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SYMMETRY AND ASYMMETRY IN THE CONSTRUCTION OF 'ELEMENTS' IN THE *TIMAEUS**

I. OUTLINE AND INTRODUCTION

In the last seventy years, aspects of symmetry have become very significant in the physical sciences. Many of the available introductions to this symmetry theory[1] refer, directly or indirectly, to the so-called 'Platonic Solids' of the *Timaeus* as prototypes. In the present article I intend to show that there is also a connection in the reverse direction, since similar symmetry arguments have relevance to a problem which has concerned commentators on the *Timaeus* for some time. At *Tim.* 54A–55C the bodies of the four 'elements' are constructed by the Demiurge from sub-units, equilateral triangles and squares, which are themselves made up of smaller triangles, Plato's *stoicheia*. The problem with this construction is that the numbers of *stoicheia* specified for the sub-units appear to be larger than necessary. Cornford[2] suggested that the *Timaeus* recipe is actually that for the second member in a series of equilateral triangles (squares) of increasing size, not for the first member, and proposed a rationalization for this. Although this rationalization has not been generally accepted,[3] and Taylor[4] had earlier given reasons for preferring the Platonic construction, Cornford's proposed structures have been used widely. The question as to why the Demiurge uses his particular constructions and not the simpler versions remains. According to Brisson,[5] 'Il est extrèmement difficile de répondre a cette question.' Zeyl[6] thinks that there is 'something of a mystery,' and the problem is considered to be 'wirklich erklärungsbedürftig' by Böhme.[7] The difficulty seems worth investigating.

In this paper I contend that the 'superfluity' of triangles is only apparent; all those specified are indeed required for the smallest sub-units, so long as the symmetry of the final body to be constructed is taken into account at earlier stages. This condition sets requirements on the symmetry of the two-dimensional sub-units, equilateral triangles and squares, and hence on the construction of larger versions. So long as the symmetry principles are followed, the construction of the larger sub-units can be shown

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[1] See the Appendix.

[2] F. M. Cornford, *Plato's Cosmology,* (London, 1937), especially 234–9.

[3] Although D. J. Zeyl, *Timaeus,* (Indianapolis, 2000), lxix, provides cautious support to Cornford at this point. I use Zeyl's translation, unless otherwise specified, and at most points I discuss arguments, at least initially, as presented in the accompanying commentary.

[4] A. E. Taylor, *A Commentary on Plato's Timaeus* (Oxford, 1928), 374.

[5] L. Brisson, *Même et l'autre dans la structure ontologique du 'Timée' de Platon: un commentaire systématique du 'Timée' de Platon* (Sankt Augustin, 1994²), 364. See also Brisson and Meyerstein, *Inventing the Universe* (Albany, 1995), 46: 'It has never been explained why Timaeus needed six right-angled triangles to make an equilateral triangle when two would have suffice(d).'

[6] Zeyl (n. 3) , lxviii, n. 141.

[7] G. Böhme, *Platons theoretische Philosophie* (Stuttgart, 2000), 304.

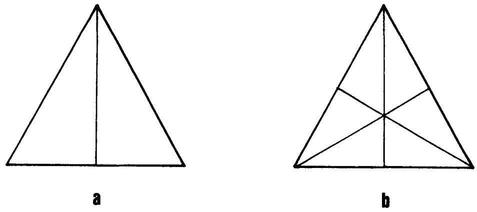


FIGURE 1. Constructions of the simplest equilateral triangle from the 'most excellent' (*Tim.* 54A) scalene right-angled triangle. (a) According to Cornford (234–8); (b) According to Timaeus (54D–E).

to be in accord with the *Timaeus* text. Fully symmetric three-dimensional bodies can be created from these sub-units and from no others; any choice of sub-unit other than those specified by Plato generates bodies with lower symmetries than the true ones.

Furthermore, if the specification is complied with, a unique construction for each of the prototype bodies follows. In contrast, there are *many* possible ways to assemble any particular body if other units are used. This leads to a built-in multiplicity for each size of 'element' , which seems inimical to Plato's thought. In the proposals of Cornford,[8] structures that are additional to those that can be assembled using the original Platonic description fail these tests of symmetry and of uniqueness.

II. THE 'MYSTERY'

The elements of Fire, Air, Water and Earth are associated in the *Timaeus* with four of the five three-dimensional regular polyhedra ('solids'). In a careful but very condensed description it is shown how these can all be constructed from elementary triangles, the scalene right-angled triangle, nowadays specified as having angles of 30° and 60°, for the tetrahedron, octahedron, and icosahedron, and the isosceles right-angled triangle (45°) for the cube.

I deal mainly with the three polyhedra with equilateral triangular faces. Figure 1b shows the assembly of the primitive equilateral triangle which Timaeus specifies at 54A–E. Figure 1a shows the apparently simpler alternative which is suggested by the name *hêmitrigônon*. The 'mystery' can be specified briefly as: why does the text specify 1b, and not 1a? The answer proposed by Cornford[9] is that 1b is intended as a generalized specification of an equilateral triangle which is intermediate in size between the small 1a and higher members of a set of increasingly larger ones, each constructed from the minimum possible number of half-equilateral triangles.

Although it is clear that Plato requires many sizes of each element, there is no evidence in the text for this particular proposal of Cornford, but his series has been widely adopted, *inter alia* by Friedländer,[10] Vlastos,[11] Gregory,[12] and Zeyl.[13] The first two of these authors show in some detail three-dimensional structures which can be constructed using the Cornford series of equilateral triangles, and these illustrations

[8] Cornford (n. 2), 238. [9] Cornford (n. 2), 234.
[10] P. Friedländer, *Plato*, translated from the German by Hans Meyerhoff (Princeton, 1958).
[11] G. Vlastos, *Plato's Universe* (Oxford, 1975), 69–79. The diagrams here are reproductions of those in n. 10; they are quoted in the text, rather than the originals, since they are probably more readily accessible.
[12] A. Gregory, *Plato's Philosophy of Science* (London, 2000), 196–200.
[13] Zeyl (n. 3), lxix.

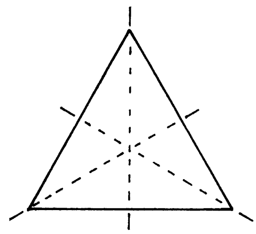


FIGURE 2. A simple equilateral triangle, showing the positions of the three reflection or mirror planes.

can be consulted, bearing in mind the *caveats* that will become apparent, to extend the range of examples which will be given here.

More than a century ago it was suggested by Wilson[14] that the advantage of 1b over 1a 'is that the former division is symmetrical and the latter is not'. The response of Cornford,[15] 'this is true, but why is this important?', seems to have buried arguments based on symmetry, at least for the English-speaking world.[16] I intend to show not only that this is important, but that it is crucial.

III. THE ARGUMENT IN TWO DIMENSIONS

The symmetry of an object is completely specified by the set of symmetry operations which convert the configuration of the object to an equivalent, indistinguishable configuration. This may appear trivial, but merely by specifying this symmetry fully, some surprising results appear; for example, despite their different shapes, it can readily be shown[17] that two of Plato's polyhedra, the octahedron and the cube, have identical symmetries, and the same is true for the dodecahedron and icosahedron pair.

For a simple equilateral triangle, one variety of such a symmetry operation is a rotation through 120°. (A more detailed presentation of this argument is given in the Appendix.) Three such operations bring us back to the starting configuration, so the rotation axis is referred to as threefold. Another operation, which relates to the obvious 'left–right' symmetry, is that of an imaginary mirror perpendicular to the plane of the triangle—this 'reflects' the left half into the right, and vice versa. There are three such 'mirror' or 'reflection' planes, whose positions are shown by dashed lines on Figure 2 ; the intersection of these gives the position of the rotation axis.

The triangle in Figure 2 is a complete, undivided triangle, but it is clear that the mirror plane positions match exactly the positions of the join lines in the composite triangle of Figure 1b. If we superimpose the two figures, there is no difference except for the join lines. It is shown in the Appendix that the symmetries of the simple undivided and composite triangles are therefore identical.

[14] J. C. Wilson, *On the Interpretation of Plato's Timaeus: Critical Studies* (London, 1889), 49, also cited by Taylor (n. 3), 374.
[15] Cornford (n. 2), 217, n. 2.
[16] A similar comment to Wilson's, but concerning the vertices of the polyhedra, appears as a footnote in E. M. Bruins, 'La chimie du Timée', *Revue de Métaphysique et de Morale* 56 (1951), 269–82, 279, n. 2. For a comment concerning axial symmetry, see n. 23 below.
[17] See references in the Appendix.

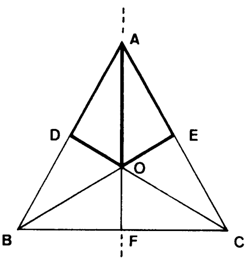


FIGURE 3. Construction of the triangle shown in Figure 1b. The first stage, the quadrilateral AEOD formed from two elementary triangles *kata diametron* is emphasized, and the position of the mirror plane which generates this from a single triangle is indicated by the dashed line. The second stage can be described as the operations of the threefold axis on AEOD. Taylor's unit (see n. 24) OADOCFO also has mirror symmetry and generates the complete triangle through the rotation operations. Lettering follows Zeyl (n. 3), 45.

The simple equilateral triangle is described by Speusippus,[18,19] writing only a little later than Plato, and probably influenced by him[20] at this point: *πρῶτον γάρ ἐστι τρίγωνον τὸ ἰσόπλευρον, ὃ ἔχει μίαν πως γραμμὴν καὶ γωνίαν· λέγω δὲ μίαν, διότι ἴσας ἔχει· ἄσχιστον γὰρ ἀεὶ καὶ ἑνοειδὲς τὸ ἴσον.* If it is possible to make any connection at all between the language of the Academy and that used nowadays, this seems to be close to a modern description of the symmetry of an equilateral triangle,[21] and of 1b, Plato's unit.

In terms of symmetry, any individual one of the elementary 'half-equilateral' triangles of Figure 1b can be used to generate the complete sixfold unit by operating on it with the reflection and rotation operations. This has similarities to the recipe of Timaeus (55D–E): first, two triangles are joined *kata diametron*. Most commentators take this to mean joining them with a common hypotenuse to create a quadrilateral. There are two ways to do this; one of these produces a rectangle,[22,23] but with no mirror symmetry. The alternative is to join them as the unit[24] AEOD in Figure 3 which does have mirror symmetry. In modern terms, operation by the mirror generates this odd-looking unit from a single triangle. Operation by the threefold axis on this unit now generates the complete equilateral triangle; in the language of Timaeus, 'this is done three times'. The first step of the assembly implied mirror

[18] Quotation in Iamblichus (attrib.), *Theologumena Arithmeticae*; text with English translations in Thomas, *Greek Mathematics* 1 (Boston, MA, 1939), 75, and L. Tarán, *Speusippus of Athens: A Critical Study with a Collection of the Related Texts and Commentary* (Leiden, 1981), 141. Taylor (n. 4), 370 has emphasized the significance of this quotation; his translation reads 'The first triangle is the equilateral which has in a sense only one side and one angle; I say one, because they are equal, for the equal is always undivided and unitary.'

[19] Speusippus can use 'one' in the modern sense of the first integer, unlike Plato and Aristotle. See the discussion in J. M. Dillon, *The Heirs of Plato: A Study of the Old Academy (347–274 B.C.)* (Oxford, 2003), 44–51, and Tarán (n. 18), 277.

[20] Tarán (n. 18), 286.

[21] Although Speusippus does call this triangle 'undivided' (*aschiston*, 'indivisible' in Thomas, n. 18), which suggests that he might have some difficulty with the argument for the equivalence of the undivided and (appropriately) divided triangles.

[22] For a fuller discussion, see Brisson (n. 5), 364.

[23] The phrase which follows, at *Tim.* 54E2, 'their short sides converge upon a single point as center', may be meant partly to avoid the rectangle confusion.

[24] See also Zeyl (n. 3), 45; Cornford (n. 2), 217, n. 1.

symmetry, and it seems that Plato may now be thinking in terms of the rotational symmetry in this second step, even though this language does not exist for him. A recent comment by Brisson[25] supports this idea: 'On peut cependant penser que, dans le cas du carré et dans celui du triangle équilatéral, Platon veut trouver un centre de symétrie axiale' (the square is discussed below).

There is an alternative interpretation of the first part of the recipe for the equilateral triangle,[26] but this does not change the symmetry argument. For either interpretation, the Timaeus recipe makes perfect sense when expressed in the modern language of symmetry.

Returning to Figure 1a, the triangle proposed by Cornford as the minimum unit, it is clear that this does *not* have the threefold rotational symmetry required to specify a truly symmetric equilateral triangle. So much symmetry has been lost that the only operation possible is that of the single mirror plane along the join line. We can now compare the effects of the two types of symmetries represented in 1a and 1b as we build up the polyhedra.

IV. THE ARGUMENT IN THREE DIMENSIONS

The primitive tetrahedron can be constructed from four equilateral triangle sub-units, following the recipe indicated at 54E. The commentators have very little to say about this final assembly process, but sometimes illustrate the end result with a diagram; I will return to some aspects of the presentation of such diagrams during the course of this paper. Popper has presented an analysis of some implications of the *Timaeus* text here, which he believes has been misunderstood.[27]

In the context of the symmetry argument, it is interesting that Popper makes no commitment about the construction of the equilateral triangles: by leaving them undivided, no symmetry problems arise. It is convenient to start by considering the assembly of four such 'blank' undivided triangles. Because the undivided units have the full triangular symmetry, all possible relative orientations of the separate units as they are brought together, including Popper's, give the same result: there is only one possibility for the final regular tetrahedron.

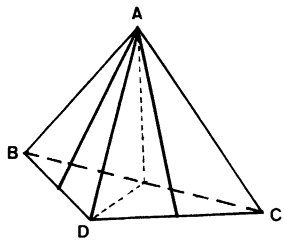
The same applies to the assembly of four divided equilateral triangles of the type shown in Figure 1b. Such a tetrahedron, despite the sectioning of the faces, has the full symmetry of the undivided regular one (see Appendix).

[25] L. Brisson, 'À quelles conditions peut-on parler de "matière" dans la Timeé de Platon?', *Revue de Métaphysique et de Morale* (2003), 14.

[26] Taylor (n. 4), 374–5, disagrees strongly with the interpretation described above. He thinks that *kata diametron* does not mean adjacent hypotenuses but the creation of diagonals across a quadrilateral ACFD (see Figure 3). This also generates a unit which has mirror symmetry and is converted into the full triangle by the operation of the threefold axis. As far as the present argument about symmetry goes, this works as well as the alternative; indeed, if this is correct it provides better support for the rotation axis concept, since it is difficult to see how the second stage can physically be 'done three times' if triangle pairs are joined to one another only at a mathematical point.

However his quadrilateral requires the construction of a line (DF) which is *not* a part of the triangle, and which will cut across other triangles in the final sub-unit. In contrast, the unit AEOD preferred by Zeyl and Cornford already has a quadrilateral formed once the triangles are brought together, and the diagonal of this is the common hypotenuse. Taylor seems to take up this position because for him it is 'unthinkable' that the quadrilateral ('trapezium') could be considered by Timaeus.

[27] K. R. Popper, 'Plato, *Timaeus* 54E–55A', *CR* 20 (1970), 4–5.



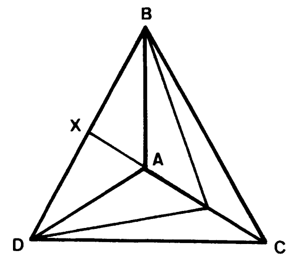


FIGURE 4. Two different tetrahedra made up by assembling type 1a triangles. (a) In perspective, following Vlastos (n. 11), but see text; (b) in projection, with join lines now set for maximum symmetry. Heavy lines outline the three triangles meeting at apex A, which points towards the viewer.

In describing the regular tetrahedron, Speusippus uses remarkably similar language to that above for the equilateral triangle: *γίνεται γάρ πως ἡ μὲν πρώτη πυραμὶς μίαν πως γραμμὴν τε καὶ ἐπιφάνειαν ἐν ἰσότητι ἔχουσα.*[28] Symmetry, in the present-day sense, is clearly of great importance to him, and presumably to others in the Academy. Even though our language is not available, the 'one-ness' appears to indicate something of our 'fully symmetric'.

The 'one-ness' needs to be borne in mind as we consider trying to assemble a set of four sub-units of the type in Figure 1a. The most obvious symmetry element of a regular tetrahedron, when it is drawn or imagined as sitting on a side, is the 'vertical' threefold rotation axis which passes through this triangular base. But a base like 1a has lost its threefold symmetry—here can be no 'vertical' axis for this tetrahedron. One of the important symmetry properties of the regular tetrahedron is that it has *four* threefold axes, but in going through the above argument successively it is clear that *all of these have been destroyed* by using 1a-type triangles. There is simply no way to construct a truly regular tetrahedron with these triangles.

This is illustrated in Figure 4a, which shows a perspective view of one of the possibilities. This is essentially the figure shown by Friedländer,[29] and reproduced by Vlastos[30] in his Figure 2. However, on the original figure, only the front join lines, which I have shown with heavy lines, are indicated. The present point about the low symmetry is therefore obscured. When the dashed join lines on the back faces are added, it can be seen that there is only a single reflection plane left, which includes these join lines. Any alternative orientation of one of the three sides which meet at the apex A would destroy even this low symmetry. Figure 4b shows a view of a different tetrahedron, this time in projection, so that the apex A is now at the centre of the figure. If the join line on the base is aligned with CX, the maximum possible symmetry with 1a triangles (two mirror planes and various axes) is reached.

In the terms used by Speusippus, it can be seen, using Figure 4, that a tetrahedron built from 1a triangles no longer has 'one edge'; some edges have two triangle joins at the mid-points, others one, and others have none. The tetrahedron no longer has 'one angle', since at the vertices some planes are bisected by joins, others are not. Neither he nor Plato are likely to have found such irregular tetrahedra acceptable.

[28] nn. 18–20. Taylor (n. 3) translates this as 'it has in a sense one edge and one angle in equality, like a number one'.

[29] n. 10. [30] n. 11.

There is yet another problem: not only is this tetrahedron irregular, it is very far from unique, as Figure 4 illustrates. The 1a triangle forming the base has three distinguishable orientations, and the same is true of each of the other three sides. Since each side can be oriented independently, there are $3 \times 3 \times 3 \times 3 = 81$ possible combinations. These come in sets of equivalent structures, but there are still many different sorts of this element. These structures are difficult to draw adequately in two dimensions, but three-dimensional models can readily be made, then as now,[31] and the points I have been trying to make with the symmetry arguments are immediately obvious on such models. Since Timaeus is setting up the generation of 'elements', though he disapproves of the term,[32] it is hardly likely that he would have allowed a scheme in which each of his 'elements' has a built-in multiplicity, when 'order is in every way better than disorder' (30A) and 'likeness is incalculably more excellent than unlikeness' (33B). These constraints apply to the *kosmos* at the cosmic scale; why should they be relaxed at the microscopic scale?

It would be convenient to have a name for these multiple 'sorts' of 'elements' which at first sight appear to be one structure, but actually have many different join line patterns. In modern chemistry, the two words 'allotrope' and 'isotope' distinguish particular aspects of elements as they are now understood (see Section VII), but neither has much connection with this point in the development by Timaeus. Although the polyhedra assembled from sub-units have been called atoms by Gregory,[33] they have more in common with the modern idea of molecules. Friedländer refers to 'the elementary molecules of Plato',[34] and Rex to 'Molekulartheorie'.[35] Chemists use the word 'isomers' for different forms of a molecule of the same formula; this description fits the various tetrahedra, such as 4a and 4b, very well, and will be used again below.

The constructions of the octahedron and the icosahedron follow the pattern established for the tetrahedron. The proof that assembly of type 1b triangles gives a perfect octahedron follows, as with the tetrahedron, from the fact that the symmetry of the set of join lines of the composite body is the same as that of the regular octahedron. For example, the mirror planes of the triangles are maintained in the symmetric octahedron, and they convert this set of joins into itself, but only so long as type 1b triangles are used.[36] In contrast, Figure 5a shows an octahedron built up from triangles of type 1a, with orientations chosen to maintain the maximum possible symmetry. This is similar to Figure 3 of Vlastos.[37] At first sight this may look like a symmetric octahedron—it certainly has a 'vertical' fourfold axis, and this is at the intersection of mirror planes as in the regular octahedron. However, drawings like this can be deceptive. The eye concentrates on the vertical axis only, but if this were a truly

[31] Although this is somewhat easier now. Anyone with paper, pencil, ruler and 60° set-square, or some Euclidean construction with a pair of compasses, can draw the nets of these figures; scissors and some adhesive can transform these into respectable models of the solids. It is even easier to use the child's toy 'Magnetfix', in which steel balls are clamped together by coloured bar magnets, and the polyhedra almost self-assemble. The 'isomers' referred to below can then be symbolized by using different-coloured magnets to represent edges with zero, one, or two joins.

[32] At 48B–C the word *stoicheion* has been rejected, and it has been used for the elemental half-equilateral triangle at 54D. I refer to 'elements' when discussing the various bodies, but elements when referring to modern usage.

[33] Gregory (n. 12), 198.

[34] (n. 10), 250.

[35] F. Rex, 'Die älteste Molekulartheorie', *Chemie in unserer Zeit* 23 (1989), 200–6.

[36] The effects of mirror planes are probably the easiest to visualize, but *all* the operations transform the set of joins into equivalent configurations. More detail is given in the Appendix.

[37] n. 11. The original figure does not show all the join lines.

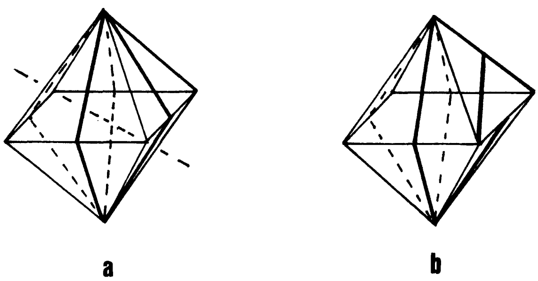


FIGURE 5. 'Isomers' (see text) of the primitive octahedron made up using six triangles of the form 1a. The body is drawn as a 'wire frame', but the joins on the back faces are shown with dashed lines. The front join lines have been emphasized. (a) This has fourfold symmetry, but *only* along the vertical axis, and along the other two axes, one of which is marked, there is only twofold symmetry. (b) One triangle has been rotated before joining on to the others, and the 'octahedron' now has no symmetry.

regular octahedron, it would have a set of *three* such fourfold axes through opposite vertices. Two of these have disappeared, and have been replaced by twofold rotations, one of which is indicated.

The orientation shown also concentrates attention away from the four threefold axes which should run through opposite faces. All of these 1a-type triangles have lost their threefold symmetry, so all four of these axes are missing, and the true symmetry of an octahedron has been lost. Nevertheless 5a looks, and is, quite symmetric, since the triangle orientations have been chosen to maximize symmetry. However, there is no requirement to choose this particular set of orientations, and only one of these orientations needs to be changed to destroy the symmetry completely, as shown in Figure 5b. As with the tetrahedron, there is a multiplicity of possible orientations for the assembled octahedron, this time $3^6 = 729$, and again what is supposed to be one element consists of very many different isomers, two of which are shown in Figure 5.

The symmetries of the tetrahedron and the octahedron are disrupted by incorrect symmetries on the faces, and this is equally true of the icosahedron, which can only have full symmetry if constructed from 1b triangles. There are even more possible isomers if 1a triangles are used here; a brief discussion is provided in the Appendix.

V. LARGER TRIANGLES AND POLYHEDRA

At 57C–D it is made clear that there are various sizes of triangle, and Cornford argued convincingly that this cannot mean that there are different sizes of elementary half-equilateral triangles.[38] Instead, he has proposed that the larger equilateral

[38] Cornford (n. 2), 230–3. He notes that the alternative, of a series of differently sized elementary triangles, *stoicheia*, for example A,B,C, . . . requires that 'A-fire' can only transform to 'A-water' and 'A-air', and so on. There is no evidence in the text for this very restrictive condition; Plato seems to assume that transformations are perfectly general. There is a further problem with differently sized *stoicheia* which Cornford does not raise. Unless the numbers of grades of fire, of water, and of air are exactly the same, then there are redundant possibilities. There might perhaps be 'X-air' and no 'X-fire'. It would then need to be explained why 'A-air' and 'A- fire' interconvert but 'X-air and 'X- fire' cannot. The alternative of exact equivalence of the numbers in the grades seems very unlikely when the numbers of actual examples given are compared; see Brisson and Meyerstein (n. 5), 53.

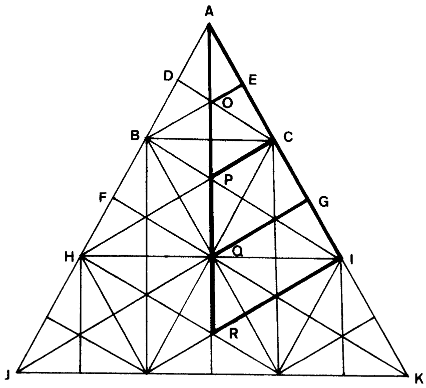


FIGURE 6. A superposition of the first three equilateral triangles of the series S (ABC, AHI, and AJK) showing how two parts of the *Timaeus* text both apply to them. Successive members may be considered to be built up by combining increasing numbers of equilateral triangles of the 1b type. Alternatively, increasing numbers of the smallest scalene triangle, of the type AOE, can be combined to generate larger scalenes APC, and so on, as indicated by the heavy lines. These larger *stoicheia* satisfy the conditions of 57C–D; they vary according to their *sustasin*. These scalenes can then be combined according to 54D–E, generating the larger equilaterals.

triangles are all built up using a single size of half-equilateral. He has gone further in taking the 1a triangle as the simplest equilateral unit; I have shown that this does not allow the construction of properly symmetric bodies, but that Plato's triangle, 1b, does. This latter is therefore the smallest unit, and it is necessary to examine the symmetries of the larger triangles and the bodies constructed from them.

Larger equilateral triangles can be built up by assembling several smaller ones in a symmetric array. There is a simple (**S**) series of these, in which the numbers of component 1b triangles are given by the squares of successive integers, 1, 4, 9, etc., and the numbers of elementary half-equilaterals are six times as great. However, an alternative and more complex (**C**) series has been proposed by Cornford, beginning with the 1a triangle. This **C** series generates equilateral triangles in which the numbers of half-equilaterals are 2, 6, 8, 18, 24. The arguments for the **C** series have been criticized by Pohle,[39] who demonstrates that the supposed advantages of **C** over **S** disappear as we move to the limit of large numbers.

Just as the first member of **S** (Figure 1b) has the full symmetry of an undivided triangle, so does the next, which is made up from four of 1b (that is, the fifth member of **C**, with 24 half-equilaterals); this symmetry is maintained through the **S** series, since in each case the join lines of the composite unit have the full equilateral triangular symmetry. Figure 6 illustrates this; the differences in weight of the lines should be ignored for the moment. ABC, with threefold symmetry at O, has been dealt with in Section III. The next member AHI, which can be considered as made up of four 1b-like equilaterals, has threefold symmetry at P, with the full complement of mirror planes. Similar comments apply to the symmetry at Q for triangle AJK (with nine of 1b). R is the centre for the fourth member of **S** (not shown), and a similar

[39] W. Pohle, 'The mathematical foundations of Plato's atomic physics', *Isis* 62 (1971), 36–47. The designations **C** and **S** are taken from this paper. His illustration for **S** shows empty (undivided) triangles, but it is equally relevant to 1b triangles; in his later analysis, this is his *Series III;* his *Series II* corresponds to my (**C–S**) series below.

pattern repeats throughout the series. Fully symmetric, unique polyhedra can therefore be built from these **S**-series triangles.

The **C**-series includes the **S**, but the additional equilateral triangles in **C**, which can be described as the series **(C–S)**, all have lower symmetry; they can be considered to be assemblies of 1a triangles. This **(C–S)** series *cannot* be used to build symmetric polyhedra.[40] Many different tetrahedra and so on can be built from 1a triangles (see Section IV), the first member of **(C–S)**. The next member of this **(C–S)** series, with eight half-equilaterals, has the same problem, but in addition there is a non-uniqueness in the triangle itself, since all four 1a triangular components can have their own orientation; there are isomers of the triangle as well as of the polyhedra. Indeed different authors draw this composite triangle in different ways.[41] It is surprising that this does not seem to have rung alarm bells earlier.

In addition to the advantage of symmetry, the **S** series has a very important property which relates to the text of Timaeus. The only specification given for the construction of the equilateral triangle is that at 54D5-E3, for the 1b triangle, but *no* specification is given for the building of larger equilaterals. However, at 57C8-D3 we are told that the *stoicheia* vary according to their *sustasin*, which generates triangles of different sizes. Cornford analyses this passage in detail, and remarks that the meaning of *sustasin* has to be active, a 'putting together', and translates as 'construction'. Zehl agrees with this, and Zekl translates this as 'Bildung'. So far I have considered members of the **S** series as assemblies of equilaterals, but they can equally well be considered to be assemblies of successively larger scalene triangles, all of which follow Plato's construction as used for Figure 1b, at 54D5-E3. The scalene triangles themselves are put together from increasing numbers of the first *stoicheion*. These numbers also follow the sequence of squares of integers: 1, 4, 9, . . .

This is illustrated in Figure 6, in which the heavy lines mark out successively larger composite scalene triangles. The first member, AOE, is as in Figure 3; six of these are assembled to form ABC according to the Timaeus recipe. Similarly six of APC (each with four components of the type AOE) form AHI, the second member. Six of AQG (nine components) form the third member AJK, and six of ARI, each with sixteeen components, will generate the fourth. Thus all of the **S** series equilateral triangles are consistent both with 57C7–D3 and with 54D5–E3, and also satisfy the symmetry condition.[42]

The **(C–S)** series, in contrast, can only be reconciled with 57C7-D3 if the pattern of assembly of the scalene triangles in Figure 1a is adopted, and this cannot be justified from the text. It is not simply that the 1a triangle is not in agreement with the specification of the equilateral at 54D–E; *all* the other members of this **(C–S)** series have the

[40] Bruins (n. 16) has discussed a possible mechanism for interchange between the different bodies. His ingenious scheme requires some very complex sectioning of large polyhedra in order to generate the next smaller ones. Much of this complication arises because he is carrying out an interconversion between a **(C–S)** series body and an **S**-series body. Since only **S**-series bodies are allowed by symmetry, his scheme can be greatly simplified. In particular, further division of the minimal equilateral triangles or squares defined in the Timaeus text becomes unnecessary—these units can remain *a-tomon* throughout the interconversions.

[41] Compare Cornford (n. 2), 238, Vlastos (n. 11), 75, and Gregory (n. 12), 200.

[42] Gregory (n. 12), 298, n. 44, has commented that there may be a problem with *sustasin*, since it is not clear whether this refers to the putting together of the stoicheic triangles in the sense of their internal composition or to their external arrangement with other stoicheics into what he calls complexes, the equilateral triangles. Within the **S** series, this problem disappears; the construction can be set out in either formulation, and both can be picked out by eye on Figure 6.

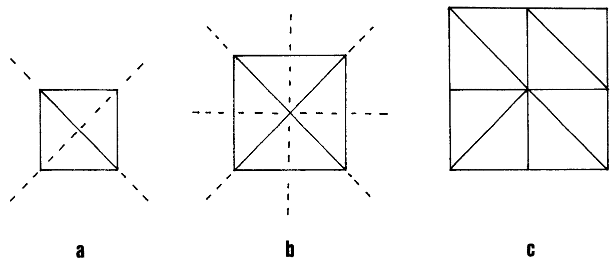


FIGURE 7. Construction of squares from isosceles right-angled triangles. (a) The simplest, according to Cornford ([n. 2], 238). The two symmetry planes are shown; the fourfold axis of the true square has become twofold. (b) The construction according to *Tim.* 55B–C, with a fourfold axis and four mirror planes. (c) the third member of Cornford's series, with a construction which is different from the usual one, but equally valid.

same problem; they can be rejected for this reason, as well as for the symmetry reasons given above.

Up to now the Cornford proposal for larger bodies, the **C** series, has had general approval among writers in English, even if with some qualification and muted criticism. Given this, the reaction from Brisson,[43] who referred to this construction system as 'la tentative louable, mais hypothétique et inutile, de F.M.Cornford', is at least surprising. I contend that, although there are indeed serious problems with the complete set of Cornford's suggested structures, which have been set out in detail above, the subset of these which is represented as the **S** series is by no means 'inutile'; it satisfies the conditions set out by Plato as well as those demanded by symmetry.

VI. THE CUBE

There are no new symmetry principles here. However, Plato has to use a different elementary triangle, the isosceles right-angled triangle, in order to create the two-dimensional sub-unit, the square. Again Cornford[44] takes the Timaeus construction at 55B–C, shown in Figure 7b, to be the second member of a series (**C**) which begins with a 'simpler' one (Figure 7a). It should be clear that in 6a most of the symmetry operations of a square are absent.[45] A 'cube' built from these cannot have the true symmetry of this body, and as with the polyhedra built from triangles of the (**C–S**) series, there will be many isomers, differing in the pattern of join orientations, for each of these 'cubes'. In contrast the joins in 7b have the true symmetry of the square, and the unique cube built from such squares has the true symmetry of the cube.

The argument is not exactly parallel to that of sections III–V for squares of increasing size. The above discussion of **C** and **S** series still applies, and the **S** series squares, of which the first is 7b, generate unique, symmetric cubes. The discussion of the construction of differently sized *stoicheia* in section V can be applied equally to the isosceles triangles forming these squares. However, the higher members of the (**C–S**) series can, if desired, be made more symmetric than those for the triangle series.

[43] Brisson (n. 5), 391, n. 3. [44] Cornford (n. 2), 233, 238.

[45] The comment of Brisson (n. 25) about rotational symmetry can be applied to 7b, but not to 7a, which has only a twofold axis; a true square has a fourfold axis.

Illustrations[46] show the third member as if built up from four of 7a, but with orientations adjusted appropriately so that the diagonals of the complete square are formed by the joins, giving a fully symmetric square, with a fourfold axis and four mirror planes. If this procedure is followed for the square, a symmetric cube, which for this particular square is unique, is generated.[47] However, there is no requirement that the alignments of the 7a component squares must be adjusted to maintain a high symmetry of the larger square. Figure 7c shows the third member, but now with an arbitrary adjustment of orientation—three are as 7a, and one has been rotated. We cannot build a symmetric cube from this, and there will be many isomers. Thus already at the two-dimensional stage, there can be different sorts of square as there can be different triangles, and the diversity is multiplied as we assemble cubes.

VII. EXCURSUS: *TOPOS* OR *TROPOS*?

At 58C–D Plato associates very different properties with different sizes of polyhedra. In modern understanding, different atoms of the same element are indicated by the name 'isotope'. The *topos* here is position in the modern periodic table of the elements. It is unfortunate that this name has been applied inappropriately, even though with some caution, by Friedländer[48] to Plato's different types of the same 'element' which have different numbers of triangles in their constituent polyhedra, and hence are of different sizes. This usage has been followed by Vlastos.[49]

The periodic table is a classification according to property, mostly but not exclusively chemical property, and isotopes of an element, with the same position in the table, have almost the same properties. In contrast, Plato's different versions of an 'element', such as fire at 45B–D, and water and 'liquifiable waters' at 58D–59C, have very different properties, so 'isotope' conveys the wrong meaning. Gregory[50] defines 'atom' to mean an individual polyhedron. He then uses 'isotope' in the captions to figures describing the Cornford triangular constructions, which are *components* of a polyhedron, and in modern terms, this would imply 'isotopes' of a nuclear constituent. If 'isotope' is to have any meaning at all here, it must be reserved for the polyhedra. It is probably inadvisable to associate new words with ancient concepts, but using one that carries a totally wrong connection to Plato's meaning is likely to lead to serious misunderstandings.

There is a well-established word, 'allotrope', for different forms of an element. These different forms can have notably different chemical properties, most spectacularly with life-giving oxygen and life-destroying (bactericidal) ozone. Where there are molecular forms, as in this example, allotropes differ in the numbers of constituent atoms, so this word is reasonably close in meaning. If a modern term is necessary, allotrope is preferable to isotope, and may even have the advantage in this context of conveying a reminder that the various forms can undergo *alloiôsis*.

[46] Cornford (n. 2), 238; Vlastos (n. 11), 79.

[47] The interconversion example given by Bruins (n. 16) involves a **(C–S)** series cube which can be drawn in this way, so that the point of n. 40 is not immediately obvious; he leaves the interconversions of the other three bodies, where it is inevitable that asymmetric bodies will be generated, as an 'exercise for the reader'.

[48] Friedländer (n. 10), 255. It is surprising that he does this, since at 250 he refers to 'molecules', and isotopes cannot have any meaning in describing molecules.

[49] Vlastos (n. 11), 72.

[50] Gregory (n. 12), 200.

VIII. FINAL REMARKS

Various components of my symmetry-based analysis have appeared previously, sometimes in rather indirect form. Wilson[51] and Taylor[52] point out the symmetry of the 1b construction, Cleary[53] suggests that 'considerations of geometrical symmetry may be responsible for the more complex construction', and Brisson's comment on the rotation centre[54] echoes Taylor's remarks[55] about the 'centre of gravity' and Böhme's similar comments[56] about 'Gleichgewichtsfiguren'. Bruins notes the asymmetry of the corners of a tetrahedron if the Timaeus specification is not used.[57] However, these various suggestions have not been followed through, and the importance of symmetry has been largely ignored, at least by writers in English. By a curious irony, Cornford's dismissal of a symmetry argument[58] came at a time when the scientific implications of symmetry, formalized as group theory, were being worked out and the first textbooks describing this work were beginning to appear.

It is still something of a mystery why it has taken so long for a revival of the idea, but part of the answer may lie in the inadequacy of two-dimensional diagrams to represent three-dimensional relationships. In Zeyl,[59] the polyhedra are introduced, before the discussion of the Platonic construction, by diagrams which are already reduced in symmetry by lines drawn on the surfaces which face towards the viewer. This helps to emphasize the three-dimensionality of the figures, but it has the unintended side-effect that when the polyhedra made up of the divided triangles are drawn, as in Vlastos,[60] these diagrams fail to emphasize the difference in symmetry from bodies made up of undivided triangles. This is particularly true of the simplest unit, the tetrahedron, which is surprisingly difficult to visualize without a model.[61] Because of these visual complications, it may have been easy to miss something which has long been available in published diagrams.

Plato's apparently mysterious construction of his elemental polyhedra makes sense in the light of the modern concept of symmetry, and proposed constructions of the equilateral triangles or squares which do not follow his instructions simply do not work; this applies for all sizes of polyhedra. These same principles of symmetry can be seen for any particular case by building the appropriate models, or, with sufficient imagination, by visualizing them in three dimensions. The individual points concerning symmetry that have been noted by the authors mentioned above, and the symmetry relationships that can be seen on models, are different aspects of the underlying principles which are treated rigorously in the theory of point groups. Bruins[62] comments: 'Si Timée divise le carré par la tracé des deux diagonales, et le triangle équilatéral en six triangles partiels, c'est qu'il y a une nécessité fondamentale pour ce faire.' In the words of the referee for this paper, the Demiurge uses precisely the procedures he needs to get the results aimed at.

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[51] Wilson (n. 14). [52] Taylor (n. 4), 374.
[53] J. J. Cleary, in Tomás Calvo and Luc Brisson (edd.), *Interpreting the Timaeus-Critias: Proceedings of the IV Symposium Platonicum: Selected Papers* (Sankt Augustin,1997), 245.
[54] Brisson (n. 25). [55] Taylor (n. 4), 374. [56] Bruins (n. 7).
[57] Bruins (n. 16). [58] Cornford (n. 15). [59] Zeyl (n. 3), lxvii.
[60] Vlastos (n. 11), 74–7.
[61] As an example, consider Figure 4b, which has higher symmetry than 4a.
[62] Bruins (n. 16), 277, n. 2.

APPENDIX: SYMMETRY OPERATIONS

Many accessible introductions to symmetry theory are now available. The following simplified treatment uses the approach of Shriver and Atkins,[63] with some modifications; references to more advanced treatments are given there. The essence of this approach is to examine the complete set of symmetry *operations* associated with some particular physical unit, often a molecule. This set forms a 'group', and knowledge of the mathematical properties of any group allows wide-ranging rigorous conclusions to be drawn about the physical properties of the unit concerned.

The operations are best considered as transformations of the unit into exactly equivalent configurations; they are associated with particular symmetry *elements*, such as an axis of rotation, but it is the set of operations which constitutes the group. For the present purpose the formal machinery of group theory is unnecessary, but it is useful to be able to appreciate the operations which form the group of the equilateral triangle, and how these extend into the three-dimensional solids.

(a) Operations in two dimensions

Figure 8a shows an equilateral triangle to which a 'marker' point has been added; at this stage it is merely a reminder of the results of the operations which we will consider. One possible operation which generates an equivalent configuration is a rotation about an axis, perpendicular to the paper, through 120°. The result is shown in 8b; the marker has moved, but we have an exactly equivalent configuration. This can be repeated (8c). A third repetition regenerates the original configuration; the axis is called 'threefold'. The same mark can also be used to indicate a particular point on the surface of the triangle, and then, in order to maintain the symmetry, we have to mark all the equivalent positions generated by the rotations (see Figure 8d).

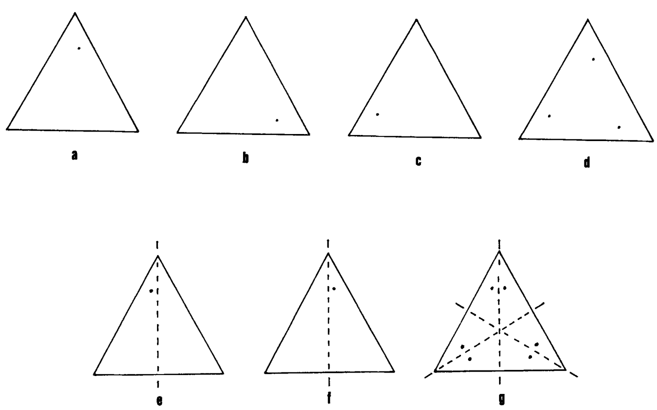

FIGURE 8. Illustration of symmetry operations upon configurations of a simple undivided equilateral triangle with a temporary marker. Rotation operations: (a) initial state, (b) after a (clockwise) rotation through 120°, (c) after a further rotation, (d) marker now a permanent part of the triangle, maintaining the rotational symmetry. Reflection operations: (e) initial state, (f) operated on by the mirror plane indicated by the dashed line, (g) as (d), but now symmetric with respect to all operations in two dimensions.

[63] D. F. Shriver and P. W. Atkins, *Inorganic Chemistry* (Oxford, 1999[3]), 117–18.

In addition to these rotation operations, it is clear that the triangle as drawn in (8a–c) also has left–right symmetries. The corresponding operation is that of an imaginary mirror plane, again perpendicular to the paper, indicated by the dashed line, and again we use a temporary marker (8e). Unlike the rotation, this is not a physically possible operation in space. The operation of the mirror transforms the marked point, and all others in the left-hand half, into corresponding points in the right-hand half, and vice versa (8f). There are two other such planes running through the other two vertices. All three of these intersect at the same point, which is also the rotation centre. The effect of this complete set of operations is indicated by the sixfold mark on Figure 8g.

The triangle used in Figure 8a–g is a complete, undivided triangle, but it is clear that the reflection planes in 8g match exactly the positions of the join lines in the composite triangle of Figure 1b. We can apply the operations described above to this composite triangle also. Except for the join lines, the composite triangle behaves exactly as the undivided triangle. To illustrate this, consider the general point marked in Figure 8g—it could be at any position within the small scalene triangle, so every point that is not along a join line is transformed by the operations into a set of six equivalent points. Now consider the join lines themselves as an 'object'. This 'object' has exactly the same symmetry elements, and operations, as the equilateral triangle. The reflection plane of Figure 8e, for example, transforms its coincident join line (AOF on Figure 3) into itself, and interchanges the other two (DOC becomes EOB). Thus the complete composite triangle behaves in exactly the same way as an undivided equilateral triangle. When the composite triangle was assembled, it acquired the full symmetry of the undivided triangle; Plato's construction, Figure 1b, is equivalent to an undivided equilateral triangle.

(b) The tetrahedron

The symmetry elements of the original equilateral triangles are now present in, and operate on, the entire tetrahedron which has been assembled from them. However, since Plato is only concerned with faces, and not with the interior, we can restrict ourselves to transformations on faces. So long as 1b triangles are used, any one of the threefold rotation operations carries out the same transformations as before on the triangle to which it is perpendicular, and therefore transforms this set of joins into an equivalent configuration. For the other faces it rotates one face into the next, and generates a new set of joins in exactly the same position as those which were in this position before the operation. Overall, the set of operations associated with any one of the triangular faces transforms the complete set of joins on all faces into itself. The tetrahedron has more operations than those of the separate triangles, but these additional operations also transform the set of joins into itself.

However, any attempt to generate tetrahedra using 1a triangles fails to generate the full symmetry of the regular tetrahedron; examples have been shown in Figure 4.

(c) The octahedron and icosahedron

The octahedron constructed with 1b triangles has the full octahedral symmetry. For example, the threefold axes of the triangles become the four threefold axes of the octahedron, running through opposite pairs of faces. These rotate the individual join lines in these faces into one another, and rotate the other four faces, with their join lines, into each other. The mirror planes of the triangles remain, and new ones are

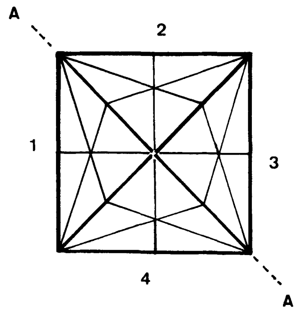

FIGURE 9. A regular octahedron made up from 1b triangles, seen from above (in projection). The heavy lines are the outlines of the equilateral triangles 1–4. Line A–A shows the position of a second fourfold axis and of a mirror plane.

created by the assembly of the octahedron. Figure 9 shows an octahedron of 1b triangles in projection on to a plane perpendicular to a fourfold axis. Faces 1–4 are facing upwards, and corresponding faces 1′–4′ face downwards, away from the viewer. A single operation of the fourfold axis rotates 1, with its join lines, into 2, and so on. Line A–A represents one of the new mirror operations, which reflects 1 into 2, but with a different orientation from that produced by the fourfold axis. AA also represents a fourfold axis; a single operation of this converts 1 into 2, again with a different orientation, and 2 into 2′, and so on. The plane of the paper is a mirror plane, whose operation converts 1 into 1′, and so on. With 1a triangles, the highest possible symmetry is that shown in Figure 5a.

The full symmetry of the icosahedron is quite complex; again it is maintained by 1b triangles but destroyed by 1a triangles. There are even more isomers than for the simpler bodies; it should be relatively easy to see a few of these by inspecting, for example, the first item in Figure 4 of Vlastos,[64] and imagining triangles to be rotated as is illustrated here at 5b for the octahedron. The icosahedron shown by Vlastos, like the octahedron built from 1a triangles in his Figure 3, is drawn with a choice of triangle orientations such that symmetry is preserved along the most obvious dimension, the vertical,[65] but most of the operations of the true regular icosahedron have been lost, such as those of the other five fivefold axes. However , if 1b triangles are used, as in the second component of Figure 4 in Vlastos,[66] the operations of the threefold and fivefold axes and the mirror planes now carry out transformations on the joins exactly as they do on the rest of the triangular surfaces; only with 1b triangles can we generate a fully symmetric icosahedron having only one isomer.

Note added in proof

Further work confirms the analysis of the **S** series of triangles, but shows that some higher members of the **(C–S)** series also have the correct symmetry. This will be described elsewhere.

[64] Vlastos (n. 11), 76.
[65] So long as it is assumed that the missing joins on the back faces maintain this symmetry.
[66] Vlastos (n. 11), 76.