DEEP FRIED LOGIC

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Abstract. There is a natural story about what logic is that sees it as tied up with two operations: a 'throw things into a bag' operation and a 'closure' operation. In a pair of recent papers, Jc Beall has fleshed out the account of logic this leaves us with in more detail. Using Beall's exposition as a guide, this paper points out some problems with taking the second operation to be closure in the usual sense. After pointing out these problems, I then turn to fixing them in a restricted case and modulo a few simplifying assumptions. In a followup paper, the simplifications and restrictions will be removed.

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1. Introduction

There is a tight connection between logic and theory building. Theory building, in turn, is tightly connected to our search for as-complete-as-possible accounts of various phenomena. In a pair of recent papers, Jc Beall has fleshed out these connections as follows:

> When we form a theory of some phenomenon we throw a bunch of sentences into the theory, namely, all of those sentences that we think are true about the phenomenon. In turn, we require a theory that reflects not just our thrown-in truths; we require a theory that reflects all of the true consequences of the theory's claims. And this is the job for a closure relation: a relation that 'completes' the set of truths by adding all sentences that are consequences of the theory according to the relation. [6, p. 5]

As this passage makes clear, on the understanding Beall is advancing theory building is bundled up with two operations: the *throw in some stuff* operation and the *closure* operation. The closure operation takes a theory (intuitively the background theory) and another theory (intuitively the foreground theory) and somehow mashes them together to give us a third theory (the closure of the given foreground theory under the given background theory). Fixing a particular background theory, closure (the operation) gives rise to a closure *relation*. Explicitly, if b is a background theory, b-closure is the relation that holds between the theory t and the sentence ϕ just when ϕ is in the closure of t under b. We connect all of this to our search for as-complete-as-possible accounts by saying that ϕ is in the closure of t under b just when there are no b-counterexamples to the inference from t to ϕ. Thus the b-closure of t contains everything we can, modulo b, safely infer from t.

The connection to logic is now straightforward: logic is the minimal closure relation. Beall puts it in this way:

> Logical consequence… plays the role of universal closure relation – or universal basement-level closure relation – involved in all of our true theories[.] [7, p.3]

So (unpacking a bit) Beall's answer to the question 'what is logic?' is that logic is the relation that holds between the theory t and the sentence ϕ just when for any background theory b, ϕ is in the closure of t under b. Stated more colloquially, logic is about what follows from what (that is, what is contained in the closure of what) no matter what (that is, no matter what we close with respect to). Beall calls this answer 'very traditional and very familiar'. This strikes me as correct, but I still think the answer is wrong.

Here's what I'm willing to grant: logic, in at least one of its more philosophically central and important senses, is about what follows from what no matter what. I also think that the right way to flesh this out involves two operations, one of which is the 'throw some stuff together' operation and the other of which is the operation at play when we mash together background and foreground theories. But I take issue with the use of *closure* for the second operation.

The basic thought is this: however it is that theories behave under closure, this behavior should be the result of features of the theories themselves. But the only features theories have is the sentences they contain. So how theories behave under closure is determined by (certain of their) sentences. To capture this, we stipulate the existence of a particular connective (which we will write '$\rightarrow$' and read 'entails') that internalizes the instructions a theory contains about how it should behave under closure. We will refer to sentences that have '$\rightarrow$' as their main logical operator as *entailments*.

More concretely, write '$cl(t_1, t_2)$' for 'the closure of t_2 under t_1'. We can then characterize $\rightarrow$ semantically by the following clause:

- $\phi \rightarrow \psi$ is in t_1 iff ψ is in $cl(t_1, t_2)$ whenever ϕ is in t_2.

Now recall that in Tarski's terminology (see e.g. [42]), a *closure operator* is a function F from sets of sentences to sets of sentences that satisfies the following three conditions:

- $X \subseteq F(X)$
- $F(X) = F(F(X))$
- If $X \subseteq Y$, then $F(X) \subseteq F(Y)$.

The two-place function cl naturally gives rise to the various one-place functions $cl(t_1, -) : t_2 \mapsto cl(t_1, t_2)$. These functions can easily be seen as functions from sets of sentences to sets of sentences. Since we call $cl(t_1, t_2)$ 'the closure of t_2 under t_1', it's natural to expect $cl(t_1, -)$ to be a genuine closure operator. But absent some strong assumptions, it isn't!

Definition. (I) is the set of sentences in the language of the theory t that have the form $\phi \rightarrow \phi$. (W) is the set of sentences in the language of the theory t that have the form $(\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi)$.

Theorem 1. *$X \subseteq cl(t, X)$ for all sets of sentences X in the language of the theory t iff* (I) $\subseteq t$.

Theorem 2. *$cl(t, X) \supseteq cl(t, cl(t, X))$ for all sets of sentences X in the language of the theory t iff* (W) $\subseteq t$.

We leave the proofs of Theorems 1 and 2 to the reader. What they show is that assuming that the operators $cl(t, -)$ are genuine closure operators is (somewhat sneakily) a substantive assumption about the nature of theories – it rules out, by fiat, theories that don't contain every instance of $\phi \rightarrow \phi$, or which don't contain every instance of $(\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi)$.

But surely such theories are possible.[1] Such theories may even be the true and complete-as-possible account of some phenomena. For example, both [43] and [22] give theories lacking members of (W) or members of (I) as a theory of truth in a self-referential language. And given that there *are* such theories, if we restrict our attention to theories that *do* contain every instance of (I) and every instance of (W), then we aren't doing logic. After all,

[1] This isn't a novel observation. See, for example, [35], for further discussion of the connection between contraction (that is to say, (W)) and closure.

logic should be about what follows from what *no matter what* – that is, no matter the background theory we use. Thus, any artificial restriction to which theories we consider – e.g. a restriction to only considering those theories that contain (I) or contain (W) – keeps us from doing genuine logic.

Once we allow theories that don't contain (I) or (W), though, what we've been calling 'closure' isn't closure in Tarski's sense. So much the worse for closure – it turns out we can live without it just fine. Seeing where doing so takes us is the aim of this paper.

In light of the fact that we're living without closure, we'll write $t_1 \circ t_2$ for what we had been writing $cl(t_1, t_2)$. Inspired by [41] we will read '$t_1 \circ t_2$' as 'the application of t_1 to t_2'. With this change, we have a new proposal for what logic is, which we will here contrast with Beall's 'very traditional and very familiar' (VTVF) proposal: rather than paying attention only to the *closure* of one theory under another, we will instead (and more generally) pay attention to the *application* of one theory to another. Mechanically, we will accomplish our goal by defining a semantics where the basic objects are theories. We will recognize two operations on theories: theories can be *extended* and one theory can be *applied* to another. To keep the technical details to a minimum and the paper short(ish), we will make two simplifying assumptions:

- We take application to be commutative and associative;
- We will deal with negation via the what's known as *the American Plan*.[2] Thus, sentences are allowed take any of the following semantic values: True ($\{1\}$), False ($\{0\}$), Both True and False ($\{0, 1\}$), or Neither True nor False ($\emptyset$). In addition, we will treat negation in an essentially classical way: $\neg\phi$ is true iff ϕ is false and is false iff ϕ is true.

It's important to acknowledge that adopting these assumptions leaves us short of fully correcting the issues we've identified so far. For example, the assumption that application is commutative and associative is tantamount to requiring that every theory contain every instance of the following two formulas:[3]

$$\text{(Asn)} \quad \phi \to ((\phi \to \psi) \to \psi)$$
$$\text{(B)} \quad (\phi \to \psi) \to ((\psi \to \rho) \to (\phi \to \rho))$$

So making the simplifying assumptions I've identified above is in fact making a version of the very error I'm claiming to correct.[4]

I think the account I'm providing is, in spite of this, worthwhile. I think this for several reasons. First, while I didn't make it explicit above, Beall is also committed to accepting every instance of (Asn) and of (B). So, the account I'm presenting reduces the errors in Beall's account without introducing new errors. Second, the broad outlines of an account that makes *none* of these errors are visible in the account I give. So, both as a first approximation and as a proving ground, it's useful. Nonetheless, in a followup paper to this one, I will explore what happens if we drop these assumptions. The followup will also extend the

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[2] The terminology here comes from [28]. For recent discussions of the merits of Australian versus American-flavored theories, see [12].

[3] 'Asn' is short for 'assertion', which is the name given to this particular family of sentences in, e.g. [2]. The name 'B' comes from the combinatory logic tradition; see e.g. [13].

[4] There's a subtlety here that I'm glossing over. I say that the assumptions I make are *tantamount to* requiring these sentences in every theory because an examination of the completeness proof I give will show that in fact we do allow theories that lack these sentences. What's *not* allowed are theories that contain, e.g. ϕ, but which do not also contain $(\phi \to \psi) \to \psi$. But, again, such theories seem clearly possible. This extra detail obfuscates the underlying message that the assumptions I'm making do in fact commit me to an error of the exact same sort as I'm accusing Beall of making, so I'm relegating it to a footnote

account given here to the more robust setting of first-order logic. For now, however, the assumptions will be useful, and the restriction to the propositional level will be enough work to keep us busy.

So that's what we'll be doing in this paper: spelling out (modulo a few restrictions and simplifying assumptions) what logic qua universal theory-building toolbox *is* once we ditch assumptions about theories containing things like (I) and (W). But there are other reasons for being interested in this paper as well. The first of these is that it provides a relevance-free motivation for a fairly strong relevant logic.

It turns out that the logic this account leads us to is (in a sense to be spelled out in §6) equivalent to the well-known relevant logic RW. But, as you'll notice, there's no mention made of relevance in our motivation. This should be encouraging news for relevant logicians, especially those of a philosophical bent. The traditional motivations for studying these logics, after all, have taken something of a beating over the years.[5] A recent example of this can be found in the following passage:

> The problem for relevant logics is that there are far too many of them and, as such, there is a lack of definition in the concept of relevance. If we take relevance as meaning relatedness, which is its immediately intuitive concept, this is, by itself, not a suitable concept upon which to base a logic as it is too vague. Relevance, as determined in its sharper form by the variable-sharing property . . . has been taken as a necessary condition for a good logic, but not a sufficient one, leaving a plethora of systems to consider. The strong relevant logics such as R, satisfying this property, are based on technical criteria such as the neatness in the presentation of their natural deduction systems rather than on a specific logical concept. [16]

Complaints like this are ubiquitous – if not in print, then at least in conversation – wherever relevance is brought up.[6] What's more is that I think the complaint is correct – relevance, in any of its traditional forms, is just not enough for us to build a logic around.

But relevance logicians needn't worry. One of the things this paper does is show that taking *relevance* seriously isn't the only way to end up taking *relevant logics* seriously. Indeed, relevant logics show up quite naturally (once we give up on question-begging refusals to admit theories that don't contain (I) or (W)) from considering logic as involved with theory building.

And, if (a) understanding logic qua universal theory-building toolbox and (b) having a look at a relevance-free motivation for a strong relevant logic *still* aren't enough to capture your attention, here's one final reason to care about the paper: it fills in a notable lacuna in the literature.

To say more, we need a bit of background. As I just mentioned, the logic we end up with in the end is a relevant logic. Semantic theories for relevant logics come in two flavors: Australian and American. Australian-flavored theories use a two-valued semantics and interpret negation using the Routley star. American-flavored theories use a four-valued semantics and interpret negation in a more classical way.

Semantic theories for relevant logics also come in two broad mathematical frameworks: operational and relational. Operational semantic theories evaluate formulas on a structure consisting of a class of indices of some sort together with a binary operation, sometimes together with a further binary relation. Relational semantic theories evaluate formulas on a

[5] A useful critical overview of 'the traditional motivations' (such as they are) can be found in §2 of [24].

[6] They are in fact also ubiquitous in print, see e.g. [18] or [19] just to start. A recent argument for the alternative view that relevance *is* a coherent notion can be found in [17].

structure consisting of a class of indices together with one or more ternary relations instead. These two divisions give us a natural way to divide relevant semantic theories into four families: Australian-flavored relational theories, Australian-flavored operational theories, American-flavored relational theories, and American-flavored operational theories.

Australian-flavored relational theories are the best known of the lot. They originated in Richard Routley and Bob Meyer's seminal works [37], [38], and [39]. Textbook treatments of these theories can be found in, e.g. [27] and [31]. Australian-flavored operational theories are less well known, but have also been influential. Among possible works to cite as examples of this approach, [21] has had the most direct influence on this paper, so is probably best. Next are the American-flavored relational theories. These theories are a bit less popular in technical applications than the other theories, but there is a small industry surrounding them nonetheless, as witnessed by, e.g. [33] and [26] and the citations included in those works.[7]

Of course, this leaves us with a conspicuous absence: prior to this paper, there have been, so far as I am aware, no philosophical analyses of theories that are simultaneously both American-flavored and operational. In fact, such theories are barely mentioned in the literature at all; the most I've found is the brief mention made in [36] of some of the difficulties such theories present. The semantics I give here is exactly this: an American-flavored operational semantics. Because of this, I call the theory *deep fried* semantics: the *operation* that produces the most paradigmatically *American flavors* is surely the operation of deep frying.[8] So this paper, in addition to presenting a theory of independent philosophical interest, also fills in a noticeable gap in the literature.

So there you have it: three reasons to care about the paper. The remainder of the paper is organized as follows. In the next section, we examine the formal objects – bunches – that will play the role of theories in the formal system we develop. In §3, we build our theory-based semantics. §4 provides a proof-theoretic account of the logic defined by the semantics. §5 contains soundness and completeness results. §6 discusses the logic and compares it to other systems.

2. Bunches

Before beginning, we need to address a misleading oversimplification hinted at in the introduction. In the story we told there, there were exactly two roles for theories to play – foreground and background. But that cannot be correct. Theories, on the story we're telling, can be applied to one another. The result is another theory. This theory, in turn, can be applied to further theories or have other theories applied to it. Thus, there's much more variety than the foreground/background distinction captures. A typical background theory, for example, will itself be the result of applying a (backbackground?) theory to a (forebackground?) theory. And so on. This messiness was, in a sense, already unavoidable once closure came onto the scene. With application in tow, it's just a bit more obvious. Regardless, our first task is to build formal objects – we call them *bunches* – that can play the role of the complicated, messy things that *theories* can be.

To reflect our assumption that there are two ways of building theories (the 'lump things together' way and the 'apply this to that' way) we will distinguish two different types of

[7]Having cited [33], I am obliged to include a footnote pointing out that it contains errors, though these are helpfully located and partially corrected in [34].

[8]The (extremely) motivated reader might stop at this point and try building deep fried semantics on her own. Hint: modify the theory of [21] using the 'Americanizing' tricks found in [30], [32], and [33].

bunches: I-bunches (I for *Intensional*) and E-bunches (E for *Extensional*). We formally define I-bunch, E-bunch, and bunch by simultaneous recursion as follows:

- Any sentence is an atomic I-bunch.
- Atomic I-bunches are I-bunches.
- Any set of I-bunches is an E-bunch.
- I-bunches and E-bunches (and nothing else) are bunches.
- If X and Y are bunches, then $(X; Y)$ is an I-bunch.[9]

We will draw our sentences (atomic I-bunches) from a propositional language with the connectives $\neg$, $\wedge$, and $\rightarrow$ as primitive and with $\vee$ given its usual definition in terms of $\neg$ and $\wedge$. The formation rules for this language are as expected.

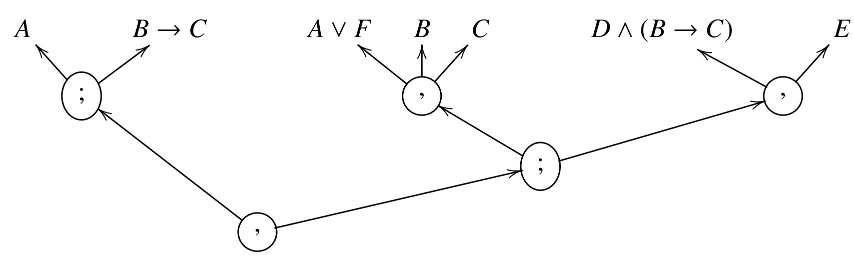
To prevent a proliferation of parentheses and set braces, we will adopt some notational conventions. First, in both bunches and sentences, we drop outermost parentheses in the usual way. Second, if E_1 and E_2 are E-bunches, then E_1, E_2 will mean $E_1 \cup E_2$. On the other hand, if I_1 and I_2 are both I-bunches, then I_1, I_2 will mean $\{I_1, I_2\}$. Finally, if E is an E-bunch and I is a I-bunch, then E, I and I, E will both mean $E \cup \{I\}$. A final note about E-bunches: since E-bunches are sets, they inherit the usual identities that sets enjoy – e.g. $\{\Gamma, \Delta\}$ and $\{\Delta, \Gamma\}$ and $\{\Gamma, \Gamma, \Delta\}$ are all the same E-bunch.

Recall from the introduction that the VTVF proposal was tightly focused on the issue of b-closure: ϕ is a logical consequence of t iff ϕ is in the b-closure of t for every t iff there are no b-counterexamples to the inference from t to ϕ. In terms of bunches, this means that the VTVF proposal restricts attention inferences whose antecedents are bunches of the form $b; t$. But it's unclear why logic should restrict attention to this sort of bunch. So in this paper we'll be more ecumenical and allow bunches of all shapes and sizes to have a say in our logic. Luckily, it turns out that this more general project is, for technical reasons, much simpler anyways. But that task is for §3. Before we get there, it's worthwhile to spend a moment discussing how to *picture* bunches (which we do in the next subsection) and how to think about the relationship between bunches and theories (which we do in §2.2).

2.1. **Picturing Bunches.**

The best way to picture bunches is to see them as a certain type of labeled tree. Formally, we take trees to be a pair consisting of a set of vertices V and set of directed edges E satisfying some conditions we won't bother writing down. A *labeled* tree is a tree together with a function from V to some set of labels.

Bunches can be pictured as labeled trees $\langle V, E, f \rangle$ with the feature that for each leaf vertex v, $f(v)$ is a sentence, and for each non-leaf vertex w, $f(w)$ is either the comma or the semicolon. As an example, the following tree:

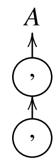


Corresponds to this bunch:

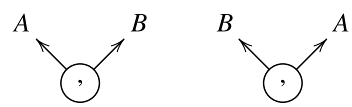
$$(A; B \rightarrow C), ((A \vee F, B, C); (D \wedge (B \rightarrow C), E))$$

[9]This definition is lifted from [31].

It's important to note that the correspondence highlighted here is not perfect. For example, consider the following labeled tree:



Intuitively, this would correspond to $\{\{A\}\}$. But in order for $\{X_i\}_{i\in I}$ to be a bunch, each X_i is required to be an I-bunch. So since $\{A\}$ is an E-bunch, $\{\{A\}\}$ isn't a bunch at all. Thus some labeled trees of the sort we're considering don't correspond to bunches. On the other hand, the following trees, while distinct, both correspond to the bunch $\{A, B\}$:



Clearly we could avoid these problems by adding conditions to rule out the first sort of problem and by working with appropriate sorts of equivalence classes of trees to deal with the second problem. But I won't bother to do this because it won't matter for anything that happens in this paper. We'll never, as it turns out, have to explicitly refer to the bunches-as-trees interpretation – I've mentioned it only because it helps one develop the right sort of intuitions for how to think about, manipulate, and work with bunches.

2.2. **Bunches and Theories.** While it's useful to be able to picture bunches, it is perhaps more useful yet to understand what they're supposed to represent. For this purpose it's best to think of bunches as recipes for building theories. At this point, we will take a very *logician's* view of theories and think of them as sets of sentences – though perhaps, as in our completeness proof, they need to be sets of sentences with nice features.[10]

To that end, we can think of X, Y as saying 'follow the X-recipe, then follow the Y-recipe, then take the union of the resulting theories'. Of course, the union of two theories need not be a theory (depending on what features we require of theories). So perhaps we need to quibble a bit here and say that the recipe given by 'X, Y' is actually the above followed by 'then find the smallest theory containing the resulting set.'

In any event, we can easily imagine using this operation in our investigations. Suppose, for example, I want to construct the complete theory of all the stuff on my desk. Suppose also that what's on my desk right now are piles of red papers and piles of blue papers. Then I might go about the task by fist constructing a complete theory of all the red-paper stuff, then constructing a complete theory of all the blue-paper stuff, then lumping the two theories together and seeing what theory is generated. The result ought to capture all there is to know about what's on my desk.

On the other hand, we can think of $X; Y$ as saying 'apply X to Y', which we take to mean 'using X as your background theory, see what theory Y gives you'. A historically relevant caricature of an example of this way of generating theories is the following. Suppose Y contains my observations of the location of Mercury. We can imagine applying two different background theories to Y: relativistic physics and Newtonian physics. In either case, the

[10]There are natural ways to extend this to more robust notions of theoryhood. A particularly promising route would be to include in our theories not only an account of what sentences are true, but an account of what counts as evidence (that is, what counts as a proof) and when two pieces of evidence are identical (that is, when two proofs are the same). It's fairly natural to see much of the work in algebraic logic as giving us this sort of account. See, for example, the discussion in the first few sections of [40].

result is a theory that tells us where Mercury ought to be in the future. The fact that the two theories are *different* is, of course, a fairly important fact in the history of science.

3. Deep Fried Semantics

We'll now turn to semantics. For reasons explained in the introduction, I call the theory I build here *deep fried semantics*. We begin with some definitions:

Definition. A deep fried premodel is a 4-tuple $\langle T, \circ, \sqsubseteq, v\rangle$ with

- T a set of of indices we call *theories*;
- $\circ$ a binary operation on T;
- $\sqsubseteq$ a binary relation on T; and
- v a function from theories to functions from atomic formulas to $\{\{1\}, \{0\}, \{0, 1\}, \emptyset\}$.

Intuitively, we want $\sqsubseteq$ to track containment of one theory in another and we want $\circ$ to track application of one theory to another. For this to work, we have to impose some conditions.

Definition. A deep fried model (hereafter just a model) is a deep fried premodel $\langle T, \circ, \sqsubseteq, v\rangle$ such that

C1: $\sqsubseteq$ is a partial ordering.
C2: $\circ$ is $\sqsubseteq$-monotonic: if $s \sqsubseteq t$, then $v \circ s \sqsubseteq v \circ t$ and $s \circ v \sqsubseteq t \circ v$.
C3: (Atomic heredity) if $s \sqsubseteq t$ and q is atomic, then $v(s)(q) \subseteq v(t)(q)$.
C4: (Associativity) $s \circ (t \circ u) = (s \circ t) \circ u$.
C5: (Commutativity) $s \circ t = t \circ s$.

It's worth pausing to explain the conditions. C1 is clearly what we'd expect given that $\sqsubseteq$ is meant to mimic containment of one theory in another. Similarly, if $\circ$ is supposed to track application, then C2 is also what we'd expect: If t contains s, then when we apply v to t we get everything we would get by applying v to s (and potentially more). Similarly, if t contains s, then applying t to v gets us everything we would get by applying s to v (and potentially more).

C3 is straightforward: it simply forces the $\sqsubseteq$ relation to track atomic truth and atomic falsity. C4 and C5 are, as mentioned, simplifying assumptions that will be done away with in future work.

3.1. Semantics 2: Truth.

Next up we need to give a recursive definition of satisfaction-at-a-theory-in-a-model ($\vDash_1$) and antisatisfaction-at-a-theory-in-a-model ($\vDash_0$). This definition will have to extend a bit further than might be expected because we need to define these concepts not just for every sentence, but for every bunch. Altogether this means we will need twelve different semantic clauses – one pair for each of the following: atomic sentences, negations, conjunctions, entailments, I-bunches and E-bunches. Six of the twelve clauses strike me as uncontroversial and will be adopted without comment. Throughout this subsection, let $M = \langle T, \circ, \sqsubseteq, v\rangle$ be an arbitrary model. Given this,

- If q is atomic then $M, t \vDash_1 q$ iff $1 \in v(t)(q)$;
- If q is atomic then $M, t \vDash_0 q$ iff $0 \in v(t)(q)$;
- $M, t \vDash_1 \neg\phi$ iff $M, t \vDash_0 \phi$;
- $M, t \vDash_0 \neg\phi$ iff $M, t \vDash_1 \phi$;
- $M, t \vDash_1 \phi \wedge \psi$ iff $M, t \vDash_1 \phi$ and $M, t \vDash \psi$;
- $M, t \vDash_0 \phi \wedge \psi$ iff $M, t \vDash_0 \phi$ or $M, t \vDash_0 \psi$;

What remains is to decide how to define truth and falsity for entailments and bunches. For entailments, our course has been set: in the introduction we stipulated that $\phi \to \psi$ should be in the theory t just if, given any theory u that contains ϕ, when we apply t to u, the result is a theory that contains ψ. But theories are supposed to be the *true* and complete-as-possible account of a given phenomenon, so if a sentence is *in* a given theory it ought to be, relative to that theory, true. Thus $\phi \to \psi$ should be true at the theory t just if, given any theory u that makes ϕ true, when we apply t to u, the result is a theory that makes ψ true. In fact, we should probably extend this slightly: since we're putting falsity on the same footing as truth, true entailments should be required to treat both truth and falsity correctly. Codifying this in our semantics gets us the following clause:

- $M, t \vDash_1 \phi \to \psi$ iff for all u, (i) if $M, u \vDash_1 \phi$ then $M, t \circ u \vDash_1 \psi$ and (ii) if $M, u \vDash_0 \psi$, then $M, t \circ u \vDash_0 \phi$.

The clause for falsity of entailments is less complex: an entailment is false at t when the falsity of its consequent is compatible with the truth of its antecedent. This means that there are theories – one that makes the antecedent true, one that makes the consequent false – and applying one to the other results in a theory that agrees with t.[11] We codify this as follows:

- $M, t \vDash_0 \phi \to \psi$ iff there are u and v so that $u \circ v \sqsubseteq t$ and $M, u \vDash_0 \psi$ and $M, v \vDash_1 \phi$.

The clauses for E-bunches are easy. The E-bunch $\{X_i\}_{i \in I}$ is meant to be the theory built by simply taking on each of the individual theories X_i. This leads us to the following clauses:

- $M, t \vDash_1 \{X_i\}_{i \in I}$ iff for all $i \in I$, $M, t \vDash_1 X_i$.
- $M, t \vDash_0 \{X_i\}_{i \in I}$ iff for some $i \in I$, $M, t \vDash_0 X_i$.

Finally, we turn to I-bunches. '$\circ$' is the representative, in our formal semantics, of the operation of *applying* one theory to another. Since the semicolon similarly tracks application, the following semantic clause is the natural choice:

- $M, t \vDash_1 X; Y$ iff there are u and v so that $u \circ v \sqsubseteq t$ and $M, u \vDash_1 X$ and $M, v \vDash_1 Y$.

The falsity clause for I-bunches requires a bit more thought. It helps to think in terms that will be useful in our completeness proof. While the semantics is designed to capture the behavior of the sorts of theories we actually build when we're investigating the world, theories, as we've been discussing them so far, are actually no more than the abstract sites at which sentences are evaluated for truth or falsity. Of course, as might be expected, in the completeness proof in §5, we will work with *formal theories*, which are sets of sentences that contain all of their syntactic consequences.

Looking ahead to that discussion, suppose t is a formal theory and suppose we define truth and falsity for formal theories in the following way:

- $t \vDash_1 \phi$ just if $\phi \in t$ and $t \vDash_0 \phi$ just if $\neg\phi \in t$;
- If s is an E-bunch, say $t \vDash_1 s$ just if $t \vDash_1 \sigma$ for all $\sigma \in s$ and say $t \vDash_0 s$ just if $t \vDash_0 \sigma$ for some $\sigma \in s$.

Finally, suppose we define $u; v$ to be the smallest formal theory containing every β such that $\alpha \to \beta \in u$ and $\alpha \in v$.

With all of this setup in hand, suppose $t \vDash_0 u; v$. Then t must falsify some member of $u; v$. Given what we've said this means that there is an entailment $\phi \to \psi \in u$ with $\phi \in v$ and $\neg\psi \in t$. The question to ask is how we would 'detect' this from outside t – that is, by

[11]Dave Ripley, in correspondence, has helpfully pointed out an oddity that arises here: the clause we've adopted for the conditional leaves *compatibility* essentially synonymous with *agreement*. But surely this is incorrect: I can recognize that, e.g. theories u and v are compatible *with one another* even if they disagree with my current theory t. This criticism has merit, but addressing it completely is both beyond the scope of this paper and, in any event, unnecessary since it arises only as an artifact of our simplifying assumptions.

just observing the way t interacts with other theories, rather than by looking inside t to see what sentences it contains. Here's how: supposing t really does contain such a sentence, suppose w_1 is any theory that makes true everything u does. Then w_1 will make true the entailment $\phi \to \psi$. So since t falsifies ψ, when we apply w_1 to t the resulting theory must falsify ϕ. But we said ϕ was a member of v. So then $t \circ w_1$ falsifies v. On the other hand, if w_2 makes true every member of v, then in particular w_2 makes true ϕ. But since t falsifies ψ, it then follows from the falsity clause for entailments that $t \circ w_2$ falsifies $\phi \to \psi$. Since we said $\phi \to \psi$ was a member of u, it follows that $t \circ w_2$ falsifies u. Altogether, what this suggests is the following clause:

- $M, t \vDash_0 X; Y$ iff for all u, (i) if $M, u \vDash_1 Y$ then $M, t \circ u \vDash_0 X$ and (ii) if $M, u \vDash_1 X$, then $M, u \circ t \vDash_0 Y$.

3.2. **Semantics 3: Validity.** As is probably expected, if X is a bunch and ϕ is a sentence, then we say $X \vDash \phi$ when there is no counterexample to the inference from X to ϕ But more needs to be said about what, exactly, should count as a counterexample.

Let's start with the easy stuff: whatever else counterexamples might be, they're the kind of thing that can rule out a sequent. Sequents, on the view under discussion, have the form $X \succ \phi$, where X is a bunch and ϕ is a sentence.[12] What remains is to settle conditions under which a sequent has been ruled out.

The standard view would have it that a sequent is ruled out by anything that satisfies X without also satisfying ϕ. But it seems to me this is too narrow. The four valued semantics we've built treats falsity as being semantically 'on a par with' truth.[13] So sequents should be ruled out just as easily by mishandling falsity as by mishandling truth. The sequent $X \succ \phi$ would seem to mishandle falsity, in turn, if ϕ can be 'antisatisfied' (that is, falsified) without X also being antisatisfied. With this broadened view of counterexamples in hand, we can now define validity.

Definition. Given a bunch X and a sentence ϕ, we will say that the sequent $X \succ \phi$ is *valid* (and write $X \vDash \phi$) iff for every deep fried model M and every $t \in M$, if $M, t \vDash_1 X$, then $M, t \vDash_1 \phi$ and if $M, t \vDash_0 \phi$, then $M, t \vDash_0 X$.

4. Deep Fried Proof Theory

Semantics is well and good, but digging through the deep fried models is an inefficient and messy way to find out which sequents are valid. So we'll now turn to giving a systematic procedure for actually *finding* valid sequents. As usual, we'll do this proof-theoretically. Since bunches and the machinery associated with bunches are quite unfamiliar, it's reasonable to worry that the proof theory will be rather awkward. But this worry turns out to be unfounded.[14] Here are our operational rules:

[12]An anonymous referee asks why we don't examine the more general sort of sequents that have bunches on both sides. The simple answer is that I felt burdening the reader with bunches was already a bit of a stretch without further burdening her with having to have them on both sides. This isn't to say the proposal is without merit. A similar idea that has been painfully understudied by philosophers can be found in [11].

[13]This follows a general pattern seen in, e.g. [20], [10], or (more briefly, but also with more technical breadth), [36].

[14]The particular rules used here are largely pilfered (without shame) from [31].

¬ rules	
$\dfrac{X; A \succ B \quad (\text{or } A \succ B) \qquad Y \succ \neg B}{X; Y \succ \neg A \quad (\text{or } Y \succ \neg A)}\ \neg\mathrm{I}$	$\dfrac{X \succ \neg\neg A}{X \succ A}\ \mathrm{DNE/DNI}$
∧ rules	
$\dfrac{X \succ A \qquad Y \succ B}{X, Y \succ A \wedge B}\ \wedge\mathrm{I}$	$\dfrac{X \succ A \wedge B \qquad Y(A, B) \succ C}{Y(X) \succ C}\ \wedge\mathrm{E}$
∨ rules	
$\dfrac{X \succ A \quad (\text{or } X \succ B)}{X \succ A \vee B}\ \vee\mathrm{I}$	$\dfrac{X \succ A \vee B \qquad X, A \succ C \qquad X, B \succ C}{X \succ C}\ \vee\mathrm{E}$
→ rules	
$\dfrac{X; A \succ B}{X \succ A \to B}\ \to\mathrm{I}$	$\dfrac{X \succ A \to B \qquad Y \succ A}{X; Y \succ B}\ \to\mathrm{E}$

The system also has three structural rules.[15] The first two require no comment:

$$\dfrac{X; Y \succ A}{Y; X \succ A}\ \mathrm{com} \qquad \dfrac{X; (Y; Z) \succ A}{(X; Y); Z \succ A}\ \mathrm{assoc}$$

The third requires a definition:

Definition. We define $\mathrel{\text{✂}}$ to be the smallest relation satisfying the following three conditions:

(i) $X \mathrel{\text{✂}} X$ for all bunches X.
(ii) If $X \mathrel{\text{✂}} Y$ and $Y \mathrel{\text{✂}} Z$, then $X \mathrel{\text{✂}} Z$.
(iii) If $X \subseteq Y$ are E-bunches, then $Z(X) \mathrel{\text{✂}} Z(Y)$.

We read '$X \mathrel{\text{✂}} Y$' as 'X is a pruning of Y'.[16]

Our final structural rule says that the system is pruning monotonic: if $X \succ A$ and $X \mathrel{\text{✂}} Y$, then $Y \succ A$. When we use this rule, we will label it $\mathrel{\text{✂}}$K. As usual we take everything of the form $A \succ A$ as an axiom. A deep fried *proof* (from here on just a proof) is a finite tree with finite sequents at each node, all of whose leaves are axioms and with the feature that each internal node, together with its parents, is an instance of one of our inference rules. We write $X \vdash A$ as shorthand for the claim 'for some finite $X' \mathrel{\text{✂}} X$ there is a proof ending at the sequent $X' \succ A$'.

Before moving on, we mention the usual three theorems:

Theorem 3 (Cut Admissibility)**.** *If $X \vdash A$ and $Y(A) \vdash B$, then $Y(X) \vdash B$.*

Theorem 4 (Soundness)**.** *If $X \vdash A$, then $X \vDash A$.*

Theorem 5 (Completeness)**.** *If $X \vDash A$, then $X \vdash A$.*

The proof of Theorem 3 is unilluminating, so relegated to an appendix. Theorem 4 and Theorem 5 are proved in the next section, after which we turn to a discussion of some interesting features of deep fried logic.

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[15] Note that since E-bunches are literally sets, we don't need to add structural rules governing e.g. repetition or associativity for E-bunches.

[16] The pruning relation, and the name 'pruning' for this relation, are due to Stephen Read; again see [31].

5. Deep Fried Metatheory

This section is the technical heart of the paper. As such, it can safely be skipped by the reader who lacks an interest in technical matters and is comfortable accepting Theorem 4 and Theorem 5.

Since many of the proofs I give rely on it, it's worthwhile to verify that we can actually can do structural induction on bunches. It turns out that defining a measure of complexity for bunches that shows this is possible is a bit harder than one would expect. But here is one way to do so:

Definition. We define the complexity, $\kappa(\Gamma)$ of the bunch Γ as follows:

- If Γ is a sentence, then $\kappa(\Gamma) = 1$.
- If $\Gamma = \Gamma_1; \Gamma_2$, then $\kappa(\Gamma) = \sup(\kappa(\Gamma_1), \kappa(\Gamma_2)) + 1$
- If $\Gamma = \{\Gamma_i\}_{i \in I}$, then $\kappa(\Gamma) = \sup_{i \in I}(\kappa(\Gamma_i) + 1)$.

Worth noting is that κ is an *ordinal* valued function. In the remainder of the document, whenever we do induction on bunches, we are implicitly doing induction on this measure of complexity. To see that this will work – that is, that this definition will do the job of allowing us to do structural induction on bunches – see Appendix A.

5.1. Soundness.

Lemma 1 (Heredity)**.** *If $a \sqsubseteq b$ and $M, a \vDash_1 X$, then $M, b \vDash_1 X$ and if $M, a \vDash_0 X$, then $M, b \vDash_0 X$.*

Proof Sketch. By induction on the complexity of the bunch X. Since atomic bunches are formulas, the base case requires a separate induction on the complexity of the *formula* X. Both inductions are straightforward, so left to the reader. □

Lemma 2. *If $S \subseteq T$ are E-bunches then if $M, t \vDash_1 X(T)$, then $M, t \vDash_1 X(S)$ and if $M, t \vDash_0 X(S)$ then $M, t \vDash_0 X(T)$.*

Proof. By induction on the complexity of X. The base case, when $X(S) = S$ and $X(T) = T$, is obvious, as is the case when $X(T) = X_1, X_2(T)$. Suppose $X(T) = X_1(T); X_2$. Then $M, t \vDash_1 X(T)$ iff there are u and v with $u \circ v \sqsubseteq t$ and $M, u \vDash_1 X_1(T)$ and $M, v \vDash_1 X_2$. But then by the inductive hypothesis, $M, u \vDash_1 X_1(S)$ as well. So $M, t \vDash_1 X_1(S); X_2$ which is to say $M, t \vDash_1 X(S)$.

Now suppose $M, t \vDash_0 X(S)$. Let $M, u \vDash_1 X_2$. Then $M, t \circ u \vDash_0 X_1(S)$, so by the inducive hypothesis, $M, t \circ u \vDash_0 X_1(T)$. On the other hand, if $M, u \vDash_1 X_1(T)$, then by the inductive hypothesis, $M, u \vDash_1 X_1(S)$ as well. So $M, u \circ t \vDash_0 X_2$. Altogether this shows that $M, t \vDash_0 X(T)$.

The last case, when $X(T) = X_1(T); X_2$, is essentially the same, so left to the reader. □

Lemma 3. *If $X \between Y$ and $M, t \vDash_1 Y$, then $M, t \vDash_1 X$ and if $M, t \vDash_0 X$, then $M, t \vDash_0 Y$.*

Proof. By the previous Lemma, using a straightforward induction on $\between$. □

Corollary 1. *If $X \between Y$ and $X \vDash \phi$, then $Y \vDash \phi$.*

Proof. Immediate from the previous lemma. □

We now have all the tools we need to prove Theorem 4.

Proof of the Soundness theorem. Our goal is to show that if $X \vdash A$, then $X \vDash A$. We will first show that if X is finite and $X \vdash A$ then $X \vDash A$. The proof is a standard induction on the length of the proof of $X \succ A$. If $X \succ A$ has a proof of length one, then $X \succ A$ is actually $A \succ A$, and the result is trivial. From here we proceed by considering the various options for the last rule applied in the proof. We only deal with the interesting cases here, these being $\neg$I, $\wedge$E and the $\rightarrow$ rules.

The $\neg$I case: We deal only with the more complicated version of the rule; the other version can be dealt with in essentially the same way. To begin, let $X; A \vDash B$ and $Y \vDash \neg B$.

Suppose $M, t \vDash_1 X; Y$. Then there are u and v with $u \circ v \sqsubseteq t$ so that $M, u \vDash_1 X$ and $M, v \vDash_1 Y$. Since $M, v \vDash_1 Y$ and $Y \vDash \neg B$, it follows that $M, v \vDash_1 \neg B$. So $M, v \vDash_0 B$. Thus, since $X; A \vDash B$, we have that $M, v \vDash_0 X; A$. So since $M, u \vDash_1 X$, it follows that $M, u \circ v \vDash_0 A$. Since $u \circ v \sqsubseteq t$, we then see that $M, t \vDash_0 A$, so then $M, t \vDash_1 \neg A$, as required.

Now suppose $M, t \vDash_0 \neg A$. To show that $M, t \vDash_0 X; Y$ we must show that if $M, u \vDash_1 Y$, then $M, t \circ u \vDash_0 X$ and if $M, u \vDash_1 X$, then $M, u \circ t \vDash_0 Y$.

For the first of these, suppose $M, u \vDash_1 Y$. Then since $Y \vDash \neg B$, we see that $M, u \vDash_1 \neg B$. So $M, u \vDash_0 B$. But then since $X; A \vDash B$, we see that $M, u \vDash_0 X; A$. Since $M, t \vDash_0 \neg A$, $M, t \vDash_1 A$. So $M, t \circ u \vDash_0 X$ as required.

For the second, suppose $M, u \vDash_1 X$. Note that since $M, t \vDash_0 \neg A$, $M, t \vDash_1 A$. Thus $M, u \circ t \vDash_1 X; A$. So since $X; A \vDash B$, $M, u \circ t \vDash_1 B$. It follows that $M, u \circ t \vDash_0 \neg B$, so since $Y \vDash \neg B$, we see that $M, u \circ t \vDash_0 Y$ as required.

The $\wedge$E case: To begin, let $X \vDash A \wedge B$ and $Y(A, B) \vDash C$. We first prove by induction on the structure of Y that if $M, t \vDash_1 Y(X)$, then $M, t \vDash_1 Y(A, B)$ (call this *the first bit*) and if $M, t \vDash_0 Y(A, B)$, then $M, t \vDash_0 Y(X)$ (call this *the second bit*). Clearly it follows from these that $Y(X) \vDash C$ for any Y.

In the base case, $Y(A, B) = (A, B)$ and $Y(X) = X$. For the first bit, suppose $M, t \vDash_1 X$. Then $M, t \vDash_1 A \wedge B$. So $M, t \vDash_1 A$ and $M, t \vDash_1 B$. It follows that $M, t \vDash (A, B)$ as required. For the second bit, suppose $M, t \vDash_0 (A, B)$. Then $M, t \vDash_0 A$ or $M, t \vDash_0 B$. In either case $M, t \vDash_0 A \wedge B$. So $M, t \vDash_0 X$ as required.

Next suppose that $Y(A, B) = Y_1(A, B), Y_2$ and $Y(X) = Y_1(X), Y_2$. For the first bit, suppose $M, t \vDash_1 Y_1(X), Y_2$. Then $M, t \vDash_1 Y(X)$ and $M, t \vDash Y_2$. By the inductive hypothesis, since $M, t \vDash_1 Y_1(X)$, it follows that $M, t \vDash_1 Y_1(A, B)$. Thus $M, t \vDash_1 Y_1(A, B), Y_2$ as required. For the second bit, suppose $M, t \vDash_0 Y_1(A, B), Y_2$. Then either $M, t \vDash_0 Y_1(A, B)$ or $M, t \vDash_0 Y_2$. In the first case, the inductive hypothesis gives that $M, t \vDash_0 Y_1(X)$. Thus in either case, $M, t \vDash_0 Y_1(X), Y_2$, as required.

The cases where $Y(A, B) = Y_1(A, B); Y_2$ and where $Y(A, B) = Y_1; Y_2(A, B)$ are essentially the same. So we deal only with the former. In this case we also have that $Y(X) = Y_1(X); Y_2$. For the first bit, suppose $M, t \vDash_1 Y_1(X); Y_2$. Then there are u and v with $u \circ v \sqsubseteq t$ and $M, u \vDash_1 Y_1(X)$ and $M, v \vDash_1 Y_2$. Since $M, u \vDash_1 Y_1(X)$, it follows by the inductive hypothesis that $M, u \vDash_1 Y_1(A, B)$. So $M, t \vDash Y_1(A, B); Y_2$ as required.

For the second bit, suppose $M, t \vDash_0 Y_1(A, B); Y_2$. Let $M, u \vDash_1 Y_1(X)$. Then by the inductive hypothesis, $M, u \vDash_1 Y_1(A, B)$. So $M, u \circ t \vDash_0 Y_2$. Now let $M, u \vDash_1 Y_2$. Then $M, t \circ u \vDash_0 Y_1(A, B)$. Thus, by the inductive hypothesis, $M, t \circ u \vDash_0 Y_1(X)$. So $M, t \vDash_0 Y_1(X); Y_2$.

The $\rightarrow$I case: To begin, let $X; A \vDash B$. Suppose $M, t \vDash_1 X$. Let $M, u \vDash_1 A$. Then $M, t \circ u \vDash_1 X; A$. So $M, t \circ u \vDash_1 B$. Now let $M, u \vDash_0 B$. Then $M, u \vDash_0 X; A$. So $M, t \circ u \vDash_0 A$.

Now suppose $M, t \vDash_0 A \to B$. Then there are u and v with $u \circ v \sqsubseteq t$ and $M, u \vDash_0 B$ and $M, v \vDash_1 A$. So $M, u \vDash_0 X; A$. It follows that $M, u \circ v \vDash_0 X$. Thus by heredity $M, t \vDash_0 X$ as well.

The →E case: To begin, let $X \vDash A \to B$ and $Y \vDash A$. Suppose $M, t \vDash_1 X; Y$. Then there are u and v with $u \circ v \sqsubseteq t$ and $M, u \vDash_1 X$ and $M, v \vDash_1 Y$. So $M, u \vDash_1 A \to B$ and $M, v \vDash_1 A$. Thus $M, u \circ v \vDash_1 B$. So by heredity $M, t \vDash_1 B$.

Finally suppose $M, t \vDash_0 B$. Let $M, u \vDash_1 X$. Then $M, u \vDash_1 A \to B$. So $M, u \circ t \vDash_0 A$. It follows that $M, u \circ t \vDash_0 Y$. On the other hand, if $M, u \vDash_1 Y$, then $M, u \vDash_1 A$. Thus $M, t \circ u \vDash_0 A \to B$. It follows that $M, t \circ u \vDash_0 X$.

To complete the proof, suppose $X \vdash \phi$. Then, by definition of $\vdash$, for some finite $X' \text{✂} X$ we have that $X' \vdash \phi$. By the above, $X' \vDash \phi$. Thus, by Lemma 3, $X \vDash \phi$ as well. □

5.2. **Completeness.** Now we turn to completeness. The deep fried system is tailor-made for a Henkin-style canonical model construction, so that's the approach we'll take.[17] We begin with a few standard definitions:

- An E-bunch of sentences is called a *formal theory* when for all sentences ϕ, if $X \vdash \phi$, then $\phi \in X$.
- An E-bunch of sentences Π is *prime* when $\phi \vee \psi$ is in Π only if either ϕ is in Π or ψ is in Π.
- An E-bunch of sentences Π is *disjunctively closed* when $\phi \vee \psi \in \Pi$ whenever ϕ is in Π and ψ is in Π.
- The *disjunctive closure* of an E-bunch of sentences X is the smallest disjunctively closed E-bunch of sentences containing X.
- $ap(\Pi_1, \Pi_2) = \{\beta : \alpha \to \beta \in \Pi_1 \text{ and } \alpha \in \Pi_2\}$. $ap(\Pi_1, \Pi_2)$ is called the *application* of Π_1 to Π_2.
- $cb(X) = \{\phi : X \vdash \phi\}$. $cb(X)$ is called the *consequence bunch* of the bunch X

The canonical model m_c is the 4-tuple $\langle T, ap, \subseteq, v\rangle$ where T is the set of formal theories, ap and $\subseteq$ are as expected or defined, and v is defined as follows:

$$v(t)(q) = \begin{cases} \{1\} & \text{if } q \in t \text{ but } \neg q \notin t \\ \{0\} & \text{if } \neg q \in t \text{ but } q \notin t \\ \{0,1\} & \text{if both } q \in t \text{ and } \neg q \in t \\ \emptyset & \text{otherwise} \end{cases}$$

Before moving on to the actual completeness proof, it's worth discussing something that the keen-eyed reader is likely to notice anyways: non-prime formal theories play a very small role in any of the proceedings. In fact the key Lemma – Lemma 12 – holds *only* for prime formal theories. So it's natural to wonder why, in the canonical model, T is the set containing *all* formal theories. The answer is straightforward: even if both Π_1 and Π_2 are prime theories, $ap(\Pi_1, \Pi_2)$ may not be. Thus, for ap to be well-defined, non-prime theories are required.[18]

[17] See, e.g. [23].

[18] It's easy to see why application of one prime formal theory to another might fail to produce a prime formal theory. Suppose t_1 is the smallest prime formal theory containing $A \to (B \vee C)$. Let t_2 be the smallest prime formal containing A. Then $ap(t_1, t_2)$ will contain $B \vee C$, but will contain neither B nor C. The 'natural' solution is to kludge together ap and $\subseteq$ into a ternary relation that holds between formal theories a, b, and c just when $ap(a, b) \subseteq c$. But this strikes me as paying lip service to technical convenience while actively sabotaging intelligibility and philosophical value, so I won't do it. This isn't to say that ternary relation semantics is generally silly. I think it's just silly given the motivation of this particular project. For other projects, it might be perfectly viable – see [8] for more.

Now to the actual proof. We will periodically appeal to the following facts, whose verification we leave to the reader:

Fact 1: If $\phi_1, \ldots, \phi_n \vdash \psi$, then $\phi_1 \wedge \cdots \wedge \phi_n \vdash \psi$.
Fact 2: $\phi_1 \to \psi_1, \ldots, \phi_n \to \psi_n \vdash (\phi_1 \wedge \cdots \wedge \phi_n) \to (\psi_1 \wedge \cdots \wedge \psi_n)$.
Fact 3: If $\psi \vdash \rho$ then $\phi \to \psi \vdash \phi \to \rho$.
Fact 4: $\phi \to \psi \dashv\vdash \neg\psi \to \neg\phi$
Fact 5: $\phi \vdash \neg\psi \to \neg(\phi \to \psi)$
Fact 6: If $\alpha \vdash \beta$, then $\neg\beta \vdash \neg\alpha$.
Fact 7: $\psi \vdash (\psi \to \phi) \to \phi$
Fact 8: $\rho \vdash \psi \to ((\rho \to (\psi \to \phi)) \to \phi)$
Fact 9: $\psi \to \phi \vdash (\rho \to \psi) \to (\rho \to \phi)$
Fact 10: $\neg(\phi \vee \psi) \dashv\vdash \neg\phi \wedge \neg\psi$
Fact 11: $\neg(\phi \wedge \psi) \dashv\vdash \neg\phi \vee \neg\psi$

Lemma 4. *If t and u are formal theories, then $ap(t, u)$ is a formal theory.*

Proof. Suppose $ap(t, u) \vdash \phi$. It follows from the definition of '$\vdash$' that there are $\beta_1, \ldots, \beta_n \in ap(t, u)$ such that $\beta_1, \ldots, \beta_n \vdash \phi$. So by Fact 1, $\beta_1 \wedge \cdots \wedge \beta_n \vdash \phi$. Since each $\beta_i \in ap(t, u)$, there are corresponding $\alpha_i \in u$ with $\alpha_i \to \beta_i \in t$. Since t is a formal theory, Fact 2 then gives that $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to (\beta_1 \wedge \cdots \wedge \beta_n) \in t$. Fact 3 then gives us that $(\alpha_1 \wedge \cdots \wedge \alpha_n) \to \phi \in t$. But Fact 1 gives that $\alpha_1 \wedge \cdots \wedge \alpha_n \in u$. So $\phi \in ap(t, u)$. □

Lemma 5. *For any bunch X, $cb(X)$ is a formal theory.*

Proof. Suppose $cb(X) \vdash \phi$. Then there are $\psi_1, \ldots, \psi_n \in cb(X)$ so that $\psi_1, \ldots, \psi_n \vdash \phi$. So by Fact 1, $\psi_1 \wedge \cdots \wedge \psi_n \vdash \phi$. But for each i, $X \vdash \psi_i$. So by repeated application of $\wedge$I, $X \vdash \psi_1 \wedge \cdots \wedge \psi_n$. So by cut, $X \vdash \phi$. Thus $\phi \in cb(X)$. □

Lemma 6. *ap is commutative and associative.*

Proof. Suppose $\phi \in ap(s, t)$. Then for some $\psi \in t$, $\psi \to \phi \in s$. But $\psi \vdash (\psi \to \phi) \to \phi$ by Fact 7. So since $\psi \in t$ and t is a formal theory, $(\psi \to \phi) \to \phi \in t$ as well. Thus since $\psi \to \phi \in s$, $\phi \in ap(t, s)$. So $ap(s, t) \subseteq ap(t, s)$. The converse containment is established in much the same way.

For associativity, suppose $\phi \in ap(ap(t, s), r)$. Then for some $\psi \in r$, $\psi \to \phi \in ap(t, s)$. Thus, for some $\rho \in s$, $\rho \to (\psi \to \phi) \in t$. But by Fact 8 $\rho \vdash \psi \to ((\rho \to (\psi \to \phi)) \to \phi)$. So since $\rho \in s$ and s is a formal theory, $\psi \to ((\rho \to (\psi \to \phi)) \to \phi) \in s$ as well. So since $\psi \in r$, $(\rho \to (\psi \to \phi)) \to \phi \in ap(s, r)$. And then commutativity plus the fact that $\rho \to (\psi \to \phi) \in t$ gives that $\phi \in ap(t, ap(s, r))$, establishing that $ap(ap(t, s), r) \subseteq ap(t, ap(s, r))$. The converse containment is left as an exercise for the reader; Fact 9 is the key observation. □

Lemma 7. *The canonical model is a model.*

Proof. Obvious, given Lemmas 4-6. □

Lemma 8 (Lindenbaum)**.** *Suppose Δ is closed under disjunction and t is a formal theory with $t \cap \Delta = \emptyset$. Then there is a prime formal theory $t' \supseteq t$ with $t' \cap \Delta = \emptyset$.*

Proof. Entirely standard, so relegated to an appendix. □

Lemma 9. *If $X \nvdash \phi$, then there is a prime formal theory $p \supseteq cb(X)$ with $\phi \notin p$.*

Proof. By Lemma 5, $cb(X)$ is a formal theory. Letting Δ be the disjunctive closure of $\{\phi\}$, the Lindenbaum Lemma immediately gives the result. □

Lemma 10. *If $X_1 \vdash \phi_1$ and $X_2 \vdash \phi_2$, then $X_1; X_2 \vdash \neg(\phi_1 \to \neg\phi_2)$.*

Proof. Let Θ_1 be a proof of $X_1 \succ \phi_1$ and let Θ_2 be a proof of $X_2 \succ \phi_2$. Then the following is a proof of $X_1; X_2 \succ \neg(\phi_1 \to \neg\phi_2)$ as required:

$$\dfrac{\dfrac{\dfrac{\phi_2 \to \neg\phi_1 \succ \phi_2 \to \neg\phi_1 \qquad \dfrac{\Theta_2}{X_2 \succ \phi_2}}{\phi_2 \to \neg\phi_1; X_2 \succ \neg\phi_1}\,{\to}\mathrm{E}}{X_2; \phi_2 \to \neg\phi_1 \succ \neg\phi_1}\,\mathrm{com} \qquad \dfrac{\Theta_1}{X_1 \succ \phi_1}}{\dfrac{X_2; X_1 \succ \neg(\phi_2 \to \neg\phi_1)}{X_1; X_2 \succ \neg(\phi_2 \to \neg\phi_1)}\,\mathrm{com}}\,\neg\mathrm{I}$$

□

Definition. If X is a finite bunch, the characteristic formula of X – written $cf(X)$ – is defined as follows:

- If X is a formula, $cf(X) = X$.
- If $X = X_1, X_2$, then $cf(X) = cf(X_1) \wedge cf(X_2)$.
- If $X = X_1; X_2$, then $cf(X) = \neg(cf(X_1) \to \neg cf(X_2))$.

We then have the following result:

Lemma 11. *$X \vdash cf(X)$ for all finite bunches X.*

Proof. By induction on the complexity of X. If X is a formula, the result is obvious. If $X = X_1, X_2$, then by the inductive hypothesis, $X_1 \vdash cf(X_1)$ and $X_2 \vdash cf(X_2)$. So by $\wedge$I, $X_1, X_2 \vdash cf(X_1) \wedge cf(X_2)$ as required. If $X = X_1; X_2$, then by the inductive hypothesis, $X_1 \vdash cf(X_1)$ and $X_2 \vdash cf(X_2)$. It follows by Lemma 10 that $X_1; X_2 \vdash \neg(cf(X_1) \to \neg cf(X_2))$. □

Corollary 2. *If X is a finite bunch and $Y(X) \vdash \phi$, then $Y(cf(X)) \vdash \phi$*

Proof. By cut, using the previous lemma. □

Lemma 12. *If t is a prime formal theory, then $m_c, t \vDash_1 X$ iff $cb(X) \subseteq t$ and $m_c, t \vDash_0 X$ iff $\neg\psi \in t$ for some $\psi \in cb(X)$.*

Proof. By a simultaneous induction on the complexity of X in both parts of the result. For the base case, where X is a sentence, we proceed by induction on complexity again. If X is an atomic sentence, the result follows from the definition of the valuation v in m_c. We examine only the more complicated inductive steps here, leaving the remaining steps to the reader:

$\wedge$0: Let $m_c, t \vDash_0 \phi \wedge \psi$. Then either $m_c, t \vDash_0 \phi$ or $m_c, t \vDash_0 \psi$. Suppose without loss of generality that $m_c, t \vDash_0 \phi$. Then by the inductive hypothesis, there is a $\rho \in cb(\phi)$ with $\neg\rho \in t$. But clearly $\phi \wedge \psi \vdash \phi$, so since $\rho \in cb(\phi)$, $\rho \in cb(\phi \wedge \psi)$ as well. Thus ρ is an element of $cb(\phi \wedge \psi)$ whose negation is in t as is required.

Now suppose that for some $\rho \in cb(\phi \wedge \psi)$, $\neg\rho \in t$. Then by Fact 6, $\neg(\phi \wedge \psi) \in t$. But by Fact 10, $\neg(\neg\phi \vee \neg\psi) \vdash \neg\neg\phi \wedge \neg\neg\psi$. From this a quick argument that we leave to the reader gives that $\neg(\neg\phi \vee \neg\psi) \vdash \phi \wedge \psi$. Thus, since $\neg(\phi \wedge \psi) \in t$, Fact 6 gives that $\neg\neg(\neg\phi \vee \neg\psi) \in t$, and thus that $\neg\phi \vee \neg\psi \in t$. Since t is prime, it follows that $\neg\phi \in t$ or $\neg\psi \in t$. So by the inductive hypothesis, either $m_c, t \vDash_0 \phi$ or $m_c, t \vDash_0 \psi$. In either event, $m_c, t \vDash_0 \phi \wedge \psi$.

→1: Suppose $cb(\phi \to \psi) \subseteq t$. Then in particular $\phi \to \psi \in t$. Let $m_c, u \vDash_1 \phi$. Then by the inductive hypothesis, $cb(\phi) \subseteq u$. Again, we have that in particular $\phi \in u$. It follows from our two particular observations that $\psi \in ap(t, u)$. So $cb(\psi) \subseteq ap(t, u)$, and thus by the inductive hypothesis, $m_c, ap(t, u) \vDash_1 \psi$. Now let $m_c, u \vDash_0 \psi$. Using a by-now-familiar argument this gives that $\neg\psi \in u$. Since $\phi \to \psi \in t$ and $\phi \to \psi \vdash \neg\psi \to \neg\phi$, it follows that $\neg\psi \to \neg\phi \in t$ as well. Thus $\neg\phi \in ap(t, u)$. So by the inductive hypothesis $m_c, ap(t, u) \vDash_0 \phi$. Altogether this gives us that $m_c, t \vDash_1 \phi \to \psi$.

Now suppose $cb(\phi \to \psi) \nsubseteq t$. Then $\phi \to \psi \notin t$. Let $u = cb(\phi)$. By Lemma 5, u is a formal theory, and by the inductive hypothesis, $m_c, u \vDash_1 \phi$. Suppose $\psi \in ap(t, u)$. Then for some $\alpha \in u$, $\alpha \to \psi \in t$. Since $\alpha \in u$, $\phi \vdash \alpha$. Thus we can argue as follows:

$$\cfrac{\cfrac{t \vdash \alpha \to \psi \qquad \phi \vdash \alpha}{t; \phi \vdash \psi}\;{\to}\mathrm{E}}{t \vdash \phi \to \psi}\;{\to}\mathrm{I}$$

So $\phi \to \psi \in t$, which is a contradiction. Thus $\psi \notin ap(t, u)$. So by the inductive hypothesis, $m_c, u \vDash_1 \phi$ but $m_c, ap(t, u) \nvDash_1 \psi$. So $m_c, t \nvDash_1 \phi \to \psi$.

→0: Let $m_c, t \vDash_0 \phi \to \psi$. Then there are u and v with $ap(u, v) \subseteq t$ and $u \vDash_0 \psi$ and $v \vDash_1 \phi$. Skipping a few familiar steps, the inductive hypothesis then gives that $\neg\psi \in u$ and $\phi \in v$. But then by Fact 5, since $\phi \in v$, $\neg\psi \to \neg(\phi \to \psi) \in v$ as well. So $\neg(\phi \to \psi) \in ap(u, v) \subseteq t$. Thus there is an element of $cb(\phi \to \psi)$ whose negation is in t as is required.

Now suppose $cb(\neg(\phi \to \psi)) \subseteq t$. Let $b = cb(\phi)$ and $c = cb(\neg\psi)$. I claim that $ap(c, b) \subseteq t$. Clearly it follows from this and the inductive hypothesis that $m_c, t \vDash_0 \phi \to \psi$.

To see that $ap(c, b) \subseteq t$, suppose $\gamma \in ap(c, b)$. Then for some β we have that $\beta \to \gamma \in c$ and $\beta \in b$. So $\neg\psi \vdash \beta \to \gamma$ and $\phi \vdash \beta$. Thus by →E, $\neg\psi; \phi \vdash \gamma$. So by →I, $\neg\psi \vdash \phi \to \gamma$. So by Fact 4 and cut, $\neg\psi \vdash \neg\gamma \to \neg\phi$, and thus $\neg\psi; \neg\gamma \vdash \neg\phi$. By commutativity, then, $\neg\gamma; \neg\psi \vdash \neg\phi$. So $\neg\gamma \vdash \neg\psi \to \neg\phi$. Thus by Fact 4 (this time in the other direction), $\neg\gamma \vdash \phi \to \psi$. So by Fact 6, $\neg(\phi \to \psi) \vdash \gamma$. Thus, since $\neg(\phi \to \psi) \in t$, $t \vdash \gamma$. Since t is a formal theory, it follows that $\gamma \in t$, and thus $ap(c, b) \subseteq t$.

This completes the base case of our induction on the complexity of the bunch X. We now turn to the inductive steps:

The E-bunch Case: Suppose $X = \{X_i\}_{i \in I}$ is an E-bunch and suppose $m_c, t \vDash_1 X$. Then $m_c, t \vDash X_i$ for all $i \in I$, so by the inductive hypothesis, $cb(X_i) \subseteq t$ for all $i \in I$. Let $X \vdash \phi$. Then there is a finite $X' \mathrel{\text{✂<}} X$ so that $X' \vdash \phi$. Thus there are $i_1, \ldots, i_n \in I$ and finite bunches $X'_{i_1} \mathrel{\text{✂<}} X_{i_1}, \ldots, X'_{i_n} \mathrel{\text{✂<}} X'_{i_n}$ so that $X'_{i_1}, \ldots, X'_{i_n} \vdash \phi$. By repeated applications of the corollary to Lemma 10, we see that $cf(X'_{i_1}), \ldots, cf(X'_{i_n}) \vdash \phi$. By Lemma 10, each $X'_{i_j} \vdash cf(X'_{i_j})$. By ✂<K, then, we have that $X_{i_j} \vdash cf(X'_{i_j})$. So since $cb(X_i) \subseteq t$ for all $i \in I$, each $cf(X'_{i_j})$ is in t. Thus by ✂<K, since $cf(X'_{i_1}), \ldots, cf(X'_{i_n}) \vdash \phi$, we also have that $t \vdash \phi$, and since t is a formal theory it follows that $\phi \in t$. Thus $cb(X) \subseteq t$.

Now suppose $m_c, t \nvDash_1 X$. Then for some $i \in I$, $m_c, t \nvDash_1 X_i$. Thus by the inductive hypothesis, $cb(X_i) \nsubseteq t$. So for some ϕ, we have both $X_i \vdash \phi$ and $\phi \notin t$. But if $X_i \vdash \phi$, then $\{X_i\} \vdash \phi$ as well. And then since $\{X_i\} \mathrel{\text{✂<}} X$, it follows that $X \vdash \phi$ as well. Thus $cb(X) \nsubseteq t$.

Next suppose $m_c, t \vDash_0 X$. Then for some X_i, $m_c, t \vDash_0 X_i$. Thus by the inductive hypothesis, there is a $\phi \in cb(X_i)$ with $\neg\phi \in t$. But by a similar argument to the one in the previous paragraph, we can see that $\phi \in cb(X)$ as well, giving a sentence that is a consequence of X, but whose negation is in t as is required.

Finally, suppose there is a ϕ with $\phi \in cb(X)$ but $\neg\phi \in t$. By a similar argument as in the first paragraph of this case, since $\phi \in cb(X)$, there are $i_1, \ldots, i_n \in I$ and finite bunches $X'_{i_1} \prec X_{i_1}, \ldots, X'_{i_n} \prec X'_{i_n}$ so that $cf(X'_{i_1}), \ldots, cf(X'_{i_n}) \vdash \phi$. Thus by Fact 1, $cf(X'_{i_1}) \wedge \cdots \wedge cf(X'_{i_n}) \vdash \phi$. So by Fact 6, $\neg\phi \vdash \neg(cf(X'_{i_1}) \wedge \cdots \wedge cf(X'_{i_n}))$. So since $\neg\phi \in t$, it follows that $\neg(cf(X'_{i_1}) \wedge \cdots \wedge cf(X'_{i_n})) \in t$. So by repeated application of Fact 11, $\neg cf(X'_{i_1}) \vee \cdots \vee \neg cf(X'_{i_n}) \in t$ as well. So since t is prime, for some i_j, $cf(X'_{i_j}) \in t$. Thus by the inductive hypothesis, $m_c, t \vDash_0 X'_{i_j}$. So by Lemma 3, $m_c, t \vDash_0 X_{i_j}$. It follows that $m_c, t \vDash_0 X$.

The I-bunch Case: Suppose $X = X_1; X_2$ is a I-bunch and suppose $m_c, t \vDash_1 X$. Then there are u and v with $ap(u, v) \subseteq t$ and $m_c, u \vDash_1 X_1$ and $m_c, v \vDash_1 X_2$. It follows from the inductive hypothesis that $cb(X_2) \subseteq v$. Suppose $\phi \in cb(X)$. Then $X_1; X_2 \vdash \phi$. So there is a finite $X'_2 \prec X_2$ so that $X_1; X'_2 \vdash \phi$. By the corollary to Lemma 10, it follows that $X_1; cf(X'_2) \vdash \phi$. So $X_1 \vdash cf(X'_2) \to \phi$, and thus $cf(X'_2) \to \phi \in cb(X_1) \subseteq u$. Thus since $cf(X'_2) \in cb(X_2)$ by Lemma 10 and $\prec$K and since $cb(X_2) \subseteq v$, it follows that $\phi \in ap(u, v)$. So $cb(X_1; X_2) \subseteq t$.

Now suppose that $cb(X_1; X_2) \subseteq t$. Let $u = cb(X_1)$ and $v = cb(X_2)$. By the inductive hypothesis, $m_c, u \vDash_1 X_1$ and $m_c, v \vDash_1 X_2$. Notice that if $\phi \to \psi \in u$ and $\phi \in v$, then $\psi \in cb(X_1; X_2) \subseteq t$. So $ap(u, v) \subseteq t$. Thus $m_c, t \vDash_1 X_1; X_2$.

Suppose $m_c, t \vDash_0 X_1; X_2$. Then for all u, if $m_c, u \vDash_1 X_1$, then $m_c, ap(t, u) \vDash_0 X_2$ and if $m_c, u \vDash_1 X_2$, then $m_c, ap(u, t) \vDash_0 X_1$. Let $u = cb(X_2)$. By the inductive hypothesis, $m_c, u \vDash_1 X_2$. So $m_c, ap(t, u) \vDash_0 X_1$. Thus by the inductive hypothesis, there is a sentence ϕ so that $\phi \in cb(X_1)$ but $\neg\phi \in ap(t, u)$. So by the definition of cl, there is a sentence ψ with $\psi \in u$ and $\psi \to \neg\phi \in t$. Clearly if $\psi \to \neg\phi \in t$, then $\neg\neg(\psi \to \neg\phi) \in t$ as well. But since $X_2 \vdash \psi$ and $X_1 \vdash \phi$, Lemma 10 gives that $X_2; X_1 \vdash \neg(\psi \to \neg\phi)$. Thus clearly $X_1; X_2 \vdash \neg(\psi \to \neg\phi)$ as well. So $\neg(\psi \to \neg\phi)$ is an example of a sentence that is in $cb(X_1; X_2)$ whose negation is in t, as is required.

Finally, suppose $\neg\phi \in t$ for some $\phi \in cb(X_1; X_2)$. Let $m_c, u \vDash_1 X_1$. By the inductive hypothesis, it follows that $cb(X_1) \subseteq X_1$. By an argument similar to the one in the first paragraph of this case, we see that since $\phi \in cb(X_1; X_2)$ there is a finite $X'_2 \prec X_2$ with $X_1 \vdash cf(X'_2) \to \phi$. Thus $cf(X'_2) \to \phi \in u$. Thus by Fact 4, $\neg\phi \to \neg cf(X'_2) \in u$ as well. So since $\neg\phi \in t$, $\neg cf(X'_2) \in ap(u, t)$. By Lemma 10, and $\prec$K, $X_2 \vdash cf(X'_2)$. Thus $cf(X'_2)$ is a sentence in $cb(X_2)$ whose negation is in $ap(u, t)$. Thus by the inductive hypothesis, $m_c, ap(u, t) \vDash_0 X_2$. Mutatis mutandis, the same argument shows that if $m_c, u \vDash_1 X_2$, then $m_c, ap(t, u) \vDash_0 X_1$. So $m_c, t \vDash_0 X_1; X_2$.

□

We now have all the tools we need to prove the completeness theorem:

Proof of the Completeness Theorem. Suppose $X \nvdash \phi$. Then by Lemma 9, there is a prime formal theory p with $cb(X) \subseteq p$ and $\phi \notin p$. It then follows by Lemma 11 that $m_c, p \vDash_1 X$ but $m_c, p \nvDash_1 \phi$. So $X \nvDash \phi$. Contraposing gives the result. □

6. Discussion and Auxiliary Results

Now that we have both syntax and proof theory for deep fried logic on the table and have seen that they match up, we return to our philosophical analysis of the system. An important note: as mentioned in the introduction, the system we've presented here is similar in many important ways to the system that results from making none of the simplifying assumptions we've made to keep the formal results manageable. In particular, the results presented in this system carry over, mutatis mutandis, to results about a non-commutative, non-associative system with an Australian-style negation. Proving this claim is beyond the scope of this paper, however, so will have to wait for its sequel.

6.1. **FDE.** We began the paper with a discussion of Jc Beall's recent work on logic and the task of the theorist. In this work, Beall argues that, for (what amounts to) the extensional fragment of our system, the weak formal entailment relation knows as FDE is the correct logic. For details on FDE, I refer the reader to [1] or [10] (for philosophical details) or [29] or [9] (for textbook treatments). We will write $\vdash_{\text{FDE}}$ for FDE-consequence. My first goal is to show that I'm in agreement with Beall about everything he takes a stand on.

Definition. Say that a formula is *purely extensional* when it contains no occurrences of '$\to$'. In turn, say that a bunch is purely extensional when (a) every formula in it is purely extensional and (b) the bunch itself contains no semicolons. Finally, say that a sequent is purely extensional when both its lefthand and righthand sides are purely extensional.

Theorem 6. *FDE is sound and complete with respect to one-element deep fried models.*

Proof. See e.g. [29]. □

Lemma 13. *If $X \succ \phi$ is purely extensional and $X \nvDash \phi$, then there is a one-element countermodel to $X \succ \phi$.*

Proof Sketch. Since $X \nvDash \phi$, it follows from our soundness theorem that $X \nvdash \phi$. Thus by Lemma 9, there is a prime formal theory $p \supseteq cb(X)$ with $\phi \notin p$. Thus by Lemma 11 $m_c, p \vDash_1 X$ but $m_c, p \nvDash_1 \phi$. But since $X \succ \phi$ is purely extensional, no other points in the canonical model play a role in these facts. So in fact p, taken as a one-element model, satisfies X and antisatisfies ϕ. □

Theorem 7. *If $X \succ \phi$ is purely extensional, then $X \vdash \phi$ iff $X \vdash_{\text{FDE}} \phi$.*

Proof Sketch. Suppose $X \succ \phi$ is purely extensional and $X \nvdash \phi$. Then by completeness, $X \nvDash \phi$. So by Lemma 13 there is a one-element deep fried countermodel to $X \succ \phi$. Thus $X \succ \phi$ has an FDE countermodel. So $X \nvdash_{\text{FDE}} \phi$. On the other hand, if $X \nvdash_{\text{FDE}} \phi$, then there is an FDE-countermodel to $X \succ \phi$. But an FDE countermodel just is a one-element deep fried countermodel. So $X \nvdash \phi$. □

Philosophical Takeaway. Beall's preferred account of logic – FDE – is exactly the *extensional fragment* deep fried logic. This leaves the disagreement between Beall and I (to the extent that there is one) at the level of vocabulary. More concretely, we disagree about whether the intensional connective '$\to$' is properly part of our logical vocabulary (with logic understood in the sense at hand).

Clearly this isn't the time or place to settle the question of which vocabulary should be counted as logical. Nonetheless, I think even on Beall's own grounds, there's a clear case to be made for including entailment as logical.

In brief, the case is this: in [7], Beall claims that logical vocabulary 'is the vocabulary that figures in all of our true (and complete-as-possible theories)'. Theories that differ only in how they interact with other theories nonetheless genuinely differ – that is, are different theories. Since how theories interact with one another is determined by entailments, it follows that to fully characterize a theory, one must specify which entailments it contains. It follows that entailment figures in all our theories, true or otherwise, and thus belongs to the properly logical vocabulary.

6.2. **Classical Logic.** It's worthwhile to also situate deep fried logic with respect to classical logic (CL), since this will probably be more helpful for most readers than situating it with respect to FDE was.

Definition. If ϕ is a purely extensional sentence, define $\iota(\phi)$ and $\varepsilon(\phi)$ as follows:

- For atomic sentences q, $\iota(q) = q \wedge \neg q$ and $\varepsilon(q) = q \vee \neg q$.
- $\iota(\neg\phi) = \iota(\phi)$.
- $\varepsilon(\neg\phi) = \varepsilon(\phi)$.
- $\iota(\phi \wedge \psi) = \iota(\phi \vee \psi) = \iota(\phi) \vee \iota(\psi)$.
- $\varepsilon(\phi \wedge \psi) = \varepsilon(\phi \vee \psi) = \varepsilon(\phi) \wedge \varepsilon(\psi)$.

We extend ι (but not ε) to finite purely extensional bunches by saying that $\iota(\{\phi_1, \dots, \phi_n\}) = \iota(\phi_1) \vee \cdots \vee \iota(\phi_n)$. With these functions on hand, it's straightforward to extract from [4] or [3] a proof of the following result:

Theorem 8. *If $X \succ \phi$ is purely extensional, X is finite, and $X \vdash_{\mathrm{CL}} \phi$, then $\varepsilon(\phi), X \vdash_{\mathrm{FDE}} \phi \vee \iota(X)$.*

Before applying this result, it's worth first noting what it says about the relation between FDE and classical logic. Personifying the logics, what Theorem 8 tells us is that whenever classical logic says that ϕ follows from X, FDE hedges its bets and instead says that *if* we assume excluded middle for every atomic in ϕ, then *either* (as classical logic thought), ϕ follows from X *or* X contains an inconsistency. This explains why Beall refers to (a version of) Theorem 8 as a 'classical collapse' result: it shows, in essence, that FDE is just a *more cautious version* of classical logic.[19]

We put Theorem 8 to work in the following result:

Theorem 9. *If $X \succ \phi$ is purely extensional and $X \vdash_{\mathrm{CL}} \phi$, then there is a finite $X' \subseteq X$ so that $\varepsilon(\phi), X \vdash \phi \vee \iota(X')$.*

Proof Sketch. First, since classical logic is compact, $X \vdash_{\mathrm{CL}} \phi$ only if there is a finite $X' \subseteq X$ so that $X' \vdash_{\mathrm{CL}} \phi$. It follows by Theorem 8 that if $X \vdash_{\mathrm{CL}} \phi$, then for some finite $X' \subseteq X$, $\varepsilon(\phi), X' \vdash_{\mathrm{FDE}} \phi \vee \iota(X')$. But then by Theorem 7, $\varepsilon(\phi), X' \vdash \phi \vee \iota(X')$. Thus, by ✂K, $\varepsilon(\phi), X \vdash \phi \vee \iota(X')$. $\square$

Philosophical Takeaway. The extensional fragment of deep fried logic is also just a cautious version of classical logic. But deep fried logic goes further than classical logic by containing non-extensional vocabulary. It thus has a pair of complementary virtues: it's more cautious than classical logic, but also says more. In brief, deep fried logic says more than classical logic, and also says it better.

6.3. **Relevance and Contraction-freedom.** Finally, we'll show that deep fried logic is equivalent to a certain well known relevant logic. We begin with a definition pulled from [14]:

Definition. RW is the smallest set of sentences containing all instances of the following axioms that is closed under the rules R1 and R2:

A1: $\alpha \to \alpha$

A2: $(\alpha \wedge \beta) \to \alpha$

A3: $(\alpha \wedge \beta) \to \beta$

A4: $((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma))$

[19]For more on classical collapse results, see [4] and [5].

A5: $(\alpha \wedge (\beta \vee \gamma)) \to ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

A6: $\neg\neg\alpha \to \alpha$

A7: $(\alpha \to \neg\beta) \to (\beta \to \neg\alpha)$

A8: $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

A9: $\alpha \to ((\alpha \to \beta) \to \beta)$

R1: $\dfrac{\alpha, \alpha \to \beta}{\beta}$

R2: $\dfrac{\alpha, \beta}{\alpha \wedge \beta}$

As it turns out, RW and deep fried logic are tightly connected – so much so that you might be tempted to call them the same logic.[20] The result that makes this connection clear is Theorem 10. Before we get to it, we need some machinery.

Definition. Let $\theta(\phi)$ be a formula with a highlighted occurrence of the formula ϕ as a subformula. We say that the highlighted occurrence is a positive occurrence or a negative occurrence by following these rules:

- ϕ is positive in ϕ.
- If ϕ is positive in $\theta_1(\phi)$ and θ_2 is a formula, then ϕ is positive in $\theta_1(\phi) \wedge \theta_2$, in $\theta_2 \wedge \theta_1(\phi)$, and in $\theta_2 \to \theta_1(\phi)$.
- If ϕ is positive in $\theta_1(\phi)$ and θ_2 is a formula, then ϕ is negative in $\neg\theta_1(\phi)$ and in $\theta_1(\phi) \to \theta_2$.
- If ϕ is negative in $\theta_1(\phi)$ and θ_2 is a formula, then ϕ is negative in $\theta_1(\phi) \wedge \theta_2$, in $\theta_2 \wedge \theta_1(\phi)$, and in $\theta_2 \to \theta_1(\phi)$.
- If ϕ is negative in $\theta_1(\phi)$ and θ_2 is a formula, then ϕ is positive in $\neg\theta_1(\phi)$ and in $\theta_1(\phi) \to \theta_2$.

Lemma 14. *If $\phi \to \psi$ is a theorem of RW then*

- *If ϕ is positive in $\theta(\phi)$, then $\theta(\phi) \to \theta(\psi)$ is a theorem of RW; and*
- *If ϕ is negative in $\theta(\phi)$, then $\theta(\psi) \to \theta(\phi)$ is a theorem of RW.*

As an example, $\alpha \to ((\alpha \to \beta) \to \beta)$ is a theorem of RW. Thus, applying the construction in the first part of the Lemma to the formula $p \vee (q \to \alpha)$ in which α occurs positively, we see that $(p \vee (q \to \alpha)) \to (p \vee (q \to ((\alpha \to \beta) \to \beta)))$ is also a theorem of RW, as the reader can check. The proof of Lemma 14 is a tedious induction on the complexity of the context θ, which we leave to the (motivated) reader.

Theorem 10. *If $X \vdash \phi$ and X is finite, then $cf(X) \to \phi$ is a theorem of RW.*

Proof Sketch. By induction on the length of the proof of $X \succ \phi$. The difficult step to deal with is the case involving $\wedge$E which requires an induction on the complexity of the bunch Y. Lemma 14 plays a key role here. □

Philosophical Takeaway. RW is a contractionless relevant logic. Thus, by Theorem 10, deep fried logic is contractionless and relevant. That deep fried logic has either of these features should be somewhat surprising, since neither relevance nor contraction-freedom (nor any of their usual surrogates) played a role in motivating it.[21]

[20] For more on RW, the canonical reference is [2]. A deep study of RWQ – the quantified extension of RW – can be found in [25].

[21] It's worth noting that the same thing is true of Beall's account: FDE is also a relevant logic, but Beall's motivation for examining it is in no way motivated by relevance.

Conclusion

Logic, in at least one of its more philosophically central and important senses, is about what follows from what no matter what. One standard way to flesh this out is in terms of theory-building: logic is about what a complete-as-possible theory (no matter its subject) should contain, given its construction. But when we look at how we can construct theories, there are two operations to consider: we can throw more sentences into the theory, and we can apply one theory to another. Given some mild simplifications, deep fried logic spells out exactly what logic amounts to, given all of this.

As it turns out, deep fried logic is contractionless and relevant. Relevantists should take this as good news. After all, the concept of relevance (and its suitability for grounding a logic) has taken a beating in recent years. What I've provided here is a backdoor route to taking relevance logics seriously. And that's a reason to be excited about deep fried logic even if the notion of logic as universal theory-building toolkit isn't.

References

[1] A. R. Anderson and Nuel D Belnap. First degree entailments. *Journal of Symbolic Logic*, 36(3):520–521, 1971.

[2] Alan R. Anderson and Nuel D. Belnap. *Entailment: The Logic of Relevance and Neccessity, Vol. I*. Princeton University Press, 1975.

[3] Jc Beall. Multiple-conclusion LP and default classicality. *Review of Symbolic Logic*, 4(2):326–336, 2011.

[4] Jc Beall. LP+, K3+, FDE+, and their 'classical collapse'. *Review of Symbolic Logic*, 6(4):742–754, 2013.

[5] Jc Beall. Free of detachment: Logic, rationality, and gluts. *Noûs*, 49(2):410–423, 2015.

[6] Jc Beall. There is no logical negation: True, false, both, and neither. *Australasian Journal of Logic*, 14(1):Article no. 1, 2017.

[7] Jc Beall. The simple argument for subclassical logic. *Philosophical Issues*, 28(1):30–54, 2018.

[8] Jc Beall, Ross T. Brady, J. Michael Dunn, A. P. Hazen, Edwin D. Mares, Robert K. Meyer, Graham Priest, Greg Restall, David Ripley, John Slaney, and Richard Sylvan. On the ternary relation and conditionality. *Journal of Philosophical Logic*, 41(3):595–612, 2012.

[9] Jc Beall and Shay Allen Logan. *Logic: The Basics (2nd Edition)*. Routledge, 2017.

[10] Nuel D Belnap. A useful four-valued logic. In J. Michael Dunn and G. Epstein, editors, *Modern Uses of Multiple-Valued Logic*. D. Reidel, 1977.

[11] Nuel D Belnap. Display logic. *Journal of philosophical logic*, 11(4):375–417, 1982.

[12] Francesco Berto and Greg Restall. Negation on the australian plan. *Journal of Philosophical Logic*, 48(6):1119–1144, 2019.

[13] Katalin Bimbó. *Combinatory Logic: Pure, Applied, and Typed*. Taylor & Francis, 2012.

[14] Ross T. Brady. Natural deduction systems for some quantified relevant logics. *Logique et Analyse*, 27(108):355–378, 1984.

[15] Ross T. Brady. The Gentzenization and decidability of RW. *Journal of Philosophical Logic*, 19(1):35–73, 1990.

[16] Ross T. Brady. Some concerns regarding ternary-relation semantics and truth-theoretic semantics in general. *IfCoLog Journal of Logics and their Applications*, 4(3):755–781, 2017.

[17] Ethan Brauer. Relevance for the classical logician. *Review of Symbolic Logic*, pages 1–22, forthcoming.

[18] John P. Burgess. Relevance: A fallacy? *Notre Dame Journal of Formal Logic*, 22(2):97–104, 1981.

[19] B. J. Copeland. The trouble anderson and belnap have with relevance. *Philosophical Studies*, 37(4):325–334, 1980.

[20] J. Michael Dunn. Intuitive semantics for first-degree entailments and 'coupled trees'. *Philosophical Studies*, 29(3):149–168, 1976.

[21] Kit Fine. Models for entailment. *Journal of Philosophical Logic*, 3(4):347–372, 1974.

[22] Rohan French. Structural reflexivity and the paradoxes of self-reference. *Ergo, an Open Access Journal of Philosophy*, 3, 2016.

[23] Leon Henkin. The completeness of the first-order functional calculus. *Journal of Symbolic Logic*, 14(3):159–166, 1949.

[24] Alison Duncan Kerr. A plea for KR. *Synthese*, pages 1–25, forthcoming.

[25] Shay Allen Logan. Notes on stratified semantics. *Journal of Philosophical Logic*, 48(4):749–786, 2019.

[26] Edwin D. Mares. 'Four-valued' semantics for the relevant logic R. *Journal of philosophical logic*, 33(3):327–341, 2004.
[27] Edwin David Mares. *Relevant Logic: A Philosophical Interpretation*. Cambridge Univeristy Press, 2004.
[28] Robert K. Meyer and Errol P. Martin. Logic on the Australian plan. *Journal of Philosophical Logic*, 15(3):305–332, 1986.
[29] Graham Priest. *An Introduction to Non-Classical Logic: From If to Is*. Cambridge University Press, 2008.
[30] Graham Priest and Richard Sylvan. Simplified semantics for basic relevant logics. *Journal of Philosophical Logic*, 21(2):217–232, 1992.
[31] Stephen Read. *Relevant Logic: A Philosophical Examination of Inference*. Blackwell, 1988.
[32] Greg Restall. Simplified semantics for relevant logics (and some of their rivals). *Journal of Philosophical Logic*, 22(5):481–511, 1993.
[33] Greg Restall. Four-valued semantics for relevant logics (and some of their rivals). *Journal of Philosophical Logic*, 24(2):139–160, 1995.
[34] Greg Restall and Tony Roy. On permutation in simplified semantics. *Journal of Philosophical Logic*, 38(3):333–341, 2009.
[35] David Ripley. Contraction and closure. *Thought: A Journal of Philosophy*, 4(2):131–138, 2015.
[36] Richard Routley. The American plan completed: Alternative classical-style semantics, without stars, for relevant and paraconsistent logics. *Studia Logica*, 43(1-2):131–158, 1984.
[37] Richard Routley and Robert K. Meyer. The semantics of entailment. Truth, Syntax, and Modality, pages 199–243. North-Holland Publishing Company, Amsterdam, 1972.
[38] Richard Routley and Robert K. Meyer. The semantics of entailment II. *Journal of philosophical logic*, 1(1):53–73, 1972.
[39] Richard Routley and Robert K. Meyer. The semantics of entailment III. *Journal of philosophical logic*, 1(2):192–208, 1972.
[40] Robert AG Seely. Hyperdoctrines, natural deduction and the beck condition. *Mathematical Logic Quarterly*, 29(10):505–542, 1983.
[41] John Slaney. A general logic. *Australasian Journal of Philosophy*, 68(1):74–88, 1990.
[42] Alfred Tarski. *Logic, Semantics, Metamathematics*. Oxford, Clarendon Press, 1956.
[43] Elia Zardini. Truth without contra(di)ction. *Review of Symbolic Logic*, 4(4):498–535, 2011.

Appendix A. Complexity

The most natural way to define complexity for bunches is by saying that the complexity of a bunch is the supremum of the lengths of its branches (thinking of the bunch as a tree). Call this the complexity-as-sup-of-lengths-of-branches definition.

To see that this definition is problematic, first let $\phi_1, \phi_2, \ldots$ be an infinite never-repeating list of formulas. Let $B_1 = \phi_1$ and for $n \geq 1$ let $B_{n+1} := B_n; \phi_{n+1}$. Each B_n is a I-bunch, so we can form the E-bunch S_ω containing all of the B_n's. On the complexity-as-sup-of-lengths definition, the complexity of S_ω would, as expected, be ω. But this raises a problem. Consider the bunch $S_\omega; S_\omega$. Intuitively, $S_\omega; S_\omega$ is a *more* complex bunch than S_ω. Unfortunately, if we accept the complexity-as-sup-of-lengths-of-paths definition, $S_\omega; S_\omega$ and S_ω will *both* have complexity ω. So they are, counter to intuition, equally complex. And this isn't *just* a problem because it violates our intuitive ideas about complexity. It's a problem because we will eventually need to do structural induction on bunches, and doing so requires having on hand some measure of complexity that guarantees both of the following:

(i) The complexity of $\{X_i\}_{i \in I}$ is strictly greater then the complexity of any of the X_i's; and
(ii) The complexity of $X_1; X_2$ is strictly greater than the complexity of X_1 and strictly greater than the complexity of X_2.

Giving a measure of complexity that has both these features isn't hard, but it does require some caution. Here's how I'll do it. Define the ordinal valued complexity function κ by the following recursive clauses:

- If Γ is a sentence, then $\kappa(\Gamma) = 1$.

- If $\Gamma = \Gamma_1; \Gamma_2$, then $\kappa(\Gamma) = \sup(\kappa(\Gamma_1), \kappa(\Gamma_2)) + 1$
- If $\Gamma = \{\Gamma_i\}_{i \in I}$, then $\kappa(\Gamma) = \sup_{i \in I}(\kappa(\Gamma_i) + 1)$.

I take it to be clear that if we define the complexity of Γ to be the ordinal $\kappa(\Gamma)$, then we will have given a definition that satisfies condition (ii). To see that this definition also satisfies condition (i), first note that, $\kappa(\{\Gamma_i\}_{i \in I}) \geq \kappa(\Gamma_i)$ for all $i \in I$. Now suppose $\kappa(\{X_i\}_{i \in I}) = \alpha + 1$ is a successor ordinal. Then, $\alpha + 1 \geq \kappa(X_i) + 1$ for each $i \in I$. Thus $\alpha \geq \kappa(X_i)$ for each $i \in I$. So $\kappa(\{X_i\}_{i \in I}) = \alpha + 1 > \kappa(X_i)$ for each $i \in I$, as required.

On the other hand, suppose $\kappa(\{\Gamma_i\}_{i \in I}) = \lambda$ is a limit ordinal. Clearly we still have that $\lambda \geq \kappa(\Gamma_i) + 1$ for all $i \in I$. Also since $\kappa(\Gamma_i) + 1$ is a successor ordinal, λ and $\kappa(\Gamma_i) + 1$ are distinct. So $\lambda > \kappa(\Gamma_i) + 1$ for all $i \in I$; thus $\lambda \geq \kappa(\Gamma_i)$ for all $i \in I$. But for $\{\Gamma_i\}_{i \in I}$ to be a bunch, each Γ_i must be a I-bunch. And it follows from the definition of κ that the complexity of a I-bunch is always a successor ordinal. So since λ is a limit ordinal, λ and $\kappa(\Gamma_i)$ are always different. It follows that $\kappa(\{\Gamma_i\}_{i \in I}) = \lambda > \kappa(\Gamma_i)$ for all $i \in I$, as required.

So the definition has what features we need it to have for us to be able to do structural induction on bunches. It also lets us point out something interesting. First, note that it's easy to construct, for each ordinal α, a bunch Γ_α with $\kappa(\Gamma_\alpha) = \alpha$. Also note that if $\kappa(\Gamma) \neq \kappa(\Delta)$ then $\Gamma \neq \Delta$. Thus the mapping $\alpha \mapsto \Gamma_\alpha$ is an injection from the ordinals into the bunches. It follows that there are at least as many bunches as there are ordinals. So it looks like the complexity involved in a bunch-based proof theory will *vastly* outstrip the complexity of a set-based proof theory. But this in fact isn't the case. As we proved in Theorem 10, RW and deep fried logic are (in the specified sense) equivalent. In [15], it's been shown that RW is decidable. Thus, as a brief argument that we leave to the reader will show, deep fried logic is also decidable. I leave it to the motivated reader to see if there's anything philosophically interesting to say further in this regard.

Appendix B. Proof of Cut Admissibility

Here we prove that if $X \vdash A$ and $Y(A) \vdash B$, then $Y(X) \vdash B$. The proof is by induction on the length of the proof of $Y(A) \succ B$. If the proof of $Y(A) \succ B$ has length 1, then $Y(A) = A = B$ the result is trivial. Now suppose that whenever there are proofs of $X \succ A$ and $Y'(A) \succ B'$ and the length of the second proof is at least two but no more than k, there is also a proof of $Y'(X) \succ B'$. Let Π be a proof of $Y(A) \succ B$ with length $k + 1$. We consider three of the possibilities for the last rule applied in Π, leaving the other possibilities to the reader.

If $B = \neg B'$ and the last rule applied in Π is $\neg$I, we consider two subcases:

$$\text{Subcase i: } \left(\frac{\Pi}{Y(A) \succ B}\right) = \left(\frac{\dfrac{\Pi_1}{Y_1(A); B' \succ C} \qquad \dfrac{\Pi_2}{Y_2 \succ \neg C}}{Y_1(A); Y_2 \succ \neg B'}\right)$$

$$\text{Subcase ii: } \left(\frac{\Pi}{Y(A) \succ B}\right) = \left(\frac{\dfrac{\Pi_1}{Y_1; B' \succ C} \qquad \dfrac{\Pi_2}{Y_2(A) \succ \neg C}}{Y_1; Y_2(A) \succ \neg B'}\right)$$

Both are straightforward: in the first subcase, we can apply the inductive hypothesis to the lefthand side; in the second we can apply it to the righthand side. Either way, we end up with a proof of $Y(X) \succ B$.

If the last rule applied in Π is $\wedge$E, again we consider two subcases:

$$\text{Subcase i: } \left(\frac{\Pi}{Y(A) \succ B}\right) = \left(\frac{\dfrac{\Pi_1}{Y_1(A) \succ C_1 \wedge C_2} \qquad \dfrac{\Pi_2}{Y_2(C_1, C_2) \succ B}}{Y_2(Y_1(A)) \succ B}\right)$$

Subcase ii: $\left(\dfrac{\Pi}{Y(A) \succ B}\right) = \left(\dfrac{\dfrac{\Pi_1}{Y_1 \succ C_1 \wedge C_2} \qquad \dfrac{\Pi_2}{(Y_2(C_1, C_2))(A) \succ B}}{(Y_2(Y_1))(A) \succ B}\right)$

The division of the two subcases concerns whether the highlighted occurrence of A in $Y(A)$ is in Y_1 or in Y_2. In the first subcase, it's in Y_1 and the inductive hypothesis clearly applies, after which application of $\wedge$E again gives the desired result. In the second subcase, it's in Y_2. Since A still occurs in $Y(A) = Y_2(Y_1(A))$, A must not be C_1 or C_2. In $Y_2(Y_1)(A)$, the highlighted occurrence of the bunch C_1, C_2 has been removed and Y_1 has been inserted in its place. The occurrence of A remains unchanged. With the notation now explained, it's clear that the inductive hypothesis applies and that applying $\wedge$E will again give the desired result.

Finally, if the last rule applied in Π is $\mathrel{\text{✄}}$K, we again consider two subcases:

Subcase i: $\left(\dfrac{\Pi}{Y(A) \succ B}\right) = \left(\dfrac{\dfrac{\Pi_1}{Y'(A) \succ B} \qquad Y'(A) \mathrel{\text{✄}} Y(A)}{Y(A) \succ B}\right)$

Subcase ii: $\left(\dfrac{\Pi}{Y(A) \succ B}\right) = \left(\dfrac{\dfrac{\Pi_1}{Y' \succ B} \qquad Y' \mathrel{\text{✄}} Y(A)}{Y(A) \succ B}\right)$

In the first case, it's clear both that the inductive hypothesis applies and that $Y'(X) \mathrel{\text{✄}} Y(X)$. Applying $\mathrel{\text{✄}}$K then gives the desired result.

In the second subcase, we don't even need the inductive hypothesis: if $Y' \mathrel{\text{✄}} Y(A)$, then $Y' \mathrel{\text{✄}} Y(X)$ as well. So we can simply use a different instance of $\mathrel{\text{✄}}$K to get the desired result.

Appendix C. Proof of the Lindenbaum Lemma

The proof is entirely standard, but it's worth reminding ourselves how it goes: first, we enumerate all the disjunctions in our language. We then go through them one by one. For each disjunction our formal theory proves, we add to our formal theory whichever disjunct is safe to add to it. Since this may mean that we're adding new disjunctions to our formal theory, we will have to go through again. And again. But each time, we add less complex disjunctions. So after doing this at most countably infinitely many times, the process terminates.

Explicitly, let $A_i \vee B_i$ be the ith disjunction. Define sets of sentences t^i_j as follows:

- $t^0_0 = t$
- If $t^i_j \nvdash A_j \vee B_j$, then $t^i_{j+1} = t^i_j$
- If $t^i_j \vdash A_j \vee B_j$, then

$$t^i_{j+1} = \begin{cases} t^i_j \cup \{A_j\} & \text{if } cb(t^i_j \cup \{A_j\}) \cap \Delta = \emptyset \\ t^i_j \cup \{B_j\} & \text{otherwise} \end{cases}$$

- $t^{i+1}_0 = \displaystyle\bigcup_{j=0}^{\infty} t^i_j$

We then define $t' = \displaystyle\bigcup_{i=0}^{\infty} t^i_0$. It's clear that $t \subseteq t'$, and seeing that t' is a prime formal theory isn't hard. What remains is to show that $t' \cap \Delta = \emptyset$.

The proof is by contradiction. Suppose $t' \cap \Delta \neq \emptyset$. Let $i_0 = \inf\{i : \text{for some } i, cb(t^i_j) \cap \Delta \neq \emptyset\}$ and $j_0 = \inf\{j : cb(t^{i_0}_j) \cap \Delta \neq \emptyset\}$. By construction, $j_0 \neq 0$ and by minimality, $t^{i_0}_{j_0-1} \neq t^{i_0}_{j_0}$.

So $t^{i_0}_{j_0} \vdash A_{j_0-1} \vee B_{j_0-1}$. Clearly both $cb(t^{i_0}_{j_0-1} \cup \{A_{j_0-1}\})$ and $cb(t^{i_0}_{j_0-1} \cup \{B_{j_0-1}\})$ intersect Δ, since if they did not, then by construction $t^{i_0}_{j_0}$ also wouldn't, contrary to our assumptions. Thus there are D_1 and D_2 in Δ so that $t^{i_0}_{j_0}, A_{j_0-1} \vdash D_1$ and $t^{i_0}_{j_0}, B_{j_0-1} \vdash D_2$. Thus, let Θ_1 be a proof of $t^{i_0}_{j_0}, A_{j_0-1} \succ D_1$ and let Θ_2 be a proof of $t^{i_0}_{j_0}, B_{j_0-1} \succ D_2$. Then the following is a proof of $t^{i_0}_{j_0-1} \succ D_1 \vee D_2$:

$$\dfrac{t^{i_0}_{j_0-1} \succ A_{i_0-1} \vee B_{i_0-1} \qquad \dfrac{\dfrac{\Theta_1}{t^{i_0}_{j_0-1}, A_{i_0-1} \succ D_1}}{t^{i_0}_{j_0-1}, A_{i_0-1} \succ D_1 \vee D_2} \qquad \dfrac{\dfrac{\Theta_2}{t^{i_0}_{j_0-1}, B_{i_0-1} \succ D_2}}{t^{i_0}_{j_0-1}, B_{i_0-1} \succ D_1 \vee D_2}}{t^{i_0}_{j_0-1} \succ D_1 \vee D_2}$$

Thus $t^{i_0}_{j_0-1} \vdash D_1 \vee D_2$, so $D_1 \vee D_2 \in cb(t^{i_0}_{j_0-1})$. But since Δ is closed under disjunctions, this means that $cb(t^{i_0}_{j_0-1}) \cap \Delta \neq \emptyset$, contradicting the minimality of i_0 and j_0.