Category Theory is a Contentful Theory[†]

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ABSTRACT

Linnebo and Pettigrew present some objections to category theory as an autonomous foundation. They do a commendable job making clear several distinct senses of 'autonomous' as it occurs in the phrase 'autonomous foundation'. Unfortunately, their paper seems to treat the 'categorist' perspective rather unfairly. Several infelicities of this sort were addressed by McLarty. In this note I address yet another apparent infelicity.

The subject of this paper is the comments in [Linnebo and Pettigrew, 2011] concerning the *contentfulness* of William Lawvere's axiomatic system CCAF.[1] For details of this system itself the reader is encouraged to consult [Lawvere, 1966] or [Lawvere, 1963]. No technical details from these expositions will be needed, however.

Linnebo and Pettigrew are willing to admit 'CCAF asserts the existence of certain categories and describes some of the functors between them.' However, as they correctly point out, this by itself is insufficient for CCAF to serve as a foundation of mathematics.

Specifically, Linnebo and Petigrew highlight that a necessary condition for CCAF to serve as a foundation of mathematics is that it be contentful, and 'if it is to make a contentful assertion, we need to be able to identify its subject matter — namely, categories — independently of the theory'. The authors provide us an illustrative example of a non-contentful theory to help make this objection clearer.

> [A lack of content] would be the problem, for instance, with the theory that consists of the following sentence: 'the mome raths outgrabe'. The reason that this theory lacks content is that there is no way of identifying, independently of the

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[1]CCAF is often assumed to abbreviate either 'category of categories and functors' or 'category of categories as foundation'. In Lawvere's thesis, however, the system normally called CCAF appears under the heading 'category of categories and adjoint functors'.

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theory, what mome raths are, or what it is to outgrabe. Without identifying the subject matter of the theory in such a way, the statements of the theory do not make contentful assertions. [2011, p. 231]

Presumably (and the authors are certainly correct here) a theory that runs afoul of this objection (which I will call the Contentful Theory Objection — CTO) cannot serve as an autonomous foundation for mathematics.

Of course, a proponent of CCAF can respond to CTO by simply providing a CCAF-free identification of the subject matter of category theory. The problem, the authors hold, is that this cannot be accomplished without running afoul of what they call the Logical Dependence Objection (LDO): if any theory T is to provide an alternative foundation for mathematics to the foundation provided by set theory, it must not be the case that T 'depend[s] logically on a prior theory of classes and functions in order to ground [its] existential assertions'. That is, if one cannot identify what the content of T is without appealing to classes and functions, then T does not qualify as an *autonomous* foundation for mathematics. In the case of CCAF, Linnebo and Pettigrew seem to suggest that any attempt to avoid CTO by providing a CCAF-free identification of the subject matter of category theory will, in the process of describing this subject matter, make appeal to a prior theory of classes and functions. That is, they claim any attempt to avoid CTO is bound to run afoul of LDO. In the remainder of this note I show this is simply false.

First let us be clear on the content of LDO. As it applies to CCAF it is fairly clear: if we claim the objects studied by CCAF are, *e.g.*, 'things like the category with groups for objects and group homomorphisms as arrows', then the identification we have made is logically dependent on a prior set theory since groups and group homomorphisms are defined set-theoretically.

It is less clear what LDO has to say about set theory itself. Presumably set theory is contentful; so there is some way of identifying its subject matter independent of set theory itself. Also, since if anything is an independent foundation for mathematics, set theory is, one would hope this identification would not also run afoul of LDO.

So what, then, *is* the content of set theory? Presumably it is the obvious:

> Set theory is about collections of things. $\hspace{4em}$ (1)

Thus if set theory manages to avoid CTO without running afoul of LDO, it must be the case that I have a set-theory-free way W of understanding what 'a collection of things' is, and W itself must not depend on prior understanding of some mathematical theory. Let us examine how the set-theory supporter might go about supplying such a W.

To begin, she could claim 'collections of things' simply *is* sufficiently definite to serve as the content of set theory. This amounts to claiming that

(a) The notion of 'collection' needs no further explanation to be understood,[2] and

(b) Using this notion we can identify, independently of set theory, the content of set theory.

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[2]Note this claim seems extremely dubious in light of the antinomies of naïve set theory — we will not address this point in this note.

If we allow set theory this recourse, surely CCAF can help itself to a similar one; the CCAF theorist can then easily avoid CTO without running afoul of LDO by claiming

> Category theory is about ways of combining two things to make a third.   (2)

Actually, the phrase 'ways of combining two things to make a third' only suffices as a *partial* specification of the subject matter of CCAF. More specifically, CCAF is a first-order theory about those ways of combining two things to make a third that satisfy the following two conditions:

(a) Whenever the combination of a with b (which we will write $a \circ b$) and the combination of b with c (which we will write $b \circ c$) are defined, then both $a \circ (b \circ c)$ (that is, the combination of a with $b \circ c$) and $(a \circ b) \circ c$ (that is, the combination of $a \circ b$ with c) are defined and further these two combinations of things are actually the same thing; and

(b) It must admit, for each 'combinee', both a left-combination identity and a right-combination identity.

Of course CCAF is supposed to be a first-order theory of a category of categories; so to those unfamiliar with category theory it may not be obvious at first that 'ways of combining two things to make a third' could be its subject matter. After all, 'ways of combining two things to make a third' makes mention of neither objects nor arrows. One might thus wonder where its categorial content is supposed to come from.

These worries are easily laid to rest — it has long been known (actually since the beginning of category theory in [Eilenberg and Mac Lane, 1945]) that objects are superfluous to the definition of a category; their role can be played by arrows. When characterizing categories in an object-free way, the only axioms that matter are those specifying that the composition of arrows be associative and that it admit, for each arrow, a left and a right identity element. Categories viewed from this perspective are nothing more than a type of algebra of arrows — that is, a way of combining two things (arrows) to make a third (their composite). Since the object-free characterization of categories is in fact precisely the characterization that underlies CCAF, it is especially appropriate to use here. I turn now to examining a few objections to the idea that (2) is sufficiently definite to serve as the subject matter of category theory.

Objection 1: One might see the word 'things' in (2) and object that 'ways of combining two things to make a third' is not sufficiently definite to characterize anything until we have been told what the membership structure of the 'thing-space' is. This seems to be what Linnebo and Pettigrew address in one of their footnotes:

> the proponent of CCAF as a foundation may complain that a category need not involve a *set* of objects and a *set* of arrows but rather a *collection* or *aggregate* of objects and a *collection* or *aggregate* of arrows. But this will not buy him much time, since our best theory of collections or aggregates or pluralities of any sort is set theory. [2011, p. 232, note 5]

But why must the CCAF suporter specify at all what structure is formed by the relation '*x* is an arrow of category *y*' just because the set-theory supporter is in the habit of doing so? It seems perfectly coherent to study 'ways of combining things' without needing information about the structure formed by the membership pattern that holds between the things being combined and their totality.

For example: one can easily imagine building physical instantiations of associative ways of combining two things to make a third that admit left and right identities. These would be machines that take in two things and output a third, and which behave as they ought to in order to model this description. A trained mechanic could probably build many such machines, repair such machines when they broke, identify when two such machines were essentially the same, *etc*. All of this could be done without a moment's thought being given to what structure was being instantiated by relation '*x* is a possible input to machine *y*', or whether there is such a structure at all — the mechanic could work perfectly well with the relation '*x* can combine with something to produce an output' instead, for example.

Perhaps an alternative argument will make this clearer: if we take as primitive the membership relation, then it can appear the CCAF supporter has failed to specify the membership structure of her space of arrows.[3] But by the same token, if we take as primitive 'ways of combining two things to make a third', then it can appear as if the set theorist has similarly failed to specify the combination structure of her spaces of elements. So when one begins either from the *assumption* that '*x* is a member of *y*' is primitive or that 'combining *x* with *y* gives *z*' is primitive, those structures specified only in terms of the other relation seem insufficiently specified.

Objection 2: One could perhaps object that, even if 'ways of combining two things to make a third' is sufficiently definite to define the content of *some* theory, nonetheless in order to single out those ways that are associative and admit left and right identities (which we must do to specify the content of CCAF in particular) one must have a prior understanding of sets. Thus, even if we can understand 'ways of combining two things to make a third' without relying on a prior theory of collection and membership, we nonetheless need such a theory to be able to state which particular ways we are interested in as CCAF theorists.

But we can state versions of the usual category axioms specifying the associativity of composition and the existence of identity morphisms without quantifying over collections at all as follows:

Let W be a way of combining two things to make a third. Let $a \circ_W b$ stand for the result of combining a and b (in this order) in way W. Then we say W is an

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[3]This may *appear* to be the case, but it is at least debatable whether it *is* the case. One can make perfectly coherent the notion of a discrete category within CCAF; and can use this notion to define the words 'set', 'membership', and the like (again, the interested reader is urged to consult Lawvere's work for the details). Nonetheless, arguing for the claim that CCAF admits a membership relation that is as robust as the membership relation present in, say, ZFC is not relevant to the details of this particular paper.

associative way of combining two things to make a third that admits left and right identities if

- Whenever $a \circ_W b$ and $b \circ_W c$ are defined, so are $a \circ_W (b \circ_W c)$ and $(a \circ_W b) \circ_W c$ and these two are equal.
- For every a there are things 1_{sa} and 1_{ta} so that

  $a \circ_W 1_{sa} = a$ and for any b if either of $b \circ_W 1_{sa}$ or $1_{sa} \circ_W b$ is defined then it equals b; and

  $1_{ta} \circ_W a = a$ and for any b if either of $1_{ta} \circ_W b$ or $b \circ_W 1_{ta}$ is defined then it equals b.[4]

Since we can thus explain which ways of combining two things to make a third are the associative ones that admit both a left and a right identity for each element without even *quantifying over* collections, surely we can understand this without *relying on a theory of* collections.

Altogether, then, it seems that if we justify set theory's status as an autonomous foundation by claiming (1) is sufficiently definite to serve as the subject matter of set theory, then we can also justify CCAF as an autonomous foundation by claiming (2) is sufficiently definite to serve as the subject matter of that theory.

The proponent of set theory may at this point wish to go down another road altogether and say that the reason (1) is an acceptable identification of the content of set theory is because it relies only on purely logical notions. That is, 'a collection of things' can be understood by simply grasping what it means for an object to hold a particular property — the collection of the blahs is just all those xs for which 'x is a blah' is a truth.

If this is the road the set theorist goes down, it is extremely difficult to see how she will block the CCAF theorist from taking the same path. A proponent of CCAF can do this most easily by pointing out that 'ways of combining two things to make a third' can be perfectly well explained in terms ternary relations. Thus, if the set theorist is allowed to claim set theory as an autonomous foundation because the content of set theory can be grasped using pure logic, then it seems the CCAF theorist should be allowed to make the same claim — unless the set theorist has some reason to claim that properties are logical while ternary relations are not.

Of course, Linnebo's and Pettigrew's arguments are closely related to arguments in [Feferman, 1977] that have been revisited repeatedly. A summary of 'the ways in which Feferman's [1977] arguments have been used (and misused) in the philosophical literature' can be found in [Landry, 2013]. While the argument above may appear merely to contribute to this collection of uses (but hopefully not to the misuses), it should be pointed out that one can read in the argument I have offered an *agreement* with Feferman's basic point that 'the general concepts of operation and collection have logical priority with respect to structural notions'. Admittedly, the agreement is rather tenuous, as the theory proposed above seems to do Feferman one better by relying

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[4]It may appear that these two by themselves leave open the possibility that the identities are not unique. However, if 1_{sa} and $1'_{sa}$ were two right identities for a, then we would have that $1_{sa} = 1_{sa} \circ_W 1'_{sa} = 1'_{sa}$. Thus, right identities are unique. A similar argument shows left identities also to be unique.

only on a previous understanding of the general concept of an operation. Nonetheless, I can agree to 'the general concepts of operation and collection' having logical priority while still maintaining that this leaves set theory and category theory on equal footings; the difference between the two cases amounts only to the following: where the operation assumed in set-theoretic foundations is a binary membership relation, the operation assumed in category-theoretic foundations is a ternary composition relation. Thus, if set-theoretic foundations are to be labeled as autonomous, category theoretic ones should be as well.[5]

The lesson to be learned from all this is that remaining silent sometimes is a perfectly fine answer. The mere fact that the things the set theorist likes to talk about are things the category theorist does not find interesting does not make the category theorist's contributions dependent on the set theorist's — not everyone has to have something to say about sets.

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[5]Feferman, of course, famously rejects both set-theoretic and category-theoretic foundations. So he might be willing to make this concession without seeing it as giving up very much.