delta : \psi) \# \omega^{\deg(\phi)+1})$. But then $[\Delta', p]$ has lower order than Π. This contradicts the induction hypothesis.

$\square$

Theorem 1.5.14. *If $\Gamma : \phi$ is valid then $\Gamma \models \phi$.*

Proof. A straightforward induction on the complexity of ϕ. If ϕ is atomic, it follows directly from theorem 1.5.13. If ϕ is $\psi_0 \wedge \psi_1$, $\psi_0 \vee \psi_1$ or $\psi_0 \rightarrow \psi_1$; suppose that $[\Gamma, \phi]$ is a counterexample. Use 1.5.12 to get a counterexample $[\Gamma', p]$, and then apply theorem 1.5.13 to get a contradiction. $\square$

1.5.5 Doing without the conditional

For the verificationist meaning-theory in § 1.3) we had to use boundary rules; for the pragmatist meaning-theory developed here this has not been necessary. Clearly, the availability of the conditional and *modus ponens* is the reason for this. For consider the language of $\vee$. In this language the sequent $: p \vee q$ is valid! For in the absence of the conditional, nothing except p follows (canonically) from p and nothing except q follows (canonically) from q.

If we extend our definition of a canonical argument to encompass boundary rules between atomic formulæ this example is easy to deal with: by

adding boundary rules of the form $p : r$, $q : r$ we witness the invalidity of $: p \lor q$. This example is interesting for another reason as well. In the proof of theorem 1.5.6 (that canonical arguments are closed under uniform substitution) we relied on the presence of conditionals in the language. This example shows that in the absence of conditionals we may have failure of substitution. It is natural to conjecture that as long as we have boundary rules the class of valid sequents will be closed under substitution. I will prove this in part 2 (theorem 2.2.13).

This is an interesting example for a third reason: it shows that when we add new connectives to a language $\mathcal{L}$, sequents from $\mathcal{L}$ which were valid prior to the extension may become invalid after the extension. From the perspective of an adherent of molecularity like Dummett this would be deeply problematic. Whether a sequent $\Gamma : \phi$ is valid should turn only on the meanings of the terms which figure in Γ, ϕ and not on the meaning of connectives which are not in Γ, ϕ. It turns out that once we have boundary rules we can never make a sequent invalid by adding new vocabulary (propositions 2.2.15 and 2.2.16.)

2

General Rules

In the previous chapter I've established that intuitionistic logic is complete with respect to both the intuitionistic introduction and elimination rules. This amounts to a justification of the inversion principle in terms of the theory of canonical arguments. In this chapter I will consider arbitrary introduction and elimination rules.

In order to set the stage for what follows, it will be useful to consider two alternative accounts of harmony.

Dummett (1991, p. 219) considers what he calls "total harmony". This notion is defined as follows. Let $\mathcal{L}$ be a language and λ a connective not in $\mathcal{L}$. Let L be the logic associated with $\mathcal{L}$ and L' be the logic associated with the language $\mathcal{L} \cup \{\lambda\}$. The rules for the connective λ are *totally harmonious* relative to $\mathcal{L}$ iff whenever $\phi \in \mathcal{L}$ and ϕ is provable in L' then ϕ is provable in L.

There are two problems with this account of harmony. First, the relativity to background language. A connective λ can be harmonious relative to a language $\mathcal{L}$ but be inharmonious with respect to the language $\mathcal{L}'$. Ideally, we would like an *absolute* account of harmony. Second, and more seriously, the connectives λ and λ' may be individually harmonious with respect to $\mathcal{L}$, while being jointly inharmonious.

Another account of harmony takes a language $\mathcal{L}$ (generated by the connectives $\lambda_0, \lambda_1, \ldots, \lambda_n$) to be harmonious when every proof in $\mathcal{L}$ can be normalized. But the normalizability of a language $\mathcal{L}$ is a *global* feature of the language. Normalizability need not be preserved when we add new vocabulary to the language.[1]

Is the account of harmony in terms of verificationist and pragmatist meaning-theories free of these defects?

In this part I will define the class of general introduction and elimination rules; I will extend the notion of canonical argument to take into account general introduction and elimination rules and I will show that the result of adding a general introduction (elimination) rule to language always results in a conservative extension of that language.

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[1] For a more sustained discussion of problems like these see (Steinberger, forthcoming)

The second question concerns what we may call the "maximality of intuitionistic logic". It has been intimated that intuitionistic logic is the strongest logic which can be given a proof-theoretic meaning theory.[2] I will here show that any connective which can be given a proof-theoretic meaning theory can be *defined* in terms of the intuitionistic connectives $\wedge, \vee, \rightarrow, \perp$.

If one thinks that the proof-theoretic account of the meanings of the logical constants *itself* is part of an argument against classical logic, this result is of particular importance. It's all well and good to note that classical logic is not in harmony with the intuitionistic introduction (elimination) rules, but this does not rule out that classical logic is in harmony with some other introduction (elimination) rules.[3]

2.1 General Introduction Rules

What is an introduction rule? In general, an introduction rule for a connective λ is an *argument-schema* such that the conclusion of the argument-schema has λ as its dominant operator. Here, I will only consider introduction rules which are *pure*, *simple* and *single-ended* (Dummett, 1991, pp. 256–8). That is,

1. an introduction rule for the connective λ is not simultaneously an elimination rule for a (different) connective λ' (single-endedness);

2. only λ figures in an introduction rule for the connective λ;[4]

---

[2] (Peregrin, 2008; Zucker and Tragesser, 1978; and Schroeder-Heister, 1983, 1985) develop this idea in different ways. Due to considerations of space I will not compare the present approach to theirs.

[3] As noted above this is not my view. The proof-theoretic account of the meanings of the logical connectives has a more modest goal. In any case, the results are of significant technical interest.

[4] This demand is perhaps excessive; we should be able to allow that the introduction rules

$$\frac{\ldots \Delta^i_j, \Sigma^i_j : \phi^i_0, \phi^i_1, \ldots, \phi^i_{n_i} \qquad \Delta^i_{j+1}, \Sigma^i_{j+1} : \phi^i_0, \ldots, \phi^i_{n_i} \cdots \qquad \Delta^{i+1}_0, \Sigma^{i+1}_0 : \phi^{i+1}_0, \ldots, \phi^{i+1}_{n_{i+1}} \cdots}{\Delta^0_0, \ldots, \Delta^k_{m_k} : \lambda(\Sigma^0_0, \Sigma^0_1, \ldots, \Sigma^0_{m_0}, \Sigma^k_{m_{k-1}}, \Sigma^k_{m_k}, \phi^0_0, \phi^0_1, \ldots, \phi^k_{n_k - 1}, \phi^k_{n_k})}$$

Figure 2.1.1: A general introduction rule

3. any connective figuring in an introduction rule figures as the dominant operator in the conclusion of that rule.

In order to improve readability I'll introduce the following notation. We say that a sequent $\Sigma : \phi_0, \phi_1, \ldots, \phi_n$ is valid iff each of the sequents $\Sigma : \phi_0, \ldots, \Sigma : \sigma_n$ is valid. A general introduction rule then has the form depicted in figure 2.1.1.

Note that a general introduction rule is *wholly schematic*: no logical connectives can make an essential appearance in the premisses of the introduction rule.

It will be convenient for typographical reasons to write general introduction rules as Prawitz-style natural deduction rules. For those purposes, I'll use

$$\begin{array}{c} \Sigma \\ \vdots \\ \phi_0, \phi_1, \ldots, \phi_n \end{array}$$

to be short for

$$\begin{array}{ccc} \Sigma & \Sigma \ldots & \Sigma \\ \vdots & \vdots & \vdots \\ \phi_0 & \phi_1 \cdots & \phi_n \end{array}$$

are partially ordered, so that rules higher in the order have premisses which essentially involve a connective which is governed by a rule occurring earlier in the ordering. It would be of interest to extend the framework to deal with ordered sets of connectives.

Some examples may help. Note first that the introduction rules for $\wedge, \vee$ and $\rightarrow$ are all general introduction rules according to the above definition.[5]

The following is an introduction rule for the bi-conditional.

$$
\dfrac{\overline{\phi}^{\ 1} \qquad \overline{\psi}^{\ 2}}{\begin{array}{cc} \vdots & \vdots \\ \psi & \phi \end{array}}
$$
$$
\dfrac{\psi \qquad \phi}{\phi \leftrightarrow \psi}\ {}_{1,2}
$$

The following is a general introduction rule for a what looks like the compound connective:

$$(\phi_0 \vee \phi_1 \rightarrow \psi) \wedge (\phi_2 \wedge \phi_3 \rightarrow \theta \wedge \sigma)$$

$$
\dfrac{\overline{\phi_0}^{\ 0} \qquad \overline{\phi_1}^{\ 1} \qquad \overline{\phi_2,\phi_3}^{\ 2} \qquad \overline{\phi_2,\phi_3}^{\ 3}}{\dfrac{\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ \psi & \psi & \theta & \sigma \end{array}}{\lambda(\phi_0,\phi_1,\phi_2,\phi_3,\psi,\theta,\sigma)}\ {}_{0,1,2,3}}
$$

Definition 2.1.1. A connective λ is associated with l-many introduction rules of the form depicted in figure 2.1.1 subject to the condition that any immediate subformula ϕ of $\lambda(\bar{\phi})$ occurs in at least one introduction rule for λ.

If the connective λ is associated with l introduction rules, we'll write the l'th introduction rule in the following way:

---

[5]The rule for $\perp$, on the other hand, is not wholly schematic. We, in effect, treat $\perp$ as the conjunction of all *atomic* sentences. Being an atomic sentence is not a schematic condition.

$$\frac{\begin{array}{ccc} \dots \Sigma_j^{l,i} & \Sigma_{j+1}^{l,i} \dots & \Sigma_0^{l,i+1} \dots \\ \vdots & \vdots & \vdots \\ \dots \phi_0^{l,i}, \phi_1^{l,i}, \dots, \phi_{n_i}^{l,i} & \phi_0^{l,i}, \dots, \phi_{n_i}^{l,i} \dots & \phi_0^{l,i+1}, \dots, \phi_{n_{i+1}}^{l,i+1} \dots \end{array}}{\lambda(\Sigma_0^{l,o}, \Sigma_1^{l,o}, \dots, \Sigma_{m_o}^{l,o}, \Sigma_{m_{k-1}}^{l,k}, \Sigma_{m_k}^{l,k}, \phi_0^{l,o}, \phi_1^{l,o}, \dots, \phi_{n_k-1}^{l,k}, \phi_{n_k}^{l,k})}$$

To improve legibility we'll often write $\lambda(\bar{\Sigma}, \bar{\phi})$ for $\lambda(\Sigma_0^o, \dots, \Sigma_k^k, \phi_o, \dots, \phi_l)$.

2.1.1 CANONICAL ARGUMENTS

We again have to give a definition of canonical arguments relative to boundary rules. We define the notions "Π is an S-canonical argument" and "$\Gamma: \phi$ is an α, S-valid sequent" simultaneously by recursion. As before α_S is the closure of the set of verifiers α under the boundary rules in S.

Definition 2.1.2. Let α be a set of verifiers; and let S be a collection of boundary rules. Let Π be an argument with conclusion $\alpha: \phi$. Π is an S-*valid canonical argument* iff either

1. Π is the one-place argument $\alpha^-, p^i: p$ for p atomic and $p^i \in V_p$.

2. $p^i \in \alpha_S$, for some $p^i \in V_p$.

3. ϕ is $\lambda(\Sigma_0^o, \Sigma_1^o, \dots, \Sigma_{m_o}^o, \Sigma_{m_{k-1}}^k, \Sigma_{m_k}^k, \phi_0^o, \phi_1^o, \dots, \phi_{n_k-1}^k, \phi_{n_k}^k)$ and

   (a) the last rule applied in Π is λ-introduction; and

   (b) for all $i \leq k, j \leq m_k$, the sequent $\Delta_j^i, \Sigma_j^i: \phi_o^i, \dots, \phi_{n_i}^i$ is α, S-valid.

Definition 2.1.3. A sequent $\Gamma: \phi$ is α, S-*valid* iff either

1. it is $\Gamma^-, p^i: p$ for p atomic and $p^i \in V_p$; or

2. $p^i \in \alpha_S$ for some $p^i \in V_p$; or

3. for all $\beta \supset \alpha$, and all $S' \supset S$: if for all $\psi \in \Gamma$ there is a β, S'-valid canonical argument Π_ψ with conclusion $\beta \colon \psi$, then there is a β, S'-valid canonical argument Π with conclusion $\beta \colon \phi$.

We now go on to give a general characterization of the connectives given by general introduction rules.

Theorem 2.1.4. *Let λ be a connective associated with l introduction rules. We then have $\alpha \Vdash_S \lambda(\Sigma_0^0, \ldots, \Sigma_{m_k}^k, \phi_0^0, \ldots, \phi_{n_k}^k)$ iff for some $l_0 \leq l$ and for all $\beta \supseteq \alpha$ and for all i, j and all $S' \supset S$ if $\beta \Vdash_{S'} \Sigma_j^{l_0,i}$, then $\beta \Vdash_{S'} \phi_0^{l_0,i}, \phi_1^{l_0,i}, \ldots, \phi_1^{l_0,i}$ where $\Sigma_j^i = \Sigma_j^{l_0,i}$ and $\phi_j^i = \phi_j^{l_0,i}$ for all i, j.*

Proof. Suppose first that $\alpha \Vdash_s \lambda(\Sigma_0^0, \Sigma_1^0, \Sigma_n^n \ldots, \phi_0^0, \ldots \phi_{k_n}^n)$. Pick a canonical proof Π witnessing this. Π ends with an application of λ-introduction. Suppose that the introduction rule applied was

$$
\begin{array}{ccc}
\ldots \Sigma_j^{l_0,i} & \ldots \Sigma_{j+1}^{l_0,i} & \Sigma_0^{l_0,i+1} \\[4pt]
\vdots & \vdots & \vdots \\[4pt]
\phi_0^{l_0,i}, \phi_1^{l_0,i}, \ldots, \phi_{n_{l_0,i}}^{l_0,i} & \phi_0^{l_0,i}, \ldots \phi_{n_{i_0,i}}^{l_0,i} & \phi_0^{l_0,i+1}, \ldots, \phi_{n_{l_0,i+1}}^{l_0,i+1}
\end{array}
$$
$$
\overline{\qquad\lambda(\Sigma_0^{l_0,0}, \ldots, \Sigma_{m_{l_0,k}}^{l_0,k}, \phi_0^{l_0,0}, \phi_1^{l_0,1}, \ldots, \phi_{n_{l_0,k}}^{l_0,k})\qquad}
$$

That means that for each $\beta \supseteq \alpha$, and $S' \supset S$ if $\beta \Vdash_{S'} \Sigma_j^{l_0,i}$ then $\beta \Vdash_{S'} \phi_0^{l_0,i}, \ldots \phi_{n_{l_0,i}}^{l_0,i}$. This is what we require to show.

For the other direction, suppose that for all $\beta \supseteq \alpha$, $S' \supset S$ and for all i, j, if $\beta \Vdash_{S'} \Sigma_j^{l_0,i}$, then $\beta \Vdash_{S'} \phi_0^{l_0,i}, \phi_1^{l_0,i}, \ldots, \phi_{n_{l_0,i}}^{l_0,i}$ for some l_0. Then by the induction hypothesis we have that the sequents $\alpha, \Sigma_j^{l_0,i} \colon \phi_0^{l_0,i}, \phi_1^{l_0,i}, \ldots, \phi_{n_{l_0,i}}^{l_0,i}$ are α, S valid. This shows that the premisses of the l_0'th introduction rule

for λ are α, S-valid, and $\lambda(\Sigma_0^{l_0,0}, \ldots, \Sigma_{m_{l_0,k}}^{l_0,k}, \phi_0^{l_0,0}, \ldots, \phi_{n_{l_0,k}}^{l_0,k})$ now follows by λ-introduction. $\square$

2.1.3 Maximality of intuitionistic logic

Definition 2.1.5. Let $\mathcal{L}$ be a language containing the intuitionistic connectives $\wedge, \vee, \rightarrow, \bot$ as well as some general connectives $\lambda_0, \lambda_1, \ldots, \lambda_n$. We define the translation $*$ as follows. Let $\lambda(\Sigma_0^0, \Sigma_1^0, \ldots, \Sigma_{m_0}^0, \Sigma_{m_{k-1}}^k, \Sigma_{m_k}^k, \phi_0^0, \phi_1^0, \ldots, \phi_{n_k-1}^k, \phi_{n_k}^k)^*$ be

$$\bigvee_{l_0 \le l} (\bigwedge_i (\bigvee_j \bigwedge \Sigma_l^{l_0,i} * \rightarrow \bigwedge_{r \le n_i} \phi_r^{l_0,i} *))$$

(Here we assume that this is written out using binary conjunctions and disjunctions.)

Theorem 2.1.6. *Let $\mathcal{L}$ be as in the definition above. And let $\wedge, \vee, \rightarrow, \bot$ be governed by the usual intuitionistic introduction rules. Then $\Gamma \Vdash_S \phi$ iff $\Gamma * \Vdash_S \phi*$.*

Proof. Note first that if suffices to show the result for the special case where Γ is a collection of atomic formulæ α. For suppose that $\Gamma \Vdash_S \phi$. Now let $\beta \supset \alpha$ and $S' \supset S$ be given such that $\beta \Vdash_{S'} \Gamma*$. Then $\beta \Vdash_{S'} \Gamma$ by the special case and hence $\beta \Vdash_{S'} \phi$, since $\Gamma \Vdash_S \phi$. But then $\beta \Vdash_{S'} \phi*$ by the special case again, which is what we require to show. The converse direction is proved similarly.

So suppose that ϕ is $\lambda(\Sigma_0^0, \Sigma_1^0, \ldots, \Sigma_{m_0}^0, \Sigma_{m_{k-1}}^k, \Sigma_{m_k}^k, \phi_0^0, \phi_1^0, \ldots, \phi_{n_k-1}^k, \phi_{n_k}^k)$ and let $\alpha \Vdash_S \lambda(\Sigma_0^0, \Sigma_1^0, \ldots, \Sigma_{m_0}^0, \Sigma_{m_{k-1}}^k, \Sigma_{m_k}^k, \phi_0^0, \phi_1^0, \ldots, \phi_{n_k-1}^k, \phi_{n_k}^k)$. Then we may assume that the proof has the following form:

$$\dots \Sigma_j^{l_o,i} \qquad\qquad \dots \Sigma_{j+1}^{l_o,i} \qquad\qquad \Sigma_o^{l_o,i+1}$$

$$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$$

$$\frac{\phi_o^{l_o,i},\phi_1^{l_o,i},\dots,\phi_{n_{l_o,i}}^{l_o,i} \qquad \phi_o^{l_o,i},\dots\phi_{n_{i_o,i}}^{l_o,i} \qquad \phi_o^{l_o,i+1},\dots,\phi_{n_{l_o,i+1}}^{l_o,i+1}}{\lambda(\Sigma_o^{l_o,o},\dots,\Sigma_{m_{l_o,k}}^{l_o,k},\phi_o^{l_o,o},\phi_1^{l_o,1},\dots,\phi_{n_{l_o,k}}^{l_o,k})}$$

By the induction hypothesis we have that $\Sigma_j^{l_o,i}* \Vdash \phi_o^{l_o,i}*,\dots,\phi_{n_i,i}^{l_o,i}*$ for all i,j.

It follows intuitionistically that $\bigwedge \Sigma_j^{l_o,i}* \to \bigwedge \phi^{l_o,i}*$ for each i,j. It then follows intuitionistically that $\bigvee_j \bigwedge \Sigma_j^{l_o,i}* \to \bigwedge \phi^{l_o,i}*$. It then follows intuition-istically that $\bigwedge_i(\bigvee_j \bigwedge \Sigma_j^{l_o,i}* \to \bigwedge \phi^{l_o,i})*$. Finally

$$\bigvee_{l_o \le l}(\bigwedge_i(\bigvee_j \bigwedge \Sigma_j^{l_o,i}* \to \bigwedge_{r \le n_i} \phi_r^{l_o,i}*))$$

follows by repeated application of $\vee$-introduction.

The other direction is dealt with as follows. Suppose $\beta \Vdash_S \bigvee_{l_o \le l}(\bigwedge_i(\bigvee_j \bigwedge \Sigma_j^{l_o,i}* \to \bigwedge_{r \le n_i} \phi_r^{l_o,i}*))$. Find a canonical proof witnessing this. The canonical proof ends with applications of $\vee$-introduction. So we can assume that we have a canonical proof of, e.g., $(\bigwedge_i(\bigvee_j \bigwedge \Sigma_j^{l_o,i}* \to \bigwedge_{r \le n_i} \phi_r^{l_o,i}*))$. Since the last rule applied here is $\wedge$-introduction, for each i we have to have a canonical proof of $\bigvee_j \bigwedge \Sigma_j^{l_o,i}* \to \bigwedge_{r \le n_i} \phi_r^{l_o,i}*$. And that means that for each $\beta_o \supset \beta$ and each $S' \supset S$, if $\beta_o \Vdash_{S'} \bigvee_j \bigwedge \Sigma_j^{l_o,i}*$, then $\beta_o \Vdash_{S'} \bigwedge_{r \le n_i} \phi_r^{l_o,i}*$. In other words, for each j if $\beta_o \Vdash_{S'} \Sigma_j^{l_o,i}*$, then $\beta_o \Vdash_{S'} \phi_r^{l_o,i}*$, for each $r \le n_i$. But that means that the argument

$$\Sigma_j^{l_o,i}*$$

$$\phi^{l_o,i}*,\dots,\phi_{n_i}^{l_o,i}*$$

is β,S-valid for each i,j. It then follows by the induction hypothesis that

$$\Sigma_j^{l_0,i}$$

$$\phi^{l_0,i}, \ldots, \phi_{n_i}^{l_0,i}$$

is β, S-valid for each i, j. $\beta \Vdash_s \lambda(\Sigma_0^{l_0,0}, \ldots, \Sigma_{m_{l_0,k}}^{l_0,k}, \phi_0^{l_0,0}, \phi_1^{l_0,1}, \ldots, \phi_{n_{l_0,k}}^{l_0,k})$ now follows by λ-introduction. □

This result just shows that any generalized connective λ in a language $\mathcal{L}$ is definable in terms of connectives $\wedge, \vee, \rightarrow, \bot$ governed by the intuitionistic introduction rules *in the language $\mathcal{L}$*. This result does *not* show any generalized connective is definable in terms of connectives obeying only the rules of intuitionistic logic. For it could be that in the presence of the connective λ the connectives $\wedge, \vee, \rightarrow, \bot$ obey a different logic than intuitionistic logic. This is not so, however, as is shown by the next two results.

Let $\mathcal{L}$ be a language generated by some connectives over the atomic sentences $p_0, p_1, \ldots$ and let the logic of the language $\mathcal{L}$ be generated by the introduction rules for the connectives of $\mathcal{L}$ in accordance with the procedures above and let $\Vdash$ be the forcing relation associated with this logic. Now let $\mathcal{L}'$ be a language extending $\mathcal{L}$ with some new connectives (but no new atomic sentences), and let $\Vdash'$ be the forcing relation associated with this logic.

Proposition 2.1.7 (Conservativity). *If Γ, ϕ are in the language $\mathcal{L}$ and $\Gamma \Vdash'_S \phi$, then $\Gamma \Vdash_S \phi$.*

Proposition 2.1.8 (Monotonicity). *Suppose that Γ, ϕ are in the language $\mathcal{L}$ and that $\Gamma \Vdash_S \phi$. Then $\Gamma \Vdash'_S \phi$.*

These propositions are proved simultaneously by induction.

Proof of proposition 2.1.7. Note first that if suffices to prove that if $\alpha \Vdash'_S \phi$, with $\phi \in \mathcal{L}$, then $\alpha \Vdash_S \phi$. For suppose that $\Gamma \Vdash'_S \phi$ with $\Gamma, \phi \subset \mathcal{L}$. Now let α and $S_o \supset S$ be such that $\alpha \Vdash_{S_o} \Gamma$. By monotonicity, we then have $\alpha \Vdash'_{S_o} \Gamma$ and hence $\alpha \Vdash'_{S_o} \phi$. The special case then gives us that $\alpha \Vdash_{S_o} \phi$, which is what we require to show.

So suppose that $\alpha \Vdash'_S \phi$. The proof is by induction on ϕ. If the canonical argument witnessing this is an axiom or a boundary rule, it's immediate that $\alpha \Vdash_S \phi$. So suppose that ϕ is $\lambda(\bar{\Sigma}, \bar{\psi})$ and that the final rule applied is λ-introduction. The proof then takes the form:

$$\begin{array}{ccc} \dots \Sigma^i_l & \Sigma^i_{l+1} & \Sigma^{i+1}_o \\ \vdots & \vdots & \vdots \\ \phi^i_o, \phi^i_1, \dots, \phi^i_{n_i} & \phi^i_o, \dots, \phi^i_{n_i} & \phi^{i+1}_o, \dots, \phi^{i+1}_{n_{i+1}} \end{array}$$
$$\overline{\qquad \lambda(\Sigma^o_o, \Sigma^o_1, \dots, \Sigma^k_{m_k}, \phi^o_o, \phi^o_1, \dots, \phi^k_{n_k-1}, \phi^k_{n_k}) \qquad}$$

It then suffices to show that $\alpha, \Sigma^i_j \Vdash_S \phi^i_o, \dots, \phi^i_{n_i}$ for each i and j. So let $\beta \supset \alpha$ and $S_o \supset S$ be given and suppose that $\beta \Vdash_{S_o} \Sigma^i_j$. Then by monotonicity we have $\beta \Vdash'_{S_o} \Sigma^i_j$. Hence $\beta \Vdash'_{S_o} \phi^i_o, \dots, \phi^i_{n_i}$. The induction hypothesis now gives us that $\beta \Vdash_{S_o} \phi^i_o, \dots, \phi^i_{n_i}$.

$\square$

Proof of proposition 2.1.8. Again it suffices to show the result for the special case $\alpha \Vdash_S \phi$. For suppose that $\Gamma \Vdash_S \phi$. Now let β, S_o be such that $\beta \Vdash'_{S_o} \Gamma$. By conservativity we have $\beta \Vdash_{S_o} \Gamma$ and hence $\beta \Vdash_{S_o} \phi$. $\beta \Vdash'_{S_o} \phi$ now follows by the special case.

We show by induction on the order of the canonical argument witnessing $\alpha \Vdash_S \phi$, that $\alpha \Vdash'_S \phi$.

If the argument is an axiom or a boundary rule there is nothing to show. Suppose then that the argument ends with an application of λ-introduction. The argument then takes the form:

$$\begin{array}{ccc}
\ldots\Sigma_l^i & \Sigma_{l+1}^i & \Sigma_0^{i+1} \\
\vdots & \vdots & \vdots \\
\phi_0^i,\phi_1^i,\ldots,\phi_{n_i}^i & \phi_0^i,\ldots,\phi_{n_i}^i & \phi_0^{i+1},\ldots,\phi_{n_{i+1}}^{i+1} \\
\hline
\end{array}$$
$$\lambda(\Sigma_0^0,\Sigma_1^0,\ldots,\Sigma_{m_k}^k,\phi_0^0,\phi_1^0,\ldots,\phi_{n_k-1}^k,\phi_{n_k}^k)$$

Let $\beta \supset \alpha$ and $S_0 \supset S$ be given. It suffices to show that if $\beta \Vdash_{S_0} \Sigma_j^i$ then $\beta \Vdash_{S_0} \phi_0^i,\ldots,\phi_{n_i}^i$. For then $\alpha,\Sigma_j^i \Vdash'_S \phi_0^i,\ldots,\phi_{n_i}^i$, for each i,j.

$$\alpha \Vdash'_S \lambda(\Sigma_0^0,\Sigma_1^0,\ldots,\Sigma_{m_k}^k,\phi_0^0,\phi_1^0,\ldots,\phi_{n_k-1}^k,\phi_{n_k}^k)$$

follows by λ-introduction.

But this is straightforward. For suppose that $\beta \Vdash'_{S_0} \Sigma_j^i$. Then $\beta \Vdash_{S_0} \Sigma_j^i$ by conservativity. Hence $\beta \Vdash_{S_0} \phi_0^i,\ldots,\phi_{n_i}^i$. $\beta \Vdash'_{S_0} \phi_0^i,\ldots,\phi_{n_i}^i$ follows by the induction hypothesis. $\qquad\square$

Taken together these results show that intuitionistic logic is the strongest logic which can be validated by a verificationist meaning theory based on general introduction rules. I now turn towards establishing similar results for pragmatist meaning theories.

2.2 GENERAL ELIMINATION RULES

A general elimination rule is depicted in figure 2.2.1. The premisses $\Delta_j^i,\Sigma_j^i\colon \psi_i$ are the *critical* minor premisses. The premisses $\Gamma_i,\Theta_i\colon p$ are non-critical mi-

$$\frac{\Gamma \colon \lambda(\Sigma_0^0, \ldots, \Sigma_{m_k}^k, \psi_0, \ldots, \psi_k, \Theta_0, \ldots, \Theta_n) \qquad \ldots \Delta_j^i, \Sigma_j^i \colon \psi_j^i \ldots \qquad \Gamma_0, \Theta_0 \colon p \ldots \Gamma_n, \Theta_n \colon p}{\lambda(\Sigma_0^0, \ldots, \Sigma_{m_k}^k, \psi_0, \ldots, \psi_k, \Theta_0, \ldots, \Theta_n), \Gamma, \Delta_0, \ldots, \Delta_m, \Gamma_0, \ldots, \Gamma_n \colon p}$$

Figure 2.2.1: A General Elimination Rule

nor premisses. I'll often write $\lambda(\bar{\Sigma}, \bar{\phi}, \bar{\Theta})$ for $\lambda(\Sigma_0^0, \ldots, \Sigma_{m_k}^k, \phi_0, \ldots, \phi_k, \Theta_0, \ldots, \Theta_n)$.

A connective λ is associated with any finite number of elimination rules. We demand that any formula which occurs immediately in $\lambda(\bar{\Sigma}, \bar{\psi}, \bar{\Theta})$ occurs in at least one minor premiss in one of the elimination for λ.

Some examples of general elimination rules may help. Note that Σ_j^i can be empty. This ensures that the standard elimination rules for $\wedge, \vee, \rightarrow, \bot$ count as generalized elimination rules.

Here are the general elimination rules for the biconditional:

$$\frac{\Gamma \colon \phi \leftrightarrow \psi \qquad \Delta_0 \colon \phi \qquad \Gamma_0, \psi \colon p}{\Gamma, \Delta_0, \Gamma_0 \colon p} \text{ and } \frac{\Gamma \colon \phi \leftrightarrow \psi \qquad \Delta_0 \colon \psi \qquad \Gamma_0, \phi \colon p}{\Gamma, \Delta_0, \Gamma_0 \colon p}$$

Here's the general elimination rule for the compound connective $((\phi \vee \psi) \rightarrow \theta) \rightarrow \sigma_0 \vee \sigma_1$.

$$\frac{\Gamma \colon \lambda(\phi, \psi, \theta, \sigma_0, \sigma_1) \qquad \Delta_0, \phi \colon \theta \quad \Delta_1, \psi \colon \theta \qquad \Gamma_0, \sigma_0 \colon p \quad \Gamma_1, \sigma_1 \colon p}{\Gamma, \Delta_0, \Delta_1, \Gamma_0, \Gamma_1 \colon p}$$

2.2.1 CANONICAL ARGUMENTS

We extend the definition of canonical argument and valid sequent. The definitions are complicated somewhat by the need to accommodate boundary rules.

Definition 2.2.1. The *degree* of a formula $\lambda(\Sigma_0, \ldots \Sigma_m, \psi_0, \ldots \psi_m, \Theta_0, \ldots \Theta_n)$ is $\max \{\deg(\Sigma_0), \ldots, \deg(\Sigma_m), \deg(\psi_0), \ldots, \deg(\psi_m), \deg(\Theta_0), \ldots, \deg(\Theta_n)\} + 1$.

The *degree* of a sequent is the maximum of the degree of the formulæ occurring in the sequent. The order of a formula ϕ is $\omega^{\deg \phi}$. The order of a sequent is the Hessenberg sum of the orders of the formulæ occurring in the sequent. The order of an argument is the order of the final sequent in the argument.

We now extend the definition of canonical argument to encompass general elimination rules and boundary rules. In this case we can treat boundary rules as relating atomic sentences. It will in fact be easiest to treat the boundary rules as sequents.

Definition 2.2.2. A *boundary sequent* is a sequent $\Gamma : p$, where Γ is a (possibly empty) multiset of atomic sentences.

Definition 2.2.3. S is a collection of *boundary rules* iff S is a collection of boundary sequents which are closed under *cut*: if $p_0, \ldots p_i, \ldots, p_n : p \in S$ and $q_0, \ldots, q_m : p_i \in S$ then $p_0, \ldots, q_0, \ldots, q_m, \ldots, p_n : p \in S$. If S, S' be two collections of boundary rules. We will write S, S' for the least collection of boundary rules containing the both S and S'.

Let $\mathcal{L}$ be a language and let $\mathcal{R}_{\mathcal{L}}$ be some general elimination rules for the connectives in $\mathcal{L}$; let S be a collection of boundary rules. We'll define the notions of S-canonical argument and S-valid sequent simultaneously by recursion.

Definition 2.2.4 (Definition of Valid Canonical Argument). An argument Π is S-canonically valid if

1. Π is the one-step argument $\Gamma : p$, where $p \in \Gamma$; or

2. Π is the one-step argument $\Gamma, \Gamma_o : p$ where $\Gamma_o : p$ is a boundary sequent in S; or

3. Π has a principal premiss $\Gamma : \phi$, such that $\phi \in \Gamma$ and

   (a) If ϕ is $\lambda(\Sigma_o^o, \dots, \Sigma_{m_k}^k, \psi_o, \dots, \psi_k, \Theta_o, \dots, \Theta_n)$, each critical sequent $\Delta_j^i, \Sigma_j^i : \psi_i$ is S-valid and for each non-critical sequent $\Gamma_i, \Theta_i : p$ the subargument Π_i to $\Gamma_i, \Theta_i : p$ is S-canonical.

Definition 2.2.5 (Definition of Valid Sequent). A sequent $\Gamma : \phi$ is $S{-}valid$ iff either

1. The sequent is $\Gamma, p : p$; or

2. The sequent is $\Gamma', \Gamma_o : p$ where $\Gamma_o : p$ is a boundary sequent in S; or

3. for every collection of boundary rules S_o and every Δ and p and every S_o-valid canonical argument Π (of degree at most $\deg(\phi)$) such that

   (a) the conclusion of Π is $\Delta, \phi : p$; and

   (b) the principal major premiss of Π is $\Delta', \phi : \phi$, for some $\Delta' \subseteq \Delta$;

4. there is a S_o, S-canonical argument Π' such that the conclusion of Π' is $\Gamma, \Delta_* : p$, where Δ_* is a subset of Δ^m for some m.

We'll now write $\Pi(\Gamma \vdash_{\phi,S}^{\mathcal{R}} p)$ for the claim that Π is an S-canonical argument with principal premiss ϕ and conclusion $\Gamma, \phi : p$. We'll write $\Gamma \vdash_{\phi,S}^{\mathcal{R}} p)$ for the claim there is Π such that $\Pi(\Gamma \vdash_{\phi,S}^{\mathcal{R}} p)$. We'll write $\Gamma \Vdash_S^{\mathcal{R}} p$ for the claim that there is $\phi \in \Gamma$ such that $\Gamma \setminus \{\phi\} \vdash_{\phi,S}^{\mathcal{R}} p$. Finally we'll write $\Gamma \Vdash_S^{\mathcal{R}} \phi$ for the claim that $\Gamma : \phi$ is S-valid. When the language and the set of rules is

clear from context I'll drop the superscripts and write simply $\Gamma \vdash_{\phi,S} p$ etc.
For typographical reason I will often write $\Gamma \Vdash^S_\phi$ for $\Gamma \Vdash_{\phi,S}$.

Observation 2.2.6. *If $\Gamma \Vdash_S \phi$ and $S_0 \supset S$, then $\Gamma \Vdash_{S_0} \phi$.*

Proof. Left to the reader. □

There are now two tasks ahead of us. The first is to show a cut-elimination result analogous to theorem 1.5.2 above. The second is to show that when we augment a language $\mathcal{L}$ governed by the general elimination rules $\mathcal{R}$ with a new connective λ governed by the general elimination rule R that $\Vdash^{\mathcal{R} \cup R}_S$ is a conservative extension of $\Vdash^{\mathcal{R}}_S$ for each S.

I have not been able to obtain these results in their full generality, but I have been able to obtain these results when we restrict our attention to elimination rules where the Σ_i in the critical minor premisses are empty. Henceforth when I speak of a general elimination rule I mean rules of this form. For definiteness, such rules take the following form:

$$\frac{\Gamma : \lambda(\psi_0, \ldots, \psi_k, \Theta_0, \ldots, \Theta_n) \qquad \ldots \Delta_0, : \psi_0 \ldots \Delta_k : \psi_k \qquad \Gamma_0, \Theta_0 : p \ldots \Gamma_n, \Theta_n : p}{\Gamma, \Delta_0, \ldots, \Delta_k, \Gamma_0, \ldots, \Gamma_n : p}$$

I conjecture that the results hold for the general case as well.

Connectives governed by rules which have critical minor premisses are *conditional-like* or *conditional*. Such connectives will play a very important rôle in what follows because they allow us to imitate the behavior of conditionals. Let λ be a connective one introduction rule of which has m critical minor premisses, and n non-critical minor premisses. λ can imitate the

intuitionistic conditional as follows.

$$\lambda(\overbrace{\phi,\ldots,\phi}^{m},\overbrace{\psi,\ldots,\psi,\ldots,\psi,\ldots,\psi}^{n}) \tag{2.2.1}$$

serves to imitate a conditional $\phi \to \psi$. Note that this formula has the same degree as a regular conditional $\phi \to \psi$. These "mock conditionals" will be very important for establishing both cut-elimination and the conservativity properties. To improve legibility I will write mock conditionals like (2.2.1) above simply as $\phi \longrightarrow \psi$.

2.2.2 CUT-ELIMINATION

The strategy for proving cut-elimination is essentially the same as above. As before we have to establish some technical lemmas.

Lemma 2.2.7. *Let Π be a canonical argument with principal premiss ϕ. Let $p_0, p_1, \ldots, p_n$ be $n+1$ distinct atomic formulæ that occur in Π. Let $q_0, \ldots, q_n$ be $n+1$ distinct atomic formulæ such that no q_j occurs in Π. Let Π' be obtained from Π by uniformly substituting the atomic formulæ $\bar{q}$ for the atomic formulæ $\bar{p}$. Then Π' is a canonical argument with principal premiss $\phi[\frac{\bar{q}}{\bar{p}}]$.*

Proof. Analogous to the proof of lemma 1.5.4. (Left to the reader.) □

Lemma 2.2.8. *Let S be a set of boundary rules and let $S_0 \supset S$. Suppose*

$$\Gamma, [\phi_i \longrightarrow q_i]_{i \in I}, [q_i]_{i \in I} \Vdash_{S_0} \psi$$

where each q_i is not $\perp$ and does not occur in any of the ϕ_i, ψ or Γ. Suppose

further that no sequent which is in S_o but not in S contains an atomic formula which occurs in Γ, ϕ_i or ψ. Then $\Gamma \Vdash_S \psi$.

Corollary 2.2.9. *Suppose $\Gamma \Vdash_{S_o} \phi$ where $S_o \supset S$ is such that it only differs from S over sequents which contain formulæ which are in neither Γ nor ϕ. Then $\Gamma \Vdash_S \phi$.*

Proof of lemma 2.2.8. As is the case with the analogous lemma 1.5.5 it suffices to prove the case where ψ is the atomic. The proof is by induction on the order of the canonical argument Π witnessing that

$$\Gamma, [\phi_i \longrightarrow q_i]_{i \in I}, \dots, [q_i]_{i \in I} \Vdash_{S_o} p$$

The base cases are straightforward. If Π takes the form $\Gamma', p \colon p$, then, by assumption, p isn't one of the q_i. If Π takes the form $\Gamma', \Gamma_o \colon p$, where $\Gamma_o \colon p$ is a boundary rule in S_o then this boundary rule has to be in S, for the boundary rules which are in S_o but not in S don't contain p.

The case where the final rule applied in Π is a non-conditional elimination rule follows immediately by the induction hypothesis. In the case, where the last rule applied is a conditional-like elimination rule there are two cases.

First, the principal major premiss is $\phi_i \longrightarrow q_i$, for some i. Without loss of generality we may assume that the argument looks like this:

$$\frac{\phi_i \longrightarrow q_i \colon \phi_i \longrightarrow q_i \qquad \Delta_o \colon \phi_i \qquad \Gamma_o, [\phi_j \longrightarrow q_j]_{j \in I \setminus \{i\}}, [q_j]_{j \in I \setminus \{i\}} \colon p}{\phi_i \longrightarrow q_i, \Delta_o, \Gamma_o, [\phi_j \longrightarrow q_j]_{j \in I \setminus \{i\}}, [q_j]_{j \in I \setminus \{i\}} \colon p}$$

The induction hypothesis then gives us that $\Gamma_o \colon p$ is S valid; the result follows by weakening.

Suppose then that the principal premiss is some other conditional-like statement. By familiar reasoning it suffices to check that the critical minor

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premisses are valid. Suppose that $\Delta, [\phi_i \longrightarrow q_i]_{i \in I}, [q_i]_{i \in I} : \psi$ is a critical minor premiss which is S_0-valid. We have to show that $\Delta : \psi$ is S-valid. Now let S_1, Δ_0, q_0 be such that $\Delta_0 \vdash^{S_1}_{\psi} q_0$.

Let $\bar{r}$ be the collection of atomic sentences such that if $r \in \bar{r}$ then r occurs in S_0 but not in S. Consider now $\bar{r}, \bar{q}_i$. Let $\bar{r}'$ be some distinct atomic constants which occur in neither $\Delta, \Delta_0, \phi_i, \psi, q_0, S_1, S_0$. By lemma 2.2.7 we then have that $\Delta_0[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}] \Vdash^{S_1[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}]}_{\psi} q_0[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}]$. Since $\Delta, [\phi_i \longrightarrow q_i]_{i \in I}, [q_i]_{i \in I} \Vdash_{S_0} \psi$ we get that $\Delta, [\phi_i \longrightarrow q_i]_{i \in I}, [q_i]_{i \in I}, \Delta_0[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}] \Vdash_{S_0, S_1[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}]} q_0[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}}]$. Since $S_0, S_1[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}]$ only differs from $S, S_1[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}]$ over sequents which contain no formula occurring in $\Delta, \Delta_0[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}], q_0[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}]$ the induction hypothesis applies and we get that $\Delta, \Delta_0[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}] \Vdash_{S, S_1[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}]} q_0[\genfrac{}{}{0pt}{}{\bar{r}'}{\bar{r}, \bar{q}_i}]$. Substituting $\bar{r}, \bar{q}_i$ for $\bar{r}'$ we then obtain $\Delta_0, \Delta \Vdash_{S, S_1} q$ which is what we're required to show. □

Proposition 2.2.10. *Let L be the language generated by the connectives $\lambda_0, \ldots, \lambda_n$ where the rules $\mathcal{R}$ governing the λ_i are all non-conditional. Suppose that $\Gamma \Vdash_S p$. Then there is a proof $\Pi(\Gamma \Vdash_S p)$ such that if $\phi_0, \phi_1, \ldots, \phi_n$ is a sequence of principal major premisses occurring in Π such that ϕ_0 is a top-most principal major premiss and ϕ_n is the last principal major premiss in Π and such that ϕ_{i+1} occurs lower in the proof for each $i < n$, then $\deg(\phi_i) \leq \deg(\phi_{i+1})$, for all $i < n$.*

Proof. It suffices to show how we can permute principal major premisses upwards. The following example shows how we can do that. (Here $\lambda_0(\Theta_0, \Theta_1)$ is, in effect $\bigwedge \Theta_0 \wedge \bigwedge \Theta_1$ and $\lambda_1(\Sigma_0, \Sigma_1)$ is, in effect, $\bigvee \Sigma_0 \vee \bigvee \Sigma_1$.)

$$\frac{\lambda_0(\Theta_0, \Theta_1): \lambda_0(\Theta_0, \Theta_1) \qquad \dfrac{\lambda_1(\Sigma_0, \Sigma_1): \lambda_1(\Sigma_0, \Sigma_1) \qquad \Gamma_0, \Theta_0, \Sigma_0 : p \qquad \Gamma_1, \Sigma_1 : p}{\Gamma_0, \Gamma_1, \lambda_1(\Sigma_0, \Sigma_1), \Theta_0 : p}}{\Gamma_0, \Gamma_1, \lambda_1(\Sigma_0, \Sigma_1), \lambda_0(\Theta_0, \Theta_1) : p}$$

is transformed into

$$\dfrac{\lambda_0(\Theta_0,\Theta_1)\colon \lambda_0(\Theta_0,\Theta_1) \qquad \Gamma_0,\Theta_0,\Sigma_0\colon p}{\Gamma_0,\lambda_0(\Theta_0,\Theta_1),\Sigma_0\colon p}$$

$$\dfrac{\lambda_1(\Sigma_0,\Sigma_1) \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \Gamma_1,\Sigma_1\colon p}{\Gamma_0,\Gamma_1,\lambda_1(\Sigma_0,\Sigma_1),\lambda(\Theta_0,\Theta_1)\colon p}$$

$\square$

Corollary 2.2.11. *Let $\mathcal{L}$ be a language without conditional connectives. Let $\Pi(\Gamma \Vdash_S p)$ be a canonical proof. Then there is a canonical proof $\Pi'(\Gamma' \Vdash_S p)$ with $\Gamma' \subset \Gamma$ such that every formula occurring in Π' is a subformula of a principal major premiss in Π' or else is the conclusion of a boundary rule where the antecedent only contains subformulæ of Γ'.*

Proof. Immediate from proposition 2.2.10 $\qquad\qquad\qquad\qquad\qquad\qquad \square$

Theorem 2.2.12. *Let $\mathcal{L}$ be the language generated by the connectives $\lambda_0,\dots,\lambda_n$. Let the connectives $\{\lambda_i\}_{0\leq i\leq n}$ be associated with general elimination rules $\mathcal{R}$. Let $L_{\mathcal{R}}$ be the logic generated by the rules $\mathcal{R}$. Then $L_{\mathcal{R}}$ has cut-elimination in the following strong sense.*

If $\Gamma,\phi \Vdash_S \psi$ and $\Delta \Vdash_{S_0} \phi$ then $\Gamma,\Delta \Vdash_{S,S_0} \psi$.

Proof. We will again prove the result for the special case where ψ is an atomic formula p. For suppose that the result holds for that special case. Now suppose that $\Gamma,\phi \Vdash_S \psi$ and $\Delta \Vdash_{S_0} \phi$ Let Δ_0 and S_1 be such that $\Delta_0 \vdash_{\psi,S_1} p$ (with $\deg(\Delta) \leq \deg(\psi)$). Then $\Gamma,\Delta_0,\phi \Vdash_{S,S_1} p$. By the special case of cut we then get that $\Gamma,\Delta,\Delta_0 \Vdash_{S,S_0,S_1} p$ which is what we're required to show.

So let's prove the special case. The proof is by induction on the degree of the cut formula ϕ with a subsidiary induction on the order of the canonical argument Π witnessing that $\Gamma,\phi \Vdash p$.

Consider first the case where ϕ is not principal in Π. Without loss of generality we may then assume that Π looks like this:

$$\frac{\lambda_0(\psi,\Theta_0,\Theta_1)\colon \lambda_0(\psi,\Theta_0,\Theta_1) \qquad \Delta_0,\phi\colon \psi \qquad \Gamma_0,\Theta_0\colon p \quad \Gamma_1,\Theta_1\colon p}{\lambda_0(\psi,\Theta_0,\Theta_1),\Delta_0,\Gamma_0,\Gamma_1,\phi\colon p}$$

For if the cut-formula ϕ occurs in either of the non-critical minor premisses the induction hypothesis applies immediately. We then have to show that $\Delta,\Delta_0 \Vdash_{S,S_0} \psi$. This is now routine. Suppose that S_1 and Γ,q are such that $\deg(\Gamma) \leq \deg(\psi)$ and $\Gamma \vdash_{\psi,S_1} q$. Then since $\Delta_0,\phi \Vdash_S \psi$, we get that $\Gamma,\Delta,\phi \Vdash_{S,S_1} q$. A canonical argument witnessing this has lower order than Π; it follows by the induction hypothesis that $\Gamma,\Delta,\Delta_0 \Vdash_{S,S_0,S_1} q$ which is what we have to show.

Suppose then that ϕ is principal in Π. The case where the proof is an axiom of the form $\Gamma,p\colon p$ is straightforward. $\Gamma,\Delta \Vdash_{S,S_0} p$ follows by weakening from $\Delta \Vdash_{S_0} p$.

Suppose next that Π is a one-step argument of the form $\Gamma,p_0,\dots,p_i,\dots,p_m\colon p$ where $p_0,\dots,p_i,\dots,p_m\colon p$ is a S-boundary rule and $\Delta \Vdash p_i$.

We use induction on the argument Π_0 witnessing that $\Delta \Vdash p_i$. If Π_0 is an axiom $\Delta',p_i\colon p_i$ the result is immediate. If Π_0 is by a S_0 boundary rules $q_0,\dots,q_n\colon p_i$, the result follows from the fact that $p_0,\dots,p_{i-1},q_0,\dots,q_n,p_{i+1},\dots,p_m\colon p$ by definition is a S,S_0-boundary rule.

Otherwise, suppose that the principal major premiss in Π_0 has λ as the dominant connective for some λ, and the final rule applied is λ-elimination. Without loss of generality, we may assume that Π_0 looks like this:

$$\frac{\lambda(\theta,\Theta)\colon \lambda(\theta,\Theta) \qquad \Delta'\colon \theta \qquad \Gamma',\Theta\colon p_i}{\Delta',\Gamma',\lambda(\theta,\Theta)\colon p_i}$$

The canonical argument to $\Gamma',\Theta\colon p_i$ is S_0-valid and of lower order than the order of Π_0. The induction hypothesis thus applies and we get that

$\Gamma', \Gamma', \Theta, p_0, \ldots, p_{i-1}, p_i, \ldots, p_m \Vdash_{S, S_0} p$. Since $\Delta'_{S, S_0} \theta$ The following proof then witnesses that $\Gamma, \Delta \Vdash p$.

$$\frac{\lambda(\theta, \Theta): \lambda(\theta, \Theta) \qquad \Delta': \theta \qquad \Gamma', \Theta, \Gamma, p_0, \ldots, p_{i-1}, p_{i+1}, \ldots p_m: p}{\Gamma, p_0, \ldots, p_{i-1}, p_{i+1}, \ldots p_m, \Delta: p}$$

(Recall that Δ is $\Gamma', \Delta', \lambda(\theta, \Theta)$.)

Suppose then that ϕ is a complex formula and that ϕ is principal in the proof Π. The proof splits into two cases depending on whether the language $\mathcal{L}$ has conditional connectives or not.

If $\mathcal{L}$ does not have conditional connectives we use proposition 2.2.10 and corollary 2.2.11 to get a canonical argument Π_0 witnessing $\Gamma, \phi \Vdash p$ such that, Π_0, the only occurrences of the cut-formula ϕ is as principal major premiss in arguments Π_ϕ such that Π_ϕ has degree less than or equal to ϕ. The conclusion of such an argument Π_ϕ will be of the form $\Gamma^{\Pi_\phi}, \phi: r$. Since $\Delta \Vdash_{S_0} \phi$ it follows that $\Gamma^{\Pi_\phi}, \Delta \Vdash_{S, S_0} q$. Let Π'_ϕ witness this. Replace each argument Π_ϕ in Π_0 with Π'_ϕ; in this way we obtain a canonical argument witnessing $\Gamma, \Delta \Vdash_{S, S_0} p$.

Suppose, then, that $\mathcal{L}$ has a conditional connective λ_0. We will need to use mock-conditionals; as above we will write these $\phi \longrightarrow \psi$.

Without loss of generality, we can assume that ϕ is $\lambda(\psi, \Theta_0, \Theta_1)$. Suppose that $\Pi(\Gamma, \lambda(\psi, \Theta_0, \Theta_1) \Vdash_S p)$ with $\lambda(\psi, \Theta_0, \Theta_1)$ principal in Π. Suppose further that $\Delta \Vdash_{S_0} \lambda(\psi, \Theta_0, \Theta_1)$.

Π takes the following form:

$$\frac{\lambda(\psi, \Theta_0, \Theta_1): \lambda(\psi, \Theta_0, \Theta_1) \qquad \Delta_0: \psi \qquad \Gamma_0, \Theta_0: p \quad \Gamma_1, \Theta_1,: p}{\lambda(\psi, \Theta_0, \Theta_1), \Delta_0, \Gamma_0, \Gamma_1, \Phi: p}$$

If the degree of Π is less than or equal to $\lambda(\psi, \Theta_0, \Theta_1)$, $\Gamma, \Delta \Vdash_{S_0, S} p$ follows immediately from $\Delta \Vdash_{S_0} \lambda(\psi, \Theta_0, \Theta_1)$.

So suppose that Π is of strictly greater degree than $\lambda(\psi, \Theta_0, \Theta_1)$. We will first construct a set Δ^+ with $\deg(\Delta^+) \leq \deg(\lambda(\psi, \Theta_0, \Theta_1))$ and a set of boundary rules S_1 such that $\Delta^+ \vdash^{S_1}_{\lambda(\psi, \Theta_0, \Theta)} q$.

1. Let $P_{\Theta_i} = \{p_\theta : \theta \in \Theta_i\}$ be a set of fresh atomic constants for $i = 0, 1$.

2. Let $P_{\Gamma_i} = \{p_\gamma : \gamma \in \Gamma_i\}$ be another set of fresh constants for $i = 0, 1$.

3. Let $\Theta_i \Longrightarrow P_{\Theta_i} = \{\theta \longrightarrow p_\theta : \theta \in \Theta_i\}$.

Let Δ^+ be $\psi, \Theta_i \Longrightarrow P_{\Theta_i, i=0,1}, P_{\Gamma_i, i=0,1}$.

We let S_1 be S_0 and in addition the rules

1. $P_{\Gamma_0}, P_{\Theta_0} : q$ and

2. $P_{\Gamma_1}, P_{\Theta_1} : q$ where q is a fresh constant.

The following proof witnesses that $\Delta^+ \vdash^{S_1}_{\lambda(\psi, \Theta_0, \Theta_1)} q$.

$$\frac{\lambda(\psi, \Theta_0, \Theta_1) \qquad \psi : \psi \qquad \Theta_i \Longrightarrow P_{\Theta_i}, \Theta_i, P_{\Gamma_i} : q}{\lambda(\phi, \Theta_0, \Theta_1), \Delta^+) : q}$$

Since $\deg(\Delta^+) \leq \deg(\lambda(\psi, \Theta_0, \Theta_1))$ and $\Delta \Vdash_{S_0} \lambda(\phi, \Theta_0, \Theta_1)$ we get that $\Delta, \Delta^+ \Vdash_{S_0, S_1} q$. Let Π_0 be a canonical proof witnessing that $\Delta, \Delta^+ \Vdash_{S_0, S_1} q$. We will use this proof to construct a proof witnessing $\Gamma, \Delta \Vdash_{S, S_0} p$.

First, we transform the proof Π_0 into the proof Π_1 as follows. If $\theta \longrightarrow p_\theta, \sigma : \sigma$ is a principal major premiss occurring in Π_0 we replace this premiss with $\sigma : \sigma$. Suppose that $\Delta', P'_{\Theta_i}, \Theta_i \Longrightarrow P_{\Theta_i}, P'_{\Gamma_i} : \sigma$ is a critical minor premiss in Π_0. (Where $P'_{\Gamma_i} \subset P_{\Gamma_i}$ and $P'_{\Theta_i} \subset P_{\Theta_i}$.) Then σ does not contain any formula from P_{Θ_i} or P_{Γ_i}: if it did, there would be formulæ containing fresh constants in the antecedent position of Δ, and that's impossible. But then the conditions of lemma 2.2.8 are satisfied, and we get that $\Delta' : \sigma$ is S_1 valid.

So we can replace any critical minor premiss $\Delta', P_{\Phi'}, P'_{\Theta_i}, \Theta_i \implies P_{\Theta_i}, P'_{\Gamma_i} : \sigma$ in Π_o with $\Delta' : \sigma$. Let Π_1 be the resulting proof.

In Π_1 all formulæ of the form $\theta \longrightarrow p_\theta$, occur as principal major premisses. Now let $T = \Gamma_o^T : q, \ldots, \Gamma_n^T : q$ be a track in Π_1. Since q does not occur on the left of the conclusion of Π_1 at all, the top node of the track T—that is, $\Gamma_o^T : q$— has to be a boundary axiom of the form $\Gamma', P_{\Gamma_i}, P_{\Theta_i}, : q$. Now since none of the p_θ occur in the conclusion of Π_o—that is, in $\Gamma_n^T : q$—except in a formula of the form $\theta \longrightarrow p_\theta$ we know that each occurrence of p_θ is discharged by an application of $\longrightarrow$-elimination with major premiss $\theta \longrightarrow p_\theta$. Let j be largest such that Γ_j^T is of the form $\Gamma', P_{\Theta_i}, P_{\Gamma_i} : q$. Then for all p_θ in P_{Θ_i}, there is a critical minor premiss $\Sigma_\theta, [\psi] : \theta$ below Γ_j^T which discharges p_θ. Moreover, since Σ_θ and θ don't contain any formulæ in P_{Θ_i} and P_{Γ_i}, we in fact have that $\Sigma_\theta \Vdash_{S_o} \theta$

Let's write $\Theta_i = \theta_o^i, \ldots, \theta_{m_i}^i$. We have $\Sigma_{\theta_j} \Vdash_{S,S_o} \theta_j$ for each $j \leq m_i$. We have that $\Gamma_i, \Theta_i \, Vdash_S p$. Since each formula in Θ_i is of lower degree than $\lambda(\psi, \Theta_o, \Theta_1)$ the induction hypothesis (applied $|\Theta_i|$-many times) gives us that

$$\Gamma_i, \overbrace{\psi, \ldots, \psi}^{|\Theta_i|}, \Sigma_{\theta_o}, \ldots, \Sigma_{\theta_{m_i}} \Vdash_{S,S_o} p$$

Since $\Delta_o \Vdash_S \psi$, and ψ is also of lower degree than $\lambda(\psi, \Theta_o, \Theta_1)$ the induction hypothesis allows us to cut again to get

$$\Gamma_i, \overbrace{\Delta_o, \ldots, \Delta_o}^{|\Theta_i|}, \Sigma_{\theta_o}, \ldots, \Sigma_{\theta_{m_i}} : q$$

Let Π^T be a canonical proof witnessing this. Replace the sequent $\Gamma_j^T : q$

with the concluding sequent of Π^T and replace the argument above this sequent with Π^T. Replace any occurrences of q on the right after Γ_j with p. Now delete all applications of $\theta \longrightarrow p_\theta$-elimination in Π_1 below $\Gamma_i \colon q$ but keep all other elimination rules. Let Π_2 be the resulting proof. Π_2 witnesses that $\Gamma, \Delta_0, \ldots, \Delta_n \Vdash_{S, S_0} p$.

$\square$

Armed with cut, we can establish that the class of canonical arguments is closed under arbitrary substitution.

Theorem 2.2.13. *Let Π be an S-canonical argument with principal premiss ϕ. Let Π' be obtained from Π by uniformly substituting the atomic formulæ $\bar{q}$ for the atomic formulæ $\bar{p}$. Then Π' is an S' canonical argument with principal premiss $\phi[\frac{\bar{q}}{\bar{p}}]$. (Where S' is the result of substituting $\bar{q}$ for $\bar{p}$ in S.)*

Proof of theorem 2.2.13. The proof is analogous to the proof of theorem 1.5.6. Suppose that Π is an S-canonical argument with principal premiss ϕ. We prove the result by induction on the order of Π. The base cases are straightforward. The induction steps for non-conditional connectives follow immediately by the induction hypothesis.

The problematic case is where the principal major premiss of Π is conditional-like. Without loss of generality, we can assume that the proof looks like this.

$$\frac{\lambda(\phi, \Theta) \colon \lambda(\phi, \Theta) \qquad \Delta \colon \phi \qquad \Gamma, \Theta \colon p}{\lambda(\phi, \Theta), \Delta, \Gamma \colon p}$$

By the induction hypothesis, we get that there is a valid canonical argument witnessing $\Gamma[\frac{\bar{q}}{\bar{p}}], \Theta[\frac{\bar{q}}{\bar{p}}] \Vdash_{S_0} p[\frac{\bar{q}}{\bar{p}}]$. (Here S_0 is the result of substituting $\bar{q}$ for $\bar{p}$ in S.) We therefore only have to show that $\Delta[\frac{\bar{q}}{\bar{p}}] \colon \phi[\frac{\bar{q}}{\bar{p}}]$ is S_0-valid.

The proof is by induction on ϕ. If ϕ is atomic there is a canonical argument Π_0 witnessing that $\Delta \Vdash_S \phi$. This canonical argument has lower order than Π. It follows by the induction hypothesis that the sequent $\Delta[\frac{\bar{q}}{\bar{p}}] : \phi[\frac{\bar{q}}{\bar{p}}]$ is S_0-valid.

Suppose, then, that ϕ is complex. We'll use a variation of the technique we used to prove substitution when dealing only with the language of intuitionistic propositional logic. Without loss of generality we can assume that ϕ has the form $\lambda(\phi_0, \Theta_0, \Theta_1)$.

Let Δ_0, q and S_1 be such that $\Delta_0 \vdash^{S_1}_{\lambda(\phi_0[\frac{\bar{q}}{\bar{p}}], \Theta_0[\frac{\bar{q}}{\bar{p}}]\Theta_1[\frac{\bar{q}}{\bar{p}}])} q$. Let Π_1 be a S_1-canonical argument witnessing this. We have to show that $\Delta[\frac{\bar{q}}{\bar{p}}], \Delta_0 \Vdash_{S_0, S_1} q$.

Since we are considering a critical minor premiss there has to be at least one conditional like connective in the language.

Now let $r_0, \ldots, r_m$ be all the atomic formulæ which occur in $\Delta_0, \lambda(\phi_0[\frac{\bar{q}}{\bar{p}}], \Theta_0[\frac{\bar{q}}{\bar{p}}], \Theta_1[\frac{\bar{q}}{\bar{p}}])$ and q. Let $r'_0, \ldots, r'_m$ be distinct atomic constants which are disjoint from all of the constants which occur in Δ and $\lambda(\phi_0[\frac{\bar{q}}{\bar{p}}], \Theta_0[\frac{\bar{q}}{\bar{p}}]\Theta_1[\frac{\bar{q}}{\bar{p}}]), S_1$ and q. Since the order of Π_1 is lower than the order of Π, the induction hypothesis applies and we get that

$$\Delta_0[\tfrac{\bar{r}'}{\bar{r}}] \vdash^{S_1[\frac{\bar{r}'}{\bar{r}}]}_{\lambda(\phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \Theta_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \Theta_1[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}])} q[\tfrac{\bar{r}'}{\bar{r}}]$$

We can assume that this proof has the form:

$$\frac{\lambda(\phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \Theta_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]\Theta_1[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]) \qquad \Delta'_0[\frac{\bar{r}'}{\bar{r}}] : \phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \qquad \Gamma'_i[\frac{\bar{r}'}{\bar{r}}], \Theta_i[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] : q[\frac{\bar{r}'}{\bar{r}}]}{\lambda(\phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \Theta_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \Theta_1[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]), \Delta'_0[\frac{\bar{r}'}{\bar{r}}], \Gamma'_0[\frac{\bar{r}'}{\bar{r}}] : q[\frac{\bar{r}'}{\bar{r}}]}$$

We now add the premises $\phi[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \longrightarrow \phi$ and $\theta \longrightarrow \theta[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]$, for each $\theta \in \Theta_i$. Let's write $\Theta_i \Longrightarrow \Theta_i[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]$ for the latter collection of premises.

It is easily seen that $\phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \longrightarrow \phi_0 \Vdash \phi_0$. By cut we then get $\Delta'_0[\frac{\bar{r}'}{\bar{r}}], \phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \longrightarrow \phi_0 \Vdash_{S_1} \phi_0$. It is also easy to see that $\Theta_i, \Gamma'_0[\frac{\bar{r}'}{\bar{r}}], \Theta_i \Longrightarrow$

$\Theta_i[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \Vdash q[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]$.

Putting all this together we obtain the following proof.

$$\frac{\lambda(\phi_0, \Theta_0, \Theta_1) \qquad \Delta_0'[\frac{\bar{r}'}{\bar{r}}], \phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \longrightarrow \phi_0 : \phi_0 \qquad \Gamma_0'[\frac{\bar{r}'}{\bar{r}}], \Theta_i \Longrightarrow \Theta_i[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \Theta_i : q[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]}{q[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]}$$

Since $\Delta \Vdash_S \lambda(\phi_0, \Theta_0, \Theta_1)$ and the above proof has degree at most $\deg(\lambda(\Sigma, \phi_0, \Theta_0, \Theta_1))$ we get that

$$\Delta, \Delta_0'[\frac{\bar{r}'}{\bar{r}}], \phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \longrightarrow \phi, \Gamma_0'[\frac{\bar{r}'}{\bar{r}}], \Theta_i \Longrightarrow \Theta_i[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \Vdash q[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \Vdash_{S, S_1[\frac{\bar{r}'}{\bar{r}}]} q[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]$$

The proof witnessing this has lower order than the proof than the proof Π with which we started. By the induction hypothesis we can therefore substitute $\bar{q}$ for $\bar{p}$ and then $\bar{r}'$ for $\bar{r}$ to get the following.

$$\Delta[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \Delta_0'[\frac{\bar{r}'}{\bar{r}}], \phi_0[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \longrightarrow \bar{\phi}[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]), \Gamma_0'[\frac{\bar{r}'}{\bar{r}}], \Theta_i[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \Longrightarrow \Theta_i[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}] \Vdash_{S_0[\frac{\bar{r}'}{\bar{r}}], S_1[\frac{\bar{r}'}{\bar{r}}]} q[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]$$

Since $\Vdash \phi \longrightarrow \phi$ for any ϕ we can use theorem 2.2.12 and the fact that $\Delta_0[\frac{\bar{r}'}{\bar{r}}] = \Delta_0'[\frac{\bar{r}'}{\bar{r}}], \Gamma_0'[\frac{\bar{r}'}{\bar{r}}], \Gamma_1'[\frac{\bar{r}'}{\bar{r}}]$ to get that

$$\Delta[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}], \Delta_0[\frac{\bar{r}'}{\bar{r}}] \Vdash_{S_0[\frac{\bar{r}'}{\bar{r}}], S_1[\frac{\bar{r}'}{\bar{r}}]} q[\frac{\bar{r}'}{\bar{r}}][\frac{\bar{q}}{\bar{p}}]$$

Substituting $\bar{r}$ for r' we then obtain

$$\Delta[\frac{\bar{q}}{\bar{p}}], \Delta_0 \Vdash_{S_0, S_1} q$$

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which is what we're required to show.

□

2.2.3 Conservativity Results

Let $\mathcal{L}$ be a language generated by some connectives $\lambda_0,\ldots,\lambda_n$ governed by some rules $\mathcal{R}$. Let $\mathcal{L}^+$ be a language extending $\mathcal{L}$ where the connectives in $\mathcal{L}^+$ but not in $\mathcal{L}$ are governed by the rules $\mathcal{R}^+$ and the connectives both in $\mathcal{L}$ and $\mathcal{L}^+$ are governed by $\mathcal{R}$. We use $\Vdash_S$ for validity in the logic generated by $\mathcal{R}$ and we use $\Vdash_S^+$ for validity in the logic generated by $\mathcal{R} \cup \mathcal{R}^+$.

We would now like to extend prove conservativity results analogous to those in section 2.1.2. The key to this is the following theorem.

Theorem 2.2.14. *Let $\mathcal{L}$ be a language generated by some connectives $\lambda_0,\ldots,\lambda_n$ governed by the general elimination rules $\mathcal{R}$. Suppose that $\Delta \nVdash_S \phi$; then there is S_0 and Δ_0 such that all the connectives in Δ_0 occur in Δ, ϕ and such that $\Delta_0 \Vdash_{\phi,S_0} p$ but $\Delta, \Delta_0 \nVdash_{S,S_0} p$.*

Using this theorem we can prove the following two propositions simultaneously by induction.

Proposition 2.2.15. *Suppose $\mathcal{L} \subseteq \mathcal{L}^+$ and that $\Gamma \vdash_{\phi,S} p$. Then $\Gamma \vdash_{\phi,S}^+ p$*

Proposition 2.2.16. *Suppose $\Gamma \vdash_{\phi,S}^+ p$ where $\Gamma, \phi \subset \mathcal{L}$, then $\Gamma \vdash_{\phi,S} p$.*

Proof of proposition 2.2.15. Suppose that Π witnesses that $\Gamma \vdash_{\phi,S} p$. The only problematic case is where there are critical subarguments. Suppose that $\Delta: \phi$ is a critical subargument in Π. Now suppose that $\Delta \nVdash_S^+ \psi$. Then by theorem 2.2.14 there is a Δ_0 and an S_0 such that $\Delta_0 \vdash_{\psi,S_0}^+ p$ but $\Delta_0, \Delta \nVdash_{S_0,S}^+ p$ where Δ_0 contains only connectives from Δ and ψ. But then $\Delta_0 \subseteq \mathcal{L}$. And

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since the argument witnessing that $\Delta_0 \vdash^+_{\psi,S_0} p$ is of lower order than Π it follows by proposition 2.2.16 that $\Delta_0 \vdash_{\psi,S_0} p$. Since $\Delta_0 \Vdash_S \psi$, it follows that $\Delta_0, \Delta \Vdash_{S_0,S} p$. But then it follows by the induction hypothesis that $\Delta_0, \Delta \Vdash^+_{S,S_0} p$. Contradiction. Hence $\Delta \Vdash^+_S \psi$. $\qquad\square$

Proof of proposition 2.2.16. Suppose that $\Gamma \Vdash^+_\phi p$, where $\Gamma, \phi \subset \mathcal{L}$. Let Π witness this. The problematic case is, again, where Π contains critical minor sequents. So let $\Delta: \psi$ be a critical minor sequent occurring in Π. We have to show that $\Delta \Vdash_S \psi$. So suppose that Δ_0, S_0 are such that $\Delta_0 \vdash_{\psi,S_0} p$. Then by proposition 2.2.15 we get that $\Delta_0 \vdash^+_{\psi,S_0} p$. Hence we get that $\Delta_0, \Delta \Vdash^+_{S,S_0} p$. By the induction hypothesis we get that $\Delta, \Delta_0 \Vdash_{S,S_0} p$. $\qquad\square$

So we will prove theorem 2.2.14. The technique is similar to the technique used in the proof of cut-elimination above (theorem section 2.2.12))

Proof of theorem 2.2.14. Suppose that $\Delta \nVdash_S \phi$. There are two cases. Suppose first that a conditional connective occurs in Δ, ϕ; we'll use this connective to create mock conditionals. As before we'll write these conditionals using $\longrightarrow$.

Now, let S_0^+, Δ_0^+ and p witness that $\Delta \nVdash_S \phi$. Assume, without loss of generality, that ϕ is of the form $\lambda(\psi, \Theta_0, \Theta_1)$. The proof Π witnessing that $\Delta_0^+ \vdash_{\phi,S_0^+} p$ looks like this:

$$\frac{\lambda(\psi,\Theta_0,\Theta_1): \lambda(\psi,\Theta_0,\Theta_1) \qquad \Gamma: \psi \qquad \Gamma_0, \Theta_0: p \quad \Gamma_1, \Theta_1: p}{\lambda(\psi,\Theta_0,\Theta_1), \Gamma, \Gamma_0, \Gamma_1: p}$$

Now let $P_{\Gamma_i} = \left\{ p_\gamma : \gamma \in \Gamma_i \right\}$ where the p_γ are fresh atomic constants. Let $\Pi_{\Theta_i} = \{ p_\theta : \theta \in \Theta_i \}$, where the p_θ are another collection of fresh atomic constants. Now let $S_0 = S_0^+ \cup \left\{ P_{\Gamma_i}, P_{\Theta_i} : q \right\}$ where q is fresh atomic constant.

Then let $\Delta_0 = \psi, \Theta_i \implies P_{\Theta_i, i=0,1}, P_{\Gamma_i, i=0,1}$. The following proof witnesses that $\Delta_0 \Vdash_{\phi, S_0} q$.

$$\cfrac{\cfrac{\theta \longrightarrow p_\theta : \theta \longrightarrow p_\theta \qquad \theta : \theta \qquad P_{\Gamma_i}, P'_{\Theta_i}, p_\theta : q}{\theta \longrightarrow p_\theta, \theta, P'_{\Theta_i}, : q}}{\vdots}$$

$$\cfrac{\lambda(\psi, \Theta_0, \Theta_1) : \lambda(\psi, \Theta_0, \Theta_1) \qquad \psi : \psi \qquad \Theta_i \longrightarrow P_{\Theta_i}, \Theta_i, : q}{\lambda(\psi, \Theta_0, \Theta_1) : \lambda(\psi, \Theta_0, \Theta_1), \psi, \Theta_i \longrightarrow P_{\Theta_i}, : q}$$

Now suppose that $\Delta_0, \Delta \Vdash_{S, S_0} q$. Let Π be a proof witnessing this. Since q does not occur in Δ_0 or Δ, q has to be introduced by means of a boundary rule. The only boundary rules which introduce q are the rules $P_{\Gamma_i}, P_{\Theta_i} : q$.

Go up each track in the proof until we find a sequent of the form $\Gamma', P_{\Gamma_i}, P_{\Theta_i} : q$. Now none of the atomic formulæ p_θ, for $\theta \in \Theta_i$ occur in Δ, Δ_0 except in the form $\theta \to p_\theta$. That means that every occurrence of a p_θ has been discharged by an instance of $\longrightarrow$-elimination. That means, in particular, that we have to have: $\Delta_\theta, [\psi] \Vdash_{S, S_0} \theta$, for each p_θ, for some Δ_θ. By lemma 2.2.8 we then get that $\Delta_\theta, [\psi] \setminus \left\{\Theta_i \implies T_{\Theta_i} \cup P_{\Gamma_i} : i = 0, 1\right\} \Vdash_{S, S_0^+} \theta$. Since $\Gamma_i, \Theta_i : p$ we then get $\Gamma_i, \overbrace{\Gamma, \ldots, \Gamma}^{n+1} \Delta_{\theta_0}, \ldots, \Delta_{\theta_n} \Vdash_{S, S_0^+} p$, by several applications of cut. But $\Gamma_i, \overbrace{\Gamma, \ldots, \Gamma}^{n+1} \Delta_{\theta_0}, \ldots, \Delta_{\theta_n} \Vdash_{S, S_0^+}$ only contains formulæ in Δ, Δ_0^+. This contradicts that $\Delta, \Delta_0^+ \nVdash_{S, S_0^+} p$.

The case where there is no conditional connectives in Δ, ϕ is dealt with as follows.

Claim: If Γ does not contain any conditionals, then there are $\Gamma_0, \ldots, \Gamma_n$ where each Γ_i only contains atomic subformulæ of Γ such that for all Δ, q, S: $\Gamma, \Delta \Vdash_S q$ iff $\Gamma_0, \Delta \Vdash_S q, \ldots, \Gamma_n, \Delta_S \Vdash_S q$

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The result follows by the claim. Suppose that $\Delta \nVdash_S \phi$, where Δ, ϕ don't contain any conditionals. Let S_o^+, Δ_o, p witness this. That is we have $\Delta_o^+ \Vdash_{\phi, S_o^+} p$ but we don't have $\Delta, \Delta^+ \Vdash_{S, S_o^+} p$. Now use the claim to obtain $\Gamma_o, \Gamma_1, \ldots, \Gamma_n$ such that each Γ_i is a set of atomic subformulæ of ϕ and such that $\Gamma_i, \Delta_o^+ \Vdash_{S_o^+} p$. Now let p_Δ and q be fresh atomic formulæ. Let S_o be the boundary rules in S_o^+ as well as the boundary rules $\Gamma_i, p_\Delta : q$. Let Δ_o be p_Δ.

We then get that $\Delta_o \vdash_{\phi, S_o} q$. Suppose then that $\Delta, \Delta_o \Vdash_{S, S_o} q$. Then since q is fresh it can only be introduced by means of a boundary rule. Go up each track until we hit something of the form $\Gamma', \Gamma_i, p_\Delta : q$. Replace these sequents with $\Gamma_i, \Delta_o^+ : p$. What results is a proof witnessing that $\Delta, \Delta_o^+ \Vdash_{S, S_o^+} p$. Contradiction.

To prove the claim, note that non-conditional connectives are basically conjunctions of disjunctions of conjunctions. The result is proved pretty much as theorem 1.5.9 above and is left to the reader.

$\square$

2.2.4 MAXIMALITY OF INTUITIONISTIC LOGIC

For verificationist meaning-theories we were able to establish not just that any extension of intuitionistic logic by connectives governed by general introduction rules is a conservative extension of intuitionistic logic: we were also able to establish that any such connective can be defined by means of the intuitionistic connectives. We can do the same here.

We'll need the following definition.

Definition 2.2.17. Let λ be a connective with k elimination rules. A formula with λ dominant has the following schematic form: $\lambda(\phi_o, \ldots, \phi_n, \Theta_o, \ldots, \Theta_m)$.

We define the functions I, J as follows. For all $l < k$

$I_\lambda(l) = \{\phi_i : i \leq n$ such that $\Delta : \phi_i$ is a critical minor premiss in the l'th elimination rule.$\}$

$J_\lambda(l) = \{\Theta_i : i \leq m$ such that $\Gamma, \Theta_i : p$ is a non-critical minor premiss in the l'th elimination rule.$\}$

We can now define the following translation.

Definition 2.2.18. Let $\lambda_0, \ldots, \lambda_n$ be some connectives governed by general elimination rules. We define the translation $*$ of formulæ containing $\lambda_0, \ldots, \lambda_n$ into formulæ built only using the intuitionistic connectives as follows.

1. $p*$ is p for every atomic formula p;

2. if $\lambda \in \{\lambda_0, \ldots, \lambda_n\}$ then $\lambda(\phi_0, \phi_1, \ldots, \phi_n, \Theta_0, \Theta_1, \ldots \Theta_m)*$ is

$$\bigwedge_{k_0 \leq k} (\bigwedge_{i \in I_{k_0}} \phi_i* \to \bigvee_{j \in J_{k_0}} \bigwedge \Theta_j*$$

   here k is the number of elimination rules governing λ; If $\Theta = \{\theta_0, \ldots, \theta_n\}$ then $\Theta*$ is $\{\theta_0*, \ldots, \theta_n*\}$.

Theorem 2.2.19. *Let L be the logic generated by the intuitionistic connectives and the connectives $\lambda_0, \ldots, \lambda_n$. Then both $\phi : \phi*$ and $\phi* : \phi$ are L-valid for all ϕ.*

Proof. The proof is by induction on ϕ. The case where ϕ is atomic is obvious. We first show that $\phi* : \phi$. Without loss of generality we can assume that ϕ has the form $\lambda(\phi, \Theta_0, \Theta_1)$ and that it's associated with a single elimination rule.

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$$\frac{\lambda(\psi,\Theta_0,\Theta_1) \qquad \Delta:\psi \qquad \Gamma_0,\Theta_0:p \quad \Gamma_1,\Theta_1:p}{\Delta,\Gamma_0,\Gamma_1,\lambda(\psi,\Theta_0,\Theta_1):p}$$

By the induction hypothesis we have $\psi:\psi*$. Hence we have $\Delta:\psi*$ by cut. By the induction hypothesis we also have $\Theta*_i:\Theta_i$. By cut we then have $\Gamma_i,\Theta_i*:p$. It is easy to see that $\psi* \to (\bigwedge\Theta_0 * \vee \bigwedge\Theta_1 *),\Delta,\Gamma_0,\Gamma_1:p$, which is what we have to show.

For the other direction, suppose that $\Gamma \vdash_{\psi*\to(\bigwedge\Theta_0*\vee\bigwedge\Theta_1*)} p$. The proof then takes the following form:

$$\frac{\to (\bigwedge\Theta_0 * \vee \bigwedge\Theta_1 *) \qquad \Delta_0:\psi* \qquad \Gamma,(\bigwedge\Theta_0 * \vee \bigwedge\Theta_1 *):p}{(\bigwedge\Theta_0 * \vee \bigwedge\Theta_1 *),\Delta,\Gamma:p}$$

By the induction hypothesis and cut we get that $\Delta:\psi$ is valid. By theorem 1.5.9 we now get that $\Gamma,\Theta_0*:p$ and $\Gamma,\Theta_1*:p$. By the induction hypothesis and cut we get that $\Gamma,\Theta_0:p$ and $\Gamma,\Theta_1:p$ are both valid. It is easy to find the canonical argument witnessing $\lambda(\psi,\Theta_0,\Theta_1),\Delta,\Gamma_0,\Gamma_1 \Vdash p$.

$\square$

2.3 Stability

So far we've considered verificationist and pragmatist meaning theories separately; it is now time to bring them together by studying Dummett's notion of *stability*. The notion of stability was originally introduced to diagnose the following sort of failing. Consider the connective $\ddot{\vee}$ which has the elimination rule $\dfrac{\Gamma:\phi\ddot{\vee}\psi \qquad \phi:\sigma \qquad \psi:\sigma}{\Gamma:\sigma}$, i.e., where we demand that σ follows from $\phi(\psi)$ alone. Dummett observes that when we add a connective $\vee$ subject to the usual disjunction introduction and elimination rules to a language already including $\ddot{\vee}$ we can derive unrestricted $\ddot{\vee}$-elimination as follows.

$$\frac{\Delta:\phi\,\ddot{\vee}\,\psi \qquad \dfrac{\phi\,:\,\phi}{\phi\,:\,\phi\vee\psi} \qquad \dfrac{\psi\,:\,\psi}{\psi\,:\,\phi\vee\psi}}{\Delta\,:\,\phi\vee\psi} \qquad \frac{\Gamma_0,\phi\,:\,\sigma \qquad \Gamma_1,\psi\,:\,\sigma}{\Delta,\Gamma_0,\Gamma_1\,:\,\sigma}$$

This gives us a failure of cut-elimination.

Intuitively, the $\ddot{\vee}$-elimination rule is unstable in the sense that the standard $\vee$-introduction rule is validated by a pragmatist meaning-theory based on $\ddot{\vee}$ elimination. But the unrestricted $\vee$-elimination rule is validated by a verificationist meaning theory based on the standard $\vee$ rules.

What's the importance of stable rules? If some connectives have unstable rules, a verificationist and a pragmatist will not agree on the logic governing those connectives; if the rules are stable, on the other hand, the verificationist and pragmatist will agree on the logic governing the connectives.

Dummett makes an interesting conjecture concerning stability. First, a definition.

Definition 2.3.1. We'll define two functions I, E

1. Let R be some collection of introduction rules for the connectives $\lambda_0,\ldots,\lambda_n$. Let $E(R)$ be all the elimination rules for $\lambda_0,\ldots,\lambda_n$ which are validated by the verificationist meaning theory for $\lambda_0,\ldots,\lambda_n$ based on R.

2. Let R be some collection of elimination rules for the connectives $\lambda_0,\ldots,\lambda_n$. Let $I(R)$ be all the elimination rules for $\lambda_0,\ldots,\lambda_n$ which are validated by the pragmatist meaning theory for $\lambda_0,\ldots,\lambda_n$ based on R.

Definition 2.3.2. 1. Let R be a collection of introduction rules for some connectives $\lambda_0,\ldots,\lambda_n$. The rules R are *stable* iff $I(E(R)) \subset L_R$.

2. Let R be a collection of elimination rules for some connectives $\lambda_0, \ldots, \lambda_n$. The rules R are *stable* iff $E(I(R)) \subset L_R$.

Definition 2.3.3. Let $J = \{\lambda_0, \ldots, \lambda_n\}$ be some connectives governed by some introduction (elimination) rules R. For each subset $I \subset \{\lambda_0, \ldots, \lambda_n\}$ let L_I be the logic generated by verificationist (pragmatist) meaning-theory based on the rules governing just the connectives in I. We use $\Vdash^I$ for the induced consequence relations. We say that $\mathcal{L}$ is in *total harmony* iff for all such $I \subset J$ if Γ, ϕ contain only connectives in I and $\Gamma \Vdash^J \phi$, then $\Gamma \Vdash^I \phi$.[6]

Dummett conjectures that "intrinsic harmony implies total harmony in a context where stability prevails" (Dummett, 1991, p. 290). Given the results obtained so far, haven't we established this conjecture? After all, we have shown that any logic induced by a verificationist or pragmatist meaning-theory is totally harmonious. Indeed, in order to prove these results we did not need to rely on the stability of the rules as all.

From the example of $\ddot{\vee}$ it should be clear that this is not what Dummett is after. The above conservativity results shows that the verificationist (pragmatist) meaning theory for the language $\mathcal{L} \cup \{\lambda\}$ is a conservative extension of the verificationist (pragmatist) meaning-theory for $\mathcal{L}$. What we want to do is to look at "mixed" meaning-theories for $\mathcal{L}$ and $\mathcal{L} \cup \{\lambda\}$ and show that the latter is a conservative extension of the former.

Definition 2.3.4. Let $\mathcal{R}$ be a collection of introduction (elimination) rules. We define the logic L_{R+} as the least set of sequents $\mathcal{S}$ such that

1. $\Gamma : \phi \in L_{R+}$ iff $\Gamma \Vdash_R \phi$ or $\Gamma \Vdash_{I(E(\mathcal{R}))} \phi$ ($\Gamma \Vdash_{E(I(\mathcal{R}))} \phi$))

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[6] In other words, $\Vdash^J$ is a conservative extension of $\Vdash^I$ for each I.

2. if $\Delta_0 : \phi_0, \dots, \Delta_n : \phi_n$ are in L_{R+} and $\dfrac{\Delta_0 : \phi_0, \Delta_1 : \phi_1 \dots \Delta_n : \phi_n}{\Gamma : \phi}$ is in $I(E(\mathcal{R}))\,(E(I(\mathcal{R})))$ or L_R then $\Gamma : \phi$ is in L_{R+}.

We can now give a more interesting formulation of Dummett's Conjecture.

Theorem 2.3.5 (Dummett's conjecture). *Let J be a collection of connectives governed by some introduction (elimination) rules R and assume that the rules R are stable. Let Γ, ϕ contain only connectives from $I \subset J$ and let R' be the rules governing only the connectives in I. Then if $\Gamma \Vdash_{L_{R^+}} \phi$ then $\Gamma \Vdash_{L_{R',+}} \phi$.*

Proof. We prove the case where R is a collection of introduction rules. Suppose that Γ, ϕ contain only connectives in I. Suppose that $\Gamma \Vdash_{R^+} \phi$. It suffices to show that $\Gamma \Vdash_R \phi$, for then the result follows by proposition 2.2.16 . The proof is by induction on the proof in L_{R+} witnessing that $\Gamma \Vdash_{L_{R+}} \phi$. If the proof is an axiom, $\Gamma : \phi$, then we either have $\Gamma \Vdash_R \phi$, in which case there is nothing to prove; or else we have $\Gamma \Vdash_{I(E(R))} \phi$. In the latter case the result follows by stability.

So suppose that the proof ends with an application of a rule $\dfrac{\Delta_0 : \phi_0, \Delta_1 : \phi_1 \dots \Delta_n : \phi_n}{\Gamma : \phi}$. By the induction hypothesis we have that $\Delta_i \Vdash_R \phi_i$ for each i.

If the rule is in R, the result is immediate. So suppose that the rule is in $I(E(R))$. Since the rules R are stable the rules is $\Vdash_R$-valid. The result follows.

$\square$

Which rules are stable? We know that the introduction and elimination rules of intuitionistic logic are stable.

Proposition 2.3.6. *1. The introduction rules of intuitionistic logic are stable;*

2. The elimination rules of intuitionistic logic are stable.

Proof. Let $\mathcal{R}$ be the standard intuitionistic introduction rules. $E(R)$ is the collection of all elimination rules which are validated by the verificationist meaning theory based on R. By theorem 1.3.22 every rule $R \in \mathcal{R}$ is an intuitionistically valid elimination rule. The standard intuitionistic elimination rules are in $E(R)$. By theorem section 1.5.14 every intuitionistically valid elimination rules is pragmatist consequence of the standard intuitionistic elimination rules.

Suppose then that $R \in I(E(\mathcal{R}))$. Suppose then that $\mathcal{R}'$ is a collection of introduction rules which are proof-theoretic consequences of the rules $\mathcal{R}$. Then the rule R is a pragmatist consequence of the standard intuitionistic elimination rules. Hence R is an intuitionistically valid introduction. But every intuitionistically valid introduction rule is a verificationist consequence of the standard introduction rule. Hence R is verificationist consequence of $\mathcal{R}$.

An analogous argument shows that the intuitionistic elimination rules are stable. $\qquad\square$

Conjecture 2.3.7. *Every general introduction (elimination) rule is stable.*

2.4 SUMMING UP

In this paper I have settled the technical questions concerning proof-theoretic meaning-theories in the style Dummett outlined in *The Logical Basis of Metaphysics*. In particular, 1. I have given a semantic proof that intuitionistic logic is validated by a verificationist meaning-theory; indeed, 2. I have shown that intuitionistic logic is *complete* with respect to the verificationist meaning-theory based on the intuitionistic introduction rules; moreover,

3. I have shown that intuitionistic logic is the strongest logic which can be introduced by means of a verificationist meaning theory. 4. I have developed the pragmatist meaning-theory sketched in *The Logical Basis of Metaphysics* and shown that a pragmatist meaning-theory can be developed on the basis of *any* collection of general elimination rules. 5. If it is developed on the basis of the intuitionistic elimination rules, we can justify intuitionistic logic; indeed, 6. intuitionistic logic is complete with respect to the pragmatist meaning-theory based on the intuitionistic elimination rules; moreover, 7. intuitionistic logic, is the strongest logic which can be justified by a pragmatist meaning-theory based on almost general elimination rules.

It would be of interest to show that cut-elimination holds for wholly general elimination rules. It would also be of great interest to extend these results to first-order logic; it would be particularly interesting to develop a theory of general quantifiers.

Part II

Metaphysics

3

Is the Vagueness Argument Valid?

3.1 Introduction

The vagueness argument, as I will understand it here, purports to establish the following answer to the *Special Composition Question*:[1]

Universalism for every class C there is an object x such that each member of C is a part of x and every part of x shares a part with some member

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[1] Roughly, what are the conditions under which some objects compose some further object (Inwagen, 1990, pp. 21–32). We'll disregard tense throughout; this will not matter.

of C.

Sider (2001) gives the following three premisses (cf. Sider, 1997, pp. 217–9):[2]

P1 If not every class has a fusion, then there must be a pair of cases connected by a continuous series such that in one, composition occurs, but in the other, composition does not occur (Sider, 2001, p. 123).

P2 In no continuous series is there a sharp cut-off in whether composition occurs (Sider, 2001, p. 124).

P3 In any case of composition, either composition definitely occurs, or composition definitely does not occur (Sider, 2001, p. 125).

The argument is seemingly straightforward. Suppose composition is restricted. Then by (P1) we find a pair of cases c_0 and c_n such that in c_0 composition definitely occurs and in c_n composition definitely does not occur and such that they are connected by a continuous series $c_1, c_2, \ldots, c_{n-1}$. By (P3), in every c_i composition definitely occurs or definitely does not occur. Then there has to be a j such that $\langle c_j, c_{j+1} \rangle$ is a sharp cut-off. This contradicts (P2). Hence composition always occurs and so (UNIVERSALISM) is true.

While (UNIVERSALISM) is commonly supposed to be counterintuitive, my goal in this paper is not to quarrel with (UNIVERSALISM) nor with any of the premisses.[3] I will rather show that, contrary to appearances, the vagueness argument is invalid.[4] Initially, however, the invalidity seems easily plugged.

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[2]Sider intends to improve on the argument in (Lewis, 1986, pp. 211–2). (Sider, 2001, pp. 134–139) extends the argument to argue in favor of arbitrary diachronic fusions; the extended argument doesn't raise any further problems.

[3]See (Korman, 2010) for an exhaustive survey of how to deny one of (P1), (P2) or (P3). (see also Varzi, 2005) for an earlier overview of the options.

[4]Nolan (2006, p. 724) notes that the argument is not formally valid, pointing out that Sider's formulation leaves it open that the end points are outside the continuous series. By

While this is not so, seeing that this is the case will force us to think hard about the interplay between metaphysical modality and determinacy. The issues are fairly subtle, and the vagueness argument gives us a fairly concrete illustration of what's at stake.

3.1.1 OVERVIEW

I'll proceed as follows. In § 3.2 I define some central notions, in particular I define the central notion 'composition's occurring in a case'. Having defined it, a potential ambiguity in Sider's premisses becomes apparent; in § 3.3 I'll present non-equivocal versions of the premisses. In §§ 3.3.2–3.3.3 I show that Sider's arguments only establish readings of the premisses on which the vagueness argument is not valid. One way of witnessing that the argument is invalid is by holding that it can be contingent whether composition occurs. In § 3.4 I discuss a (very strong) sense of contingency required to make this out. In § 3.5 I consider deflationary views of existence and argue that on such views its being contingent whether composition occurs is exactly what we should expect. In § 3.6 I discuss the notion of an "arbitrary" cut-off and raise the worry that the deflationary view is *itself* committed to arbitrary cut-offs; in § 3.6.2–3.6.4 I address this worry. In § 3.6.5 I then show how we can witness the invalidity of the argument without committing to the possibility contingent composition. In § 3.7 I deal with some loose threads and in § 3.8 I discuss the notion of supervenience. In § 3.9 I consider some versions of the argument which *are* valid. We now have to reject some of the premisses, but the cost of doing so should no longer appear great. I end

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adding a premiss to that effect the validity of Sider's argument is restored. The invalidity pointed out in this paper is not of this enthymematic variety.

with some methodological morals (§ 3.10).

3.2 Technicalities

Sider introduces the quasi-technical notion of a *case* as follows:

> A 'case of composition' ('case' for short) [...] [is] [...] a possible situation involving a class of objects having certain properties and standing in certain relations.[5] (Sider, 2001, p. 122)

The notion of a *sharp cut-off* is defined as follows:

> By a 'sharp cut-off' in a continuous series I mean a pair of *adjacent* cases in a continuous series such that in one, composition definitely occurs, but in the other, composition definitely does not occur[6]. (Sider, 2001, p. 123)

The problem is the notion of 'composition occurring in a case'. Immediately after having given the definition of 'case' Sider writes:

> We will ask with respect to various cases whether composition occurs; that is, whether the class in the case would have a fusion. (Sider, 2001, p. 122)

But we're not told what it means for "the class in the case to have a fusion". To fix on terminology, let's use $c_0, c_1, \ldots$ as variables over cases and $C_0, C_1, \ldots$ as variables over classes (with the understanding that class C_i is the class

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[5]Some points of clarification. 1. For Sider a case of composition is not necessarily a case in which composition occurs. I will reserve the label 'case of composition' for cases in which composition *does* occur, using the simple 'case' for the neutral notion. 2. We will assume that the relations which figure in a case are non-mereological. 3. As I will use the notion in this paper, a case may obtain in more than one world. 4. Cases need not be detailed. For instance, a case may consist of some objects $x, y, z \ldots$ and the relation *being between 1 and 2 meters from any of the other objects*. 5. However, cases are precise in that for any proposition p the case either settles p or does not settle p but 6. if a case does not settle that p it need not be the case that the case settles that not p.

[6]cf. Sider, 1997, p. 216.

associated with case c_i); we use O for 'obtains' and Fcx for 'the class C of objects associated with c has x as their fusion'.[7]

That the notion is not defined might appear unproblematic. There are only two serious candidate definitions.[8] We can define the notion of composition's occurring in a case using either counterfactuals or strict conditionals; that is, we have the following two options.

cc-cf if c were to obtain there would be an x such that the class C composed x $(Oc \,\square\!\!\rightarrow \exists x Fcx)$.

cc-sc Necessarily, if c obtains then there is an x such that the class C composes x $(\square(Oc \rightarrow \exists x Fcx))$.

For definiteness, I will frame the argument in terms of strict conditionals.[9]

There are two possibilities for defining 'c is a case of non-composition'.

ncc It's not necessary that if (c obtains then there is x such that the class C composes x. $(\neg\square(Oc \rightarrow \exists x Fcx))$

cnc Necessarily, if c obtains, then there is no x such that class C composes x. $(\square(Oc \rightarrow \neg\exists x Fcx))$

The correct definition is (cnc). As the mnemonics may suggest, (ncc) defines that c is not a case of composition; that is, the case c is not such that whenever it obtains the class C associated with it has a fusion. This does not define that

---

[7] Couldn't a case be associated with two classes? In one sense of 'case', sure; but we'll take cases to be individuated so finely that they are associated with only one class. At the cost of complicating our formalism we could allow cases to be associated with several classes.

[8] Though see § 3.7.2 below.

[9] That formulating the argument in terms of counterfactuals makes no a difference will be made clear in § 3.7.

c is a case of *non*-composition, that is, that the class *c* is such that whenever it obtains, the associated class *C* fails to have a fusion.

We're now in a position to present precise versions of Sider's premisses.

3.3 THE ARGUMENT

It will prove convenient to slightly change the presentation. Sider casts his argument as a *reductio* of the proposition that composition is restricted. I will assume that we have fixed on a continuous series of cases $c_0, c_1, \ldots, c_n$. I'll build the assumption that composition is restricted into (P1) and recast Sider's argument as a purported demonstration that (P1), (P2), (P3)—so understood—are jointly inconsistent. This is unproblematic: Sider's argument is successful if the triad is inconsistent; conversely, if there is a model of the triad Sider's *reductio* fails.

3.3.1 THE PREMISSES

We have seen how to define 'composition's occurring in a case' and 'composition's not occurring in a case', but Sider's argument turns on the notion of 'composition's *definitely* occurring in a case'. How are we to define this notion? We'll introduce an operator 'determinately' (*D*) to express this.[10] The presence of the determinacy operator gives us, unsurprisingly, a lot of choice about how to formalize the premisses, the main question being where to place the *D*-operator(s).[11]

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[10]I choose this over 'definitely' since it has fewer epistemic overtones.

[11]In order to make things a bit more manageable, I'll make the following simplifying assumptions. We assume that for all ϕ, $D\Box\phi$ is equivalent to $\Box D\phi$; we also assume that $D\Box(Oc \leftrightarrow DOc)$ is valid; finally, we assume the Barcan and Converse Barcan Formulæ for determinacy. Not all these principles are plausible, but assuming them only strengthens Sider's argument so it's unproblematic to assume them here.

We have the following options for (P1).

P1A $\quad \Box(Oc_0 \to D\exists xFc_0x) \land \Box(Oc_n \to D\neg\exists xFc_nx)$

P1B $\quad D\Box(Oc_0 \to \exists xFc_0x) \land D\Box(Oc_n \to \neg\exists xFc_nx)$

P2 has the following readings:

P2A $\quad \neg(\Box(Oc_i \to D\exists xFc_ix) \land \Box(Oc_{i+1} \to D\neg\exists xFc_{i+1}x))$

P2B $\quad \neg(D\Box(Oc_i \to \exists xFc_ix) \land D\Box(Oc_{i+1} \to \neg\exists xFc_{i+1}x))$

Finally, and critically, for (P3) we should distinguish these options.

P3A $\quad \Box(Oc \to (D\exists xFcx \lor D\neg\exists xFcx))$

P3B $\quad D\Box(Oc \to \exists xFcx) \lor D\Box(Oc \to \neg\exists xFcx)$

P3C $\quad \Box(Oc \to D\exists xFcx) \lor \Box(Oc \to D\neg\exists xFcx)$

The principles (P1A), (P2A), (P3A) are *intraworld* determinacy principles.
(P3A), e.g., only commits us to holding that *within* each world it is either
determinate that the class C associated with c has a fusion or it is determinate
that it does not have a fusion. (P1B), (P2B) and (P3B), on the other hand,
are *interworld* determinacy principles. They concern not whether things
are determinate *within* one world but whether the structure of modal space
itself is determinate: they concern the determinacy of *modal* facts. (P3B), e.g.,
commits us to the claim that either it's determinate that in every world (in
which c obtains), C has a fusion or that it's determinate that in every world
(in which c obtains) C does not have a fusion. What about (P3C)? It has an
interestingly mixed status, about which more in § 3.6.5.

Which variants of these premisses are supported by Sider's arguments in favor of his (P1), (P2), (P3)? Little turns on the difference between (P1A) and (P1B). We'll (P2) more fully in § 3.6; for now, suffice it to say that given our assumptions about the interplay of $\Box$ and D, (P2A) entails (P2B), so let's just grant Sider (P2A). The important questions concerns the argument for (P3).

1. Does Sider's argument for (P3) establish (P3B)?

2. Does it establish (P3C)?

3.3.2 Sider's argument for (P3)

Sider argues as follows in favor of (P3).

> Now surely if (P3) can be violated, then it could be violated in a 'finite' world, a world with only finitely many concrete objects. That would mean that some *numerical sentence*—a sentence asserting that there are exactly n concrete objects, for some finite n—would be indeterminate. (Sider, 2001, p. 127)

Numerical statements can be expressed in the language of first-order logic with identity; so, Sider argues, if "any numerical statement is to be indeterminate in truth-value, it must be because one of the logical notions is vague." (Sider, 2001, p. 127) Since, so Sider holds, no logical expression is vague, (P3) has to be true.

There are, of course, many places to attack this argument.[12] Set that aside: *even* if this argument works it only establishes (P3A) and not (P3C) (much less (P3B)).

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[12]Recently, several authors have argued that we can have indeterminate composition without having indeterminate existence (hence without count-indeterminacy) (see e.g., Carmichael, forthcoming; Noonan, 2010; Donnelly, 2009).

To see this consider what has to be the case in order for there *not* be vagueness in how many things there are. What is needed is that the following *might*-counterfactuals be false for each *n*:

ID Were *c* to obtain, then it might be that it's not determinate whether there are exactly *n* concrete things.

That (ID) is false is entailed by the following:

$$\Box(Oc \rightarrow D\exists_n xCx \lor D\neg\exists_n xCx) \tag{3.3.1}$$

And the following is sufficient for (3.3.1) to hold

LD For any case *c*, and any class *C* associated with *c*: in any world in which the situation *c* obtains, there determinately is a fusion of the class *C* or there determinately is not a fusion of the class *C*

(LD) suffices to justify (P3A), but it does not justify (P3C) (or (P3B) for that matter).

3.3.3 MODELS

The following sets are consistent:

1. {(P1B), (P2A), (P3A)}

2. {(P1B), (P2B), (P3A)}

3. {(P1B), (P2B), (P3C)}

To see that {(P1B), (P2A), (P3A)} and {(P1B), (P2B), (P3A)} are consistent is straightforward.[13]

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[13]To see that {(P1B), (P2B), (P3C)} is consistent takes a bit of work, so I'll postpone discussion of this until § 3.6.5.

Here's a model. We start off with some cases $c_0, c_1, \ldots, c_k$ such that in any world in which a case c_i (with $i \leq k$) obtains it's determinate that C_i has a fusion. Then we have some cases $c_{k+1}, c_{k+2}, \ldots, c_l$ such that in some world in which c_i (with $k + 1 \leq i \leq l$) obtains C_i determinately has a fusion and in the other worlds in which c_i obtains C_i determinately does not have a fusion. Finally we have some cases $c_{l+1}, c_{l+2}, \ldots, c_n$ such that in every world in which a c_i with ($l + 1 \leq i \leq n$) obtains C_i determinately does not have a fusion. Since (P3A) holds, for each i from k through l, c_i is a case of *contingent composition*; that is, in some worlds in which c_i obtains there determinately is a fusion of the class C_i whereas in other worlds there determinately is no fusion of the class C_i. Since in this model (P1A), (P2A) (and (P2B)) and (P3A) are true and these are the premisses which are justified by Sider's argument, we have shown that the vagueness argument is invalid.

Objection: in this model it is contingent whether composition occurs, and that is completely unacceptable! If we add as a further premiss that composition cannot be contingent we save the argument.

As we will see in § 3.6.5 this in fact does not save the argument. Before we get there, however, let me first defend the idea that it may be contingent whether or not composition occurs in a case.[14]

3.4   The Contingency of Composition

Of course, the claim that it may be contingent whether composition occurs isn't itself preposterous: a case may be insufficiently detailed to settle whether its associated class has a fusion or not. Suppose, e.g., that whether

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[14]While this has been defended by (Cameron, 2007), the defense given here is different.

the class C has a fusion or not hinges on whether the objects in C are in contact to degree at least r. A case which consists of the objects in class C and the relation *being in contact to degree at least r'* (where $r' < r$) is not going to necessitate that the class C has a fusion or that class C does not have a fusion.

What appears preposterous is that one can have included *everything* relevant to whether composition occurs in a case, yet the case still doesn't settle whether composition occurs or not.[15] Let us call such cases *maximal cases*. When I say that composition is contingent I mean that there are maximal cases c such that the obtaining of c neither necessitates that composition occur nor necessitates that composition does not occur.

This is a much stronger view than just holding that facts about composition are *brute*. Markosian, e.g., holds that composition is brute in the sense that if the class C has a fusion, then that fact does not obtain in virtue of other (non-mereological) facts (Markosian, 1998, p. 215). He nevertheless accepts that facts about composition globally supervene on facts about the arrangement of simples (Markosian, 1998, pp. 215–6). That is, there cannot be two worlds which are exactly alike non-mereologically but differ mereologically. If composition is contingent in the above sense this rules out the global supervenience of facts about composition on the facts about the simples.

(Cameron, 2007) notwithstanding, this view seems indefensible: can't we

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[15]In describing the continuous series Sider says that "each case in the series is extremely similar to its immediately adjacent cases in all respects that might be relevant to whether composition occurs; qualitative homogeneity, spatial proximity, unity of action, comprehensiveness of causal relations etc." (Sider, 2001, p. 123). Clearly, Sider intends these cases to include everything which is relevant to whether composition occurs or not.

just shrug it off? Indeed, isn't the invalidity obviously *enthymematic*: why not just add the premiss that composition isn't contingent (equivalently: that whether composition occurs supervenes on facts about the simples)?

Later (§ 3.6.5) we will see this obvious fix in fact doesn't work. First, however, I'll argue that there need be nothing mysterious about contingent composition even in the above strong sense.

3.5 EXISTENCE DEFLATED

3.5.1 EXISTENCE ON THE CHEAP

Suppose we have the following view about existence, or what it takes for something to exist. We make a distinction between *basic* and *non-basic* objects. For *basic* objects, the notion of existence is primitive in the sense that the only answer to the question "what is it for basic object x to exist?" is: "there is such a thing as x". For *non-basic* objects, on the other hand, an answer of the following form can be given.

NBE What it is for non-basic object x to exist is for (the) basic objects $x_0, x_1, \ldots$ to stand in relations $R_0, R_1, \ldots$.

The existence of a non-basic object *just consists in* certain basic objects' standing in certain relations.

In the mereological case, we can take the basic objects to be mereological simples, the non-basic objects to be complex objects, in particular, fusions of simple objects. For simplicity, we'll take all talk about basic objects to take the form: 'case c obtains', assuming that cases only concern basic objects and relations between basic objects.

3.5.2 'Just consists in'

What does 'just consists in' mean here? We could spell this out in terms of *grounding*: the basic objects $x_0, x_1, \ldots$ standing in relations $R_0, R_1, \ldots$ grounds the existence of the object x.[16] We could also cash it out in terms of analytic or conceptual entailment: it's a conceptual truth that if the objects $x_0, x_1, \ldots$ stand in relations $R_0, R_1, \ldots$ then the object x exists. For present purposes we don't need to invoke such metaphysically committing machinery; the only thing we need is the following.

1. There are (necessarily) true conditionals of the forms "If c obtains, then C determinately has a fusion" and "If d obtains then, D determinately does not have a fusion". Let's call such conditionals *positive (negative) postulates* for C.

2. It is true that there is a fusion of class C if there is a case c such that c obtains and such that "If c obtains, then C determinately has a fusion" is a positive postulate for C.

3. It is *false* that C has a fusion if there is a case c such that c obtains and such that "If d obtains then, D determinately does not have a fusion" is negative postulate for C.

Supposing that this is the case, we may then say that *what it is* for there to be a fusion of the class C *just is* for c_0 to obtain or c_1 to obtain or ...; what it is for there *not* to be a fusion of the class C is for d_0 to obtain or d_1 to obtain or ....

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[16] For more on the notion of grounding see e.g., Fine, 2012, forthcoming; Schaffer, 2009.

Since we have to specify *both* truth *and* falsity conditions we shouldn't expect that the positive and negative postulates for the claim that a certain class has a fusion results in an exhaustive division of the maximal basic cases. Why could this happen? We'll say more about this in § 3.6.3; for now we can think of it as follows. It might be *vague* whether a class c is a class of composition; the class c might be a borderline case of composition, in which case it figures in neither a positive nor a negative postulate.

3.5.3 BASIC AND NON-BASIC WORLDS

It will be helpful to think of this in terms of possible worlds. Let us distinguish between *basic* and *non-basic* worlds. Basic worlds are (or encode) collections of basic objects and relations amongst basic objects. We may think of basic worlds as settling all questions in a basic, fundamental language—the language of fundamental metaphysics, as it were.

While basic worlds are tailored for the needs of metaphysics, there is no reason to expect that they can fill all the rôles possible worlds have been found useful for; in particular, there is no reason to think that they are particularly useful for semantic purposes. For the purposes of semantics we may well need entities which have opinions about questions which aren't settled by the basic worlds. Non-basic worlds fill that role. We may think of a non-basic world as consisting of an underlying basic world w together with a collection of postulates governing the non-basic vocabulary. Together this settles which non-basic sentences are true in w.

How does this apply to the mereological case we are considering? If there are maximal basic cases which don't settle whether composition occurs or

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not, doesn't that mean that (P3A) is false?

3.5.4 Non-basic worlds and determinacy

Call a non-basic world *w complete* iff every sentence is either true or false in w. There is no particular reason to have non-basic worlds be complete. A particular application may demand that they be complete in a particular respect; that does not mean that they have to be complete in every respect. If the postulates governing the non-basic vocabulary does not exhaustively partition the basic cases, then the non-basic worlds generated by those postulates will, in general, not be complete.

For some applications we want complete non-basic worlds. For instance, in order to make (P3A) true our non-basic worlds have to settle all questions about which fusions exist. There is a very natural way of extending the class of non-basic worlds in such a way that every non-basic world w has a determinate opinion about whether a certain class C has a fusion in w.[17]

Suppose c is a C-maximal case which doesn't necessitate that C has (hasn't) a fusion. Let w be a non-basic world in which c obtains, and extend w to two non-basic worlds w_0, w_1 such that in w_0, c obtains and C determinately has a fusion and in w_1, c obtains and C determinately does not have a fusion. Saying that only one is a real possibility is arbitrary; *ex hypotesi* there is nothing which determines that composition does not occur in one of the cases.

Why do we say that C *determinately* has a fusion in the world w_0 when nothing basic in w_0 necessitates that C has a fusion? We don't, in general,

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[17]A subtler construction is sketched in § 3.6.5

have to say this: when we extend a non-basic world w which is incomplete with respect to which things are red, to a world w' which is complete with respect to which things are red, we don't have to make it the case that if something in w' is red, it is determinately red. We make an exception for existence and parthood: which things exist and which things are parts of which things is something which has to be determinately settled in extending non-complete world to a complete one. Do we *have* to treat existence as always determinate? We don't, but there is no dialectical problem in treating it as always determinate. This just amounts to granting Sider what he needs for his argument for premiss (P3).

On this picture accepting contingency of composition doesn't commit us to positing a layer of brute compositional facts. As far as basic worlds go, there are certain cases such that there are *no facts* about whether composition occurs in those cases or not. When we want to extend our class of worlds such that every world has an opinion about whether composition occurs or not, we are forced to treat composition as contingent, but this is just a result of how we choose to construct our worlds; it does not correspond to any deep feature of reality. It is precisely because there is nothing basic which makes it the case that C has (hasn't) a fusion in w which allows us to extend w both to w_0 and w_1. We cannot go wrong: where there are no facts, there are no facts to answer to.

3.5.5 Penumbral Counterfactuals

One reason for wanting complete non-basic worlds is simply to show that the vagueness argument is invalid. There is, however, a less opportunistic reason

for wanting complete worlds: they are useful for capturing "penumbral connections". In particular, they come in handy for dealing with what we can call "penumbral counterfactuals".

Suppose c_i and c_{i-1} are (maximal) cases such that it is not required that composition occur and it is not required that composition does not occur for the classes C_i, C_{i-1}. And suppose that the cases c_i, c_{i-1} both actually obtain. In this case we would be correct in asserting the following counterfactual.

PCF-T If c_i were to obtain and C_i have a fusion, then c_{i-1} would still obtain and C_{i-1} would also have a fusion.

(PCF-T) is to be read as follows. In evaluating the antecedent we are not to envisage c_i occurring in a (slightly) different way so as to be a case which necessitates that C has a fusion; rather, we are to envisage c_i obtaining in exactly the way it does, but that it so occurs with C's having a fusion.

In order to account for this on a possible-worlds semantics for counterfactual conditionals we need worlds to have an opinion about things which aren't settled by what is basic in that world. For in every basic world in which c_i (c_{i-1}) obtains it is neither true nor false that C_i (C_{i-1}) has a fusion.

Note that we cannot account for the truth of (PFC-T) by holding that its antecedent is necessarily false. For let c_k be a case which also obtains and which is definitely a case of non-composition. Consider now,

PCF-F If c_i were to obtain and C_i have a fusion, then c_k would obtain and C_k would have a fusion.

(PCF-F) is false. To get the truth-conditions for these (and similar) counterfactuals right, we need to have worlds in which non-basic sentences which are

not determined to be true (false) by the basic facts are true as well as worlds in which they are false.

Such counterfactuals might appear to be of little interest—they may seem frivolous even—so one could argue that little stock should be put getting them right. This is mistaken. The impression that such counterfactuals are uninteresting is caused, I think, by how we set up our toy model. In particular, we assumed that we have a basic language *at hand* and that we can describe the differences between the cases $c_0, c_1, \ldots$ in *basic* terms. In real-world cases this is unrealistic.

At a given stage in the development of our language the only way of getting at some basic phenomenon might be by way of non-basic descriptions. And occasionally we may have to describe basic phenomena using non-basic sentences the constituents of which are undetermined by basic facts. The case of "penumbral counterfactuals" mentioned above illustrates this. We might be given two cases such that neither is a case of (non-)composition, but such that we can say that if one were to be a case of composition then the other one would be as well. We certainly think that there are basic features of the cases which make these counterfactuals true, but we may have no way of describing these features in basic vocabulary. In particular, we might think that there is some basic similarity relation that we're trying to get at, but we may have no way of getting at it except through the non-basic counterfactuals.[18]

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[18]This point applies generally. Three-valued logics are often faulted for not being able to deal with penumbral connections. All statements which are in the penumbra are given the same truth-value, so there is no way of distinguishing "if it's burgundy, it's red" from "if he's bald, he's poor". But why care about getting this right? We care about getting this right because penumbral connections track non-vague relations and we may have no way of getting at the non-vague relations except by using vague language.

3.5.6 MATTERS DIALECTICAL

Sider certainly does not accept a deflationary view of existence, nor did he accept that parthood is not a fundamental notion. Indeed, most defenders of restricted composition themselves are opposed to deflationary views of existence (see e.g., Merricks, 2005; Inwagen, 1990). What, then, is the dialectical importance of this deflationary view of existence?

Well, the vagueness argument is a problem for everyone, not just those who deny that existence can be lightweight; and there are costs to denying each premiss. If the argument is invalid, one doesn't have to worry about this. Moreover, once we know that the argument is invalid, opponents of unrestricted composition have more options than was previously thought.

In this connection, it is worth mentioning that instead of taking a deflationary attitude towards existence one could take a deflationary attitude towards (metaphysical) possibility. One could, e.g., take the view that what's possible is just what's consistent with our use of the word 'possible'. If we adopt such a view, every fact about how the actual world is can be treated as substantial. In particular, facts about which complex objects there are, are as substantial as the facts about which simple objects there are. Moreover, one could hold that facts about which objects there exist are always determinate.

What is not substantial are the facts about what is possible. As long as our conventions about the use of the word 'possible' don't settle whether a case c is a case of composition or a case of non-composition, then c would have to be a case of contingent composition, since it is consistent that c goes either way. On this view, too, one will verify (P1B), (P2B) and (P3A), but not

(P3B).[19]

3.6 Arbitrariness

(P2) is supposed to capture the intuition that a cut-off in a series of com-
position would be "arbitrary". One might think that the view considered
so far doesn't do any better on this count. Just as the idea that there is a
sharp cut-off between cases of composition and cases of non-composition
is problematic, the idea that there is a sharp cut-off between cases of com-
position and cases of contingent composition appears problematic, and for
the same reasons. How could there be a sharp cut-off between the cases of
(non)composition and the cases of contingent composition? Wouldn't such a
cut-off be as arbitrary as a sharp cut-off between cases of composition and
cases of non-composition?

3.6.1 Intraworld Arbitrariness

We should distinguish two types of arbitrariness. The first is *intraworld*:
for a particular world w, why is the cut-off between cases of composition
and non-composition, *in w*, at $\langle c_i, c_{i+1} \rangle$? The second is *interworld*: in a
continuous series of cases $\langle c_0, c_1, \ldots, c_n \rangle$, why is the cut-off between the cases
of composition (non-composition) and the cases of contingent composition
where it is—at $\langle c_i, c_{i+1} \rangle$, say?

   If we take a deflationary view about existence, the answer to the first
question is simple: because the world w is the world it is. When we construct
complete non-basic worlds one of the things we do is to settle, for each world,

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[19]Thanks to Bryan Pickel for suggesting this view. The subtler construction in § 3.6.5 can
also be appropriated by those who want to deflate modality.

whether composition occurs for the class C associated with case c. In the case where c necessitates that composition occurs (does not occur), there is an answer to the question: what makes it the case that w is a world in which composition occurs in case c? The answer: what it *is* for class C (not) to have a fusion *just is* for case c_0 to occur or for case c_1 to occur or .... If c is not a necessitating case, there is no interesting answer to the question what makes w a world in which c has a fusion as opposed to a world in which c does not have a fusion: w just is constructed that way.

3.6.2 Interworld arbitrariness and Sharp Cut-Offs

Interworld arbitrariness is a different matter: it does seem arbitrary that there would be a sharp cut-off between the cases of composition and the cases of contingent composition. How could our linguistic usage be such as to determinately single out cut-off in a continuous series?

One might try to defend oneself by giving the following argument.

"The reason that it is contingent whether composition occurs", one might say, "is that our linguistic usage is such that certain cases are determined to be cases of composition and certain other cases are determined to be cases of non-composition. In the middle there are no facts either way. By extending the basic worlds to non-basic worlds we make this middle area an area of contingent composition. What's preposterous about the idea that there is a sharp cut-off between cases of composition and cases of non-composition is the idea that our usage settles all the questions in that middle area."

"But", one continues, "that there is a sharp cut-off between the cases of composition and the cases of contingent composition *is* determined: that

the cut-off is where it is, is determined by the very fact that that's where our usage (together with the world) starts not settling whether composition occurs. More briefly: if there is no fact which necessitates that composition occurs, then it is necessary that there is no fact which necessitates that composition occurs."

This response is beside the point. The objection we are considering focuses on how usage can *determinately* single out an interworld cut-off, that is, a cut-off between the cases of composition and the cases of contingent composition. When one says that a cut-off between cases of composition and cases of contingent composition would be arbitrary, one should be understood as saying that there is no way for our *use* of the words determinately to single out a particular cut-off between the cases of composition and contingent composition. And we can accept this while accepting that if $c_0, c_1, \ldots, c_n$ is a continuous series, then there is a cut-off between the cases of composition and the cases of contingent composition; furthermore, not only is there a cut-off in the series, it is necessary that the cut-off is where it is. How can we claim this?

We can claim this by lifting to the modal case a standard supervaluationist move. Consider what the supervaluationist says about excluded middle. Since the supervaluationist wants to retain excluded middle, the way he captures the claim that there can be no sharp cut-off in a sorites series does not lead to a revision of classical logic;[20] it leads rather to the introduction of determinacy operators. Suppose, e.g., we're given a sorites series of bald

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[20]This oversimplifies. On global supervaluationism all classical validities are retained, but the consequence relation isn't classical. For more on global and local supervaluationism see e.g., (Varzi, 2007) and (Asher, Dever, and Pappas, 2009).

men: $a_0, a_1, \ldots, a_n$. While the claim that there is a cut-off in this series is, in effect, a penumbral truth, it's false that there is a *determinate* cut-off: that is, there is no i such that a_i is determinately bald, but a_{i+1} is determinately not bald.

Similarly, the claim that it's necessary that the cut-off between the cases of composition and the cases of non-composition necessarily is where it is, could be taken as a penumbral truth. (Indeed, if we want to validate S5 we're forced to accept this.) The claim that there is no sharp cut-off should be stated using determinacy operators. So while we will accept that there is a case c such that c is the cut-off between composition and contingent composition and that it's necessary that this cut-off is at c we will deny that it's determinately necessary that the cut-off is at c.

Indeed, we have compelling independent grounds for denying that necessity entails determinacy; in fact, we're forced to deny the entailment the minute we consider languages with both modal and determinacy operators. For consider a borderline bald man. The supervaluationist will insist on excluded middle, so either he's bald or he isn't. Equally, the supervaluationist should insist that either he's actually bald or he is actually not bald. But if he's actually bald, he's necessarily actually bald; and if he's actually not bald he's necessarily actually not bald. So either he's necessarily actually bald or he's necessarily actually not bald. If necessity entails determinacy, we get that either he's determinately actually bald or determinately actually not bald, and hence we get that he's determinately bald or he's determinately not bald. That contradicts that he's borderline bald.[21]

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[21] In this argument there are, in effect, several applications of disjunction-elimination. Notoriously, disjunction-elimination isn't a valid rule in global supervaluationism. (This

3.6.3 Whence the lack of sharp cut-offs?

Why, on a deflationary view, should expect there *not* to be a determinate cut-off between the cases of composition and the cases of contingent composition? Recall the picture of the construction of non-basic worlds from the basic worlds. There is no indeterminacy—or so we assume—in which basic worlds there are. And once we have settled what the postulates are, we have fixed a unique non-basic world and there is no indeterminacy in how we extend this non-basic world to a complete non-basic world. The lack of sharp cut-offs therefore has to come from its being vague what the postulates are.

There are at least two types of indeterminacy which can arise here. First, there might be a determinate list of criteria for when composition occurs, but it is indeterminate whether something satisfies the criteria on the list to sufficient degree. Second, there might be indeterminacy in what's on the list. The resulting types of indeterminacy are rather different—e.g., the latter does not give rise to sorites series[22]—but for present purposes the difference will not matter. What matters is that either form of vagueness will lead to vagueness in what the postulates are.

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is an instance of the often noted fact that not all classical *rules* of inference are valid in supervaluationism, though all classical *validities* are supervaluationally valid.) However, in the applications of disjunction-elimination here we only appealed to logical validities as side premisses in the minor arguments. An instance of disjunction elimination where the minor arguments only have logical validities as side-premisses is valid even on global supervaluationism. (There is of course no problem for local supervaluationism.)

[22] Thanks to Ned Hall here.

How should we describe the above situation in terms of possible worlds? As before we start out with fixed set of basic worlds. Since we now have allowed it to be indeterminate what is necessitated to be the case by the basic facts, each basic world a is now associated with a *range* of partial non-basic worlds W_a. Each $w \in W_a$ represents one precisification of the property *settled by the basic facts of a*. On this space of partial worlds there are defined two relations R_0 and R_1. R_0 is the accessibility relation for metaphysical modality; R_1 is the accessibility relation for determinacy. R_1 is (at least) reflexive; R_0 is an equivalence relation. We make the obvious definitions:

- $\Box \phi$ is true at w iff ϕ is true at all w' such that $w R_0 w'$;

- $D\phi$ is true at w iff ϕ is true at all w' such that $w R_1 w'$.

In order for this to work out we have to impose the following restrictions on the interaction of R_0, R_1.

Let W_a be the class of partial non-basic worlds associated with a basic world a and let W_b be the set of non-basic worlds associated with a (different) basic world w. We'll insist that if $w \in W_a$ and $w' \in W_b$, then it's *not* the case that $w R_1 w'$; informally, if w is based on one basic world and w' is based on another basic world, then w' cannot be a precisification of w (and *vice versa*.)

We also have to demand that if $w, w' \in W_a$ then it's *not* the case that $w R_0 w'$, unless $w = w'$; why is that? w and w' are precisifications of the *same* world, and so they differ on what's determined by the basic facts, but agree on the basic facts. It follows that one world, w say, thinks that basic fact p necessitates non-basic fact q, but w' disagrees. Since w' disagrees there is

a world w'' such that $w'R_0w''$ and p is true at w'' but q is false at w''. Since R_0 is an equivalence relation, if w were to stand in R_0 to w', then w would stand in R_0 to w'' contradicting that w thinks that p necessitates q.

We'll also insist that for all $w \in W_a$ there is $w' \in W_b$ such that wR_0w'. The reason is as follows. The basic worlds a, b are mutually accessible, so if ϕ is a complete basic description of b, it should be true, in a, that determinately possibly ϕ. If a precisification of the world a ruled out all precisifications of the world b, then there would be a precisification of the world a according to which it would be impossible that ϕ.

What we have, then, is a set of basic worlds W, and for each basic world a a set W_a of partial worlds which represent the precisifications of the property *made true by the basic facts of a*. In order to get the classical modal validities we now have to extend these partial worlds to complete worlds. And we have to do this while respecting the relations R_0, R_1 which are already defined. We can do this by a simpleminded extension of the technique suggested in § 3.5. If we take our initial collection of partial worlds to describe a continuous series $c_0, c_1, \ldots, c_n$ we'll then end up verifying (P1A), (P2A) and (P3A). Rather than go into detail on this, let me sketch a more interesting construction.

3.6.5 Another way of blocking the argument

We are finally in a position to consider (P3C). This premiss (partly) expresses that the mereological facts supervene on non-mereological facts. While we have seen that (P3C) isn't justified by Sider's argument for (P3) it is open to us to hold that it's a penumbral truth that the mereological facts supervene

on the basic facts.

Suppose it's not determinate whether c is a case of composition or not. Consider a basic world a in which c obtains and consider a partial non-basic world w_a in which C neither has nor doesn't have a fusion. As we consider more precise extensions of w_a one of the things we make more precise is the property *made true by the basic facts of a*, or more formally, *necessitated by the obtaining of c*. If we insist on treating the supervenience of the mereological on the non-mereological as a penumbral truth, what we have to do is this. Once we have extended w_a to a world w_a^+ in which C has got a fusion, then no world w_a' which extends w_a^+ accesses a world w_b in which the obtaining of c does not necessitate that C has a fusion. As long as we ensure that w_a is also extended to a world w_a^- in which C is necessitated not to have a fusion, we avoid verifying (P3B) whilst verifying (P3C). (For more details see § A.1.)

3.7 Loose Threads

3.7.1 Counterfactual excluded middle

So far I've defined 'c is a case of composition' by means of strict conditionals. Defining it by means of counterfactuals doesn't make any difference unless we have counterfactual excluded middle. The reason is that Sider's argument for (P3) cannot establish more than

P3A-CF $Oc \, \Box\!\!\rightarrow (D\exists xFcx \lor D\neg\exists xFcx)$

If we have counterfactual excluded middle, or more generally the inference rule:

$$\frac{\phi \, \Box\!\!\rightarrow (\psi \lor \theta)}{(\phi \, \Box\!\!\rightarrow \psi) \lor (\phi \, \Box\!\!\rightarrow \theta)}$$

this will get us the analogue of (P3C)

P3C-CF $(Oc \mathbin{\square\!\!\rightarrow} D\exists x Fcx) \vee (Oc \mathbin{\square\!\!\rightarrow} D\neg\exists x Fcx)$

What Sider would need, however, is

P3B-CF $D(Oc \mathbin{\square\!\!\rightarrow} D\exists x Fcx) \vee D(Oc \mathbin{\square\!\!\rightarrow} D\neg\exists x Fcx)$

And this he cannot get. Even if counterfactual excluded middle holds, the following does not hold:

$$D(Oc \rightarrow \exists x Fcx) \vee D(Oc \rightarrow \neg\exists x Fcx)$$

(Similar remarks apply to the above rule.)

3.7.2 "Drawn from different possible worlds"

We have assumed that the notion 'composition occurring in a case' has to be defined in terms of strict or counterfactual conditionals. There might be another option, however. (Sider, 2001, p. 122) holds that the cases are drawn from different possible worlds.[23] So could Sider say that composition occurs in case c if there is a fusion of class C in the world w_c from which c is drawn?

If he makes that definition the argument is no longer invalid since (P3B) is justified. More generally, the intraworld and the interworld principles now come to the same thing. But there are several problems with this suggestion.

First, we don't have access to what's true in possible worlds directly. When we evaluate what is true in a world we evaluate certain strict or counterfactual conditionals where the antecedent gives a (partial) description of some

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[23]Interestingly, (Sider, 1997, p. 216) does *not* make this demand.

features of the world in question. But then this alternative definition is just a variant of the definitions in terms of strict or counterfactual conditionals.

Second, if this objection can be circumvented, so that (P3B) becomes justified, it becomes hard to see why (P1) is justified. What we in effect are assuming now is that the mereological facts about a case are just *given* with the case. But if the mereological facts are just given with the cases, why don't the mereological facts themselves partly determine the similarity relations which underlie whether we have a continuous series of cases or not? And if the mereological facts *do* enter into the similarity relations, how could we have a continuous series? By construction there is going to be a sharp cut-off in whether composition occurs or not: why doesn't this ensure that there is no continuous series?[24]

3.8 Supervenience

Back in § 3.3.3 I remarked that it seemed easy to fix the vagueness argument: just add the premiss that mereological facts supervene on non-mereological facts. We are now in a position to see that this does not help. For to say that the mereological facts supervene on the non-mereological facts is just to say that there can be no difference in how the mereological structure of the world is without there being a difference in the non-mereological structure of the world. But that will give us principles of the form of (P3C), not of the form (P3B).

Objection: the claim that the mereological supervenes on the non-mereo-logical is stronger than that. In particular, every maximal case c is such that

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[24]For a somewhat similar argument, see (Merricks, 2005).

either it determinately necessitates that C has a fusion or it determinately necessitates that C does not have a fusion.

First, let me just note that this is just a much stronger claim than supervenience is usually taken to be. Significantly, it cannot be justified by the idea that there can be no difference with respect to the supervenient properties without there being a difference with respect to the subvenient properties.

Second, there are plenty of supervenience-claims that cannot be strengthened in this way; here's one. Baldness presumably supervenes on facts not involving baldness, moreover it supervenes on precise such facts, in the following sense.[25]

1. If Alberich is bald then (determinately) there is a precise property P such that Alberich determinately instantiates P and such that necessarily, anyone who instantiates P is likewise bald;

2. if Alberich is not bald, then (determinately) there is precise property P' such that Alberich determinately instantiates P' and such that necessarily, anyone who instantiates P' is likewise not bald.

But these claims cannot be strengthened to read:

3. If Alberich is bald then (determinately) there is a precise property P such that Alberich determinately instantiates P and such that determinately necessarily, anyone who instantiates P is likewise bald;

4. if Alberich is not bald, then (determinately) there is precise property P' such that Alberich determinately instantiates P' and such that

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[25] Already this might be too strong, if supervenience does not entail that the supervenient properties are necessitated by the subvenient ones. So much the worse for the objection.

determinately necessarily, anyone who instantiates P' is likewise not
bald.

For suppose that Alberich is borderline bald. Then it's still true that Alberich
is either bald or not bald. If we have 3 and 4 there would then determinately
be a property P such that either it's determinately necessary that if Alberich
has P then Alberich is bald or it's determinately necessary that if Alberich has
P then Alberich is not bald.) It then follows that Alberich is determinately
bald or Alberich is determinately not bald, contradicting that Alberich is
borderline bald.

3.9   IS THE ACTUAL WORLD A WORLD?

The reader may have had a nagging doubt for a while now. I have presented
the vagueness argument in terms of cases and this is not the standard way
of doing it, though here I follow (Sider, 1997, 2001). Could the problems
I've raised for the argument turn on a mere infelicity in its formulation?
The problems started when we observed that the notion of a 'class having a
fusion in a case' was not defined. Cannot we get around this problem?

Instead of envisaging that we have a continuous series of cases, let's rather
envisage one world in which we find a continuous series of cases. We then
ask, with respect to that world whether there is a fusion in each case. Since
we're talking about a particular world there is no need to consider whether
the cases necessitate that composition occur (does not occur): we can ask
directly whether the classes associated with the cases have a fusion *in* the
world in question.

I retort that we haven't specified a *particular* possible world; we have

rather specified a class of possible worlds—having in common that there is a continuous series of cases of a certain character obtaining in them. When we ask, with respect to a world, whether a certain class has a fusion in that world what is going on is the following. The "world" in question is given to us by way of certain statements. Take the conjunction of those statements. When we are asked what is true with respect to that world we evaluate strict (counterfactual) conditionals with that conjunction as antecedent. This brings us back to the situation discussed in the body of this paper.

One might be impatient with this. Cannot we create an *actual* sorites series of composition? In fact, isn't this easy? Don't we do it all the time when we create artifacts? With respect to such a series we can now state the argument without using any modal notions. ((Korman, 2010) presents the argument this way.)

This leads to the final gambit. In order to block the argument, I deny that the actual world is a possible world. By saying this I don't just mean that the actual world is concrete whereas possible worlds are abstract; I rather mean that no *one* non-basic world accurately represents the actual world. (A class of non-basic worlds which disagree with the a basic world only on the statements about which the basic world has no opinion accurately represent that world.)

What exactly to say depends on whether one allows for contingency of composition or if one instead takes the route of § 3.6.5. If one allows for contingency of composition, one can say the following. In every world which represents the actual world there is a determinate cut-off point in the series of cases. But the location of the cut-off is not *modally stable*; in some worlds

the cut-off is at $\langle C_i, C_{i+1} \rangle$, at others it is at $\langle C_j, C_{j+1} \rangle$.

If one takes the view of § 3.6.5, one instead has to say the following. For every non-basic world w which is an admissible precisification of the basic world there is a determinate cut-off in the series of cases; moreover, this cut-off is stable in the sense that in any world which w thinks is possible, the cut-off is at the same place. But the cut-off is not *determinately modally stable* in the sense that there is a different admissible precisification of the basic world w' in which the cut-off is at a different place.

There is also the question about how we are to describe this situation in a modal object language without quantification over possible worlds. I'm sympathetic to someone who says: "Look, I don't care about what's true in various possible worlds, whether basic or otherwise. Here's a continuous series of classes. The first class determinately has a fusion. The last class determinately doesn't have a fusion. What do you say about a case like that?"

To make it more concrete suppose that at a particular time t, n different watchmakers are at n different stages in the assembly of some watch and that the first watchmaker definitely has assembled the parts P_0 in such a way that they make up a watch and that the n-th watchmaker definitely hasn't yet assembled the parts P_n into a watch. Let's further assume that the parts of what would be a watch only compose something if they compose a watch. Consider now the following premisses.

P1w P_0 determinately composes some object; P_n determinately does not compose an object.

P2w It's not the case that P_i determinately composes an object and P_{i+1}

determinately does not compose an object.

P3w P_i determinately composes an object or P_i determinately does not compose an object.

This triad is inconsistent: which premiss has to go? We should give up on (P2w). But we should now be much less loath giving up (P2w). Let c_i be a maximal non-mereological description of how things are with P_i. What expresses that there cannot be a sharp cut-off isn't (P2w), but rather the following principle.

P2w′ It's not the case that (determinately necessarily (if c_i obtains, then P_i composes an object) and determinately necessarily (if c_{i+1} obtains then P_{i+1} does not compose an object)).

What we're seeing here is that we have to distinguish between two senses of determinacy. There is the sense of determinacy expressed by the compound operator $D\Box$— we may call this *external determinacy*. The notion of ϕ's being determinately necessary, by itself, is not very interesting. Consider again Alberich. No matter the state of his hair, he's going to be a borderline case of being bald—in this sense of "determinacy". For surely he's neither necessarily bald nor necessarily not bald, and hence he's neither determinately necessarily bald nor determinately necessarily not bald. But $D\Box$ allows us to formulate a more interesting notion of determinacy as follows: there is some basic case c such that the obtaining of c determinately necessitates that ϕ.

The other sense of determinacy, expressed just by D, we may call *internal determinacy*. Something is determinate in this sense if it's settled by how a

non-basic world is given. A non-basic world can be given in vague terms, so there is no reason to think that for all ϕ, it is determinate (in this sense) whether ϕ or not ϕ. In accepting (suitable versions of) (P3) we are accepting that one thing which can never be vague is how many things there are in a world, but this is compatible with its being indeterminate of a given world whether a person in that world is bald or not.

3.10 Conclusions

We've seen that the vagueness argument is invalid and that its invalidity raises a host of issues about metaphysical modality and determinacy. Let me end by drawing some broader methodological conclusions.

First, a general moral. A lot of metaphysics is done by describing a possible world and then asking what's the case in that world. This procedure has its pitfalls. When we are describing a possible world we cannot be singling out a particular possible world; we merely narrow down the class of worlds. When we are asking what's true in a world we're not asking about what's true in a particular world, rather, we're in effect evaluating certain conditionals, be they strict or counterfactual. If we don't pay attention to that it's easy fall into fallacies.

Second, the possibility of positions along the lines of this paper should make us wary of relying on intuitive verdicts about truth-values of modal sentences when doing metaphysics. In particular, we should be concerned about the gloss we put on various "dependence-judgments". Supervenience is often taken to be a minimal requirement for dependence. The type of contingency of composition envisaged in this paper shows that this is wrong:

one can have failure of supervenience but still have dependence. Second, in the presence of vagueness or indeterminacy, supervenience-claims, while true, could fail to be determinate. So even if true the supervenience claims may not be able to bear the argumentative burden one wants. This is what happens with (P3c).

4

Grounding Grounding

4.1 Introduction

The notion of ground has come to prominence in metaphysics in the last decade.[1] As I will understand it here, grounding is a special type of meta-physical *explanation*. As Fine puts it, the grounds *explain* the grounded

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[1] While (Fine, 2001) has been enormously influential, truth-maker theory is another important source of the interest in the notion. (Particularly noteworthy in the present context is Cameron's (2008a, 2010) use of truth-making in a defense of ontological minimalism.) Other philosophical locutions which play some of the same rôles are "in virtue of", "makes it the case that", "makes it true that", "metaphysically explains that", "because".

in the sense that "there is no stricter or fuller account of that in virtue of which the explandandum [the grounded] holds. If there is a gap between the grounds and what is grounded, then it is not an explanatory gap." (Fine, forthcoming)

Many have found grounding to be a very useful notion. Rosen (2010) argues that grounding is essential to giving an account of *intrinsicality* and the relationship between determinables and determinates. Schnieder (2008) holds that grounding is needed to give a proper account of truth-functionality. Fine holds that ground is the key to answering questions about what is REAL and what is FACTUAL (Fine, 2001) and Schaffer (2009) holds that metaphysics is about figuring out what grounds what.[2]

Unfortunately, grounding leads immediately to a nasty version of what we may call the *Status Problem*: for what is the status of the grounding-claims themselves? To begin with, what *grounds* grounding-claims? If ϕ is metaphysically explained by ψ, what explains that ϕ is metaphysically explained by ψ? In this paper I'll do the following. In §§ 4.2– 4.4, I discuss progressively more serious versions of the Status Problem. In §§ 4.5–4.6 I then discuss and reject some proposed accounts of what grounds grounding-claims. In § 4.7 I up the ante by presenting an antinomy of grounding turning on iterated grounding claims: if we cannot resolve this antinomy grounding should be given up. In § 4.8 I propose a solution to the Status Problem and argue that it resolves the antinomy. This solution essentially relies on Fine's notion of the empty ground.

In the remainder of the paper I present a new way of expressing grounding

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[2]It is not clear that Schaffer's notion of grounding is the same as those of the previous authors. See further fn. 7.

claims (§§ 4.9–4.11). I then (§ 4.12) show how we can construct logics for iterated grounding-claims where the grounding-operators in such a way that 1. the grounding-operators are equipped with both introduction- and elimination-rules; 2. the solution to the Status Problem suggested in § 4.8 is a theorem of the logic.[3] After considering some objections (§ 4.13) I end by presenting some open questions and drawing a metaphysical conclusion (§ 4.14).

4.2 The Status Problem: a Special Case

If grounding is a form of explanation, then what is explained is of the form 'it's the case that so-and-so'. Grounding relates facts or true propositions.[4] The world as the totality of *facts* is "layered". But some philosophers have wanted to go further. If one conceives the world as the totality of *objects* one wants the *objects* to be divided into levels, with objects at higher levels ontologically dependent on objects at lower levels. One may now hope to use grounding to impose an ordering on the objects and properties which figure in facts and propositions. The natural idea is that an object a is ontologically prior to an object b if the existence and features of b can be explained by the existence and features of a. In particular, one makes the following definitions of fundamentality.[5]

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[3]In this paper I sketch the main ideas behind the logics. For a rigorous development of these ideas see (Litland, 2011).

[4]Officially grounding is a sentential operator (see below § 4.3.1).

[5]The distinctions drawn by (Fund_F) and (Fund_O) are arguably a bit crude. (Raven, 2009, pp. 263–9) introduces a distinction between *integrals* and *augments*. An object a is *integral* if *some* fact F involving a is such any set of facts Δ which grounds F, contains a fact G in which a is a constituent. An object can be integral without being fundamental$_O$. This does not matter for present purposes: if an object in fundamental$_O$ it is integral and that's all that's needed to raise the present problem.

FUND$_F$ Fact F is *fundamental$_F$* iff there are no facts $G_0, G_1, \ldots$ that ground F

FUND$_O$ An object a is *fundamental$_O$* iff a figures in a fundamental fact F[6]

The problem is that if grounding-facts aren't themselves grounded then *every* object is fundamental$_O$! For consider some object a: either there is nothing which grounds the fact that a exists or there is something which grounds the fact that a exists. If the former, then a is fundamental$_O$ since the fact that a exists is fundamental$_F$, and a certainly figures in the fact that a exists. If the latter, suppose that the fact that a exists is grounded in the facts $F_0, F_1, \ldots$. Then this is a further fact, *viz.*, the fact that a's existence is grounded in $F_0, F_1, \ldots$. Is this fact fundamental$_F$? If it is, then a is fundamental$_O$. It follows that if there are to be any objects that are derivative—that is, which are not fundamental$_O$—then some facts about what grounds what themselves have to be grounded.[7]

What could ground grounding-facts? Before I try to answer this question, it will be useful to discuss the Status Problem in greater generality, and before I do that some clarifying remarks are in order.

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[6]What does it mean to say that a figures in a fact F? For now, think of facts as having canonical descriptions. If a name for object a occurs in a canonical description of fact F, then a figures in fact F.

[7]This is an instance of the problem called "The Collapse" by deRosset (2011). In fact, deRosset argues against the somewhat different view of (Schaffer, 2009). Schaffer operates with a relation of grounding—Schaffer-Grounding—which can link entities in many different categories. In particular, objects are amongst the relata of the grounding relation. Define an object to be Schaffer-fundamental if there is nothing which Schaffer-grounds it. deRosset argues that this makes no difference. First, he defends a *linking principle*:

LINK If the object e is Schaffer-grounded in the objects $e_0, \ldots, e_n$, then the existence and features of e can be explained by the existence and features of $e_0, \ldots, e_n$.

He then points out that it's a consequence of (LINK) that an object is Schaffer-fundamental if it figures in a fundamental$_F$ fact. This issue deserves more scrutiny than I can give it here; let me just note that I agree with deRosset's argument.

4.3 CLARIFICATIONS

4.3.1 THE FORM OF GROUNDING-CLAIMS

Above I treated grounding as a predicate of true propositions (or facts). It is, however, better to treat it as a sentential operator. Officially, I'll express grounding by means of the locution "… BECAUSE …".[8] Here 'BECAUSE' is a sentential operator that can take any number of arguments: if Δ is any number of sentences and ϕ is a sentence then 'ϕ BECAUSE Δ' is a sentence. In particular, we will allow Δ to be infinite and we will allow Δ to be $\emptyset$—the latter will be important later on. The form of a grounding-claim is "it's the case that so-and-so BECAUSE it is the case that …".

One reason for treating ground as a sentential operator is to remain neutral on the existence and nature of facts.[9] Another reason is frankly pragmatic. If we treat grounding as a relation we may be able to lay down some axioms governing its behavior but the prospects for a genuinely informative model-theory and proof-theory seem very limited. Moreover, allowing quantification over facts leads immediately to paradoxes—even when we treat grounding as a sentential operator (Fine, 2010a). To be sure, eventually we'll need a theory of grounding which can deal with quantification over facts and propositions, but if we treat grounding as a sentential operator we can at least postpone the day of reckoning.[10]

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[8]Unofficially, I will continue to speak freely of facts being grounded in order to avoid cumbersome language. Quantification into the scope of a grounding-operator gives rise to concerns. See below § 4.3.3.

[9]cf. Fine, 2001, 2012, forthcoming; and Correia, forthcoming.

[10]For the purposes of this paper, we would also need to invoke a notion of an object "figuring" in a fact or a fact "involving" some objects or properties. Quite independently of grounding such notions would seem to lead to paradox. Consider the conjunction of all facts which do not figure in themselves. This is a fact, call it F. If F does not figure in itself it has to

Fine (2012, forthcoming) introduces a slew of distinct notions of ground. For the purposes of this paper I will only need a few. First, grounding is *factive*: if it's the case that ϕ BECAUSE it's the case that Δ, then both Δ (that is each $\psi \in \Delta$) and ϕ are true.[11] We'll also only be concerned with *strict* and *full* ground. That Δ *fully* grounds ϕ means that nothing need be added to Δ in order to get an explanation of why it is the case that ϕ. *Strict* ground moves us *downwards* in an explanatory hierarchy: no fact can contribute to grounding any of its strict grounds. A conjunction, e.g., is strictly grounded in its conjuncts, but neither conjunct is (strictly) grounded in the conjunction.[12] I'll follow Fine's notation and use < for strict full ground.[13]

4.3.2 GROUNDING AS A PRIMITIVE

Grounding is commonly taken to be a primitive notion. I will follow this tradition subject to the following *caveat*. In taking grounding as primitive, one just rejects the demand that grounding be *defined* in other terms. This does not mean that facts about what grounds what aren't grounded in facts not themselves involving grounding; facts about grounding don't have to

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be one of the conjuncts. But then F does figure in itself. So F figures in itself. But something figures in a conjunction iff it figures in one of the conjuncts, is one of the conjuncts, or is the conjunction operation. F is not the conjunction operation so F figures in one of the conjuncts or is one of the conjuncts. Say it figures in conjunct F_i or is conjunct F_i. But F_i is a conjunct of F so F_i figures in F_i. Contradiction. I don't mean to suggest that this paradox cannot be blocked. I just want to illustrate how quickly paradoxes come when we start talking about facts. We're better off putting this to the side.

[11] We may take this to follow from *explanation*'s being factive. If ϕ is explained by ψ then it's the case that ψ.

[12] Fine accepts the implication that a conjunction $\phi \wedge \phi$ is strictly grounded in ϕ; Correia (forthcoming) employs a theory of "factual equivalence" to avoid this.

[13] The notion of weak ground and the various notions of partial ground are treated in (Litland, 2011).

be brute. By way of comparison, suppose we take a sentential operator for metaphysical possibility as primitive. That does not mean that we think that all or even any facts about what is metaphysically possible are brute. It is, e.g., consistent with taking metaphysical possibility as a primitive that every metaphysical possibility is grounded in some contingent feature of the actual world.[14]

The reason for bringing this up is that treating grounding as primitive does not offer an easy way out of the Status Problem. In asking what grounds a true grounding-claim, we're not asking for an *analysis* of grounding: we're just noting that if 'ϕ BECAUSE Δ' is true, then this is a truth like any other and we can ask what grounds it.

4.3.3 HIGHER-ORDER QUANTIFICATION

In stating the Status Problem I treated grounding as a relation between facts or propositions while, officially, grounding is a sentential operator. This leads to the following problem. In order to state the Status Problem on an operator view, we need some mechanism for quantifying into predicate and sentence position. For instance, in defining what it is for an object to be fundamental$_O$ we need to say something like

- a is fundamental$_O$ iff $\exists G$ Fundamental$_F$ $G(a)$

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[14]If every fact involving grounding is grounded in facts not involving grounding, does this mean that grounding is not a fundamental component of reality? The issue is subtle. If every fact involving grounding is grounded in a fact not involving grounding, there is a sense in which grounding needn't figure in a complete description of reality. On the other hand, if grounding really cannot be defined in other terms there is something which can be said about how the world is using the grounding-vocabulary which cannot be said without using it. Similarly, even if every particular fact about what's metaphysically possible is grounded in a non-modal fact, if metaphysical possibility isn't definable in non-modal terms, then there is something which can be said about how the world is using the vocabulary of metaphysical possibility which cannot be said without using it.

This makes no sense unless we can quantify into predicate position. More-over, since the grounding operators take any number of arguments we need some mechanism for quantifying into "sequence-of-sentences" position in order to define 'fundamental$_F$' .

- ϕ is fundamental$_F$ iff there is no Δ such that ϕ BECAUSE Δ

I'm going to assume that this problem can be solved in a way which is consistent with taking grounding to be a sentential operator. Let me, however, say a few things in defense of substitutional quantification. (We can also use objectual quantification over sentences together with a truth-predicate.)

I suggest that we define fundamental$_F$ as follows:

- ϕ is fundamental$_F$ iff there is no sequence of sentences Δ such that '$\Delta < \phi$' true

The definition of Fundamental$_O$ takes the following form.

- a is fundamental$_O$ iff there is a predicate P such that the sentence 'Pa is fundamental$_F$' is true.

One could object that this makes grounding and fundamentality a meta-linguistic matter. Whether ϕ is grounded or not now turns on the existence of sentences and their semantic properties, whereas grounding is supposed to concern not the sentences themselves but what they are about (Trogdon, forthcoming). I'm not worried about this criticism. If we have a deflationary view of the truth-predicate what we're saying is just that it's not the case that $\Delta_0 < \phi$ and it's not the case that $\Delta_1 < \phi$ and it's not the case that ... *and*

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so on—for each set of sentences Δ.[15] Unfortunately, we're unable to list all the sets of sentences so we have to use a truth-predicate to generalize, but that does not make what we want to say in any way meta-linguistic.

A more serious worry concerns impoverished substitution classes. Our present language may not contain sentences Δ which would witness that ϕ is not fundamental$_F$, whilst extensions of our language do. In this case we would wrongly count ϕ is fundamental$_F$.

This is a problem not just for substitutional quantification. There is some reason to think that the generality in the claim that there is nothing which strictly grounds ϕ is not *quantificational* at all. For suppose that it is impossible to quantify over *absolutely* everything. Then the generality intended by stating that there is no (strict) ground for ϕ *cannot* be captured by any type of quantifier, even quantifiers over facts or properties. After all, as the domain of quantification expands, so does the domain of facts and properties.

One way of proceeding would be by taking the statement that nothing grounds ϕ to have *schematic* element. We could then understand substitutional quantification in such a way that for *any* language $\mathcal{L}$ we can come to understand and any sequence of sentences Δ from $\mathcal{L}$, '$\Delta < \phi$' is not true (in $\mathcal{L}$) (cf. Lavine, 2000, 2006; Parsons, 2006, 2008).[16]

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[15] Arguably, we need take no stance on deflationism (see Heck, 2005).

[16] If substitutional quantification cannot do the work, we will need quantification directly into predicate and sequence-of-sentences position; first-order quantification over properties and propositions will not do the work. (Or if they can do the work, this depends on substantial metaphysical commitments.) The reason is as follows. Consider the following view about properties (and propositions). For every predicate P_0 there is a corresponding property P_1 and whenever some a is such that $P_0(a)$, a also *instantiates* the property P_1. However, the direction of explanation is as follows: it is BECAUSE $P_0(a)$ that a instantiates the property P_1, but it's not BECAUSE a instantiates the property P_1 that $P_0(a)$. On this type of view, the following situation is possible. There is no property P and no objects $a_0, a_1, \dots$ such

4.3.4 Explaining explanations

The problem of iterated grounding-claims is to say what grounds that Δ grounds ϕ; alternatively, to find Γ such that Γ explains why Δ explains ϕ. But what does this mean?

There is a use of 'explain why' on which explaining why it is the case that ϕ BECAUSE Δ does not give us iterated grounding-claims. Some of us are often asked to "explain our explanations". Most of the time, what we are asked to do in these cases is not to explain *why* an argument we gave *constituted* an explanation; rather, what we are asked to do is essentially to give the same explanation again, filling in some detail here and there, pointing out that steps that may look perplexing in fact are instances of common inferences and so on. In so doing we hopefully put our audience in a position to appreciate that the original explanation did in fact explain what it purported to explain.

Suppose we claim that Δ explains ϕ, and we provide an argument $\mathcal{E}$ to support this claim. Explaining that Δ explains ϕ in the above sense comes down to giving another explanation of ϕ. If we succeed we have of course witnessed that the claim $\Delta < \phi$ is true, but this does not amount to *explaining* why it's the case that $\Delta < \phi$.

4.3.5 Matters of Ideology

The problem as I presented it in § 4.2 above concerned ontology: unless grounding-claims themselves have grounds then every *object* is fundamental$_O$.

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that ϕ BECAUSE $a_0, a_1, \ldots$ instantiate P; nevertheless, there is a sentence ψ such 'ϕ BECAUSE ψ' is true. (See also fn. 17.)

There is an analogous—and perhaps more important—problem for ideology. While the elements of one's ontology are the objects the existence of which one is committed to, the elements of one's ideology are the *predicates* one accepts as meaningful.

Just as one may want to use grounding to impose an ordering on one's ontology one may want to use grounding to impose an ordering on one's ideology. One may, e.g., want to hold that the ideology of simple objects and their properties is more fundamental than the ideology of complex objects and their properties.

FUNDAMENTALITY$_P$ A predicate P is fundamental if there is some ϕ such that it is fundamental$_F$ that $\phi(P)$

Unless some grounding-claims are grounded we get an exactly analogous problem to the one in § 4.2 above: every predicate would be fundamental$_p$.[17] In particular, the operator BECAUSE is itself fundamental$_P$.

4.4   THE STATUS PROBLEM IN GENERAL

The problem presented in § 4.2 above is an instance of the Status Problem. This problem arises for those metaphysicians who think that some classes of statements have some desirable feature whereas other classes of statements have some undesirable feature. The generic Status Problem is: do the metaphysician's statements themselves have the desirable features? The

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[17]One cannot reduce the question which: "which predicates are fundamental$_P$?" to the question: "which properties are fundamental$_O$?". For suppose we take what we may call an Aristotelian view of properties: the existence of the *property* of being F is (strictly) grounded in there being something which is F. We may then suppose that it's not fundamental$_F$ that the property of being F exists since its existence is grounded in a's being F, for some particular a. But this does not mean that 'F' is not fundamental$_P$: it could very well be fundamental$_F$ that a is F.

version of the Status Problem we considered above concerned grounding: are the metaphysician's statements about what grounds what themselves ungrounded?[18]

It will be important to distinguish between desirable features which concern *objectivity* and desirable features which concern *irreducibility*. A meta-ethical realist and a meta-ethical expressivist disagree about the objectivity of ethical discourse. The realist thinks that moral claims report mind-independent objective facts; the expressivist thinks that moral claims (merely) express the attitudes of those who make them. But thinking that an area of discourse is objective is not the same as thinking that it's irreducible. A meta-ethical naturalist, e.g., would hold that any ethical fact, while perfectly objective, is reducible to a (very complex) naturalistic fact.

I will adapt the terminology of (Fine, 2001) to facilitate the discussion. To mark that a certain statement is objective (and true) Fine introduces a sentential operator "It's factual that"; to mark that a certain statement is irreducible he introduces a sentential operator "it's constitutive of reality that".[19]

These notions relate to each other and to ground as follows.

- If it's constitutive of reality that ϕ then it's factual that ϕ; moreover,

- if it's factual that ϕ, and it's not constitutive of reality that ϕ, then

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[18]Examples of desirable features may be: *being representational*, *carving nature at the joints*, *having irreducible vocabulary*. Some undesirable features may be: *being (merely) expressive*, *being gerrymandered*, *having reducible vocabulary*. An important recent discussion of (in effect) the Status Problem is (Sider, 2012). In ch. 7.5 he discusses it as it arises for his notion of structure. Ch. 8 discusses it as it arises for other views.

[19]Fine characteristically suggests that the notions of REALITY and FACTUALITY and GROUND have to be taken as primitive and that they cannot be defined in other terms. The problem I present here does *not* turn on these notions being taken as primitive.

there is some Δ such that ϕ is grounded in Δ and it's constitutive of reality that ψ for each $\psi \in \Delta$.[20].

We say that it's BASIC that ϕ if there is no Δ which grounds ϕ (and it's the case that ϕ). It now follows that

- if it is BASIC that ϕ and it's factual that ϕ then it's constitutive of reality that ϕ.[21]

On the other hand, if it's BASIC that ϕ it needn't be constitutive of reality that ϕ. Consider again an expressivist about ethics. An expressivist could hold that it's BASIC that one should always act so as to maximize expected utility. There is no more basic fact in virtue of which this is the case. But of course the expressivist is going to deny that it's factual that one should always so act.[22],[23]

Theological imagery is quite helpful for understanding the operators.

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[20]This is principle (c) of (Fine, 2001, p. 17)

[21]"[] any basic factual proposition will be real. For any true factual proposition is real or grounded in what is real; and so the proposition, if basic, will be real." (Fine, 2001, p. 26)

[22]Modern expressivists (e.g., Blackburn, 1984, 1998) typically allow that a disquotational truth-predicate applies to merely expressive discourse. Similarly, they should allow a trivial "it's a fact that" operator to attach to merely expressive statements. This leads to some linguistic oddities. We will then find ourselves saying things like: it's a fact that one should act so as to maximize expected utility but it is not factual that one should act so as to maximize expected utility.

[23]If it's constitutive of reality that ϕ, need it be basic that ϕ? No, if it's possible that there be a sequence $\phi_0, \phi_1, \ldots$ such that ϕ_0 BECAUSE ϕ_1 and ϕ_1 BECAUSE ϕ_2 and ϕ_2 BECAUSE $\phi_3 \ldots$ and the ϕ_i are not all grounded in some ϕ_ω and all the ϕ_i are factual, then all the ϕ_i have to be constitutive of reality since any cut-off point would be arbitrary. See (Fine, 2001, p. 27; and Fine, 2010b, pp. 174-5). for some examples of this. (For more about the possibility of "infinite descent" see Schaffer, 2003; and Cameron, 2008b). It is arguable that what we should conclude from examples like this is not that being constitutive of reality can come apart from being BASIC, but rather that the "it's constitutive of reality"-operator is not a unary operator, but rather takes a "plurality" of arguments. What's constitutive of reality isn't the ϕ_i taken individually, but rather the ϕ_i taken collectively. Thanks to Ned Hall for the suggestion. This view would go very well with a view where what is grounded (and not just the grounds) are irreducible plural. Then one could say that the ϕ_i taken collectively are ungrounded. (For more on this idea see Dasgupta, Manuscript)

When God created the world, He didn't have to ensure that every fact obtains. He only had to make sure that those facts that are constitutive of reality obtain. If it's factual that ϕ but not constitutive of reality that ϕ, ϕ does not represent reality as it intrinsically is, but reality as it intrinsically is fully grounds ϕ.

We can now draw more distinctions than we can when we only have grounding; but just as in the case of grounding, the structure is a structure on facts. Again it is tempting to go further and impose structure on the *objects* of the world. The natural definition is that an object a is REAL (or a REALLY exists) if it's constitutive of reality that $\phi(a)$ for some ϕ (Fine, 2010b, p. 172).[24]

We now get a slightly more sophisticated version of the problem in § 4.2. Philosophers who like to talk about which objects REALLY exist generally don't think that all objects REALLY exist: that would make their view boring. Such philosophers are in a bind. Consider a philosopher who believes that the Fs aren't REAL, but who believes, for each a which is F, that it's factual that a is F.[25] Now consider an F, a say. Such a philosopher would deny that it's constitutive of reality that a exists but would hold that it's factual that a exists. Since it's factual that a exists there has to be some Δ such that the existence of a is grounded in Δ. (Otherwise, it would be BASIC that a exists and since it's factual that a exists, it would be constitutive of reality that a

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[24]We cannot give a similar definition of x is factual. For suppose we think that *wrongness* is a non-factual property. We might still want to say that the proposition *Bob believes that vagrancy instantiates wrongness* is a factual proposition. If we apply the parallel definition, wrongness would then be a factual property. For a suggestion about how to define factual object see (Fine, 2001, p. 17n25).

[25]Cameron's (2008a, 2010) view that (of course) there are numbers but there aren't REALLY numbers, is an instance of such a view.

exists and so *a* would be REAL.) Consider now the fact that the existence *a* is grounded in Δ. Is it constitutive of reality that the existence of *a* is grounded in Δ? If it is, then, since *a* figures in this fact, *a* is, by definition, REAL.

So if the object *a* is not to be REAL after all, we have to deny that it's constitutive of reality that *a*'s existence is grounded in Δ; but merely denying this will not do. The problem is that even if it's not constitutive of reality that ϕ is grounded in Δ, it should at least be factual that ϕ is grounded in Δ. It would be quite awkward for a metaphysician first to complain that various areas of discourse were non-factual for then later to have to admit that the pronouncements of metaphysics themselves are non-factual. But if it's factual that ϕ is grounded in Δ but it's not constitutive of reality that ϕ is grounded in Δ, then it's not BASIC that ϕ is grounded in Δ.

Unless every object is to be REAL, grounding-facts themselves have to be grounded. If its being the case that ϕ metaphysically explains its being the case that ψ, there has to be a metaphysical explanation of why it is that its being the case that ϕ metaphysically explains its being the case that ψ.[26]

4.4.1 NO LAYERING OF OBJECTS?

The problems I have discussed so far concern attempts at reading off a fundamentality-ordering on objects from the ordering on facts given by grounding; but even if grounding-claims are ungrounded there is no problem with the fundamentality ordering on *facts*. And maybe the ordering on facts is all we need? Alternatively, maybe we can tweak the definition of the fundamentality ordering for objects so that we don't run into the problems

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[26] A version of the problem discussed in this section is also discussed in (Sider, 2012, §8. 2).

we've encountered so far?

This will not help for two reasons. To give the first argument let's consider a concrete example: mereology. (The second argument has to wait for § 4.7.) In this case a standard dialectic goes something like this. There are the mereological atoms and their properties and the facts that the mereological atoms have such-and-such properties; call these the *A*-facts. Then there are the complex objects and their properties and the facts that the complex objects have such-and-such properties; call these the *B*-facts. Now, why are there *B*-facts? All the causal work is being done by the simples and their properties, so why posit facts of type *B*?

I think unease of this sort reflects adherence to the principle that we should *minimize unexplained reality*. One should only postulate facts *P* that are both unexplained and constitutive of reality if forced to do so. If one holds that certain facts about complex objects are grounded in certain facts about simples and their properties one avoids postulating those facts about complex objects as unexplained reality.[27]

But we avoided committing ourselves to unexplained real facts about complex objects only by committing ourselves to grounding-facts. Indeed, for every fact about complex objects we explain we commit ourselves to a distinct grounding-fact. If those grounding-facts are themselves constitutive of reality, we've made a bad trade. For whatever one might think about facts

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[27] It does this in one of two ways. We can either be reductionists (in the sense of (Fine, 2001)) about facts about complex objects, and hold that the fact that a certain complex object *a* exists is not constitutive of reality, but that the existence of *a* is grounded in the existence and features of *a*'s parts. Alternatively, we could be non-reductionists: while it's constitutive of reality that *a* exists, this is not an unexplained reality. Its being constitutive of reality that *a* exists is explained by its being constitutive of reality that *a*'s parts exists and have the features they do. (Not only is ϕ grounded in ψ, the reality of ϕ is grounded in the reality of ψ.)

about complex objects, in comparison to facts about grounding it's perfectly clear what they are. Better to have unexplained real facts about complex objects than unexplained real facts about what grounds what. There is therefore little point in trying to tweak the definition of fundamental$_O$ or REAL.

4.5 An Account of Iterated Ground

deRosset (2011) and Bennett (2011) argue that if ϕ is grounded in Δ, then what grounds this is just Δ itself.[28] If one adopts this view one avoids all the problems we've considered so far. Grounding-facts are always grounded so they are not fundamental$_F$. Moreover, if ϕ is grounded in Δ and it's constitutive of reality that ψ for each $\psi \in \Delta$, then this grounding-fact is grounded in the REAL and so it's factual that Δ grounds ϕ. We're now free to use grounding to impose an ordering on our ontology and ideology.

As we will see (§ 4.8) this view is *almost* right; it does, however, have to be rejected.

The problem is that if Δ grounds ϕ, then Δ *explains* ϕ: whatever grounds that Δ grounds ϕ has to *explain* that Δ explains ϕ. And Δ by itself cannot explain why Δ explains ϕ. Let's take a concrete example. Suppose it's raining or snowing. Suppose we ask *why* it's either raining or snowing. Suppose it is BECAUSE it's raining. Then the fact that it's raining *explains* that it's raining or snowing. Suppose now that we ask: why is it the case that the fact that it's raining explains that it's raining or snowing? The mere fact that it's raining is not going to explain that! The fact that it's raining does not know anything

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[28] A similar view is tentatively suggested in (Raven, 2009).

about explanation, as it were.

Let's be a bit more precise. If we want to know why Δ explains ϕ we have to say something about the relationship between Δ and ϕ. We have to say something about what it is about Δ and ϕ and their relationship which makes Δ *apt to explain ϕ*. Let $E(\Delta; \phi)$ be a statement describing the features of Δ, ϕ and the relationship between Δ and ϕ that makes Δ apt to explain ϕ. We now get the following schematic account of what explains that Δ grounds ϕ. It is Δ and $E(\Delta; \phi)$ (taken together).

What is it about the fact that it's raining and the fact it's raining or snowing which makes the former apt to explain the latter? Whatever it is, it doesn't have anything to with the rain outside. In fact, we should go further. *No Δ* which contributes to making it the case that it's raining, could explain why the fact that it's raining explains the fact that it's raining or snowing. If the fact that it's raining does not know anything about explanation, then any facts which (partly) make it the case that it's raining don't know anything about explanation either.

We now have an abstract proposal about what grounds grounding-claims. Suppose that $\Delta < \phi$. This is grounded in Δ and $E(\Delta; \phi)$, where E describes the aptness of Δ to ground ϕ. This is completely schematic: what kind of fact is $E(\Delta; \phi)$? What grounds it? Before we go on to consider some possibilities let me first avert a possible misunderstanding.

It's crucial to observe that $E(\Delta; \phi)$ cannot be part of the explanation of ϕ—Δ by itself fully explains ϕ. If we have to add the aptness of Δ to explain ϕ to Δ in order to obtain an explanation of ϕ we're off on a regress à la Lewis Carroll (1895). For then Δ wouldn't, by itself, fully explain ϕ. We would

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have to add $E(\Delta;\phi)$. But $E(\Delta;\phi)$ and Δ could not fully explain ϕ either; we would have to add $E(E(\Delta;\phi),\Delta;\phi))$—that is, the aptness of $E(\Delta;\phi)$ and Δ to explain ϕ—to Δ and $E(\Delta;\phi)$ in order to obtain an explanation of ϕ. This would not do either, we would have to add . . . .

4.6 The Essentialist View of Iterated Ground

(Rosen, 2010, pp. 131–2; Dasgupta, 2011; and Fine, forthcoming) have made some tantalizing suggestions about what may ground grounding claims; let's call it the RDF-proposal.[29] We can take their proposals to be proposals about what $E(\Delta;\phi)$ is (or about what grounds $E(\Delta;\phi)$).

It will help to begin with a simple example. If ϕ is true, then it's true that $\phi \vee \psi$ is grounded in ϕ. (If there are uncontroversial principles of grounding, this is one.) So suppose that $\phi \vee \psi$ is grounded in ϕ: what grounds this? Rosen holds that this is grounded in

1. its being the case that ϕ

2. its being true in virtue of the nature of disjunction that for all p, q if p then p grounds $p \vee q$.[30]

'In virtue of' as it appears in (2) is not the 'in virtue of' which can be synonymous with 'grounding'. As Rosen uses it it is one of the locutions used to indicate essentialist claims in the *Logic of Essence* (Fine, 1995). We'll

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[29]I should say that Fine *may* be advocating this view of iterated ground. It is not clear to me whether Fine intends this claim about essence to form part of the *grounds* for particular grounding-claims, or whether he just wants to assert that statements of essence and ground are correlated in this way.

[30]I've changed Rosen's notation to bring it in line with the one used in this paper. Moreover, since on the present approach I treat grounding as an operator, some of Rosen's clauses aren't necessary. Their omission is of no importance to the criticism I'm giving.

adopt Fine's notation $\Box_F, \Box_G, \ldots$ for essentialist operators. A formula of the form $\Box_F \phi$ can be read: "It's true in virtue of the identity of the objects which are F that ϕ" or "it's true in virtue of the nature of the objects which are F that ϕ".

The RDF- view is quite tempting. If ϕ grounds ψ then there is a particularly intimate relationship between what it is for ϕ to be the case and what it is for ψ to be the case. It's natural to suppose that the obtaining of this intimate relationship is part of what makes it the case that ϕ grounds ψ. The RDF-proposal holds that the notion of essence gives us the tools for spelling out in what this intimate relationship consists.

I have my misgivings about this whole idea and I think there are very serious problems in formulating the view properly. I discuss these problems in §§ 4.6.1–4.6.3. These sections are a bit of a digression; the reader who's not so interested in questions about essence can safely skip ahead to § 4.6.4.

4.6.1 Canonical forms

So far we've only looked at one example. How, in general, should the essentialist thesis be formulated? We can take the general form to be the following:

ESSENTIAL ITERATION If $\Delta < \phi$ then there is a constituent y of ϕ such that $\Delta < \phi$
is grounded in

1. Δ

2. its being essential to y that Θ

Now, what's this claim Θ? Rosen and Fine thinks that it is a suitably generalized version of the grounding-claim $\Delta < \phi$—in 2 above this is $\forall p \forall q (p \to p < p \vee q)$. Dasgupta holds instead that it's (a generalized version of) the claim that Δ is "materially sufficient" for (a suitably generalized version of) ϕ. In the case of the above example the claim could be $\forall p \forall q (p \to p \vee q)$.

So far, we have only one example. Can we say something, in general, about what these generalized claims look like? There are serious problems here, and there are reasons to be skeptical that there is one general form which works in all cases (see Dasgupta, 2011). Here I'll just assume that these problems can be overcome. Why, however, do we need to generalize?

Generalization comes in for two reasons. First, because one wants to defend a version of what (Rosen, 2010, p. 131) calls *Formality*. The idea is this:

> If Fred is handsome in virtue of his symmetrical features and deep green eyes, then anyone with a similar face would have to be handsome for the same reason. Particular grounding facts must always be subsumable under general laws, or so it seems. (Rosen, 2010, p. 132)

Second, because the essentialist claims shouldn't concern particular objects. It's not essential to *disjunction* that if it's raining, then its raining or snowing is grounded in its raining. Disjunction does not know anything about rain, as it were. (For more on this point see Fine, forthcoming; and Dasgupta, 2011).

Which of the two views should be preferred? Is a grounding claim partly grounded in the essential truth of a generalized grounding claim or is it grounded in the essential truth of a generalized claim of material sufficiency? I will argue that both views fail, but if one isn't convinced of that, the view

where what's essential is that a certain generalized grounding-claim holds is preferable. The reason is that whatever grounds $\Delta < \phi$ has to ground the *asymmetry* of the relationship between Δ and ϕ. If $\Delta < \phi$, then ϕ cannot, in turn, partly ground a member of ψ of Δ. Whatever grounds that $\Delta < \phi$ has to account for this asymmetry. But the mere fact that there is a constituent y of ϕ such that it's essential to y that (a suitably generalized version of Δ) is materially sufficient for (a suitably generalized version of) ϕ, does not guarantee such asymmetry.[31]

4.6.2   The problem of constituents

There are some technical difficulties combining the idea of grounding as a sentential operator with the essentialist idea. Here's the problem.

Suppose that Alberich is treacherous or Siegfried is fearless BECAUSE Alberich is treacherous. The essentialist wants to say that this is grounded in

1. Alberich's being treacherous

2. its being essential to disjunction that for all p, if p then p grounds $p \lor q$

where the idea is that disjunction is a constituent of the claim that Alberich is treacherous or Siegfried is fearless. The problem is that disjunction just isn't a constituent of the claim that Alberich is treacherous or Siegfried is fearless in the same sense that Alberich and Siegfried are constituents of this claim—disjunction does not figure "as an object" in the claim that Alberich

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[31] There are plausibly cases where there is asymmetry in grounding but no asymmetry in claims of essence. Fine (forthcoming) gives the example of an object which exists in time. It's true in virtue of the nature of such an object that it exists *simpliciter* iff it exists *at* a time . But we may want to claim that the object exists *simpliciter* in virtue of existing at a time.

is treacherous and Siegfried is fearless. Indeed, the formalism of the logic of essence only allows objects as constituents. To deal with this problem one would have augment the logic of ground to encompass non-objectual constituents (e.g., Fregean concepts).

One way of making sense of this talk of constituents within the framework of the logic of ground is to give up the idea that grounding is a sentential operator, treating it instead as a relation between facts. Facts, in turn, are to be treated as structured entities as follows. The fact that Alberich is treacherous or Siegfried is fearless has as constituents the fact that Alberich is treacherous, the fact that Siegfried is fearless and it also has *disjunction* itself as a constituent.[32] This approach would vindicate talk of constituents. The price is giving up treating grounding as a sentential operator, and more seriously, committing to a controversial view of what facts (or propositions) are.

4.6.3 Do essences enter into the grounds?

Modulo the problem of constituents I do accept the following: it is essential to *disjunction* that if p then $p \vee q$ is grounded in p and that if q, then $p \vee q$ is grounded in p. What's much less clear to me is whether this fact about essences is part of the grounds for its being the case that Siegfried's fearlessness grounds the fact that Siegfried is fearless or Alberich is treacherous.

Rosen introduces his tentative proposal about what grounds grounding-claims by saying that "[i]n many cases, when one fact obtains in virtue

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[32]Instead of facts one could work with structured propositions. This may be preferable. For what are the constituents of the fact that Siegfried is fearless or Brünnhilde is meek? (Brünnhilde isn't meek.)

of another we can begin to explain why this grounding fact obtains by pointing to one or more constituents of those whose nature 'mediate' the connection." (Rosen, 2010, p. 130) The idea is that certain general, broadly formal, principles of grounding are true in virtue of one of the constituents of a grounding-claim. It is certainly true that if we were to know that the grounding claim we're interested in, say $\Delta < \phi$, was an instance of a general claim, say $(\Delta < \phi)*$ about grounding which in turn is essentially true, this would in one sense *explain* why the grounding-claim $\Delta < \phi$ is true. (Subsumption under a general law is often explanatory.) But would it be part of the *grounds* for $\Delta < \phi$?

It seems to me that it would not. Certainly, the particular grounding-claim is not grounded in the general grounding-claim of which it is an instance, for a general claim is always partly (strictly) grounded in its instances; if the reverse was also true we would have a circle of strict ground. And that's impossible.

Presumably, then, what grounds that $\Delta < \phi$ isn't the generalization $(\Delta < \phi)*$ of this grounding-claim but the essential truth of $(\Delta < \phi)*$. But this, too, it seems to me, is wrong. The claim $\Delta < \phi$ is a particular claim, and even though it is not an *instance* of the claim that it's essential to some constituent of ϕ that $(\Delta < \phi)*$, the latter claim is in some sense a generalization of the former, and a more particular claim should not obtain in virtue of a more general one.

This can perhaps be made more precise using Fine's notion of strict partial ground. (In what follows the quantifiers into sentence position will be restricted to range over true propositions (facts).) The general claim

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$\forall p \forall q (p < (p \lor q))$ is strictly partly grounded in the particular claim that its raining of snowing is grounded in its raining. The claim that it's essential to disjunction that $\forall p \forall q (p<(p\lor q))$ is strictly partly grounded in $\forall p \forall q (p<(p\lor q))$ because the essential truth of claim is partly grounded in the truth of the claim. But then the claim about essences is strictly partly grounded in what it is supposed to ground, contradicting the asymmetry of strict ground.

Setting this problem aside, the position is in any case subject to what we may call the *Problem of Factuality*.

4.6.4 THE PROBLEM OF FACTUALITY

A view of this form does not have the problem of leaving out the explanatory connection between the ground and the grounded: it spells out what the connection is in terms of essence. Moreover, if we adopt this proposal we no longer have a problem about groundless grounding-claims. In the case of disjunction, in particular, we can expect the following sequence of

grounding-claims:[33]

$$p < p \lor q$$

$$(p, \Box_\lor \forall p \forall q(p \rightarrow p < (p \lor q))) < p(<p \lor q)$$

$$p, \Box_\lor \forall p \forall q(p \rightarrow p < (p \lor q)), \Box_{\lor,<} \forall p \forall q(p, \Box_\lor(p \rightarrow p < (p \lor q)) < (p < (p \lor q))) <$$

$$(p, \Box_\lor \forall p \forall q(p \rightarrow p < (p \lor q))) < p < (p \lor q)$$

$$\vdots$$

$$(4.6.1)$$

The problem now is with ungrounded *grounds*. In particular, we have a problem with the essentialist claims $\Box_\lor \forall p \forall q(p \rightarrow p < (p \lor q))$: for what grounds these claims? If these claims are ungrounded, we haven't progressed very far. These essentialist claims are presumably factual, so if they're BASIC they have to be constitutive of reality. The ideology which figures in the essentialist claims will then be real ideology.

One may be fine with this for disjunction and other logical operators.[34] But in any case where one wants to reject P as a real predicate one gets into trouble. Suppose, e.g., that one wants to deny that ' ... is a fusion of the objects ... ' is a real predicate. One begins by defending grounding-claims of the form

- Its being the case that a is a fusion of the objects $a_0, a_1, \ldots$ is grounded in Δ

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[33]$\Box_{\lor,<}$ is the operator for " true in virtue of the nature of disjunction and strict grounding".
[34]But even this is problematic: it rules out the view that the world as it is intrinsically can be described using only atomic sentences.

where Δ is some condition on the objects $a_0, a_1, \ldots$ (or their parts) that does not involve the predicate ' $\ldots$ is a fusion of the objects $\ldots$'. Part of the grounds for this grounding-claim is then a claim about essences of the form:

$$\Box_? \forall x_0, x_1, \ldots, (\Phi(x_0, x_1, \ldots) \rightarrow (\Delta(x_0, x_1, \ldots) < a \text{ is a fusion of the } x_0, x_1, \ldots))$$

If this is ungrounded, then '$\ldots$ is a fusion of $\ldots$' is a real predicate after all.

Rosen and Dasgupta here take different lines. Dasgupta bites the bullet and holds that the essentialist facts themselves are ungrounded. He claims that they are facts of a special sort, calling them "autonomous". The idea is that a fact is autonomous if the question of what grounds it doesn't properly arise. Autonomous facts are then held to be irrelevant to what our real ontological and ideological commitments are. I'm sympathetic to the notion of autonomous fact, but as I think the idea is better expressed in terms of zero-grounding, I'll postpone discussion of it until § 4.8.2.

For Rosen, on the other hand, there are candidate grounds for the essentialist claims. Rosen accepts a principle of "essential grounding" (Rosen, 2010, pp. 120, 131n22).

ESSENTIAL GROUNDING $\Box_F \phi \rightarrow \Box_F \phi < \phi$

In words, if ϕ is essentially true, ϕ is true BECAUSE ϕ is essentially true. If we were guaranteed that if ϕ is essentially true, then it's essentially true that ϕ is essentially true, (ESSENTIAL GROUNDING) would ensure that each essential truth was grounded. Fortunately, it's theorem of the logic of essence that $\Box_F \phi \rightarrow \Box_F \Box_F \phi$[35] Does this solve the problem?

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[35]The theorem only holds when F is "quasi-rigid" (Fine, 1995, p. 255). But it's reasonable to assume that F will be quasi-rigid in the cases which concern us.

Let me first say that I don't find (ESSENTIAL GROUNDING) plausible. Dasgupta rejects it because he takes the essentialist claims that ground grounding-claims to be claims of *constitutive* (as opposed to *consequential* essence (Fine, 1994)). To state the constitutive essence of something is to state what it is in the most direct and immediate sense. While being human is plausible part of the constitutive essence of Socrates, being (constitutively) essentially human plausibly isn't part of his constitutive essence.

I think we should go further. Even when essence is understood as consequential essence, the principle of (ESSENTIAL GROUNDING) isn't plausible. Socrates is human, in fact, he's essentially human. But to me it does not seem right to say that Socrates is human BECAUSE he is essentially human. Rather, he is essentially human partly because he is human. Surely, the essential truth of a claim ϕ is partly grounded in the *truth* of ϕ? But if that's the case (ESSENTIAL GROUNDING) would give a violation of asymmetry.[36] What's right about (ESSENTIAL GROUNDING) will become clear in § 4.8.2.

Be that as it may. The appeal to (ESSENTIAL GROUNDING) does not help. It doesn't suffice that grounding-claims always have grounds; since grounding-claims have to be factual what we have to show is that the grounding-claims are grounded in something which is constitutive of reality. (ESSENTIAL GROUNDING) does not provide us with an answer to that question. Every claim about what's essential is grounded, but only in further claims about

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[36]Such essential predications are very interesting from a grounding theoretic point of view. Socrates's being human partly grounds Socrates's essentially human, but not *vice versa*. However, there seems to be no fact F which can be added to Socrates's being human—short of Socrates's being essentially human itself—such that Socrates' being human and F together strictly ground Socrates' being essentially human. For dealing with examples like these Fine's notions of weak ground and, in particular, strict partial ground come in handy (Fine, 2012, forthcoming).

what's essential, and if any of these claims are constitutive of reality we get unwanted REAL ideology.[37]

In fact, the problem of unwanted real ideology is but the beginning of the trouble.

4.7 The Antinomy of Grounding

To appreciate the next problem, we can forget about the details of the proposal of last section. Abstractly, the proposal at hand is that $\Delta < \phi$ is grounded in Δ and $E(\Delta; \phi)$. The problem at the end of last section was that since $\Delta < \phi$ is factual, $E(\Delta, \phi)$ has to be factual. (Something factual is only grounded in the factual.) Eventually, then, $E(\Delta; \phi)$ has to be grounded in something which is constitutive of reality.

Setting aside the worry about unwanted ideology, an initial problem is just that it's very hard to see how facts of the form $E(\Delta; \phi)$ could be grounded in anything which is constitutive of reality. Facts of this form seem not to turn on *how* the world is. They appear to be unwordly facts, in the terminology of (Fine, 2005).[38] I find this quite compelling, but we can do better. If the facts $E(\Delta; \phi)$ are grounded in facts which are constitutive of reality we get a paradox. To see this let us revisit the standard theological picture.

When God creates the world, what does He have to do? He only has to ensure that it's the case that ϕ for each ϕ that is constitutive of reality.[39] Every other fact will be grounded in facts that are constitutive of reality. In

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[37] In fact, (Essential Grounding) only helps us avoid that every object is fundamental$_O$. It does not help us avoid that every object is *integral* (see fn. 5).

[38] Dasgupta's notion of an autonomous fact is a close relative of the unwordly facts.

[39] One may even think that the only thing God *can* do is to fix the facts which are constitute of reality. Once that is done, there are no further facts to fix.

particular, if it's the case that each $\psi \in \Delta$ is constitutive of reality and $\Delta < \phi$, then the only thing God has to do to ensure that ϕ is the case is to ensure that each $\psi \in \Delta$ is the case. Contrapositive: if each $\psi \in \Delta$ is constitutive of reality and God has to do more to ensure that ϕ is the case than to ensure that each $\psi \in \Delta$ is the case, then it's not the case that $\Delta < \phi$. We have the following principle.

CP-Creation If God has to do more in order to make it the case that ϕ than to make it the case that Δ, where Δ is constitutive of reality, then it's not the case that $\Delta < \phi$.

The problem is as follows. Suppose that Δ is constitutive of reality and that $\Delta < \phi$. Then in order to ensure that ϕ is the case God had to do nothing more than to ensure that Δ is the case. Now, what makes it the case that Δ makes it the case that ϕ is (in part) $E(\Delta; \phi)$. If it is not the case that $E(\Delta; \phi)$, then it is not the case that $\Delta < \phi$.[40]

So $E(\Delta; \phi)$ has to be the case. But since it's factual that $E(\Delta; \phi)$ there has to be some Δ_0 such that each $\psi \in \Delta_0$ is constitutive of reality and such that $\Delta_0 < E(\Delta; \phi)$. But then God has to do more than merely make sure that it's the case that Δ in order to make sure that it's the case that ϕ. In particular, God has to make sure that Δ_0 obtains. By (CP-Creation) this contradicts that $\Delta < \phi$.

It's essential for this argument that it's factual that $\Delta < \phi$ and hence that it's factual that $E(\Delta; \phi)$. For if it were non-factual that $E(\Delta; \phi)$ there wouldn't

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[40]Strictly speaking, if $\mathcal{G}$ were not the case, there could be $\mathcal{G}'$ which grounds its being the case that ψ grounds ϕ. The argument below can be reformulated to take account of this. Let $\mathcal{G}_0, \mathcal{G}_1, \dots$ be all the $\mathcal{G}'$ such that if $\mathcal{G}'$ were to obtain, $\mathcal{G}'$ would ground that ψ grounds ϕ. We can now substitute "If none of the $\mathcal{G}_i$ obtain, then ψ would not ground ϕ" for "If $\mathcal{G}$ is not the case, then it is not the case that ψ grounds ϕ."

have to be any real Δ_o that strictly grounded $E(\Delta; \phi)$ and hence God wouldn't have to do more than make it the case that Δ in order to make it the case that ϕ.[41]

But grounding-claims have to be factual: what are we to do?

4.8 Zero-Grounding

4.8.1 Zero-grounding and factuality

Formally, the constraint on factuality is this (writing $\mathbf{R}, \mathbf{F}$) for "it's constitutive of reality that" and "it's factual that".

$$\mathbf{F}(\phi) \rightarrow (\mathbf{R}\phi \vee \forall \Delta \forall \psi \in \Delta \exists \Delta'(\Delta < \phi \rightarrow \Delta' < \psi \wedge \forall \sigma \in \Delta(\mathbf{R}\sigma)))$$

That is, if it's factual that ϕ then either ϕ is constitutive of reality or anything which partially grounds ϕ is fully grounded in the real. Note the following way for ϕ to be factual: the only Δ such that $\Delta < \phi$ is $\emptyset$. (Since there are no members of $\emptyset$, every member of $\emptyset$ is real.) In this case we say that ϕ is *zero-grounded*. In fact, the only way of avoiding the above paradox is to treat facts of the form $E(\Delta; \phi)$ as being zero-grounded.

Now, what does it mean to say that something is zero-grounded? One might think of the grounding-claims as providing the instructions to a machine. The machine is fed facts, and the machine churns out facts which are grounded in the facts which it is fed. A fact is *ungrounded* if the machine never churns it out unless the machine is fed it as input. A fact is *zero-*

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[41] Incidentally, this shows that we cannot define the notion of ground in terms of the notion: "In order to make it the case that . . . God only has to make it the case that . . . ". For if ϕ is (wholly) non-factual, there if nothing God could do to make it the case that ϕ, since the God cannot make what is non-factual the case.

grounded if the machine churns it out when it's fed *no* input. In theological terms: when God creates the world He only has to create the facts which are constitutive of reality. All the other facts are generated from some collection of such facts, the zero-grounded facts being generated from the empty collection of facts. Prior to God's making anything the case $E(\Delta; \phi)$ and its ilk already have to be the case. There is a sense, then, in which the view discussed in § 4.5 above is correct. If $\Delta < \phi$, then nothing more has to be *done*, in order to make it the case that ϕ, than to make it the case that Δ. But that does not mean that nothing more has to be the case in order for it to be the case that $\Delta < \phi$.

4.8.2 Zero-grounding and essence

We can now see what's right about the principle of (Essential Grounding). When it's essential (to x say) that ϕ, then its being the case that ϕ is zero-grounded. If one wants to hold that facts about essence partly ground grounding-claims, then since these facts about essence are zero-grounded the Problem of Factuality is overcome; moreover, the antinomy of grounding is also dealt with. This is of importance not just to the defender of the essentialist view. While I don't that the facts about essence themselves enter into grounding-claims, I do accept the essentialist claims. Since these claims are zero-grounded, I avoid the problem of unwanted real ideology.

This is cute, and it fits the bill; indeed, one suspects that it fits rather too well. There is a challenge here. One has to tell a story making it plausible that something is zero-grounded. What we're asking for here is not an account of the *grounds* for something's being zero-grounded. Presumably,

if something is zero-grounded this fact itself is zero-grounded.[42] Rather, what we're after is an account making it intelligible that something is zero-grounded.

In the remainder of the paper, I will do just that; in the course of doing so, I will also hopefully take some of the mystery out of the notion of the empty ground.[43] In order to do that it will be necessary to take several steps back and consider what exactly we mean when we say that grounding is a form of explanation.

4.9   GROUNDING AN EXPLANATION: THE GRAMMAR OF GROUND

One thought one may have when one first considers iterated grounding claims is that they represent some sort of category mistake. Grounding, is it were, is only defined for ordinary claims, and the result of applying the grounding operator to some ordinary claims is an extraordinary claim. If such a line could be defended all the above trouble would go away: iterated grounding-claims pose no problem because there aren't any. But why believe that it doesn't make sense to ask what grounds grounding-claims?

If one treats grounding as a sentential operator[44] (read as "BECAUSE") the sentence 'ϕ BECAUSE ψ' is just another sentence; and "(ϕ BECAUSE ψ) BECAUSE θ" is well-formed. And if grounding is a relation between facts[45] the fact that ϕ is grounded in the fact that ψ is just another fact: why couldn't we

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[42]The logics PPLG and PNLG both deliver this result. See (Litland, 2011).

[43] The possibility of zero-grounding is a strong reason for treating grounding as an operator rather than as a relation between facts. It is very hard to make sense of a notion of an empty fact.

[44]As does Fine, 2012; and Correia, forthcoming.

[45]As in Rosen, 2010; Audi, forthcoming; deRosset, 2011; Bennett, 2011; and Trogdon, forthcoming.

ask what grounds it?[46]

To ask what grounds its being the case that ϕ is to ask for a special type of explanation of ϕ. And if it's the case that Δ grounds ϕ, then that's just one more thing which can be the case, and we can ask what explains it. Why couldn't we?

The best answer to that question is that sentences of the form 'ϕ BECAUSE Δ' don't make claims at all. One has overlooked an option for how to express grounding. Since grounding is supposed to be a special sort of explanation; let's step back and consider how explanations in general are expressed. One way of expressing explanations uses relational expressions, e.g., "explains"; another way uses a sentential operator, e.g., "because". Some examples:

R Its raining or not raining is explained by its raining

O It's raining or not raining because it's raining

But there is a different way of expressing explanations: one can express explanations as *answers to questions*.

Q&A Why is it raining or not raining? It's raining.

I suggest that the basic of way of expressing the special form of (metaphysical) explanation known as grounding is by means of question-and-answer pairs:

GENERAL Q&A Why is it the case that ϕ? Because $\phi_0, \phi_1, \ldots$

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[46]While (Schaffer, 2009) treats grounding as a relation between objects in arbitrary ontological categories, if the grounding relation holds between two entities, then there is the fact that it so holds and the same problem arises.

Here 'because' does not function as a sentential operator; its job is rather to indicate that $\phi_0, \phi_1, \ldots$ are to be considered as answering the question "why is it the case that ϕ?".

In order fully to defend this view of grounding-claims we have to make the case that there is a sense of the question "Why ϕ?" such that any satisfactory answer would provide the grounds for its being the case that ϕ. (Moreover, we have to be capable of grasping this sense of the question "Why ϕ?" prior to grasping a grounding-operator.) Rather than meet this challenge head-on, I will here adopt as my working hypothesis that grounding-claims are best expressed as question-and-answer pairs and explore the benefits of this view.

One immediate benefit is that it accounts for why grounding is *factive*. If grounding-claims take the above form, then in claiming that ϕ is grounded in ψ, ϕ is presupposed and ψ is *asserted*.

Another immediate benefit is that we can now account for our difficulty in dealing with iterated grounding claims: there aren't any! An iterated grounding claim would have the form

ITERATED Q&A Why is it the case that (Why is it the case that ϕ? Because

$\phi_0, \phi_1, \ldots$)? Because $\psi_0, \psi_1, \ldots$

and this is ungrammatical.

The most important benefit, however, is the light we can now throw on the logic of ground.

4.10 Grounding and Explanation: Explanatory Arguments

In general, an answer ψ to a question "Why is it the case that ϕ?" can be shown to be correct by an argument from ψ to ϕ. When the question

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"Why is it the case that ϕ?" has the special sense relevant to grounding, the existence of a merely valid argument from ψ to ϕ does not suffice to show that ψ is an acceptable answer. For instance, the argument from $\phi \wedge \psi$ to ϕ had better not show that ϕ is grounded in $\phi \wedge \psi$. (A conjunction is grounded in the conjuncts but not *vice versa*.) The argument from ϕ to $\phi \vee \psi$, on the other hand, does show that ϕ is an acceptable answer to the question "Why is it the case that $\phi \vee \psi$?". Let's call the arguments which can back up grounding-claims *metaphysically explanatory arguments*.

Can we develop a logic of metaphysically explanatory arguments? Logic cannot say anything about particular arguments. Which particular arguments are explanatory is a material not a formal matter: it depends on the content of the premises and conclusions. Logic can tell us something about the structural relationship between various arguments, however.

Let us represent arguments as trees without infinite branches. It will be necessary to consider two types of argument.[47] There are the *strictly explanatory* arguments: if Π is a strictly explanatory argument from premises Δ to conclusion ϕ, then "because Δ" is an acceptable answer to the question "Why ϕ?". We also have the plain arguments: if Π is plain argument from Δ to ϕ, then if Δ is true, then ϕ is true—there is, however, no guarantee that Δ explains ϕ.[48]

I submit that the plain and strict arguments are related as follows:[49]

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[47]A fully adequate treatment needs to consider weakly explanatory arguments. These correspond to Fine's (2012) notion of a weak full ground. The notion of a weakly explanatory argument plays a prominent rôle in (Litland, 2011).

[48]I should note that there is nothing in the formalism which forces the plain arguments to be deductively valid. Thanks to Louis deRosset here.

[49]The notation $\dfrac{\overline{\phi_0, \phi_1, \phi_2, \ldots}^{\ 0,1,2,\ldots}}{\dfrac{\Pi}{\psi}\ 0,1,2,\ldots}$ is to be understood as follows. The top occur-

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STRICT-IS-PLAIN Any strictly explanatory argument is a plain argument.

NON-CIRCULARITY If Π is a strictly explanatory argument from premisses $\phi, \phi_1, \phi_2, \ldots$ to ϕ, then the following is a plain argument for any ψ

$$\frac{}{\phi, \phi_1, \phi_2, \ldots} \; 0,1,2,3,\ldots$$
$$\Pi$$
$$\frac{\phi}{\psi} \; 0,1,2,\ldots$$

CUT If Π_i is a strictly explanatory argument from Δ_i to ϕ_i for each i and Σ is a strictly explanatory argument to ϕ from $\phi_0, \phi_1, \ldots$ then

$$
\begin{array}{ccc}
\Delta_0 & \Delta_1 & \\
\Pi_0 & \Pi_1 & \\
\phi_0 & \phi_1 & \vdots \quad \ldots \\
 & \Sigma & \\
 & \phi &
\end{array}
$$

is a strictly explanatory argument from $\Delta_0, \Delta_1, \ldots$ to ϕ.

(NON-CIRCULARITY) can be explained as followed. Anything which strictly explains ϕ is of lower "order" than ϕ: ϕ cannot contribute to strictly explaining itself. If, *per impossibile*, ϕ did contribute to strictly explaining ϕ, then the concept of strict metaphysical explanation is incoherent, and anything would follow from no premisses whatsoever. This is what (NON-CIRCULARITY) expresses.[50]

It's crucial to note that we don't have *weakening*: if there is a strictly explanatory argument from Δ to ϕ there needn't be a strictly explanatory argument from Δ, ψ to ϕ. ψ might be irrelevant for explaining ϕ.

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rences of '$0,1,2,\ldots$' label the occurrences of the formulæ $\phi_0, \phi_1, \ldots$. The bottom occurrences indicate that those occurrences are discharged in passing to the conclusion ψ.

[50]Note that in an application of (NON-CIRCULARITY) all the premisses on which ϕ depends are discharged. We can assert ψ on the basis of no assumptions. I should note that there are technical reasons for writing the rule in this way (see Litland, 2011).

When people talk about grounding as a relation they typically take grounding to be *irreflexive*, *asymmetric* and *transitive* (see e.g., Schaffer, 2009, pp. 375–6). Those formal features of a grounding-relation are nicely explained by the above features of strictly explanatory arguments. (Cᴜᴛ) ensures that grounding is transitive. (Nᴏɴ-ᴄɪʀᴄᴜʟᴀʀɪᴛʏ) ensures that grounding is irreflexive. (Cᴜᴛ) and (Nᴏɴ-ᴄɪʀᴄᴜʟᴀʀɪᴛʏ) together ensure that grounding is asymmetric.

4.11   Exᴘʀᴇssɪᴠᴇ Lɪᴍɪᴛᴀᴛɪᴏɴs

There are, however, reasons not to rest content with this view. Not only do we *answer* particular "Why"-questions and *give* metaphysical explanations backing up those answers, we also reason *about* metaphysical explanations and assert general principles about them. For instance, we may want to say that if it's the case that $\phi \vee \psi$, then either ϕ explains that $\phi \vee \psi$ or ψ explains that $\phi \vee \psi$.

We could specify such principles meta-linguistically: "if we have a metaphysical explanation of this form, then we have a metaphysical explanation of this other form". But it is very tempting to try to "push" these meta-linguistic statements down into the object-language. The principles about metaphysical explanation we want to assert aren't tied to any one particular language. If we express the principles meta-linguistically, we only assert that the connections hold for all metaphysical explanations that can be formulated in a particular language. The structural principles governing metaphysical explanations are not parochial in this way: we surely intend the claim that metaphysical explanations are transitive as a claim about

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anything we could recognize as a metaphysical explanation and not just as a claim about the metaphysical explanations which happen to be expressible in English.

This is particularly important when we want to express that ϕ is *not* metaphysically explained by ψ. We don't want claims of this form to come out true just because no metaphysically explanatory argument from ψ to ϕ can be formulated in English.

In order to be able to reason about metaphysical explanations in the object language I will introduce grounding-*operators*. Once we have grounding-operators we can, of course, formulate iterated grounding-claims. Fortunately, it turns out that there is a natural way of introducing grounding-operators such that true grounding-claims are partly zero-grounded, just as required in § 4.8.

4.12 Grounding as an Operator

We'll introduce a sentential operator $\Rightarrow_s$. Think of $\Delta \Rightarrow_s \phi$ as an object-language correlate of the metalinguistic claim that there is a strictly explanatory argument to ϕ from Δ. We allow Δ to be of any cardinality; in particular, Δ can be $\emptyset$. Since $\Delta \Rightarrow_s \phi$ is just meant to report that there is an strictly explanatory argument to ϕ from Δ, it's clear that $\Rightarrow_s$ has to have the following introduction rule.

$$\frac{\overline{\phi_0, \phi_1, \ldots} \; 0, 1, 2, \ldots}{\begin{array}{c} \Pi \\ \phi \end{array}} \quad 0, 1, 2, \ldots, \Rightarrow_s\text{-Introduction}$$
$$\frac{}{\phi_0, \phi_1, \ldots \Rightarrow_s \phi}$$

Here Π is a strictly explanatory argument and $\phi_0, \phi_1, \ldots$ are *all and only* the premises on which ϕ depends. We treat arguments of the form depicted in the $\Rightarrow_s$-I rule as strictly explanatory.

Note that the $\Rightarrow_s$ operator isn't factive; $\Rightarrow_s$ isn't itself a grounding operator. We use $\Rightarrow_s$ to state the introduction rules for a factive grounding operator ($<$).

$$\frac{\Delta \qquad \Delta \Rightarrow_s \phi}{\Delta < \phi} \text{ <-Introduction}$$

Arguments of this form are strict. (Note the similarity to conjunction-introduction.)

Can we find matching elimination rules for $<$ and $\Rightarrow_s$? Yes: we can. A proof-theoretic inversion principle (see e.g., Read, 2010) provides a heuristic. This principle says that the elimination rule(s) for an operator λ should be such that if ϕ follows from the grounds for asserting $\lambda(\psi_0, \ldots, \psi_n)$, then ϕ should follow from $\lambda(\psi_0, \ldots, \psi_n)$ by an elimination rule. And conversely, if ϕ follows from $\lambda(\psi_0, \ldots, \psi_n)$ by an elimination rule, then ϕ has to follow from any grounds for asserting $\lambda(\psi_0, \ldots, \psi_n)$.

According to the $<$-introduction rule, we are entitled to conclude $\Delta < \phi$ from premises Δ and $\Delta \Rightarrow_s \phi$. Anything which follows from those premises must therefore follow from $\Delta < \phi$. The elimination rule for $<$ has to take the form:

$$\frac{\Delta < \phi \qquad \dfrac{\overline{\Delta}^{\,1} \qquad \overline{\Delta \Rightarrow_s \phi}^{\,2}}{\begin{array}{c}\Pi \\ \psi\end{array}}}{\psi} \text{ 1,2: <-Elimination}$$

(Note the similarity to conjunction-elimination.)

Is the resulting argument plain or explanatory? Clearly, it should only be plain. For by an application of <-Elimination we can derive θ from $\Delta < \phi$, for any $\theta \in \Delta$. But Δ's grounding ϕ should not explain any $\theta \in \Delta$. (Similar arguments show that any application of an elimination rule will result in a plain argument.)

The elimination rule for $\Rightarrow_s$ is much more interesting. The introduction rule for $\Rightarrow_s$ tells us that we are entitled to assert $\Delta \Rightarrow_s \phi$ if there is a strictly explanatory argument with premisses Δ and conclusion ϕ. So anything which follows from the existence of such an argument should follow from $\Delta \Rightarrow_s \phi$. In order to formulate this elimination rule we have to be able to assume and discharge *hypothetical arguments* as well as formulæ. Let terms of the form $\Delta \Vdash_s \phi$ stand for hypothetical strict arguments with conclusion ϕ and premisses (exactly) Δ.[51] A hypothetical argument $\Delta \Vdash \phi$ only occurs in the following type of context:

$$\frac{\Delta}{\phi} \; [\Delta \Vdash_s \phi]$$

Arguments of this form are strict.

We can now write down the following elimination rule for $\Rightarrow_s$:

$$\frac{\Delta \Rightarrow_s \phi \qquad \dfrac{\overline{\Delta \Vdash_s \phi}^{\,1}}{\begin{array}{c}\Pi\\\psi\end{array}}}{\psi} \; 1, \Rightarrow_s\text{-Elimination}$$

It can be shown that these introduction and elimination rules suffice to derive the uncontentious principles governing simple grounding. For

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[51] For an application of hypothetical arguments in a different context see Schroeder-Heister, 1984.

instance, we can establish that grounding is factive, that grounding is transitive and that nothing strictly grounds itself. In order to say anything about iterated grounding-claims we have to consider whether arguments of the form depicted in the $\Rightarrow_s$-Introduction rules are strict or merely plain.[52]

What about iterated grounding-claims? It turns out that facts of the form $\Delta \Rightarrow_s \phi$ are always zero-grounded. This is witnessed by the following derivation.

$$
\cfrac{\Delta \Rightarrow_s \phi \qquad \cfrac{\cfrac{\cfrac{\overline{\Delta}^{\,1}}{\phi}\,[\Delta \Vdash_s \phi]^2}{\Delta \Rightarrow_s \phi}\;1,\Rightarrow_s\text{-Introduction}}{\Rightarrow_s (\Delta \Rightarrow_s \phi)}\;\Rightarrow_s\text{-Introduction}}{\Rightarrow_s (\Delta \Rightarrow_s \phi)}\;2,\Rightarrow_s\text{-Elimination}
$$

It is easily shown that claims of the form $\Delta < \phi$ are grounded in Δ and $\Delta \Rightarrow_s \phi$. Since claims of the latter form are zero-grounded, we get the result that claims of the form $\Delta < \phi$ are partly grounded in Δ and partly grounded in nothing at all.

4.13 Objections and Clarifications

Objection

Doesn't this solve the Status Problem too handily? Am I saying that if somebody asks why Δ grounds ϕ that we can just answer like this: "Oh, it's because Δ and $\Delta \Rightarrow_s \phi$." And if somebody asks why it's the case that $\Delta \Rightarrow_s \phi$ we can just say "because"?

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[52] We can do better. The above system can be extended to a natural deduction system for all the grounding operators in Fine's (2012) Pure Logic of Ground and it can be shown that this system is a conservative extension of the Pure Logic of Ground.

No. I'm not saying that. If our opponent is trying to challenge us to provide a metaphysical explanation of why it is the case that ϕ BECAUSE Δ, then, I suggest, the above is indeed the correct response. But such a challenge only has a point if it is *agreed* that we are in possession of a metaphysical explanation of ϕ in terms of Δ. If our opponent doesn't agree that we have that, then he should just challenge us to produce that explanation or challenge an explanation we have given at a particular point. Any particular metaphysical explanation Π of ϕ can be challenged as an explanation of ϕ, but once it's agreed that Π metaphysically explains ϕ, *that* Π is a metaphysical explanation of ϕ is not a further source of trouble.

Objection

On your view, both $p \Rightarrow_s \neg\neg p$ and $p \Rightarrow_s p \vee q$ have the same ground: the empty one. But surely the explanations for why these claims hold differs? One of them turns on the nature of disjunction, the other one turns on the nature of negation. It is because of something about how disjunction works that p is a sufficient explanation for $p \vee q$.

First reply

In order to make someone understand that these grounding relationships hold, one has to do different things in the two cases. If somebody doesn't get that the first grounding relationship holds one has to tell him something about negation; if he doesn't understand that the second one holds, one has to tell him something about disjunction. But these facts about negation and

disjunction don't enter into the grounds for $p \Rightarrow_s \neg\neg p$ and $p \Rightarrow_s p \vee q$.[53]

Second Reply

One should accept that it's in virtue of the nature of disjunction that if p is the case, then p grounds $p \vee q$, but that it's not in virtue of the nature of negation that if p is the case, then p grounds $p \vee q$. As long as the "because something about how disjunction works" locution is understood to be a claim about essence and not ground one should accept the claim.

Objection

What justifies taking arguments having the form depicted in the $\Rightarrow_s$-introduction rule to be strictly explanatory? Why not just treat these arguments as plain?

Reply

If we treat these arguments as just plain, the solution to the antinomy doesn't follow from the logic, so that's one reason for treating the rule as giving rise to strict arguments.

More decisive is the following consideration. If we adopt the view that the central notion is 'metaphysically explanatory argument' and that the grounding operators are just meant to allow us to put reasoning about the existence of such derivations in the material mode, then the only constraint on the choice of introduction rules for the grounding operators is that they represent the logic of metaphysical explanations correctly. The logic of metaphysical explanation is represented by the logic of simple grounding-claims. In particular, the constraint is that as long as the introduction rule for

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[53]For more on this see (deRosset, 2011; and Dasgupta, 2011).

the $\Rightarrow_s$ operator gives rise to the right logic for *simple* grounding-claims, how we end up treating iterated grounding-claims is comparatively unimportant. So if we *can* treat claims of the form $\Delta \Rightarrow_s \phi$ as being zero-grounded we *can*; and given that treating them as being zero-grounded gets us out of the problem with the antinomy of grounding, we should.

This idea can be made precise as follows. Suppose that we have a logic **L** of simple grounding-claims which encodes the logic of metaphysical explanation. A logic of iterated ground is *acceptable* if it is a conservative extension of **L**. One may say that one adopts a "formalist" attitude towards the grounding operators. So is the logic we obtain by treating the $\Rightarrow_s$-Introduction rule as strictly explanatory acceptable in this sense? Yes, it is.[54,55]

Objection

If you take a "formalist" attitude to the grounding-operators, can you really object to a view like deRosset's and Bennett's? Suppose I accept everything

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[54]I prove this in (Litland, 2011). In particular, I argue that there are two ways of thinking about strict metaphysical explanation. One way of thinking about it gives us Fine's Pure Logic of Ground PLG. The other way of thinking about it gives us a certain subsystem PLG(-) of PLG. I define the logics PPLG and PNLG of iterated grounding and show that they are conservative extensions of PLG(-) and PLG respectively.

[55]There is a technical complication which I would be remiss in not mentioning. What I have claimed so far is only that claims of the form $\Delta \Rightarrow_s \phi$ are zero-grounded if true; in order to show that they are factual, I have to do more. I have to show that they are *only* zero-grounded. That is, I have to show that any ground of $\Delta \Rightarrow_s \phi$ is itself zero-grounded. It is not in general the case that if something is zero-grounded that all its grounds are in turn zero-grounded. Suppose, e.g., that the fact that Socrates is identical to Socrates is zero-grounded. Then the fact that (Socrates is zero-grounded or it's raining) is zero-grounded; but, supposing that it's raining, it's also grounded in the fact that it's raining. The fact that it's raining is, we can suppose, not zero-grounded. The technical situation is as follows. In the logics PPLG and PNLG we cannot derive that Γ is zero-grounded from the supposition that $\Gamma \Rightarrow_s (\Delta \Rightarrow_s \phi)$. We do however have the following. If $\Gamma \Rightarrow_s (\Delta \Rightarrow_s \phi)$ is derivable from a collection of simple grounding-claims Θ then Γ is in fact $\emptyset$.

you say about strictly explanatory arguments and that I, like you, want to introduce grounding-operators. What's to stop me from introducing a factive grounding-operator by means of the following rule:

$$\frac{\begin{array}{c} \Delta \\ \Pi \\ \phi \end{array}}{\Delta <^* \phi}$$

Here Π is a strictly explanatory argument and Δ are all and only the premisses on which ϕ depends. The difference between this rule and the $\Rightarrow_s$ rule is that Δ is *not* discharged in passing to the conclusion. Your technique for finding elimination rules can be applied here as well.

Reply

One cannot be stopped from proceeding in this way, and if one takes a formalist attitude it's not as if one is doing anything *wrong*. However, if one constructs one's grounding-operators in this way one squashes together what is more illuminatingly kept separate. That $\Delta < \phi$ is true turns not just on whether Δ is true, it also turns on the existence of strictly explanatory argument from Δ to ϕ. It is useful to be able to *state* this. Doing it my favored way allows us to do this; doing it your way doesn't.

Objection

If the grounding operators are object-language reflections of metaphysical explanations, why think that it's an objective matter whether Δ grounds ϕ? Why think that it's an objective matter whether Π is a metaphysical explanation of ϕ from premises Δ? Why couldn't this just be projection of our attitudes?

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Nothing I've said in this paper rules this out. Two points about this. First, if I'm right about how grounding-claims are to be expressed the question isn't the status of grounding claims, but rather the status of metaphysical explanations; this shifts the debate. Second, the sense in which it's objective or not that something is a metaphysically explanatory argument cannot be captured by means of the sentential operator "it's factual that", for if Δ metaphysically explains ϕ then the fact *that* Δ metaphysically explains ϕ is itself partly zero-grounded and so is factual by definition.

4.14   Conclusions

I'll end with an open question and a metaphysical conclusion.

The open question is more technical in nature. Starting from the suggestions in this paper one can create a satisfactory logic of iterated pure ground; but one would like to get logics which can deal with various logical connectives. It appears relatively straightforward to construct a logic which can deal with conjunctions and disjunctions. Negation, on the other hand, is a great problem. The problem comes down to saying what grounds a negated grounding claim. Plausibly $\neg(\Delta < \phi)$ is grounded either in what grounds $\neg\Delta$ or what grounds $\neg(\Delta \Rightarrow_s \phi)$ so the problem reduces to what grounds $\Delta \Rightarrow_s \phi$. I think the right answer is the following: if it's the case that $\neg(\Delta \Rightarrow_s \phi)$, then this is zero-grounded. The technical problem is to construct a nice logic which gives that result.

Finally, the metaphysical conclusion. To begin with, if grounding-claims are merely factual, 'grounding' is not real ideology. Grounding then plays

a curious rôle: the notion of grounding is essential for doing metaphysics but it is not one of the notions which have to be used in giving a description of Reality as it intrinsically is. Now, this much we would already get if grounding-claims $\Delta < \phi$ were just grounded in Δ (as suggested by deRosset and Bennett). What's distinctive about holding that grounding-claims are partly zero-grounded is what this tells us about the source of the structure of the world. The explanatory connection between the grounds and the grounded does not depend in any way on *how* Reality intrinsically is. The structure *of* the world is not *in* the world.

A

Appendix

A.1   (P1b), (P2b), (P3c) are jointly consistent

Definition A.1.1. A $\langle D, \square \rangle$-*model* is a tuple $\mathcal{M} = \langle K, R_0, R_1, D, \mathcal{I} \rangle$ where R_0 is an equivalence relation and R_1 is a reflexive relation. D is the (constant) domain of objects. And $\mathcal{I} \colon K \times \mathrm{Pred} \times D^{<\omega} \mapsto \{0, 1\}$ is an interpretation of the atomic predicates of $\mathcal{L}$.

We define $\mathcal{M}, w \models \phi$ in the obvious way. To get a more realistic model we should allow variable domains along the relation R_0.
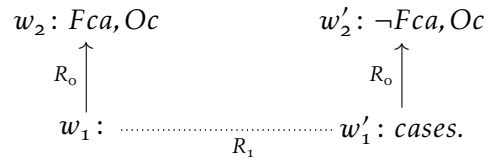
$$w_2: Fca, Oc \qquad\qquad w_2': \neg Fca, Oc$$

$$R_0 \uparrow \qquad\qquad\qquad R_0 \uparrow$$

$$w_1: \text{.................................} w_1': cases.$$
$$R_1$$

Figure A.1.1: Model of (P1B)

We have $K = \{w_1, w_2, w_1', w_2'\}$ We put $w_1 R_0 w_2$ and $w_2 R_0 w_1$ and similarly for w_1' and w_2'. We put $w_1 R_1 w_1'$. There are three states c_0, c_1, c_2 and three objects a, b, c. All the worlds model Oc_0, Oc_2 and $Fc_0 a$ and $\neg Fc_2 c$. Only the worlds w_2, w_2' model Oc_1. And $w_1 \models Fc_1 b$ whereas $w_2' \models \neg Fc_1 b$. (The essentials of the model is depicted in fig. A.1.1.)

Think of w_1 and w_1' as being two different precisifications of what is true at a basic world w. Now according to precisification w_1 at every possible world in which c_1 obtains there is a fusion of the class c_1. According to precisification w_1', at every possible world in which c_1 obtains there isn't a fusion of the class c_1.

It is easy to verify that w_1 and w_1' verify (P1B), (P3C) and (P2B).

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