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Incomplete Preference and Indeterminate Comparative Probabilities

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The notion of comparative probability defined in Bayesian subjectivist theory stems from an intuitive idea that, for a given pair of events, one event may be considered "more probable" than the other. Yet it is conceivable that there are cases where it is indeterminate as to which event is more probable, due to, e.g., lack of robust statistical information. We take that these cases involve indeterminate comparative probabilities. This paper provides a Savage-style decision-theoretic foundation for indeterminate comparative probabilities.

1. Introduction

Modern Bayesian decision theory seeks to ground statistical inference in a logical process of rational decision-making. Central to this goal is the task of specifying how rational agents organise, in a coherent manner, their probabilistic and evaluative judgments in face of uncertainties. As exemplified in classical works of Ramsey (1926), de Finetti (1937), and Savage (1954), the upshot of this approach is a representation theorem, where the decision maker's beliefs and values are characterised, respectively, by a single subjective probability measure and a personal utility function, provided that various postulates governing choice behaviour are granted.

In many classical Bayesian models of rational decision-making, the most demanding assumption is arguably the so-called "completeness axiom." This axiom mandates that an agent's preferences among possible courses of actions in any given decision situation be representable by a *complete* ordering. That is to say, in classical Bayesian decision theory it is assumed that the decision makers are 'maximally opinionated' in their choices of actions in that they are always prepared, as it is presumed, to compare and rank any two given options in any decision situations.

The completeness assumption is often questioned on the ground that the agent may, for various reasons, lack rational basis for choosing in a given pair of options a preferred one. For instance, the agent may lack robust statistical information in assessing the probabilistic nature of the events under which the acts are to be performed. In this case the decision makers face *probabilistic indeterminacy*. As a result, they are rationally justified in suspending judgments on the preferences of their pending actions: these acts are *incomparable*.

Of course, it is natural to consider incomparables in a model of decision-making. In fact, Savage himself was tempted by the idea of "analyzing preference among acts as a partial ordering, that is, ... admitting that some pairs of acts are incomparable" and this, he says, "would seem to give expression to introspective sensations of indecision or vacillation, which we may be reluctant to identify with indifference." However, the employment of incomparable acts (represented by an *incomplete* ordering) into a decision model will result in a different preferential structure from what was adopted in Savage's original framework. Savage didn't think that much can be advanced in pursuing this direction: "a blind alley" rather. He nonetheless added that "only an enthusiastic exploration could shed real light on the question." (Savage, 1972, p. 21)

In the past two decades or so we have seen a number of 'enthusiastic explorations' in this direction including the works by Dubra et al. (2004); Galaabaatar and Karni (2013); Ok et al. (2012); Seidenfeld et al. (1995), to name just a few. These efforts share a common goal of attempting to make classical decision theory more tractable and more realistic by relaxing the completeness axiom in their respective decision models. It is worth mentioning that this area of research in decision theory, i.e., modelling rational decision making without the completeness assumption, has far-reaching implications. For instance, Sen (2004, 2018) recently reiterated the importance of incompleteness in the contexts of social justice and global politics; in the field of artificial intelligence (AI), Zaffalon and Miranda (2017, 2019) also pointed out the crucial role incompleteness plays in building robust AI systems. It is thus the goal of this paper to make another 'enthusiastic exploration' in this direction, yet from a different angle.

The literature on decision-making with incomplete preferences classifies agent's inability to compare certain pair of options in decision situations as coming from two main sources: the uncertainties regarding the likelihood of the events under consideration (*probabilistic indeterminacy*) and the uncertainties about the values of the consequences of the acts available to decision makers (*evaluative indeterminacy*). The former is sometimes referred to in the economic literature as the decision maker's *indecisiveness in belief*, the latter *indecisiveness in tastes*.[1] We will follow this dichotomy in this paper. However, to focus our mind on indeterminate comparative probabilities, in what follows we consider probabilistic indeterminacy as the main source of incompleteness.

Most recent theoretic work on incompleteness (including the ones cited above) are based on the analytic framework of Anscombe and Aumann (1963). By contrast, the analysis made in the present work is set within the framework of Savage (1972), the latter is widely seen as *the* paradigmatic system of subjective decision making, on which a classical theory of personal probability – including a clearly defined notion of comparative probability – is based (cf. Remark 1.1 below). The aim is to generalise Savage's system to admit probabilistically incomparable events and to provide an axiomatisation of indeterminate comparative probability.

1. See Dubra et al. (2004) for a discussion. The notion of decisiveness in beliefs corresponds to the notion of probabilistic sophistication in Machina and Schmeidler (1992), by which the authors mean that the agent is capable of assigning precise subjective probabilities to events. As pointed out by Levi Levi (1986), the well-known paradoxes of Allais and Ellsberg are, respectively, examples of decision making with indeterminacy in values and indeterminacy in beliefs.

P1-P5		**+ P6**		**+ P7**
Qualitative probability	$\Rightarrow$	Quantitative probability & Utility for simple acts	$\Rightarrow$	Utility for all acts

Table 1: Inferential order in Savage's system.

To be more precise about the scope of this paper, recall that Savage's theory centres on a binary relation which models a decision maker's preferences over possible actions. A set of axioms is postulated on this preference relation. The culmination of the theory is a representation theorem with which an agent's preferences can be represented by expected utilities under proposed postulates. From the first five of Savage's seven postulates a *comparative* notion of subjective probability is derived which reflects the agent's qualitative probabilistic judgments over possible circumstances under which these actions are performed. With the sixth postulate, the derived qualitative probability (to be defined precisely below) is further precisified with a numerical probability measure and a personal utility function for simple acts (i.e., acts that lead to finitely many different consequences under different states). The last postulate plays the sole role of extending the utility function for simple acts to all acts (cf. Table 1).[2]

Remark 1.1. Savage's method differs from the approaches adopted by others such as Ramsey (1926) and Anscombe and Aumann (1963) in that in the latter cases the agents' subjective probabilities are derived from their personal utilities, which in turn are constructed based on some presupposed *chance mechanisms* (or, in the case of Ramsey, the notion of ethically neutral propositions, which can play the same role as an imagined unbiased coin with objective probability 1/2). This inferential order is reversed in Savage's theory of subjective utility where the decision makers' preferences over acts is taken as the only primitive notion, from which their personal probabilities and utilities are subsequently revealed. As a result of this methodological reversal, Savage's approach may appear to have some computational disadvantages in the sense that the mathematical representation theorem given in Savage's theory is considerably more involved than many of its alternatives (including Ramsey's and Anscombe and Aumann's systems), yet his theory is conceptually significant in that the system is seen as a purely *subjective* framework with no reference to *objective* probabilities.

In this paper, we generalise Savage's qualitative probability to a notion of *indeterminate* comparative probability. That is, we aim to arrive at a representation of the cases where, for two given events E and F, neither E is considered to be more probable than F nor that F is more probable than E. Our generalisation parallels the first part (P1-P5) of Savage's construction illustrated in Table 1, where we show that, under a revised set of axioms, an indeterminate comparative probability is a, what we call, *semi-qualitative probability.* This will be the main result of this paper, which can be seen as providing a decision-theoretic foundation for indeterminate comparative probability.

2. An outline of Savage's proofs can be found in Gaifman and Liu (2018). For a full exposition see Fishburn (1970).

The rest of the paper is arranged as follows. We start in the next section with some basics of Savage's system where we also provide a more precise formulation of the goal of this paper. Section 3 is devoted to a fine analysis of the sure-thing principle and its generalisation under the assumption of incompleteness, which is also a main focus of this paper. Our Savage-style axiomatisation of indeterminate comparative probability is given in Section 4. In the last section, we conclude and explore future work.

2. Savage's Theory of Qualitative Probability

2.1. Preliminaries

Recall that a *Savage decision model* is a structure of the form $(S, \mathcal{B}, X, \mathcal{A}, \succcurlyeq)$ where S is an (infinite) set of *states* of the world; $\mathcal{B}$ is a Boolean algebra on S, each element of which is referred to as an *event* in a given decision situation; X is a set of consequences; and a (Savage) *act* is a function f mapping from S to X, the intended interpretation is that $f(s)$ is the consequence of the agent's action f performed when the state of the world is in s. As a primitive notion of the model, $\succcurlyeq$ is a binary relation on the set of all acts, denoted by $\mathcal{A}$. For any $f, g \in \mathcal{A}$, $f \succcurlyeq g$ says that f is *weakly preferred to* g. Say that f is *strictly preferred to* g, written $f \succ g$, if f is weakly preferred to g but not vice versa, and that f is *indifferent to* g, denoted by $f \equiv g$, if f is weakly preferred to g and vice versa.

Definition 2.1 (fused acts)**.** For any $f, g \in \mathcal{A}$, define the *fusion* of f and g with respect to an event E (a set of states), written $f|E + g|\overline{E}$, to be such that:

$$(f|E + g|\overline{E})(s) =_{\text{Df}} \begin{cases} f(s) & \text{if } s \in E \\ g(s) & \text{if } s \in \overline{E}, \end{cases}$$

where $\overline{E} = S - E$ is the compliment of E.

In other words, $f|E + g|\overline{E}$ is the act which agrees with f on event E, with g on $\overline{E}$, and it is easily seen that $f|E + g|\overline{E} \in \mathcal{A}$. This notion of fused acts can easily be generalised for a series of acts $\{f_1, \ldots, f_n\}$ and a partition $\{E_1, \ldots, E_n\}$ of the state space such that the following is also a Savage act: $f_1|E_1 + f_2|E_2 + \cdots + f_n|E_n$. We list some straightforward properties of fused acts, which will become handy later.

Lemma 2.2. For any $E, F \in \mathcal{B}$, and for any acts $f, g \in \mathcal{A}$,

(1) $f|E + g|\overline{E} = g|\overline{E} + f|\overline{\overline{E}} = g|\overline{E} + f|E$.

(2) $\big(f|E + g|\overline{E}\big)|F + g|\overline{F} = f|E \cap F + g|\overline{E \cap F}$.

(3) $f|E + \big(f|F + g|\overline{F}\big)|\overline{E} = f|E \cup F + g|\overline{E \cup F}$.

(4) $\big(f|E + g|\overline{E}\big)|\overline{E} + g|\overline{\overline{E}} = g$.

Definition 2.3 (constant acts)**.** For any $a \in X$, an act is said to be a *constant act* with respect to consequence a, written $\mathfrak{c}_a$, if

$$\mathfrak{c}_a(s) = a \quad \text{for all } s \in S.$$

By definition, act $\mathfrak{c}_a$ 'constantly' outputs consequence a no matter which state $s \in S$ transpires. Note that constant acts play an important role in Savage's proofs. One motivation for having this type of acts is that it can be used to induce a preference ranking $\geq$ among consequences in terms of preferences $\succcurlyeq$ among acts, that is, for any $a, b \in X$, $a \geq b =_{\text{Df}} \mathfrak{c}_a \succcurlyeq \mathfrak{c}_b$. But in order to get such an induced ordering it is necessary to assume that for *any* consequence $a \in X$ there exists a constant act $\mathfrak{c}_a$. This, in fact, is an implicit assumption of Savage theory known as the "constant act assumption."

Remark 2.4. Savage's notion of constant acts is a technical construct inherited from von Neumann-Morgenstern's (vNM) notion of *degenerate lotteries* in their original utility theory: A lottery is said to be degenerate if it always yields the same (monetary) reward regardless which state obtains. However, unlike vNM's notion of degenerate lotteries, the constant acts assumption is highly problematic. Take, for instance, Savage's own famous omelette example, it is difficult to imagine what act can constantly result in a six-good-egg-omelette (a consequence) even when the sixth eggs is rotten (a state). Elsewhere (Gaifman and Liu, 2018), we addressed this issue, where we developed a new technique to show, among other things, that Savage's theory can be simplified with a weakened assumption of the existence of *two* distinct constant acts (without mandating that there is a constant act for each consequence). This paper presupposes the assumptions and proof techniques we developed there.[3]

With notions of fused acts and constant acts in hand, Savage moves to define comparative probabilities – what it means for an event to be said to be *weakly more probable* (or *no less probable*) than another in terms of preferences among acts.

Definition 2.5 (comparative probability)**.** For any events $E, F \in \mathcal{B}$, say that E is *weakly more probable* than F, written $E \succeq F$, if, for any constant acts $\mathfrak{c}_a, \mathfrak{c}_b$ with $\mathfrak{c}_a \succcurlyeq \mathfrak{c}_b$, we have

$$\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}. \tag{2.1}$$

E and F are said to be *equally probable*, written $E \equiv F$, if both $E \succeq F$ and $F \succeq E$ hold.

Intuitively, (2.1) says that the act $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E}$ is weakly preferred to $\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$ if it is weakly more probably for the former to obtain the more preferable consequence a than the latter. Savage assumes that any pairs of acts are comparable (i.e., the completeness assumption). As a consequence of this strong assumption, any two given events E, F are also taken to be probabilistically comparable. In other words, in Savage's system any events E and F must stand in one of these relations: $E \succeq F$, $E \equiv F$, or $F \succeq E$.

3. Note that although we presuppose the background assumptions and techniques adopted in Gaifman and Liu (2018), the proofs in this paper do not require them and can be read independently.

The first part of Savage's theory is to show that, under P1-P5, this comparative notion of 'weakly more probable' relation $\succeq$ defined above is a *qualitative probability*.[4]

In this paper, we will take the step of relaxing, among other assumptions in Savage's system, the completeness requirement, and consider the possibilities that two acts f, g are *incomparable* under the preference relation $\succcurlyeq$, in symbols, $f \bowtie g$. As mentioned above, we will attribute these incomparables to the probabilistic indeterminacy involved, where we take that, for a given pair of events E and F, it is possible that it is *indeterminate* that one event is more probable than the other – i.e., the case where both $E \not\succeq F$ and $F \not\succeq E$, written

$$E \bowtie F.$$

Notation. Here we adopted a systematic ambiguity by using the same symbol $\bowtie$ for both incomparable acts $f \bowtie g$ and probabilistically incomparable events $E \bowtie F$. There shall be no chance of confusion. The latter singles out the phenomenon of probabilistic indeterminacy, which is what we intend to model.

We aim to show that, under a set of revised Savage postulates (**Axioms 1-5** below), the binary relation $\succeq$ among events is a generalised notion of qualitative probability. Before proceeding, let us give some additional definitions, as well as a list of the first five of Savage's original postulates, which will be used in the rest of the paper.

2.2. Savage's postulates

One key concept in Savage's system is that of *conditional preferences* – i.e., the concept that one act is weakly preferred to another *given* the occurrence of certain event. Instead of introducing conditional preferences as a primitive notion in his system, Savage chose to define it in a roundabout way in terms of *unconditional* preferences.

Definition 2.6 (conditional preference)**.** Let E be some event, then, given acts $f, g \in \mathcal{A}$, f is said to be weakly preferred to g *given* E, written $f \succcurlyeq_E g$, if

$$\forall f', g' \in \mathcal{A}, \quad \left.\begin{array}{ll} f(s) = f'(s),\, g(s) = g'(s) & \text{if } s \in E \\ f'(s) = g'(s) & \text{if } s \in \overline{E} \end{array}\right\} \implies f' \succcurlyeq g'.$$

That is to say, f is weakly preferred to g given the occurrence of E, in symbols $f \succcurlyeq_E g$, if, for *all* pairs of acts $f', g' \in \mathcal{A}$, (i) f agrees with f' and g with g' on E, (ii) f' and g' agree with one another on $\overline{E}$, and (iii) $f' \succcurlyeq g'$. Note that this formal definition of conditional preferences is sometimes a point of confusion, especially in view of its relation to Savage's well-known sure-thing principle and its formal rendering in his system as the second postulate (P2). We shall return to this point in the next section. Meanwhile, let us provide a list of Savage's postulates for qualitative probability (as listed in the inner covers of his 1972 book).

4. A binary relation $\geq$ among events is said to be a *qualitative probability* if, for any events $E, F, G \in \mathcal{B}$, $\geq$ satisfies the following conditions: (1) $\geq$ is a weak order (a complete preorder); (2) $E \geq \varnothing$; (3) $S > \varnothing$; and (4) $E \geq F$ if and only if $E \cup G \geq F \cup G$, provided $E \cap G = F \cap G = \varnothing$.

P1 $\succcurlyeq$ is a weak order (i.e., a complete preorder).

P2 For any $f, g \in \mathcal{A}$ and for any $E \in \mathcal{B}$, $f \succcurlyeq_E g$ or $g \succcurlyeq_E f$.

P3 For any $a, b \in X$ and for any non-null event $E \in \mathcal{B}$, $\mathfrak{c}_a \succcurlyeq_E \mathfrak{c}_b$ if and only if $a \geq b$.[5]

P4 For any $a, b, c, d \in X$ satisfying $a \geq b$ and $c \geq d$ and for any events $E, F \in \mathcal{B}$, $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$ if and only if $\mathfrak{c}_c|E + \mathfrak{c}_d|\overline{E} \succcurlyeq \mathfrak{c}_c|F + \mathfrak{c}_d|\overline{F}$.

P5 For some constant acts $\mathfrak{c}_a, \mathfrak{c}_b \in \mathcal{A}$, $\mathfrak{c}_b \succ \mathfrak{c}_a$.

P1 is the "completeness axiom" which assumes that all acts are linearly ordered. Our aim is to replace this axiom with the assumption that acts are *partially ordered*. P3-P5 are usually known as 'structural axioms' which play the role of ensuring that the "weakly more probable" (Definition 2.5) is well defined and non-trivial. P2 is Savage's own formulation of his sure-thing principle using the notion of conditional preferences defined above, to which we now turn.

3. The Sure-thing Principle with Partial Ordering

Savage's sure-thing principle (STP) stems from an intuitive idea of *reasoning by cases*:

STP In deliberating the best course of actions in a given decision situation, if certain act is preferred in all possible scenarios under which the decision is to be made, then that act should be preferred *simpliciter*.

This consideration is sometimes referred to as the *dominance principle* in the literature. Savage takes this principle as fundamental to rational decision-making: "I," he says, "know of no other extra-logical principle governing decisions that finds such ready acceptance." (p. 21). However, when formulating the sure-thing principle in his model, Savage took a detour by means of his P2. As pointed out in Liu (2017), there is in fact subtle discrepancy between STP and P2. We shall briefly highlight this difference before moving on to generalise STP with partially ordered acts.

3.1. STP and P2

To simplify matters, assume that E and $\overline{E}$ are two exhaustive decision scenarios, if act f is weakly preferred to g given E or $\overline{E}$, then, by STP, f is preferred simpliciter. Given the notion of conditional preferences, the STP above can be directly translated into:

$$\left[f \succcurlyeq_E g, f \succcurlyeq_{\overline{E}} g\right] \implies f \succcurlyeq g. \tag{STP}$$

However, instead of using (STP) as the formulation of the STP, Savage adopted an alternative postulate (P2), which is stated by Savage as follows:

5. An event E is said to be a *null* if, for any acts $f, g \in \mathcal{A}$, $f \succcurlyeq_E g$.

P2 If two acts have the same consequences for some states, the preference between the two acts will not be changed if they are given new common consequences on those states where they are already in agreement and each is left unaltered elsewhere. (Savage, 1967, p. 306)

In symbols, we have that, for any acts $f, g, h, h' \in \mathcal{A}$ and for any event $E \in \mathcal{B}$,

$$f|E + h|\overline{E} \succcurlyeq g|E + h|\overline{E} \iff f|E + h'|\overline{E} \succcurlyeq g|E + h'|\overline{E}. \quad \text{(P2)}$$

Savage refers to (P2) as the formal version of the sure-thing principle in his decision model and (STP) the "loose" version. What connects the two versions of the sure-thing principle is the peculiar way how conditional preferences are defined in his system (Definition 2.6), which takes the (equivalent) form: for any $f, g \in \mathcal{A}$ and for any $E \in \mathcal{B}$,

$$f \succcurlyeq_E g =_{\text{Df}} f|E + h|\overline{E} \succcurlyeq g|E + h|\overline{E}, \text{ for all } h \in \mathcal{A}. \quad \text{(CP)}$$

As seen, there are differences between the statements of STP and P2 and that the formal definitions of (STP), (P2), and (CP) in Savage's system are very much intertwined – the construction/definition of one concept depends on that of the others. In fact, it can be shown that, solely in the presence of P1, (P2) is *not* provably equivalent to (STP) in Savage's system:

Proposition 3.1 (Liu, 2017)**.** Let $\succcurlyeq$ be a preorder on $\mathcal{A}$, then, under (CP),[6]

(1) P2 $\Longrightarrow$ STP,

(2) STP $\not\Longrightarrow$ P2.

This shows that, even though Savage's own formulation of the sure-thing principle as P2 is sufficient in bringing about STP, it is after all a strictly stronger principle. In the next section, we shall explore how the sure-thing principal can be generalised and formulated (either in the forms of STP or P2) in the presence of partially ordered acts.[7]

3.2. Incompleteness and P2

Let $\succcurlyeq$ be a binary relation on the set $\mathcal{A}$ of Savage acts. We now take the step of replacing the completeness assumption with the following weaker assumption:

Axiom 1. $\mathcal{A}$ is partially ordered by $\succcurlyeq$ (i.e., $\succcurlyeq$ is a preorder).

6. Proposition 3.1(1) also appeared as Theorem 2 in Savage (1972, p. 24). I thank an anonymous referee for highlighting this.

7. Note that Savage's P2 as listed in the inner covers of his 1972 book – i.e., for any $f, g \in \mathcal{A}$ and for any $E \in \mathcal{B}$, $f \succcurlyeq_E g$ or $g \succcurlyeq_E f$ – is different from the P2 stated in his book (and in this section) this is because the former is the consequence of the assumptions of P1, (P2), and (CP). Since in this paper we are concerned with relaxing P1, in what follows when we refer to P2 we mean the version stated in this section.

We refer to a (Savage-style) decision model $(S, \mathcal{B}, X, \mathcal{A}, \succcurlyeq)$ based on **Axiom 1** as a *partially ordered system* (or a **POS** for short). Given a partial ordering $\succcurlyeq$ on $\mathcal{A}$, the following notational conventions are observed:

$$\begin{aligned} f \equiv g &:= f \succcurlyeq g \text{ and } g \succcurlyeq f \\ f \succ g &:= f \succcurlyeq g \text{ and } g \not\succcurlyeq f \\ f \bowtie g &:= f \not\succcurlyeq g \text{ and } g \not\succcurlyeq f \end{aligned}$$

where $f \bowtie g$ says that the two acts f, g are *incomparable* with respect to $\succcurlyeq$. Then, given any two acts $f, g \in \mathcal{A}$, one of the following relations holds:

$$f \succcurlyeq g \quad \text{or} \quad f \prec g \quad \text{or} \quad f \bowtie g.$$

Now, let us consider the generalisations of (STP) for a POS. To this end, we first define the following two notions of conditional preferences:

$$f \succcurlyeq_E g \;=_{\text{Df}}\; f|E + h|\overline{E} \succcurlyeq g|E + h|\overline{E} \quad \text{for all } h \in \mathcal{A}, \tag{CP$_{\succcurlyeq}$}$$

$$f \succ_E g \;=_{\text{Df}}\; f|E + h|\overline{E} \succ g|E + h|\overline{E} \quad \text{for all } h \in \mathcal{A}. \tag{CP$_{\succ}$}$$

It is plain that these are generalisations of (CP) in a POS. In what follows let us adopt a notational shorthand of using $\text{CP}_\star$ to refer to different variants of CP above, where $\star = \succcurlyeq$ or $\succ$. Same for $\text{STP}_\star$ and $\text{P2}_\star$ below.

With different $\text{CP}_\star$ in hands, let us consider following variants of the STP, which are natural generalisations of STP in a POS:

$$\left[f \succcurlyeq_E g, f \succcurlyeq_{\overline{E}} g\right] \implies f \succcurlyeq g, \tag{STP$_{\succcurlyeq}$}$$

$$\left[f \succ_E g, f \succ_{\overline{E}} g\right] \implies f \succ g. \tag{STP$_{\succ}$}$$

Note. In presenting Savage's original system we didn't differentiate the weak ($\succcurlyeq$) and the strict ($\succ$) versions of the STP, this is because under the assumption of completeness the two versions are deductively equivalent. However, as shown below, in a POS, $\text{STP}_{\succcurlyeq}$ does not necessarily imply $\text{STP}_{\succ}$.

Lemma 3.2. Let $\succcurlyeq$ be a preorder on $\mathcal{A}$, then $\text{STP}_{\succcurlyeq} \not\Longrightarrow \text{STP}_{\succ}$

Proof. Let $S = E \cup \overline{E}$ and $X = \{a, b\}$. Consider the following four acts as illustrated in the table below.

	E	$\overline{E}$
f_1	a	a
f_2	b	a
f_3	a	b
f_4	b	b

Take a case in which the preference $\succcurlyeq$ is defined as

$$f_1 \succ f_2, f_1 \bowtie f_3, f_1 \equiv f_4, f_2 \bowtie f_3, f_2 \bowtie f_4, f_3 \succ f_4.$$

This model satisfies STP$_{\succcurlyeq}$. But, by (CP$_{\succ}$), we have $f_1 \succ_E f_4$ and $f_1 \succ_{\overline{E}} f_4$ (the latter holds by triviality), then, by STP$_{\succ}$, $f_1 \succ f_4$, which contradicts the assumption $f_1 \equiv f_4$. This shows STP$_{\succ}$ is violated in this example. □

Hence, under **Axiom 1**, it is necessary to introduce STP$_{\succ}$ and STP$_{\succcurlyeq}$ as separate principles. Now, parallel to Savage's original system, we seek to introduce variants of P2 in order to induce STP$_{\star}$ introduced above. The following are natural candidates.

$$f|E + h|\overline{E} \succcurlyeq g|E + h|\overline{E} \iff f|E + h'|\overline{E} \succcurlyeq g|E + h'|\overline{E}, \tag{P2$_{\succcurlyeq}$}$$
$$f|E + h|\overline{E} \succ g|E + h|\overline{E} \iff f|E + h'|\overline{E} \succ g|E + h'|\overline{E}. \tag{P2$_{\succ}$}$$

We now move to explore some relationships among STP$_{\star}$, CP$_{\star}$, and P2$_{\star}$ introduced above.[8] First, observe that Proposition 3.1(1) also holds for a POS. And it is easy to verify that the proof of the first claim of the proposition remains sound with all $\succcurlyeq$'s replaced by $\succ$'s. Hence we have the following.

Lemma 3.3. Let $\succcurlyeq$ be a preorder on $\mathcal{A}$, then

(1) P2$_{\succcurlyeq}$ $\implies$ STP$_{\succcurlyeq}$,

(2) P2$_{\succ}$ $\implies$ STP$_{\succ}$,

That is to say, P2$_{\star}$'s are sufficient in bringing about their respective versions of STP$_{\star}$'s in a POS. Next, we argue that, between the two versions of P2, P2$_{\succcurlyeq}$ alone is sufficient in inducing all of STP$_{\star}$.

Lemma 3.4. Let $\succcurlyeq$ be a preorder on $\mathcal{A}$, then

(1) P2$_{\succcurlyeq}$ $\implies$ P2$_{\succ}$,

(2) P2$_{\succ}$ $\not\Longrightarrow$ P2$_{\succcurlyeq}$,

Proof. In the following we provide a proof for an implication and a counter-example for a non-implication.

(1) Suppose, to the contrary, that there are f, g, h, h' such that

$$f|E + h|\overline{E} \succ g|E + h|\overline{E} \quad \text{but} \quad f|E + h'|\overline{E} \not\succ g|E + h'|\overline{E}. \tag{3.1}$$

The former implies that $f|E + h|\overline{E} \succcurlyeq g|E + h|\overline{E}$, hence, by P2$_{\succcurlyeq}$, $f|E + h'|\overline{E} \succcurlyeq g|E + h'|\overline{E}$. The latter implies that either $f|E + h'|\overline{E} \bowtie g|E + h'|\overline{E}$ or $f|E + h'|\overline{E} \preccurlyeq g|E + h'|\overline{E}$, but both are impossible given the first term in (3.1) and its consequences.

8. Here I am indebted to an anonymous referee in the formulations of STP$_{\star}$, CP$_{\star}$, and P2$_{\star}$.

(2) Again, let $S = E \cup \overline{E}$ and $X = \{a, b\}$. Consider the following four acts as illustrated in the table below.

	E	$\overline{E}$
f_1	a	a
f_2	b	a
f_3	a	b
f_4	b	b

Take a case in which the preference $\succcurlyeq$ is defined as

$$f_1 \equiv f_2, f_1 \bowtie \{f_3, f_4\}, f_2 \bowtie \{f_3, f_4\}, f_3 \bowtie f_4.$$

That is, the only comparable acts are f_1 and f_2, all other pairs of acts are incomparable. Then it is easy to see that $P2_{\succ}$ is trivially satisfied but $P2_{\succcurlyeq}$ is violated. □

Thus, as far as mandating the sure-thing principle in a POS is concerned, Lemmas 3.2, 3.3 and 3.4 establish that $P2_{\succcurlyeq}$ is all that is needed in order to bring about $STP_{\star}$:

Theorem 3.5. $P2_{\succcurlyeq} \implies [STP_{\succ}, STP_{\succcurlyeq}]$.

As seen, the relationship among $STP_{\star}$ and $P2_{\star}$ become more complicated in a POS – none of the strict versions of STP and P2 implies their respective weak version. However, as we have shown, $P2_{\succcurlyeq}$ is still sufficient in bringing about all of $P2_{\star}$ and $STP_{\star}$. Let us formally introduce $P2_{\succcurlyeq}$ as a postulate in our POS. The following axiom is stated in the fashion of Savage's P2, which is an easy consequence of $P2_{\succcurlyeq}$ and $CP_{\star}$.

Axiom 2. For any $f, g \in \mathcal{A}$ and for any event E, $f \bowtie g$ or $f \succcurlyeq_E g$ or $f \prec_E g$.

In what follows, we proceed to consider generalisations of other Savage postulates and show that a generalised notion of qualitative probabilities (without the assumption of completeness) can be defined under these revised axioms.

4. Indeterminate Comparative Probability

We now define a notion of comparative probability in a POS (cf. Footnote 4).

Definition 4.1 (semi-qualitative probability)**.** Let $\geq$ be a binary relation defined on an algebra $\mathcal{B}$ of events, $\geq$ is said to be a *semi-qualitative probability*, if, for any $E, F, G \in \mathcal{B}$, the following conditions are satisfied:

(1) $\geq$ is a preorder (i.e., $\geq$ is reflexive and transitive),

(2) $E \geq \varnothing$,

(3) $S > \varnothing$,

(4) $E \geq F$ if and only if $E \cup G \geq F \cup G$, provided $E \cap G = F \cap G = \varnothing$.

The aim is to show the comparative probability defined under our proposed axioms in a POS (Definition 4.4 below) is indeed a semi-qualitative probability.

4.1. Decisiveness in tastes on constant acts

As mentioned above, Savage adopted an implicit "constant act assumption" which says that for *every* consequence $a \in X$ there exists a constant act $\mathfrak{c}_a$ such that $\mathfrak{c}_a$ constantly outputs a regardless which state of the world transpires. That is, for any $a \in X$, there is a $\mathfrak{c}_a$ such that $\mathfrak{c}_a(s) = a$ for all $s \in S$. We found this assumption highly problematic. In fact, Savage's original theory continue to hold under the assumption of the existence of two constant acts (cf. Remark 2.4).

In this work, instead of assuming that there exists a constant act for each consequence, we assume that there exists a set of acts that are constant acts, written $\mathcal{A}_\mathfrak{c}$. Apparently, $\mathcal{A}_\mathfrak{c} \subseteq \mathcal{A}$. We take that $\mathcal{A}_\mathfrak{c}$ is non-empty (**Axiom 5** below) and that it might be a proper subset of $\mathcal{A}$ containing merely two elements.[9] Further, we identify a set $X_\mathfrak{c}$ of consequences that make up the constant acts in $\mathcal{A}_\mathfrak{c}$: $X_\mathfrak{c} =_{\text{Df}} \{a \in X \mid \mathfrak{c}_a \in \mathcal{A}_\mathfrak{c}\}$.

While, in a POS, it is possible that two acts are incomparable due to the probabilistic indeterminacy involved, we however assume that our agent is *decisive* on how constant acts in $\mathcal{A}_\mathfrak{c}$ are ranked. This is because the evaluations of constant acts requires no probabilistic considerations. In other words, while $\succcurlyeq$ only partially orders $\mathcal{A}$, it is assumed that it completely orders constant acts in $\mathcal{A}_\mathfrak{c}$. As a consequence, $\succcurlyeq$ induces a complete ordering on $X_\mathfrak{c}$ in the usual way: for any $a, b \in X_\mathfrak{c}$, $a \geq b =_{\text{Df}} \mathfrak{c}_a \succcurlyeq \mathfrak{c}_b$.

Axiom 3. For any $\mathfrak{c}_a, \mathfrak{c}_b \in \mathcal{A}_\mathfrak{c}$ and for any non-null event E,

(1) either $\mathfrak{c}_a \succcurlyeq \mathfrak{c}_b$ or $\mathfrak{c}_b \succcurlyeq \mathfrak{c}_a$,

(2) $\mathfrak{c}_a \succcurlyeq_E \mathfrak{c}_b$ if and only if $a \geq b$ where $a, b \in X_\mathfrak{c}$.

Remark 4.2. In the literature, by 'decisiveness in tastes' it usually means that the decision maker is assumed to be able to compare any options in X, that is, $\geq$ linearly orders X. In this work, we only assume $\geq$ to be a complete order on $X_\mathfrak{c}$ (or, for that matter, $\succcurlyeq$ only linearly orders $\mathcal{A}_\mathfrak{c}$ but not necessarily $\mathcal{A}$). This is a much weaker assumption than assuming that the agent is decisive in taste on all consequences in X, this is because our $X_\mathfrak{c}$ (and $\mathcal{A}_\mathfrak{c}$) may contain *as few as merely two elements*.

The following is a simple property of constant acts which holds for any POS that satisfies Axiom 1-3.

Lemma 4.3. For any $a, b \in X_\mathfrak{c}$ and for any $E \in \mathcal{B}$, if $a \geq b$ then

$$\mathfrak{c}_a \succcurlyeq \mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \mathfrak{c}_b.$$

Proof. By **Axiom 3**, $a \geq b$ iff $\mathfrak{c}_a \succcurlyeq_E \mathfrak{c}_b$. Then, by the definition of conditional preferences and $\text{P2}_\succcurlyeq$, we have that, for any $h \in \mathcal{A}$, $\mathfrak{c}_a|E + h|\overline{E} \succcurlyeq \mathfrak{c}_b|E + h|\overline{E}$. Let $h = \mathfrak{c}_b$, we get $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \mathfrak{c}_b|E + \mathfrak{c}_b|\overline{E} = \mathfrak{c}_b$. The other inequality can be similarly shown. □

9. In Gaifman and Liu (2018), we introduced a notion of *feasible* consequence. The idea is that certain consequence is incompatible with certain states under which they are considered. The six-good-egg-omelette mentioned in Remark 2.4 is an example of a consequence that is not feasible, from which no constant act should be constructed. Hence we take that $\mathcal{A}$ contains all functions from states to consequences except for those that might lead to infeasible consequences.

4.2. Indeterminacy and non-triviality

Parallel to the notion of comparative probability (Definition 2.5) in a system with completeness, we can similarly introduce a comparative notion of probability in a partially ordered system which covers probabilistically incomparable cases.

Definition 4.4 (indeterminate comparative probability)**.** Given any events $E, F \in \mathcal{B}$, say that E is *weakly more probable* than F in a partially ordered system, written $E \succeq F$, if, for any $a, b \in X_{\mathfrak{c}}$ such that $a \geq b$, we have $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$. Say that E and F are *probabilistically incomparable*, written,

$$E \bowtie F$$

if $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \bowtie \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$.

Next, we adopt the following independence postulate to guarantee that the notion of comparable probability is well defined, which says that the above definition does not depend on the choices of constant acts in use.

Axiom 4. For any $a, b, c, d \in X_{\mathfrak{c}}$ satisfying $a \geq b$ and $c \geq d$ and for any events $E, F \in \mathcal{B}$, $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$ if and only if $\mathfrak{c}_c|E + \mathfrak{c}_d|\overline{E} \succcurlyeq \mathfrak{c}_c|F + \mathfrak{c}_d|\overline{F}$.

The axiom also implies that probabilistic incomparability between two given events E and F is independent of evaluations of desirabilities of consequences, for which we also have $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \bowtie \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$ if and only if $\mathfrak{c}_c|E + \mathfrak{c}_d|\overline{E} \bowtie \mathfrak{c}_c|F + \mathfrak{c}_d|\overline{F}$. Finally, we exclude the trivial case where all consequences in $X_{\mathfrak{c}}$ are equally preferable.

Axiom 5. There exist $a, b \in X_{\mathfrak{c}}$ such that $a > b$.

We show that the comparative probability defined above is indeed a semi-qualitative probability (Definition 4.1). To this end, we first demonstrate that the following *monotonicity property* of comparative probability continues to hold in a POS.

Lemma 4.5 (monotonicity)**.** Under **Axiom 1-5**, for any $E, F \in \mathcal{B}$, if $F \subseteq E$ then $E \succeq F$.

Proof. For $a, b \in X_{\mathfrak{c}}$, assume that $a \geq b$ then by Lemma 4.3, $\mathfrak{c}_a \succcurlyeq \mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \mathfrak{c}_b$. Now, for any event E and any $h \in \mathcal{A}$, By $\mathrm{CP}_\star$ and **Axiom 2**, $\mathfrak{c}_a$ and $\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$ stands in one of the following three conditions,

(a) $\mathfrak{c}_a \bowtie \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$;

(b) $\mathfrak{c}_a|E + h|\overline{E} \prec \big(\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}\big)\big|E + h|\overline{E}$;

(c) $\mathfrak{c}_a|E + h|\overline{E} \succcurlyeq \big(\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}\big)\big|E + h|\overline{E}$.

Given Lemma 4.3 (a) is impossible. For (b), let $h = \mathfrak{c}_a$, we have, via Lemma 2.2,

$$\big(\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}\big)\big|E + \mathfrak{c}_a|\overline{E} = \mathfrak{c}_a|\overline{E} + \big(\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}\big)\big|E = \mathfrak{c}_a|\overline{E} \cup F + \mathfrak{c}_b|\overline{\overline{E} \cup F}.$$

Then again Lemma 4.3 implies that (b) cannot be the case. The remaining possibility is (c), in which case let $h = \mathfrak{c}_b$, we have, again via Lemma 2.2 and the assumption $F \subseteq E$,

$$\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \big(\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}\big)\big|E + \mathfrak{c}_b|\overline{E} = \mathfrak{c}_a|E \cap F + \mathfrak{c}_b|\overline{E \cap F} = \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}.$$

Hence, by Definition 4.4, $E \succeq F$. □

Note that, in a partially ordered system, it is possible that two events are probabilistically incomparable ($E \bowtie F$). Lemma 4.5, however, adds that probabilistic indeterminacy does not apply to the situation where one event is included in another. In this case, the former will always be considered no more probable than the latter, which accords well with our intuition.

Theorem 4.6. Let $(S, \mathcal{B}, X, \mathcal{A}, \succcurlyeq)$ be a partially ordered system. Assume that $\succcurlyeq$ satisfies Axiom 1–5, then the comparative probability relation $\succeq$ among events, defined in terms of $\succcurlyeq$, is a semi-qualitative probability.

Proof. We prove the theorem by direct verifications of the conditions in Definition 4.1.

(1) This immediately follows from Definition 4.4.

(2) In Lemma 4.5, let $F = \varnothing$.

(3) By **Axiom 5** there exist some $a, b \in X_{\mathfrak{c}}$ such that $a > b$ and $\mathfrak{c}_a \succ \mathfrak{c}_b$. Suppose, to the contrary, that $S \not\succeq \varnothing$, then, by condition (1), either $S \bowtie \varnothing$ or $\varnothing \succeq S$ holds. By condition (2), the only possibility is that $\varnothing \succeq S$. The latter implies, by definition, that $\mathfrak{c}_a|\varnothing + \mathfrak{c}_b|\overline{\varnothing} \succcurlyeq \mathfrak{c}_a|S + \mathfrak{c}_b|\overline{S}$, that is, $\mathfrak{c}_b \succcurlyeq \mathfrak{c}_a$, a contradiction.

(4) We only show here the non-trivial 'only if' direction. Let $a, b \in X_{\mathfrak{c}}$ be such that $a \geq b$, then $E \succeq F$ implies, by definition, $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \succcurlyeq \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$. Now consider $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E}$ and $\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$ and the event G and its compliment $\overline{G}$. By **Axiom 2** and $\mathrm{CP}_\star$, for any $h \in \mathcal{A}$, one of the following conditions holds,

(a) $\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E} \bowtie \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$;

(b) $\big(\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E}\big)|\overline{G} + h|\overline{\overline{G}} \prec \big(\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}\big)\big|\overline{G} + h|\overline{\overline{G}}$;

(c) $\big(\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E}\big)|\overline{G} + h|\overline{\overline{G}} \succcurlyeq \big(\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}\big)\big|\overline{G} + h|\overline{\overline{G}}$.

Let $h = \mathfrak{c}_b$, then, since $\overline{\overline{G}} = G$ and $E \cap G = F \cap G = \varnothing$, we have, via Lemma 2.2,

$$\big(\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E}\big)|\overline{G} + \mathfrak{c}_b|\overline{\overline{G}} = \mathfrak{c}_a|E \cap \overline{G} + \mathfrak{c}_b|\overline{E \cap \overline{G}} = \mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E}.$$

Similarly, $\big(\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}\big)\big|\overline{G} + \mathfrak{c}_b|\overline{\overline{G}} = \mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}$. Now, given $E \succeq F$ and $a \geq b$, by definition, the only possible case is (c). Next, in (c), let $h = \mathfrak{c}_a$, then, by Lemma 2.2,

$$\big(\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E}\big)|\overline{G} + \mathfrak{c}_a|G = \mathfrak{c}_a|G + \big(\mathfrak{c}_a|E + \mathfrak{c}_b|\overline{E}\big)|\overline{G} = \mathfrak{c}_a|E \cup G + \mathfrak{c}_b|\overline{E \cup G}.$$

Similarly, $\big(\mathfrak{c}_a|F + \mathfrak{c}_b|\overline{F}\big)\big|\overline{G} + h|\overline{\overline{G}} = \mathfrak{c}_a|E \cup G + \mathfrak{c}_b|\overline{E \cup G}$. Therefore, by definition, (c) yields $E \cup G \succeq F \cup G$.

This completes the proof of the theorem. □

5. Conclusion

This aim of this paper has been to explore a Savage-style axiomatic decision-theoretic framework for indeterminate comparative probabilities. The motivation for constructing such a model is threefold. First, as many have argued (Hájek and Smithson, 2012; Levi, 1974), probabilistic indeterminacy is more accurately reflective of the credal states of acting agents. Second, comparing to numerical probabilities, qualitative or comparative probabilities are arguably more fundamental in probabilistic reasoning (Icard, 2016) in decision situations, it is hence important to investigate how indeterminate comparative probabilities be modelled. And third, one crucial step we have taken is to adopt a partially ordered preference relation as a primitive notion in the underlying decision model. This is a meaningful generalisation toward a direction to make decision theory more realistic. For future work, it will be interesting to explore how the current system can be extended in order to arrive at a Savage-style decision-theoretic model for indeterminate quantitative probabilities.

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