Contents

Abstract

J. Schauder introduced the notion of basis in a Banach space in 1927. If a Banach space has a basis then it is also separable. The problem whether every separable Banach space has a Schauder basis appeared for the first time in 1931 in Banach's book "Theory of Linear Operations". If a Banach space X has a Schauder basis it also has the approximation property. A Banach space X has the approximation property if for every Banach space Y the finite rank operators are dense in the closed subspace of all compact operators from Y to X. Both problems where solved in the negative in 1972 by P. Enflo. His proof was almost immediately simplified by A. M. Davie, who using a probabilistic lemma constructed a separable closed subspace of ℓ_∞ without the approximation property. In this thesis we present some of the equivalent properties to the approximation property due to A. Grothendieck, and we make a detailed exposition of the proof by A. M. Davie.

Summary in English

The notion of a basis in a Banach space was introduced by J. Schauder in 1927. If a Banach space has a basis then it is also separable. The problem whether every separable Banach space has a Schauder basis appeared for the first time in 1931 in the Polish edition of Banach's book "Theory of Linear Operations".

It was clear to Banach, Mazur and Schauder that this question was related to the "approximation problem". This is mentioned in Banach's book as a remark to the chapter on compact operators. If a Banach space X has a Schauder basis it also has the approximation property, since the natural projections of X onto its finite dimensional subspaces form a bounded sequence of finite rank operators converging pointwise on X to the identity operator.

The approximation problem is equivalent to whether a Banach space has the "approximation property". A Banach space X has the approximation property if the identity operator on X is the limit for the topology of uniform convergence on compact subsets of X of a sequence of finite rank operators. The approximation property in its various forms was thoroughly analyzed by A. Grothendieck in his thesis "Produits tensoriels topologiques et espaces nucléaires" published in 1955.

But it was not until 1972, that P. Enflo solved both questions in the negative. He found a subspace of c_o, which he showed does not have the approximation property and consequently does not have a basis. Almost immediately, T. Figiel and A. M. Davie greatly simplified his proof. The later using a probabilistic lemma constructed a separable closed subspace of l_∞ without the approximation property. Moreover, they also showed that both c_o and every l_p space with $1 \leq p < \infty$, $(p \neq 2)$ has subspaces without the approximation property. In this work we present some of the equivalent properties to the approximation property due to A. Grothendieck, and we make a detailed exposition of the proof by A. M. Davie.

Sammanfattning på svenska

Begreppet bas för ett oändligtdimensionellt Banachrum introducerades för första gången av J. Schauder år 1927. Om ett Banachrum har en Schauderbas medför detta att det också är separabelt. Frågan om den omvända implikationen också gäller i alla Banachrum, vilken senare blev känd som "basproblemet" presenterades för första gången i Banachs bok 1931. Banach, Mazur och Schauder hade klart för sig att denna fråga hade ett starkt samband med "approximationsegenskapen". Ett Banach rum X har approximationsegenskapen om identitetsoperatorn på X är gränsvärdet, i topologin för likformig konvergens på kompakta delmängder av X, av en följd av operatorer av ändlig rang.

Approximationsegenskapen utforskades grundligt på femtiotalet av A. Grothendieck, som fann många ekvivalenta egenskaper till den och många konsekvenser av den. Om ett Banachrum X har en Schauderbas har den också approximationsegenskapen (man vet idag att den omvända implikation inte är sann), eftersom projektionerna av X på dess ändligtdimensionella underrum konvergerar punktvis mot identitetsoperatorn på X.

Det faktum att approximationsproblemet inte kunde lösas ledde till dess berömmelse inom Banachrumsteori och det var inte förrän 1972 som P. Enflo visade att båda frågor hade negativa svar. Han konstruerade ett underrum till c_o som inte har en bas och som inte har approximationsegenskapen. Nästan omedelbart därefter T. Figiel och A. M. Davie konstruerade ett separabelt slutet underrum till ℓ_∞ som inte har approximationsegenskapen. A.M. Davie lyckades genom användning av ett lemma från sannolikhetsläran förenkla Per Enflos bevis och bevisade dessutom att c_o och alla ℓ_p rum för $1 \leq p < \infty$, $(p \neq 2)$, har underrum som saknar approximationsegenskapen.

I detta examensarbete presenterar vi några av Grothendiecks resultat och en detaljerade presentation av A. M. Davies bevis.

Preface

My original interest for Banach space theory arose when I first took a course in Functional Analysis at the University of Lund. One of the subjects studied in the course was the relation between compact and finite rank operators in infinite-dimensional complete normed spaces. An important property for a normed space is if every compact operator on that space can be uniformly approximated by finite rank operators. That property is called the approximation property. All Hilbert spaces have the approximation property. In a Hilbert space an operator is compact if and only if it can be approximated by finite rank operators. When studying the proof for this theorem in Hilbert spaces I wondered why the same proof would not work on the more general case of Banach spaces. Later I found out that the proof was depending on the fact that for a Hilbert space we can always find a basis. There are Banach spaces though without bases and this is the reason why the proof would work only in one direction for Banach spaces, namely that the limit of a sequence of finite rank operators is compact, but not the other way around.

Trying to find out how a Banach space without the approximation property would "look like" I noticed that almost every book on Functional Analysis contained a reference to Per Enflo's [E] work "A counterexample to the approximation property in Banach spaces", originally published in Acta Mathematica in 1973. This was a very important result since it showed that the set of Banach spaces without the approximation property and consequently without a Schauder basis is not void. Even if the work was mentioned in so many references nowhere was there a hint of what is it that makes certain Banach spaces so to say "loose" the approximation property.

Naturally when time came for me to write my Masters thesis the first idea was to write about the approximation property in Banach spaces. Later I found out that the original proof by Enflo had been greatly simplified by A.M. Davie [D] in his paper "The approximation problem for Banach spaces" published also 1973 in the Bulletin of London Math. Soc. I thought that it would be of interest to present it with the special aim of trying to give the proof in a more easily understandable way. The proof presented in this work is due to Davie, but in contrast with Davie's proof, which is only three pages and written for experts in Banach space theory, here we try to give the proof with all details explaining the main ideas and in this way trying to make it accessible for a broader audience.

Chapter 1

Introduction

Banach space theory became a recognized branch of mathematics with the appearance in 1931 of Banach's book "Theory of Linear Operations". From its beginning it maintained close ties with other branches of analysis. It turned out that Banach space theory offered powerful tools to other branches of analysis. Amongst the most useful results we mention the duality theory for operators and spaces, infinite dimensional convexity and results connected to Baire's category theorem, especially the closed graph theorem.

These powerful general concepts are now well understood and appear in several books on functional analysis. They were already well understood in the forties and fifties, and at that time it seemed to many that Banach space theory was a relegated branch of mathematics without any new developments on the field. However this was not the case. The sixties, and specially the seventies and eighties saw an upsurge in the research of Banach space theory. Many old problems were solved and many new problems, as well as new ties with other branches of mathematics emerged. New and more powerful methods and directions of research appeared and one could say that these developments brought a new depth to Banach space theory.

The seventies brought something which is sometimes called the "era of counterexamples". This refers to a time when negative solutions to some older problems were found. Amongst these various problems settled in the seventies, two of the classical and most famous ones were the "basis problem " and the "approximation problem". We will try to give a brief outline of their content.

The notion of a countable infinite basis for Banach spaces was introduced by J. Schauder in 1927 and thus got its name (Schauder basis). Since a space with a Schauder basis is separable it was very natural to ask if the same is true for the converse: "Does every separable Banach space possess a Schauder basis?". This question was already raised in Banach's book (see [B], page 68) in a slightly different form: "It is not known if every separable Banach space has a basis." The answer to this question was not obvious and it gave thus rise to one of the first famous problems in Banach space theory: the "basis problem".

There are many instances in operator theory where it is convenient to represent a given linear operator as a limit of a sequence of operators with already known properties. The best investigated classes of operators are finite rank operators and compact operators. Therefore it is quite natural to ask whether every continuous linear operator can be approximated by linear operators from these classes. A Banach space has the classical "approximation property" if for every Banach space Y the set of finite rank linear operators from Y to X is dense in the space of compact linear operators from Y to X.

Banach, Mazur and Schauder had already observed that the approximation problem is closely related to the problem of existence of a basis. As we will show later (Theorem 3.15), if a Banach space X has a Schauder basis it also has the approximation property, since any compact operator can be approximated by the projections of X on the finite dimensional subspaces spanned by finitely many basis elements. Since every Banach space with a Schauder basis is separable it was also natural to ask if every separable Banach space has the approximation property? This question was also originally raised by Banach in his book (see [B], page 146). Thus Banach's comment gave rise to another famous problem in Banach space theory later called the "approximation problem". The classical approximation problem is the question whether all Banach spaces have the approximation property.

The first thoroughly detailed study of the approximation property and existence of Schauder bases was initiated by A. Grothendieck, who published in 1955 his famous work "Produits tensoriels topologiques et espaces nucléaires" [G]. In this work, which considers the more general framework of locally convex spaces, Grothendieck explained the fundamental role of the approximation problem in the structure theory of Banach spaces, and presented several problems to which the approximation property is relevant (for instance, in determining the trace of a nuclear operator).[1]Even if Grothendieck did not succeed in constructing a Banach space without a Schauder basis and without the approximation property, many of the properties and equivalent conditions to the approximation property, which contributed to its solution, are due to him.

The approximation problem was solved (in the negative) in 1972 by P. Enflo [E], who constructed the first example of a Banach space which does not have the approximation property (and consequently does not have a Schauder basis). Enflo's construction was quite complicated and it was almost immediately simplified by others. The proof we will present in this work is due to A.M. Davie [D], who, while still using the original ideas by Enflo, simplified the construction.

It is interesting to observe that subsequently in 1978 Szarek [Sza], constructed a Banach space which has the approximation property but which does not have a Schauder basis. So the two properties are not equivalent.

[1]Let X, Y be Banach spaces and $T \in \mathcal{B}(X;Y)$. If there exists a sequence $\{f_n\} \subset X^*$, a sequence $\{y_n\} \subset Y$ and a sequence $\{c_n\}$ of real numbers such that

$$\sup_n |f_n\| < \infty, \qquad \sup_n \|y_n\| < \infty, \qquad \sum_n |c_n| < \infty, \qquad \text{and}$$

$$Tx = \lim_{m\to\infty} \sum_{n=1}^{m} c_n f_n(x) y_n \text{ in } Y \text{ for every } x \in X,$$

then T is a nuclear operator on X into Y.

Chapter 2

Compact and finite rank operators

In this chapter we will introduce a few important notions needed for the rest of our work in order to study the approximation property. All these different concepts are presented in many Functional Analysis books, but we found that it would make our work more self-contained and complete by including some of these results. The first section will give a characterization of compact sets in infinite dimensional spaces and some basic properties of compact operators. The second section explains some basic facts about the topologies generated by seminorms and locally convex spaces, specifically the topology of uniform convergence on compact sets (the latter is essential for the construction of a Banach space without the approximation property). Finally we will present a historically famous result due to Grothendieck, which gives a concrete representation of the dual of some space of operators. Some concepts needed for the following chapters are also introduced.

Notation: For Banach spaces X and Y, we will use the notation $\mathcal{B}(X;Y)$ for the set of all bounded linear operators from X to Y, with the usual abbreviation $\mathcal{B}(X) = \mathcal{B}(X;X)$. We denote X^* the dual space of the Banach space X. Furthermore, in a normed space we denote by $B(x,\varepsilon)$ the closed ball of radius $\varepsilon > 0$, centered at $x \in X$.

2.1 Compact sets and compact operators

DEFINITION 2.1 (Totally bounded set) *Let X be a normed space. A subset $K \subset X$ is totally bounded if for every $\varepsilon > 0$, there exist points $x_1, x_2, \dots, x_n$ of K such that every point x of K has distance less than ε from at least one of $x_1, x_2, \dots, x_n$ of K.*

In other words K is totally bounded if, for every $\varepsilon > 0$, K can be covered by finitely many balls of radius ε and centers in K.

THEOREM 2.2 *A subset K of a Banach space X is relatively compact if and only if it is totally bounded.*

Proof:
Suppose that K is relatively compact (that is, its closure is compact, or, equivalently, every sequence in K has a convergent subsequence) and $\varepsilon > 0$ be given. We shall show that K has a finite number of points $x_1, \ldots, x_n$ such that $K \subset \bigcup_{j=1}^{n} B(x_j, \varepsilon)$. Let x_1 be any point in K. If K is contained in the set $\|x - x_1\| < \varepsilon$, we are done. Otherwise let $x_2 \in K$ be such that $\|x_2 - x_1\| \geqslant \varepsilon$. If every x in K satisfies

$$\|x - x_1\| < \varepsilon \text{ or } \|x - x_2\| < \varepsilon \tag{2.1}$$

we are again done. Otherwise, let x_3 be a point of K not satisfying (2.1). Inductively, if $x_1, \ldots, x_n$ are chosen and all points of K satisfies

$$\|x - x_1\| \leqslant \varepsilon, \cdots, \|x - x_n\| \leqslant \varepsilon \tag{2.2}$$

then K is totally bounded. If not we will have to eventually stop after a finite number of steps. Otherwise $\{x_j\}$ would be a sequence of elements of K satisfying

$$\|x_j - x_k\| \geqslant \varepsilon \text{ for } j \neq k.$$

This sequence would have no convergent subsequence, contradicting the fact that K is relatively compact.

Conversely, assume now that K is totally bounded. Let $\{x_n\}$ be a sequence of points of K. Since K is totally bounded it is covered by a finite numbers of balls of radius 1. At least one of these balls contains an infinite number of elements of $\{x_n\}$. Let B_1 be such a ball, take out all elements of $\{x_n\}$ not contained in B_1 and denote the remaining sequence by $\{x_{1,n}\}$. Similarly there is a finite number of balls of radius 1/2 which cover K. At least one of these contains infinitely many points of $\{x_{1,n}\}$. Choose one ball and denote it by B_2. Take out all the points not in B_2 and denote the remaining sequence by $\{x_{2,n}\} \subset B_2$. Continue inductively by covering K with finitely many balls of radius $1/2^n$ and set $y_n = x_{nn}$. Then for each $\varepsilon > 0$ there is an integer N large enough that all the y_j's are in a ball of radius smaller than ε for $j > N$. Thus $\{y_n\}$ is a Cauchy sequence in the Banach space X and as such must have a limit in X. □

DEFINITION 2.3 (Finite rank operators) *Let X be a Banach space. An operator $T \in \mathcal{B}(X)$ is said to be of finite rank if there are $\{x_j\}_{j=1}^N \subset X$ such that for any $x \in X$*

$$T(x) = \sum_{j=1}^{N} \phi_j(x) x_j$$

for some $\{\phi_j\}_{j=1}^N \subset X^*$. We will denote the space of all finite rank operators $T : X \to Y$, by $\mathcal{F}(X;Y)$, with $\mathcal{F}(X;X) = \mathcal{F}(X)$.

DEFINITION 2.4 (Compact operators) *Suppose X and Y are Banach spaces and B is the closed unit ball in X, $B = \{x \in X : \|x\| \leqslant 1\}$. A linear operator $T : X \to Y$ is said to be compact if the set $T(B)$ is relatively compact in Y. We will denote the set of compact linear operators by $\mathcal{K}(X;Y)$, with the convention that $\mathcal{K}(X;X) = \mathcal{K}(X)$*

Another equivalent way to define compact operators is to say that T is compact if for any bounded sequence $\{x_n\}_{n=1}^{\infty}$ in X, the sequence $\{Tx_n\}_{n=1}^{\infty}$ in Y contains a subsequence that converges to a point of Y.

THEOREM 2.5 *Let X and Y be normed spaces, and let $T \in \mathcal{K}(X;Y)$ Then T is bounded. Thus, $\mathcal{K}(X;Y) \subset \mathcal{B}(X;Y)$.*

Proof:
Since Y is a complete metric space, a subset K of Y is compact if its closure is totally bounded, which means that for every $\varepsilon > 0$, K lies within the union of finitely many open balls of radius ε. Let $B = \{x \in X : \|x\| \leqslant 1\}$. Then, if $T \in \mathcal{B}(X;Y)$ is compact, $T(B)$ is totally bounded, which automatically implies that $\|T\| < \infty$. □

THEOREM 2.6 *If S, $T \in \mathcal{K}(X;Y)$ and a, $b \in \mathbb{C}$, then $aS + bT$ is compact or in general a linear combination of compact operators is compact. Thus $\mathcal{K}(X;Y)$ is a linear subspace of $\mathcal{B}(X;Y)$.*

Proof:
Let $\{x_n\}_{n=1}^{\infty}$ be bounded in X, then since S is compact $\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\{Sx_{n_k}\}$ converges in Y. But then $\{x_{n_k}\}$ itself being bounded also has a convergent subsequence $\{x_{n_{k(i)}}\}_{i=1}^{\infty}$ such that $\{Tx_{n_{k(i)}}\}_{i=1}^{\infty}$ also converges, thus $aS + bT$ is also compact. □

THEOREM 2.7 *Let X and Y be normed spaces and $T \in \mathcal{B}(X;Y)$. Then*
(a) If T is of finite rank then T is a compact operator.
(b) If $\dim X$ or $\dim Y$ is finite then T is a compact operator.

Proof:
(a) Since T is of finite rank, the space $Z = \text{Im}(T)$ is a finite-dimensional normed space. Furthermore for any bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\}$ is bounded in Z. By the Bolzano-Weierstrass theorem this sequence must contain a convergent subsequence. Hence T is compact.
(b) If $\dim X$ is finite then $\text{rank}(T) \leqslant \dim X$, so $\text{rank}(T)$ is finite, while if $\dim Y$ is finite then clearly the dimension of $\text{Im}(T) \subset Y$ must be finite. Thus in both cases it follows from (a) that T is compact. □

We will see later in Theorem 3.19 that in a Hilbert space an operator is compact if and only if it is the limit of a sequence of finite rank operators, or in other words that $\overline{\mathcal{F}(X)} = \mathcal{K}(X)$. On Banach spaces only the if part is valid (see Theorem 2.8). Therefore there are compact operators in Banach spaces which cannot be approximated in the norm topology by finite rank operators. In this work we will explicitly construct such a Banach space.

THEOREM 2.8 *Suppose that X, Y are Banach spaces and that the sequence $\{T_n\} \subset \mathcal{K}(X;Y)$ converges to some $T \in \mathcal{B}(X;Y)$. Then T is compact. Thus $\mathcal{K}(X;Y)$ is a closed subspace of $\mathcal{B}(X;Y)$.*

Proof:
Suppose that B is the closed unit ball in X. Then B is a bounded set. Given

an $\varepsilon > 0$ let n be such that $\|T_n x - Tx\| \leqslant \varepsilon/3$ whenever $x \in B$. Since T is a compact operator $T_n(B)$ is totally bounded; thus there is a finite subset F of B such that every element of $T_n(B)$ is within distance $\varepsilon/3$ of a member of $T(F)$. It follows from a straightforward application of the triangle inequality that every member of $T(B)$ lies with distance ε of a member of $T(F)$, which implies that $T(B)$ is totally bounded and therefore T is compact. □

DEFINITION 2.9 *The convex hull of a set A in a vector space X, denoted by* conv(A)*, is the set of all convex linear combinations of members of A, that is, the set of all sums of the form*

$$\sum_{j=1}^{n} t_j x_j \qquad n \in \mathbb{N}, \qquad x_j \in A, \qquad t_j \geqslant 0, \qquad \sum_{j=1}^{n} t_j = 1.$$

Before Theorem 2.11 due to Grothendieck, which gives a characterization of compact sets in infinite-dimensional spaces, we present Mazur's compactness theorem.

THEOREM 2.10 *The closed convex hull of a compact subset K of a Banach space X is itself compact.*

Proof:
By Theorem 2.2 it is sufficient to prove that $\overline{\mathrm{conv}}(K)$ is totally bounded. Let $\varepsilon > 0$ and choose $x_1, \ldots, x_n$ in K such that

$$K \subseteq \bigcup_{j=1}^{n} B\big(x_j, \frac{\varepsilon}{4}\big) \tag{2.3}$$

Define the compact convex set

$$C = \mathrm{conv}\{x_1, \ldots, x_n\}. \tag{2.4}$$

To prove that C is compact, let $A \subseteq \mathbb{R}^n$

$$A = \Big\{(\alpha_1, \ldots, \alpha_n), \alpha_j \geqslant 0, \sum_{j=1}^{n} \alpha_j = 1\Big\}.$$

Define $f : A \to X$ by $f(\alpha_1, \ldots, \alpha_n) = \sum_{j=1}^{n} \alpha_j x_j$. Since f is continuous and A is a compact set in $\mathbb{R}^n$, its image C is also compact. Hence there are elements $y_1, \ldots, y_m$ in C such that

$$C \subseteq \bigcup_{i=1}^{m} B\big(y_i, \frac{\varepsilon}{4}\big).$$

If $w \in \overline{\mathrm{conv}}(K)$, there is a $z \in \mathrm{conv}(K)$ with $\|w - z\| < \varepsilon/4$. Thus

$$z = \sum_{n=1}^{p} \alpha_n k_n, \text{ where } k_n \in K, \alpha_n \geqslant 0, \text{ and } \sum \alpha_n = 1.$$

By (2.3), for each k_n there is a $x_{j(n)}$ with $\|k_n - x_{j(n)}\| < \varepsilon/4$. Therefore

$$\Big\|z - \sum_{n=1}^{p} \alpha_n x_{j(n)}\Big\| = \Big\|\sum_{n=1}^{p} \alpha_n (k_n - x_{j(n)})\Big\| \leqslant \sum_{n=1}^{p} \alpha_n \|k_n - x_{j(n)}\| < \frac{\varepsilon}{4}.$$

But $\sum_{n=1}^{p} \alpha_n x_{j(n)} \in C$ so there is some y_i with $\| \sum_{n=1}^{p} \alpha_n x_{j(n)} - y_i \| < \varepsilon/4$. The triangle inequality now shows that $\overline{\mathrm{conv}}(K) \subseteq \bigcup_{i=1}^{m} B(y_i, \varepsilon)$ and so $\overline{\mathrm{conv}}(K)$ is totally bounded. $\square$

THEOREM 2.11 *A closed subset K of a Banach space X is compact if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\|x_n\| \to 0$ and $K \subset \overline{\mathrm{conv}}\{x_n\}_{n=1}^{\infty}$.*

Proof:
By Mazur's compactness theorem, we see that if $\|x_n\| \to 0$ then $\overline{\mathrm{conv}}\{x_n\}_{n=1}^{\infty}$ is compact since $\{0\} \cup \{x_n\}_{n=1}^{\infty}$ is a compact set. Therefore a closed subset $K \subset \overline{\mathrm{conv}}\{x_n\}_{n=1}^{\infty}$ is compact.
To prove the other implication we let K be a nonempty compact subset of X. Since $2K = \{2x : x \in K\}$ is also compact and therefore totally bounded, there is a finite set of elements $\{x_j\}_{j=1}^{n_1}$ in $2K$ such that $2K \subset \bigcup_{j=1}^{n_1} B\big(x_j, \frac{1}{2}\big)$. Let K_1 be the union of translations of the sets $B(x_j, 1/2) \cap 2K$ for all $j = 1, \ldots, n_1$,

$$K_1 = \bigcup_{j=1}^{n_1} \Big\{ \big(B\big(x_j, \frac{1}{2}\big) \cap 2K\big) - x_j \Big\}.$$

Then K_1 is a compact subset of $B(0, 1/2)$. Pick next $\{x_j\}_{j=n_1+1}^{n_2}$ in $2K_1$ such that $2K_1 \subset \bigcup_{j=n_1+1}^{n_2} B\Big(x_j, \frac{1}{2^2}\Big)$ and define

$$K_2 = \bigcup_{j=n_1+1}^{n_2} \Big\{ \big(B\big(x_j, \frac{1}{4}\big) \cap 2K_1\big) - x_j \Big\}.$$

Then $K_2 \subset B(0, 1/4)$ and K_2 is compact. Thus there are $\{x_j\}_{j=n_2+1}^{n_3}$ of $2K_2$ such that $2K_2 \subseteq \bigcup_{j=n_2+1}^{n_3} B\Big(x_j, \frac{1}{2^3}\Big)$. Let

$$K_3 = \bigcup_{j=n_2+1}^{n_3} \Big\{ \big(B\big(x_j, \frac{1}{2^3}\big) \cap 2K_1\big) - x_j \Big\}.$$

The construction of the $\{x_j\}$ is continued in the obvious fashion. Notice that $x_n \to 0$ as $n \to \infty$.
Suppose now that $x_0 \in K$. Then there is a j_1 with $1 \leqslant j_1 \leqslant n_1$ such that $2x_0 - x_{j_1} \in K_1$; hence there is a j_2 with $n_1 \leqslant j_2 \leqslant n_2$ so that $4x_0 - 2x_{j_1} - x_{j_2} \in K_2$ and, in general,

$$x_0 - \Big(\frac{x_{j_1}}{2} + \frac{x_{j_2}}{2^2} + \cdots + \frac{x_{j_m}}{2^m} \Big) \in \frac{1}{2^m} K_m \subset B\Big(0, \frac{1}{4^m}\Big).$$

It follows that $x_0 \in \overline{\mathrm{conv}}\{x_{j_n}; n = 1, 2, \ldots\}$. Since

$$\|x_{j_n}\| \to 0 \text{ as } n \to \infty$$

the theorem is proved. □

Thus, every compact subset of a Banach space is small in the sense that it is included in the closed convex hull of a null sequence.

2.2 Seminorms

To introduce a topology in an infinite-dimensional linear space, it is sometimes necessary to make use of a system of an infinite number of seminorms. Locally convex spaces in particular can be defined through a system of seminorms satisfying the axiom of separation (see Theorem 2.13). If the system reduces to one seminorm the corresponding space is called a normed space. We shall begin with the definition of a seminorm.

DEFINITION 2.12 (Seminorm) *A real-valued nonnegative function $p(x)$ defined on a linear space X is called a seminorm on X, if the following conditions*

$$p(x+y) \leqslant p(x) + p(y) \qquad p(\alpha x) = |\alpha| p(x)$$

are satisfied for all scalars α and all $x, y \in X$.

In order to give a better idea of how a seminorm induces a topology on a linear space we present a few results which will be useful in our exposition. For the complete proofs we refer to [Y], page 24.

THEOREM 2.13 *Let a family $\{p_\gamma(x) : \gamma \in \Gamma\}$ of seminorms of a linear space X satisfy the axiom of separation:*
For any $x_0 \neq 0$, there exists $p_{\gamma_0}(x)$ in the family such that $p_{\gamma_0}(x_0) \neq 0$.
Take any finite system of seminorms of the family, say $p_{\gamma_1}(x), p_{\gamma_2}(x), \dots, p_{\gamma_n}(x)$ and any system of positive numbers $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$, and set

$$U = \{x \in X : p_{\gamma_j}(x) \leqslant \varepsilon_j \qquad (j = 1, 2, \dots, n)\}.$$

Then U is a convex, balanced (that is $x \in U$ and $|\alpha| \leqslant 1$ imply $\alpha x \in U$), and absorbing (that is for any $x \in X$ there exist $\alpha > 0$ such that $\alpha^{-1} x \in U$) subset of X. Consider such a set U as a neighborhood of the vector 0 of X and define a neighborhood of any vector x_0 as a set of the form

$$x_0 + U = \{y \in X : y = x_0 + u, u \in U\}.$$

Then the open sets of X in the topology generated by this family of seminorms is the totality $\{G\}$ of such subsets $G \subset X$ which contain a neighborhood of each of its points.

DEFINITION 2.14 (The topology of uniform convergence on compact sets) *If X and Y are Banach spaces, we let τ denote the topology on $\mathcal{B}(X;Y)$ of uniform convergence on compact sets in X. The topology τ is the (locally convex) topology on $\mathcal{B}(X;Y)$ generated by the seminorms of the form $p_K(T) = \|T\|_K = \sup\{\|Tx\| : x \in K\}$, where K ranges over the compact subsets of X. We write $(\mathcal{B}(X;Y), \tau)$.*

The topology generated by these seminorms is weaker than the one generated by the operator norm. We will give a precise characterization of this topology in Theorem 4.1 below. Recall that given a vector space X over $\mathbb{C}$ and a topology τ on X the pair (X, τ) is called a topological vector space if these two axioms are satisfied:

$$(x, y) \mapsto x + y \text{ is continuous on } X \times X \text{ into } X$$

$$(\lambda, x) \mapsto \lambda x \text{ is continuous on } \mathbb{C} \times X \text{ into } X.$$

DEFINITION 2.15 (Local basis) *A local basis of a topological vector space (X, τ) is a collection $\mathcal{B}$ of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathcal{B}$.*

DEFINITION 2.16 (Locally convex space) *A topological vector space (X, τ) is said to be locally convex if there is a basis $\mathcal{B}$ whose members are convex.*

THEOREM 2.17 *The topological vector space $(\mathcal{B}(X), \tau)$, of bounded linear operators on the Banach space X, where τ is the topology uniform convergence on compact subsets of X, is a locally convex space.*

Proof:
Let $T \in \mathcal{B}(X)$, then $\|T\|_K = \sup_{x \in K} \|Tx\|$, where K is a compact subset of X, and let $B_r^K(0) = \{T \in \mathcal{B}(X) : \sup_{x \in K} \|Tx\| < r\}$. Let $T, S \in B_r^K(0)$ and let $0 \leqslant t \leqslant 1$. Then since $\|T\|_K$ is a seminorm on $\mathcal{B}(X)$, we can use the triangle inequality to obtain

$$\sup_{x \in K} \|(t \cdot T + (1-t)S)x\| \leqslant t \sup_{x \in K} \|Tx\| + (1-t) \sup_{x \in K} \|Sx\| \leqslant r$$

Thus each of the sets $B_r^K(0)$ is convex and the collection

$$\mathcal{B} = \{B_{1/n}^K(0), n \in \mathbb{N}, K \in X\},$$

where K ranges over all the compact subsets of X is a local basis at 0, whose members are convex. $\square$

THEOREM 2.18 *Let X, Y be locally convex spaces and $\{p\}$, $\{q\}$ be the system of seminorms respectively defining the topologies of X and Y. Then a linear operator $T : X \to Y$ is continuous if and only if , for every seminorm $q \in \{q\}$, there exist a seminorm $p \in \{p\}$ and a positive number c such that:*

$$q(Tx) \leqslant cp(x) \qquad \text{for all } x \in X.$$

Specifically when $Y = \mathbb{C}$ we get as an immediate consequence of Theorem 2.18 the following result.

THEOREM 2.19 *Let X be a locally convex space, and f a linear functional on X. Then f is continuous if and only if there exists a seminorm p from the system $\{p\}$ of seminorms defining the topology of X and a number c such that*

$$|f(x)| \leqslant cp(x) \qquad \text{for all } x \in X.$$

For the proof of Theorem 2.18 see [Y], page 42.

2.3 The dual space of $(\mathcal{B}(X;Y),\tau)$

The next theorem due to Grothendieck identifies the dual space of $(\mathcal{B}(X;Y),\tau)$, where τ is the topology of uniform convergence on compact subsets of X defined above.

THEOREM 2.20 *Let X and Y be Banach spaces. Let τ be the topology on $\mathcal{B}(X;Y)$ of uniform convergence on compact subsets of X. Then every continuous linear functional $\Phi : \mathcal{B}(X;Y) \to \mathbb{C}$ on $(B(X;Y),\tau)$ can be represented as*

$$\Phi(T) = \sum_{j=1}^{\infty} y_j^*(Tx_j) \qquad \text{with} \qquad \sum_{j=1}^{\infty} \|x_j\|\|y_j^*\| < \infty, \tag{2.5}$$

and $\{x_j\}_{j=1}^{\infty} \subset X, \{y_j^*\}_{j=1}^{\infty} \subset Y^*$.

Proof:
Assume that Φ has a representation as in (2.5). We want to show that Φ is bounded. We may clearly assume that $x_j \neq 0$ for every j. Let $\{\alpha_j\}_{j=1}^{\infty}$ be a sequence of positive scalars tending to ∞ so that $\sum_{j=1}^{\infty} \alpha_j\|x_j\|\|y_j^*\| = C < \infty$. Let $K = \{x_j/\|x_j\|\alpha_j\}_{j=1}^{\infty} \cup \{0\}$. Then K is compact and

$$|\Phi(T)| = \Big|\sum_{j=1}^{\infty} y_j^*(Tx_j)\Big| \leqslant \Big(\sum_{j=1}^{\infty} \alpha_j\|x_j\|\|y_j^*\|\Big) T\Big(\frac{x_j}{\alpha_j\|x_j\|}\Big) \leqslant$$
$$C \sup_{x\in K} \|Tx\| = C\|T\|_K.$$

Thus by Theorem 2.19, Φ is a bounded linear functional on $(B(X;Y),\tau)$. Conversely, assume that Φ is a linear functional on $\mathcal{B}(X;Y)$ such that $|\Phi(T)| \leqslant C\|T\|_K$ for some constant C depending on some compact set $K \subset X$. By Theorem 2.11 we may assume that $K \subseteq \overline{\text{conv}}\{x_n\}_{n=1}^{\infty}$, where $\|x_n\| \to 0$. Let $(Y \oplus Y \oplus \cdots)_{c_0}$ denote the infinite direct sum of the Banach space Y in the sense of c_0 that is the spaces of all sequences $y = (y_1, y_2, \ldots)$, with $y_n \in Y$, for which $\lim_{n\to\infty} \|y_n\| = 0$, with the supremum norm. Let us define the operator S by

$$S : \mathcal{B}(X;Y) \to Z \subset (Y \oplus Y \oplus \cdots)_{c_0}, \qquad S(T) = (Tx_1, Tx_2, \ldots),$$

where $Z = S(\mathcal{B}(X;Y))$. Note that the norm of $S(T)$ in $(Y \oplus Y \oplus \cdots)_{c_0}$ is

$$\|S(T)\| = \sup_{n\geqslant 1} \|Tx_n\| = \|T\|_K. \tag{2.6}$$

Indeed, since $x \in K$ if and only if

$$x = \lim_{k\to\infty} \sum_{n=1}^{N(k)} \lambda_n^k x_n \text{ for some } \lambda_n^k \geqslant 0 \text{ with } \sum_{n=1}^{N(k)} \lambda_n^k = 1,$$

we have that

$$\|Tx\| \leqslant \lim_{k\to\infty} \sum_{n=1}^{N(k)} \lambda_n^k \|Tx_n\| \leqslant \sup_{n\geqslant 1} \|Tx_n\|.$$

Since $x_n \in K$ we clearly have that $\sup\limits_{n\geqslant 1}\|Tx_n\| \leqslant \|T\|_K$. Hence $\sup\limits_{n\geqslant 1}\|Tx_n\| = \|T\|_K$. From (2.6) and the continuity of Φ, we infer

$$|\Phi(T)| \leqslant C\|T\|_K = C\|S(T)\|.$$

Now we will define a functional $\Psi \in Z^*$ by $\Psi(S(T)) = \Phi(T)$. It is clear that that Ψ is linear. To see that it is well defined, suppose that $S(T_1) = S(T_2)$. Then $|\Phi(T_1 - T_2)| \leqslant C\|S(T_1) - S(T_2)\|$ which implies that $\Phi(T_1) = \Phi(T_2)$. Thus we can view Ψ as a linear functional on Z. Since $|\Phi(T)| \leqslant C\|T\|_K$ we have by our definition of Ψ that

$$|\Psi(S(T))| = |\Phi(T)| \leqslant C\|S(T)\|,$$

Thus Ψ is a bounded linear functional on Z. Because Z is a subspace of $(Y \oplus Y \oplus \cdots)_{c_0}$, by the Hahn-Banach theorem we can extend $\Psi \in Z^*$ to a linear functional on the whole $(Y \oplus Y \oplus \cdots)_{c_0}$. Since the dual of c_0 is ℓ_1 (see [Y], page 114), we have

$$(Y \oplus Y \oplus \cdots)^*_{c_0} = (Y^* \oplus Y^* \oplus \cdots)_{\ell_1}.$$

That is, there exists a sequence $\{y_n^*\}_{n=1}^{\infty}$ in Y^* so that $\sum\limits_{n=1}^{\infty}\|y_n^*\| < \infty$ and $\Phi(T) = \sum\limits_{n=1}^{\infty} y_n^* T(x_n)$. $\square$

This theorem will be used will be used later in the proof of Theorem 3.17 on the equivalence of different formulations of the approximation property.

Chapter 3

Schauder bases and the approximation property

Since Schauder bases are closely related to the approximation property in this chapter we will start by giving a short introduction about such bases and their relation with the approximation property. We will also present a proof that a Hilbert space always has the approximation property. Some equivalent formulations of the approximation property are also given.

3.1 Schauder bases in Banach spaces

The notion of algebraic basis (Hamel basis), in a finite dimensional linear space, or of orthonormal basis, in a Hilbert space, are essential tools for the study of these spaces. Therefore, if we want to investigate the structure of general Banach spaces, it is natural to try to find a corresponding notion. A way to extend the notion of algebraic bases from finite dimensional spaces to infinite dimensional ones is by using Zorn's lemma or the axiom of choice to assert the existence of a Hamel basis for any given Banach space.

DEFINITION 3.1 (Hamel basis) *A Hamel basis for a Banach space X is a set of vectors B such that very element $x \in X$ can be written in an unique way as a $finite$ linear combination of elements of B.*

But in the case of infinite dimensional spaces the existence of such a basis does not provide satisfactory information about the space, and therefore its usefulness is mainly of a theoretical value for a few reasons which we will try to present.

1- By Baire's category theorem it is easy to see that a Hamel basis in a Banach space with infinitely many linearly independent vectors cannot be countable, even if the space is separable.

2- The proof of the existence of a Hamel basis is not constructive so that in practice it is impossible to find an explicit Hamel basis for most infinite dimensional Banach spaces.

3- Besides these two disadvantages, a Hamel basis is not linked with the topology of the space. More precisely let $\{x_\alpha\}_{\alpha\in A}$ be a Hamel basis in a Banach space X, and let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of points in X converging to some point z. Each of the z_n's as well as z have finite decompositions: $z_n = \sum_{\alpha\in A} c_\alpha^{(n)} x_\alpha$, $z = \sum_{\alpha\in A} c_\alpha x_\alpha$. However most of the coefficient functionals associated to the basis are discontinuous.

In order to solve the above problems inherent to Hamel bases, it would be helpful to replace the notion of a Hamel basis, where only finite sums are used, by another one, where each vector will be represented as the sum of an infinite series. This notion of a countable infinite basis (Schauder basis) for Banach spaces was introduced by J. Schauder in 1927.

DEFINITION 3.2 (Schauder basis) *A sequence $\{x_n\}_{n=1}^\infty$ in a Banach space is called a Schauder basis of X if for any $x \in X$, there exists a unique sequence of scalars $\{a_n\}_{n=1}^\infty$ such that*

$$x = \sum_{n=1}^{\infty} a_n x_n,$$

the convergence of the series being that of the norm of X, that is

$$\lim_{N\to\infty} \left\| x - \sum_{n=1}^{N} a_n x_n \right\| = 0.$$

A sequence $\{x_n\}_{n=1}^\infty$ which is a Schauder basis of its closed linear span is called a basic sequence.

In this work when referring to a basis we always mean a Schauder basis. It is important to notice that for describing a Schauder basis one has to define the basis vectors not only as a set but also as an ordered sequence. For $x = \sum_{n=1}^{\infty} a_n x_n$ the maps $x_n^* : X \to \mathcal{C}$, defined by $x_n^*(x) = a_n$ are called the coefficient functionals associated to the basis and the projections $\{P_n\}_{j=1}^\infty$, defined by $P_n\Big(\sum_{j=1}^{\infty} a_j x_j\Big) = \sum_{j=1}^{n} a_j x_j$ are called the natural projections associated to $\{x_n\}_{n=1}^\infty$. A basis $\{x_n\}_{n=1}^\infty$ is called normalized if $\|x_n\| = 1$ for all n. Clearly, whenever $\{x_n\}_{n=1}^\infty$ is a Schauder basis of X, the sequence $\{x_n/\|x_n\|\}_{n=1}^\infty$ is a normalized basis of X.

An important characteristic of a Schauder basis is that the coefficient functionals associated with the basis are continuous. This result was proved originally by Banach (see [B], page 68). We present the proof with the help of the following lemma.

LEMMA 3.3 *Let $\{x_n\}_{n=1}^\infty$ be a normalized Schauder basis in a Banach space X, and let Y be the linear space of sequences of scalars*

$$Y = \left\{ \{a_j\} \subset \mathbb{C} : \sum_{j=1}^{\infty} a_j x_j \text{ converges in } X \right\} \tag{3.1}$$

endowed with the norm

$$|||\{a_j\}||| = \sup_n \left\| \sum_{j=1}^{n} a_j x_j \right\| = \sup_n \|S_n\|. \tag{3.2}$$

Then Y is Banach space.

Proof:
As in the statement let $S_n = \sum_{j=1}^{n} a_j x_j$. Since the sequence $\{\|S_n\|\}_{n=1}^{\infty}$ is convergent the real number $\sup_n \|S_n\|$ is finite. Since all $x_n \neq 0$, (3.2) is a norm on the linear space Y. Let $\{a_{k,n}\}_{k=1}^{\infty}$ be a Cauchy sequence in Y. This sequence will have as elements

$$\begin{aligned} a_{1,n} &= (a_{1,1}, a_{1,2}, a_{1,3}, \dots) \\ a_{2,n} &= (a_{2,1}, a_{2,2}, a_{2,3}, \dots) \\ a_{3,n} &= (a_{3,1}, a_{3,2}, a_{3,3}, \dots) \\ &\vdots \end{aligned}$$

Then for every $\varepsilon > 0$ there is a positive integer N such that

$$|||\{a_{k,n}\} - \{a_{m,n}\}||| = \sup_n \left\| \sum_{j=1}^{n} a_{k,j} x_j - \sum_{j=1}^{n} a_{m,j} x_j \right\| = \tag{3.3}$$

$$\sup_n \left\| \sum_{j=1}^{n} (a_{k,j} - a_{m,j}) x_j \right\| < \varepsilon \qquad (k, m > N). \tag{3.4}$$

Hence

$$||(a_{k,n} - a_{m,n}) x_n|| \leqslant \left\| \Big(\sum_{j=1}^{n} (a_{k,j} - a_{m,j}) x_j \Big) - \Big(\sum_{j=1}^{n-1} (a_{k,j} - a_{m,j}) x_j \Big) \right\| \leqslant$$

$$\left\| \sum_{j=1}^{n} (a_{k,j} - a_{m,j}) x_j \right\| + \left\| \sum_{j=1}^{n-1} (a_{k,j} - a_{m,j}) x_j \right\| < 2\varepsilon \qquad (k, m > N).$$

Since all $x_j \neq 0$ it follows that

$$|a_{k,n} - a_{m,n}| < \frac{2\varepsilon}{\|x_n\|} \qquad (k, m > N).$$

Consequently for each fixed $n \geqslant 1$ the sequences of scalars $\{a_{k,n}\}_{k=1}^{\infty}$ are Cauchy sequences and converge to a scalar a_n. Hence from inequality (3.4), by letting $m \to \infty$ we obtain

$$\left\| \sum_{j=1}^{n} (a_{k,j} - a_j) x_j \right\| \leqslant \varepsilon \qquad (k > N, n \geqslant 1).$$

We also note that for $n, p \geqslant 1$

$$\left\| \sum_{j=1}^{n+p} (a_{k,j} - a_j) x_j \right\| \leqslant \varepsilon \qquad \text{and} \qquad \left\| \sum_{j=1}^{n} (a_{k,j} - a_j) x_j \right\| \leqslant \varepsilon.$$

Writing

$$\sum_{j=1}^{n+p} a_j x_j = \sum_{j=1}^{n+p}(a_j - a_{k,j})x_j + \sum_{j=1}^{n+p} a_{k,j}x_j$$

we get that

$$\left\| \sum_{j=n+1}^{n+p} a_j x_j \right\| \leqslant 2\varepsilon + \left\| \sum_{j=n+1}^{n+p} a_{k,j} x_j \right\| \qquad (k > N).$$

Since the series $\sum_{j=1}^{\infty} a_{k,j} x_j$ is convergent and since X is complete, we deduce that $\sum_{j=1}^{\infty} a_j x_j$ also converges. Thus Y is a Banach space since

$$|||\{a_{k,n}\} - \{a_n\}||| \leqslant \sup_n \left\| \sum_{j=1}^{n}(a_{k,j} - a_j)x_j \right\|.$$

□

THEOREM 3.4 *Let $\{x_n\}_{n=1}^{\infty}$ be a Schauder basis in a Banach space X. Then the coefficient functionals associated with that basis are continuous.*

Proof:
Let Y be the Banach space in Lemma 3.3. For each sequence of scalars $a = (a_1, a_2, \ldots) \in Y$ define $T : Y \to X$ by

$$T(a) = T(a_1, a_2, \ldots) = \sum_{j=1}^{\infty} a_j x_j = x.$$

The operator T thus defined is a bounded linear operator because $||T(a)|| \leqslant |||a|||$, and as it maps Y bijectively onto X. By the open mapping theorem, its inverse T^{-1} is also a bounded linear operator. Let

$$f_j(x) = a_j, \text{ where } x = \sum_{j=1}^{\infty} a_j x_j.$$

Since

$$\|a_j x_j\| = \left\| \sum_{j=1}^{n} a_j x_j - \sum_{j=1}^{n-1} a_j x_j \right\| \leqslant \left\| \sum_{j=1}^{n} a_j x_j \right\| + \left\| \sum_{j=1}^{n-1} a_j x_j \right\| \leqslant 2|||a|||$$

and $a = T^{-1}x$ it follows that f_j is bounded

$$|f_j(x)| = |a_j| \leqslant \frac{2}{\|x_j\|}|||a||| \leqslant \frac{2}{\|x_j\|}\|T^{-1}\|\|x\|.$$

□

An immediate consequence of the existence of a Schauder basis is that if a Banach space X has a Schauder basis $\{x_n\}$, then it is also separable, since the set of all convergent linear combinations $\sum_{j=1}^{\infty} r_j x_j$, where the real and imaginary parts of r_j are rational numbers is a countable dense subset in X. Note that a nonseparable Banach space cannot have a Schauder basis.

The first idea which arises quite naturally in connection with the "basis problem" is to examine whether or not the various concrete separable Banach spaces occurring in practice possess a basis. We will give a few examples.

EXAMPLE 3.5 If X is c_0 or ℓ_p such that $1 \leqslant p < \infty$, then it is easy to check that the sequence $\{x_n\}_{n=1}^{\infty}$,

$$x_n = \{\delta_{nj}\}_{j=1}^{\infty}$$

of standard unit vectors of X is a basis for X. Indeed if $x = \{a_n\} \in c_0$ we have

$$\left\| x - \sum_{n=1}^{N} a_n x_n \right\| = \sup_{N+1 \leqslant n < \infty} |a_n| \to 0 \text{ for } n \to \infty.$$

For $x = \{a_n\} \in \ell_p$ we have

$$\left\| x - \sum_{n=1}^{N} a_n x_n \right\| = \left(\sum_{n=N+1}^{\infty} |a_n|^p \right)^{\frac{1}{p}} \to 0 \text{ if } n \to \infty.$$

EXAMPLE 3.6 Consider the Banach space $C[0,1]$ of all continuous scalar valued functions on the closed interval $[0,1]$ under the usual addition of functions and multiplication by scalars. Schauder, besides defining the notion of Schauder basis, also constructed a basis for $C[0,1]$. To see how such a basis for $C[0,1]$ is constructed we refer to [Meg], page 352. That $C[0,1]$ has a basis is of great interest because $C[0,1]$ plays a central role in the theory of Banach spaces. One special property of $C[0,1]$, which makes it usable in different connections is its universality amongst separable Banach spaces: every separable Banach space is isometrically isomorphic to a closed subspace of $C[0,1]$ (see [B], page 113). The fact that "every" infinite dimensional separable Banach space can be embedded in $C[0,1]$ and $C[0,1]$ has a basis was one of the immediate reasons for the appearance of the "basis problem".

EXAMPLE 3.7 The space of functions analytic on the open unit disk and continuous up to the boundary (this is the disc algebra A) has a Schauder basis. This has been proved originally by Bochkarev's [Bo] in 1974, who built a basis for the disc algebra A based on the Franklin system (i.e. the Gram-Schmidt orthogonalization of Schauder's system for $C[0,1]$ in the Hilbert space $L_2[0,1]$).

EXAMPLE 3.8 ℓ_∞, being a non-separable Banach space cannot have a Schauder basis.

3.2 The approximation property

The first systematic study of the approximation property was initiated by Grothendieck in 1955 [G]. Grothendieck found many equivalent conditions to the approximation property. One of his main motivations in studying and introducing an equivalent definition of the approximation property and in raising "the approximation problem" was the Example 3.10 below, which relates this problem to the considerably older problem of uniform approximability of compact linear operators on Banach spaces by finite rank operators. The question was originally raised in Banach's book (see [B], page 146):

"According to a remark of S. Mazur we have the following theorem: If $\{T_n\}$ is a sequence of compact linear operators, defined in a Banach space X and such that $\lim\limits_{n\to\infty} T_n x = x$ for every $x \in X$, a necessary and sufficient condition for a set $K \subseteq X$ to be compact is that the convergence be uniform. A space for which such a sequence of operators exists is separable [1]. The question of if, conversely every separable Banach space X admits such a sequence of operators, remains open".

A result which goes back to the beginnings of functional analysis asserts that the compact operators on a Hilbert space are exactly those operators which are limits in norm of operators of finite rank. One part of the assertion, namely that every $T \in \mathcal{B}(X;Y)$ for which $\|T - T_n\| \to 0$ for a suitable sequence of finite rank operators $\{T_n\}_{n=1}^{\infty} \in \mathcal{B}(X;Y)$ is compact, is true for every pair of Banach spaces X and Y. It was realized long ago that the converse is also true for many examples of Banach spaces X and Y besides Hilbert spaces, but there was some indication that there could exist Banach spaces without the approximation property. Grothendieck also showed that some common separable spaces, which appear in analysis have the approximation property, and as a matter of fact even have a Schauder basis (see [S], page 718). Historically, the motivating example is the following one.

EXAMPLE 3.10 For each positive n, define

$$P_n : \ell_2 \to \ell_2$$

by the formula

$$P_n(\{a_j\}) = (a_1, \ldots, a_n, 0, 0, \ldots).$$

Then each P_n is a bounded linear projection from ℓ_2 onto

$$\{\{a_j\} : \{a_j\} \in \ell_2, a_{n+1} = a_{n+2} = \cdots = 0\}.$$

Suppose now that K is a nonempty relatively compact subset of ℓ_2. This means that any sequence $\{\alpha_n\}_{n=1}^{\infty}$ in K has a convergent subsequence. We claim that

[1]

THEOREM 3.9 *The range of any compact operator T is separable.*

Proof:
The set $F_n = \{Tx : \|x\| \leqslant n\}$ is relatively compact and therefore separable, and hence so is the set $\cup_{n=1}^{\infty} F_n$, which is equal to the range of T. $\square$

for each positive ε, there must be a positive integer n_ε depending on the set K, but not depending on the specific elements of K, such that

$$\left(\sum_{j=n_\varepsilon+1}^{\infty} |\alpha_{k,j}|^2\right)^{1/2} < \varepsilon \tag{3.5}$$

whenever $\alpha_k \in K$, where $\alpha_{k,j}$ are the elements of the sequence $\alpha_k \in K \subset \ell_2$. We will show that if (3.5) is not true, then we could construct a sequence in K without a convergent subsequence. Let us suppose that (3.5) is false. This means that there is an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there is an $\alpha_k \in K$ with

$$\left(\sum_{j=n+1}^{\infty} |\alpha_{k,j}|^2\right)^{1/2} \geqslant \varepsilon.$$

Since we are in ℓ_2, given any $\varepsilon > 0$ and any $\alpha \in K$, there is an n_1 with

$$\left(\sum_{j=n_1+1}^{\infty} |\alpha_j|^2\right)^{1/2} < \frac{\varepsilon}{2}.$$

Start with some $\alpha_1 \in K$ and choose n_1 as above. Then, by our assumption, we can find an $\alpha_2 \in K$ and an $n_2 > n_1$ with

$$\left(\sum_{j=n_1+1}^{\infty} |\alpha_{2,j}|^2\right)^{1/2} > \varepsilon \text{ and } \left(\sum_{j=n_2+1}^{\infty} |\alpha_{2,j}|^2\right)^{1/2} < \frac{\varepsilon}{2}.$$

Continuing this process we can choose an $\alpha_3 \in K$ and $n_3 > n_2$ such that

$$\left(\sum_{j=n_2+1}^{\infty} |\alpha_{3,j}|^2\right)^{1/2} > \varepsilon \text{ and } \left(\sum_{j=n_3+1}^{\infty} |\alpha_{3,j}|^2\right)^{1/2} < \frac{\varepsilon}{2}.$$

and so on. Then we get that

$$\|\alpha_1 - \alpha_2\| = \left(\sum_{j=1}^{\infty} |\alpha_{1,j} - \alpha_{2,j}|^2\right)^{\frac{1}{2}} \geqslant$$
$$\left(\sum_{j=n_1+1}^{\infty} |\alpha_{1,j}|^2\right)^{\frac{1}{2}} - \left(\sum_{j=n_1+1}^{\infty} |\alpha_{2,j}|^2\right)^{\frac{1}{2}} > \frac{\varepsilon}{2}$$

In an exactly similar way we can show that $\|\alpha_2 - \alpha_3\| > \frac{\varepsilon}{2}$, and in general $\|\alpha_k - \alpha_m\| > \frac{\varepsilon}{2}$ for $k \neq m$. Then $\{\alpha_n\} \subseteq K$ would have no convergent subsequence, which contradicts the relative compactness of K. This completes the proof of (3.5). Hence

$$\lim_{n\to\infty} \{\sup_{\alpha \in K} \{\|(I - P_n)(\alpha)\|_2\}\} = 0.$$

Now suppose that T is a compact operator from a Banach space X into ℓ_2. Let $T_n = P_n T$ for each n. Then T_n is a bounded finite rank linear operator. Let

B_X be the closed unit ball in X centered at zero. Then $T(B_X)$ is a relatively compact set in ℓ_2, and

$$0 = \lim_{n\to\infty}\{\sup_{\alpha\in T(B_X)} \|(I-P_n)(\alpha)\|_2\} =$$
$$\lim_{n\to\infty}\{\sup_{x\in B_X}\{\|(T-T_n)(x)\|_2\} =$$
$$\lim_{n\to\infty}\|T-T_n\|.$$

This proves that every compact linear operator from a Banach space into ℓ_2 is the limit of a sequence of bounded finite rank linear operator from that Banach space into ℓ_2. Since for a Banach space X finite-rank linear operators are compact ($\mathcal{F}(X) \subset \mathcal{K}(X)$ by Theorem 2.7), another way of expressing this is to say that ℓ_2 has the following property.

DEFINITION 3.11 (Banach's approximation property) *A Banach space X has the approximation property if for every Banach space Y, the set of finite rank operators $\mathcal{F}(Y,X)$ is dense in $\mathcal{K}(Y;X)$.*

The argument that proves the following result is essentially the same as the one used in our example above for the space ℓ_2.

THEOREM 3.12 *The spaces ℓ_p for $1 \leqslant p < \infty$ have the approximation property.*

The basis property and the approximation property are closely related and in fact a Banach space X with a Schauder basis also has the approximation property (see Theorem 3.15). We would like to point out that it is usually much easier to verify that a given space has the approximation property than to construct a Schauder basis for that space. We illustrate this by considering the disc algebra A (defined in Example 3.7). As mentioned above (Example 3.7) the disc algebra has a basis. However it was not easy to construct such a basis and this was an open problem for a very long time. On the other hand it is relatively easy to verify that A has the approximation property. Indeed for $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots \in A$ let $S_n = a_0 + a_1 z + \ldots + a_n z^n$ and let $\sigma_n f = (S_1 f + S_2 f + \ldots + S_n f)/n, n = 1, 2, \ldots$. A classical result of Fejer states that $\|\sigma_n\| \leqslant 1$ for all n and that $\|\sigma_n f - f\| \to 0$ for every $f \in A$ (see [Z], page 89). This shows that the disc algebra has the approximation property.

The definition of the approximation property given above is the historical one. The definition commonly encountered today is an equivalent one due to A. Grothendieck, which we will call the "approximation property in the sense of Grothendieck". That definition has the advantage of being based on the intrinsic properties of the Banach space X in question rather than on the property that involves *every* Banach space Y as well as X and every compact operator from Y to X. The equivalence of the two definitions will be proved in Theorem 3.17. Grothendieck's definition of the approximation property is also the one which we will use in the construction of a Banach space without the approximation property.

DEFINITION 3.13 (Grothendieck's approximation property) *A Banach space X is said to have the approximation property if, for every compact* *set K in X and every $\varepsilon > 0$, there is an operator $T : X \to X$ of finite rank such that $\|Tx - x\| \leqslant \varepsilon$ for every $x \in K$.*

The above definition can be restated as: the identity operator $I : X \to X$ can be approximated, uniformly on every compact subset K of X, by linear operators of finite rank.

THEOREM 3.14 *Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$. Then X has the approximation property in the sense of Grothendieck.*

Proof:
Let $\{S_n\}$ denote the partial sum operators associated with the basis $\{x_n\}$ that is,

$$S_n x = \sum_{k=1}^{n} \alpha_k x_k \qquad \text{whenever} \qquad x = \sum_{k=1}^{\infty} \alpha_k x_k \in X.$$

Then $S_n : X \to X$ is linear and of finite rank and thus a compact operator. Thus $\|S_n\| < \infty$ and by the Banach-Steinhaus theorem $\sup_n \|S_n\|$ is also bounded, since $\lim\limits_{n\to\infty} \|S_n x - x\| = 0$ by the definition of Schauder basis. Denote this supremum by s. Let K be a compact subset of X. Since K is compact we can can cover K with finitely many balls of radius $\dfrac{\varepsilon}{2(1+s)}$ centered at $\{y_j\}_{j=1}^{m}$. Let both $x \in K$ and $\varepsilon > 0$ be arbitrary. Then there exists an y_j such that $\|x - y_j\| < \dfrac{\varepsilon}{2(1+s)}$ and a positive integer N such that $\|y_j - S_n(y_j)\| < \frac{\varepsilon}{2}$, for $j = 1, \dots, m$, whenever $n > N$. Thus

$$\|x - S_n(x)\| \leqslant \|x - y_j\| + \|y_j - S_n(y_j)\| + \|S_n(x) - S_n(y_j)\| \leqslant$$

$$\|x - y_j\|(1+s) + \|y_j - S_n(y_j)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad (n > N),$$

As $x \in K$ and $\varepsilon > 0$ have been chosen arbitrarily, we infer that

$$\lim_{n\to\infty} \sup_{x\in K} \|x - S_n(x)\| = 0$$

which completes the proof. □

In the next theorem we will show that if X has a Schauder basis then X has Banach's approximation property.

THEOREM 3.15 *Let X be a Banach space with a Schauder basis $\{x_n\}$ and let Y be an arbitrary Banach space. Then every compact linear operator $T \in \mathcal{K}(Y; X)$ can be uniformly approximated by linear operators of finite rank.*

Proof:
Let $T \in \mathcal{K}(Y; X)$ be arbitrary and let $\{S_n\}$ be the sequence of partial sum operators associated with the basis $\{x_n\}$. If B_Y is the closed unit ball in Y centered at the origin, then the set $K = \overline{T(B_Y)} \subset X$ is compact, whence, by Theorem 3.14,

$$\|T - S_n T\| = \sup_{\|y\|\leqslant 1} \|Ty - S_n Ty\| = \sup_{x\in K} \|x - S_n x\| \to 0 \text{ as } n \to \infty.$$

Since $S_n T \in \mathcal{B}(Y; X)$ is of finite rank, this completes the proof. □

3.3 Equivalent formulations of the approximation property

Our purpose in this section is to present several conditions equivalent to the approximation property, all due to Grothendieck [G].

THEOREM 3.16 *Let X be a Banach space. Then the following assertions are equivalent:*
(i) X has the approximation property in the sense of Grothendieck.
(ii) For every Banach space Y the finite rank operators are dense in $\mathcal{B}(Y;X)$ in the topology τ of uniform convergence on compact sets.
(iii) For every Banach space Y the finite rank operators are dense in $\mathcal{B}(X;Y)$ in the topology τ of uniform convergence on compact sets.

Proof:
With $X = Y$ it is clear that (ii) and (iii) imply (i), since the identity operator $I \in \mathcal{B}(X)$. To see that (i) implies (ii), let $T \in \mathcal{B}(Y;X)$. For every compact set $K \subset Y$, the set $T(K) \subset X$ is also compact in X. Hence, given $\varepsilon > 0$, we have by (i) that there is a finite rank operator $T_1 : X \to X$ such that

$$\|T_1Ty - Ty\| = \|T_1x - x\| \leqslant \varepsilon \text{ for } y \in K \text{ (or for } x \in T(K) \text{ with } x = Ty).$$

Since T_1T is of finite rank we have proved (ii). To prove that (i) implies (iii) let $T \in \mathcal{B}(X;Y)$ with $T \neq 0$, let K be a compact set in X and let $\varepsilon > 0$. By (i) there is a finite rank operator $T_1 : X \to X$ such that $\|T_1x - x\| \leqslant \varepsilon/\|T\|$ for $x \in K$. Then

$$\|TT_1x - Tx\| \leqslant \varepsilon, \qquad x \in K,$$

which proves (iii) since $T_1T \in \mathcal{F}(X;Y)$. □

THEOREM 3.17 *Let X be a Banach space. Then the following assertions are equivalent:*
(i) X has the approximation property in the sense of Grothendieck.
(ii) For each choice of $\{x_n\}_{n=1}^{\infty} \subset X$, and $\{x_n^*\}_{n=1}^{\infty} \subset X^*$ such that the sum $\sum_{n=1}^{\infty} \|x_n^*\|\|x_n\| < \infty$ and $\sum_{n=1}^{\infty} x_n^*(x)x_n = 0$ for all $x \in X$, we have $\sum_{n=1}^{\infty} x_n^*(x_n) = 0$.
(iii)For every Banach space Y, every compact $T \in \mathcal{K}(Y;X)$ and every $\varepsilon > 0$ there is a finite rank operator $T_1 \in \mathcal{B}(Y;X)$ with $\|T - T_1\| < \varepsilon$.

Proof:
Let us first prove that (i) is equivalent to (ii). By definition, (i) means that the identity operator is in the τ-closure of the subspace of finite rank linear operators $\mathcal{F}(X) \subset \mathcal{B}(X)$. By the Hahn-Banach theorem this happens if and only if every τ-continuous functional on $\mathcal{B}(X)$, which vanishes on finite rank operators also vanishes on the identity I. By Theorem 2.20, any τ-continuous functional on $\mathcal{B}(X)$ has the form

$$\Phi(T) = \sum_{j=1}^{\infty} y_j^*(Tx_j).$$

If T is of rank one then for all $z \in X$ we can write $Tz = y^*(z)x$ for some $x \in X$ and $y^* \in X^*$. If

$$\Phi(T) = \sum_{j=1}^{\infty} y_j^*(Tx_j) = \sum_{j=1}^{\infty} y_j^*(x) y^*(x_j) = 0$$

for all $x \in X$ and $y^* \in X^*$, then by the linearity of y^* we have

$$y^*\big(\sum_{j=1}^{\infty} y_j^*(x)x_j\big) = 0 \text{ for all } y^* \in X^*.$$

This implies that $\sum_{j=1}^{\infty} y_j^*(x)x_j = 0$ for all $x \in X$. Thus if (ii) holds, the previous considerations show that if Φ is a τ-continuous functional on $\mathcal{B}(X)$ which vanishes on rank one operators, then $\sum_{j=1}^{\infty} y_j^*(x_j) = 0$ or $\Phi(I) = 0$. Hence (ii) implies (i). Conversely, if (i) holds, then any τ-continuous functional on $\mathcal{B}(X)$ which annihilates finite rank operators annihilates the identity I. Clearly, Φ vanishes on finite rank operators if and only if it vanishes on operators of rank one since any operator of finite rank is a finite sum of operators of rank one. The above considerations show therefore that (i) implies (ii).

Let us now prove the equivalence between (i) and (iii). Assume that (i) holds and let $T : Y \to X$ be a compact operator. Then with B_Y denoting the closed unit ball in Y (centered at zero), $\overline{TB_Y} = K$ is a compact subset of X and hence by (i) for every $\varepsilon > 0$, there is a finite rank operator T_0 on X so that $\|T_0x - x\| \leqslant \varepsilon$ for $x \in K$. If $T_1 = T_0T$, then

$$\|T_1 - T\| = \sup_{y \in B_Y} \{\|(T_1 - T)y\|\} = \sup_{x \in TB_Y} \{\|(T_0x - x\|\} \leqslant \varepsilon.$$

Since T_1 is of finite rank, (iii) holds. Conversely, assume now that (iii) holds and let K be a compact subset of X and $\varepsilon > 0$. By Theorem 2.11 we may assume without loss of generality that $K = \overline{\text{conv}}\{x_n\}_{n=1}^{\infty}$, with $\|x_n\| \downarrow 0$ and $\|x_1\| \leqslant 1$, $\|x_n\| > 0$ for $n \geqslant 1$. Let

$$U = \overline{\text{conv}}\left\{ \frac{\pm x_n}{\|x_n\|^{\frac{1}{2}}} \right\}_{n=1}^{\infty}. \tag{3.6}$$

Clearly U is a compact convex set in X, which is symmetric with respect to the origin. Moreover, since $\|x_n\| \leqslant 1$ and $\|x_n\| \downarrow 0$ we have that $0 \in K$ and

$$x_n = \left[\|x_n\|^{\frac{1}{2}} \frac{x_n}{\|x_n\|^{\frac{1}{2}}} + (1 - \|x_n\|^{\frac{1}{2}}).0\right] \in K \text{ so that } K \subseteq U.$$

Let Y be the linear span of U in X, i.e. $Y = \bigcup_{n=1}^{\infty} nU$, and introduce in Y the norm $|||.|||$ which makes U its unit ball. That is, we define the norm of Y to be the Minkowski functional relative to the set U,

$$|||y||| = \inf\{\lambda > 0; \frac{y}{\lambda} \in U\}. \tag{3.7}$$

(The proof that $(Y, |||.|||)$ is Banach space is given in Lemma 3.18). With X and Y normed as above the formal identity map $I : Y \to X$ is compact since it maps

$U \subset Y$ (the unit ball in $(Y, |||.|||)$ into a compact subset U of $(X, \|.\|)$. Thus by (iii), given an $\varepsilon > 0$ there is a finite rank operator $T_1 \in \mathcal{F}(Y; X)$, $\{y_n^*\}_{n=1}^m \subset Y^*$ and $\{z_n\}_{n=1}^m \subset X$ a basis for $T_1(Y)$ such that

$$T_1 x = \sum_{j=1}^m y_j^*(x) z_j \text{ for all } x \in Y,$$

and

$$\Big\|T_1 x - x\Big\| = \Big\|\sum_{j=1}^m y_j^*(x) z_j - x\Big\| \leqslant \frac{\varepsilon}{2} \text{ for every } x \in K \subset U.$$

Note that the functionals $\{y_n^*\}_{n=1}^m$ are continuous with respect to the $|||.|||$-norm, but need not be continuous with respect to the $\|.\|$-norm and thus are not in general restrictions of elements of X^* to Y^*.
In order to conclude the proof it is enough to verify that there exists an $F \in \mathcal{F}(X)$ and $\{x_n^*\}_{n=1}^m \subset X^*$ with $F = \sum_{j=1}^m x_j^* z_j$ and

$$\Big\|Fx - x\Big\| = \Big\|\sum_{j=1}^m x_j^*(x) z_j - x\Big\| \leqslant \varepsilon \qquad \text{for all } x \in K \subset U.$$

This will be done once it is shown that given any $y^* \in Y^*$ and a $\delta > 0$ (in our case we take $\delta = \dfrac{\varepsilon}{2m \cdot \max \|z_j\|}$) there is an $x^* \in X^*$ such that $|y^*(z) - x^*(z)| < \delta$ for all $z \in K$. If we can find such a x^* then for elements $x_1^*, x_2^*, \dots, x_m^*$ of X^* near to $y_1^*, y_2^*, \dots, y_m^*$ of Y^* respectively, it would follow that for each z in K,

$$\begin{aligned}
&\Big\|Fz - z\Big\| = \Big\|Fz - T_1 z + T_1 z - z\Big\| \leqslant \\
&\Big\|\sum_{j=1}^m x_j^*(z) z_j - \sum_{j=1}^m y_j^*(z) z_j\Big\| + \Big\|T_1 z - z\Big\| < \\
&\sum_{j=1}^m \big(|x_j^*(z) - y_j^*(z)|\big)\|z_j\| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

By construction $\|x_n\| \leqslant 1$ and $x_n/\|x_n\|^{1/2} \in U$, which is the unit ball in $(Y, |||.|||)$. Hence $\dfrac{|||x_n|||}{\|x_n\|^{1/2}} \leqslant 1$ and thus $|||x_n||| \leqslant \|x_n\|^{1/2}$ for every $n \geqslant 1$. Consequently $|||x_n||| \to 0$. Since $x_n \to 0$ in both norms, by the continuity of y^*, we can find an n_0 such that for $n > n_0$ we have $|y^*(x_n)| < \delta/2$. Let

$$K_0 = \frac{2}{\delta}\overline{\text{conv}}\{\pm x_n\}_{n=n_0+1}^\infty \subset Y.$$

K_0 is compact in Y by Theorem 2.11 and by the compactness of the identity map from Y into X, this is the same convex hull when taken in X. Indeed, if

$$\frac{2}{\delta}\sum_{n=n_0+1}^{N(k)} \lambda_n^k x_n \to x, \text{ in } Y \text{ with } \lambda_n^k \geqslant 0 \text{ and } \sum_{n=n_0+1}^{N(k)} \lambda_n^k = 1,$$

then convergence holds also in X. Thus K_0 is contained in the convex hull in X and is also compact in X. Moreover, K_0 is convex in X, so K_0 is precisely the convex hull in X since $\frac{2}{\delta}x_n \in K_0$ for $n \geqslant n_0 + 1$. Let

$$C = \{x : x \in \operatorname{span}\{x_n\}_{n=1}^{n_0}, \text{Re } y^*(x) = 1\}.$$

Then C is a closed subset of a finite dimensional subspace of Y. Thus C closed in Y and in X.

If $x_1, x_2, \ldots, x_{n_0} \in \operatorname{Ker}(y^*)$, then by letting x^* be the zero functional in X^* it would follow that $|y^*(x_n) - x^*(x_n)| = |y^*(x_n)| < \delta/2$ for all $n \geqslant 1$ and hence for all $x \in K$. We may therefore assume that at least one of $y^*(x_j)$ with $j \in \{1, \ldots, n_0\}$ is nonzero and therefore $C \neq \emptyset$. By our definition of K_0 this implies that if $x_0 \in K_0$ then $|y^*(x_0)| < 1$ and $|y^*(x)| \geqslant 1$ when $x \in C$ so $K_0 \cap C = \emptyset$. The sets C and K_0 are disjoint and by the geometric version of the Hahn-Banach theorem (see [Meg], page 180) there is a $x^* \in X^*$ such that

$$\max\{\text{Re } x^*(z) : z \in K_0\} < \inf\{\text{Re } x^*(z) : z \in C\}.$$

Actually Re $x^*(z)$ must be constant throughout C. Indeed, note that if $x_1, x_2 \in C$, then $y^*[(nx_1) - (n-1)x_2] = 1$, so $nx_1 - (n-1)x_2 \in C$. Suppose that Re $x^*(x_1) >$ Re $x^*(x_2)$. Then

$$\begin{aligned}&\text{Re } x^*(nx_1 - (n-1)x_2) = n\text{Re } x^*(x_1) - (n-1)\text{Re } x^*(x_2)\\ &= (n-1)[\text{Re } x^*(x_1) - \text{Re } x^*(x_2)] + \text{Re } x^*(x_2) \to \infty \text{ as } n \to \infty,\end{aligned}$$

which is a contradiction. Since $0 \in K_0$ and $x^*(0) = 0$, it may be assumed that

$$\text{Re } x^*(z) = 1, \qquad z \in C.$$

It follows that for z in the linear span of $\{x_j\}_{j=1}^{n_0}$, Re $x^*(z) =$ Re $y^*(z) = 1$. Since K_0 is balanced, we deduce that for $x_0 \in K_0$,

$$|x^*(x_0)| < \inf\{\text{Re } x^*(z) : z \in C\} = 1.$$

In particular this implies $|x^*(x_n)| < \frac{\delta}{2}$ for $n > n_0$. Now, if $z \in K$, then $z \in \overline{\operatorname{conv}}\{x_n\}_{n=1}^{\infty}$ so that

$$z = \lim_{k\to\infty} \sum_{n=1}^{N(k)} \lambda_n^k x_n \text{ with } \lambda_n^k \geqslant 0 \text{ satisfying } \sum_{n=1}^{\infty} \lambda_n^k = 1.$$

We deduce that

$$\begin{aligned}&|x^*(z) - y^*(z)| = \lim_{k\to\infty} \left| \sum_{n=1}^{N(k)} \lambda_n^k \left[x^*(x_n) - y^*(x_n)\right] \right| \leqslant\\ &\lim_{k\to\infty} \sum_{n=n_0}^{N(k)} \lambda_n^k \left| x^*(x_n) + y^*(x_n) \right| < \delta\end{aligned}$$

completing the proof. □

LEMMA 3.18 *The space $(Y, |||.|||)$, where Y is the normed space defined in the proof of Theorem 3.17 with the norm given in (3.7) is a Banach space.*

Proof:
We want to prove that the metric induced by the Minkowski functional relative to the set U defined in (3.7) is complete. Suppose it is not. Let $\{z_n\}_{n=1}^{\infty}$ be a nonconvergent Cauchy sequence in $B_Y = U$. Since $\{z_n\}$ is also Cauchy in X and lies in a compact subset $U \subset X$, there is some $z \in X$ such that $\|z - z_n\| \to 0$. Now define a new sequence $w_n = z_n - z$. This sequence is nonconvergent in Y by our assumption and $\|w_n\| \to 0$ in X. It follows that there is a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and an $\varepsilon > 0$ such that $|||w_{n_k}||| > \varepsilon$ for all k. Letting

$$y_k = \frac{w_{n_k}}{|||w_{n_k}|||}$$

gives a Cauchy sequence whose elements all have norm 1 in Y, but which, when viewed as sequence in X, converges to 0. Let k_0 be an integer such that $|||y_k - y_j||| \leqslant 1/2$ whenever $j, k \geqslant k_0$. Then $2(y_{k_0} - y_k) \in U$ for $k \geqslant k_0$, because $2|||y_k - y_j||| \leqslant 1$ and U is the unit ball of Y. Since $y_k \to 0$ in X and U is closed in X it follows that $2y_{k_0} \in U$. This implies that $|||y_{k_0}||| \leqslant 1/2$, which is a contradiction. Thus the normed space $(Y, |||.|||)$ is a Banach space. □

3.4 The approximation property on Hilbert spaces

Let H be a Hilbert space. If H is separable, then it has a countable orthonormal basis. This is also a Schauder basis. Since Banach spaces with a Schauder basis are separable, we see that non-separable Hilbert spaces cannot have a Schauder basis (see [C], pages 14-17). The next result shows that any Hilbert space has the approximation property.

THEOREM 3.19 *Let H be a Hilbert space and T a linear operator in $\mathcal{B}(H)$. T is compact if and only if there is sequence of finite rank operators $T_n : H \to H$ with $\|T_n - T\| \to 0$.*

Proof:
Since finite rank operators are compact by Theorem 2.7 and the subspace $\mathcal{K}(H)$ is closed in $\mathcal{B}(H)$ by Theorem 2.8 it follows that the limit of a sequence of finite rank operators is compact. Conversely, let $B = \{x \in H : \|x\| \leqslant 1\}$ and let K be the closure of $T(B)$. Since T is a compact operator the closure of its image K is a compact subset of H, thus totally bounded. Therefore, for every $\varepsilon > 0$ we can find finitely many $y_i \in K$ with

$$K \subset \bigcup_{j=1}^{n} B(y_j, \varepsilon).$$

Let Y be the finite dimensional space generated by the y_j's and let P_Y be the orthogonal projection of $H \to Y$. Then $\|P_Y\| \leqslant 1$. The operator $T_Y = P_Y \circ T$ is of finite rank since $T_Y(H) = Y$. If $y \in B$ then there is some y_k, for $k = 1, \ldots, n$ such that $\|Ty - y_k\| \leqslant \frac{\varepsilon}{2}$. This implies that $\|P_Y(Ty - y_k)\| \leqslant \frac{\varepsilon}{2}$ or $\|T_Y y - y_k\| \leqslant \frac{\varepsilon}{2}$. From this we deduce that

$$\|Ty - T_Y y\| \leqslant \|Ty - y_k\| + \|T_Y y - y_k\| \leqslant \varepsilon.$$

Thus $\|T - T_Y\| \leqslant \varepsilon$. □

Chapter 4

A space without the approximation property

4.1 Background

The approximation problem was completely solved in the negative by P. Enflo who published his solution in May 1973. Enflo constructed an example of a closed reflexive subspace of c_0, which does neither have a Schauder basis nor the approximation property. Actually, P. Enflo proved that there exists a separable reflexive Banach space X, with a sequence $\{X_n\}$ of finite dimensional subspaces satisfying $\lim_{n\to\infty} \dim X_n = \infty$ and a constant C, such that for every finite rank operator $F \in \mathcal{B}(X)$

$$\|F - I\|_{|X_n} \geqslant 1 - \frac{C\|F\|}{\log \dim X_n}, \qquad (n = 1, 2, \ldots);$$

hence X does not have the approximation property. Closed subspaces of c_0 are good candidates for Banach spaces lacking the approximation property. This follows by results by Grothendieck and Figiel (see [F]) on the factorization of compact operators, which however is beyond the scope of this work.

Various authors subsequently simplified the construction of P. Enflo and using still the same basic ideas obtained other Banach spaces, which do not have the approximation property. The proof presented here is due to A. M. Davie. The separable Banach space X constructed by Davie is not a subspace of c_0 like Enflo's example, but a space which can be identified with a closed subspace of ℓ_∞. The existence of separable Banach spaces without the approximation property has also other applications. For example, this fact implies, that there exist Banach spaces with bases, whose dual is separable and fails to have the approximation property. This latter result can be used to show that there exists a separable Banach space X with the approximation property, having no basis and having X^* separable (see [S] II, page 308).

4.2 Idea of the construction

In this section we will present a series of results explaining the basic ideas, which will be used in the construction of a Banach space without the approximation property. We will denote by τ-topology the topology of uniform convergence on compact subsets of X (Definition 2.12) generated by the seminorms $\|T\|_K = \sup_{x\in K} \|Tx\|$, where K ranges over all the compact sets of X. Since the definition of the approximation property in the sense of Grothendieck is given in terms of uniform convergence on compact subsets, in the next theorem which is based on the Banach-Steinhaus theorem, we will start by investigating this notion.

THEOREM 4.1 *Let X be a Banach space, $\{T_n\}_{n=1}^{\infty}$ a sequence of bounded linear operators on $\mathcal{B}(X)$, $T \in \mathcal{B}(X)$ and D a dense subset of X. Then the following statements are equivalent:*
(i) $T_n \to T$ uniformly on every compact subset $K \subset X$, or $\|T_n - T\|_K \to 0$ as $n \to \infty$,
(ii) $\sup_n \|T_n\| < \infty$ and $T_n x \to Tx$ for all $x \in D$

Proof:
To prove that (i) implies (ii), let $\mathcal{M} = \{K \subset X :$ K is compact$\}$, and suppose that for every K in $\mathcal{M}$ we have $\|T_n - T\|_K \to 0$ as $n \to \infty$. Since the singleton $\{x\}$ is compact this implies that $T_n x \to Tx$ for all $x \in X$, and consequently for all $x \in D$. Besides if $\{T_n\}_{n=1}^{\infty}$ is a sequence of bounded linear operators on X converging to T on compact subsets of X, then by the compactness of $\{x\}$, the sequence of real numbers $\{\|T_n x\|\}_{n=1}^{\infty}$ is bounded for every $x \in X$, thus by the Banach-Steinhaus theorem $\sup_n \|T_n\|$ is also bounded and the implication is proved.
Let us prove that (ii) implies (i). Let $\sup_n \|T_n\| \leqslant C$. If $T_n x \to Tx$ for all $x \in D$, then by the continuity of the norm

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leqslant \sup_n \|T_n\| \|x\| = C\|x\|. \tag{4.1}$$

Now in order to prove that $\|T_n - T\|_K \to 0$, as $n \to \infty$, consider a $K \in \mathcal{M}$. Since K is compact there are y_j's in X and $m \in \mathbb{N}$, such that $K \subset \bigcup_{j=1}^{m} B(y_j, \frac{\varepsilon}{3})$. Since D is dense in X, there exist x_j's $\in D$ such that

$$\bigcup_{j=1}^{m} B(y_j, \frac{\varepsilon}{3}) \subset \bigcup_{j=1}^{m} B(x_j, \frac{\varepsilon}{2}).$$

For any $x_j \in D$ we have that $T_n x_j \to T x_j$, or equivalently, there is a $N \in \mathbb{N}$, such that

$$\|(T_n - T)x_j\| \leqslant \frac{\varepsilon}{2}, \text{ for } n \geqslant N, \qquad (j = 1, 2, \ldots, m).$$

Let x be an arbitrary element in $K \subset X$, since D is dense in X then there is an $x_j \in D$ such that

$$\|x - x_j\| < \frac{\varepsilon}{4C}.$$

Thus,

$$\sup_{x\in K}\|(T_n-T)x\| = \sup_{x\in K}\|(T_n-T)x_j+(T_n-T)(x-x_j)\| \leqslant$$
$$\sup_{x\in K}\|(T_n-T)x_j\| + \sup_{x\in K}\|(T_n-T)(x-x_j)\| \leqslant$$
$$\frac{\varepsilon}{2} + \|T_n-T\|\frac{\varepsilon}{4C} \leqslant \frac{\varepsilon}{2} + \big(\|T_n\|+\|T\|\big)\frac{\varepsilon}{4C} < \varepsilon$$

or $\|T_n - T\|_K \to 0$. □

THEOREM 4.2 *Let X be a Banach space, let D be a dense subset of X, and let $\mathcal{F}(X)$ be the subspace of all finite rank operators in $\mathcal{B}(X)$. Let $\mathcal{A}(D)$ be defined as*

$$\mathcal{A}(D) = \left\{A \in \mathcal{B}(X) : A(y) = \sum_{j=1}^{N}\Phi_j(y)x_j \text{ for } x_j \in D, y \in X \text{ and } \Phi_j \in X^*\right\}.$$

Then $\mathcal{A}(D)$ is dense in $\mathcal{F}(X)$ in the operator norm topology and consequently in the (weaker) topology of uniform convergence on compact subsets of X.

Proof:
The set $\mathcal{A}(D)$, which also appears in Enflo's proof is called the set of finite expansion operators. We have to prove that given any $F \in \mathcal{F}(X)$ and any $\varepsilon > 0$, there is an operator $A \in \mathcal{A}(D)$, such that $\|A - F\| < \varepsilon$. Let $F \in \mathcal{F}(X)$ and $y \in X$ then by Definition 2.3

$$F(y) = \sum_{j=1}^{N}\Phi_j(y)y_j, \qquad y_j \in X, \qquad \Phi_j \in X^*$$

Since D is dense in X, for each $y_j \in X$ we can choose $x_j \in D$ such that

$$\|x_j - y_j\| \leqslant \frac{\varepsilon}{N\cdot\max\|\Phi_j\|}.$$

For each $y \in X$ let

$$A(y) = \sum_{j=1}^{N}\Phi_j(y)x_j, \qquad x_j \in D, \qquad \Phi_j \in X^*.$$

Then, by definition of the operator norm

$$\|A-F\| = \sup_{\|y\|\leqslant 1}\left|\sum_{j=1}^{N}\Phi_j(y)(x_j-y_j)\right| \leqslant \sup_{\|y\|\leqslant 1}\sum_{j=1}^{N}\|\Phi_j\|.\|x_j-y_j\| < \varepsilon$$

□

As proved in Theorem 2.17 $(\mathcal{B}(X);\tau)$ is a locally convex space. By the Hahn-Banach theorem whenever Y is a subspace of $\mathcal{B}(X)$ and $T \in \mathcal{B}(X)$ such that T does not belong to the closure of Y, there is a functional $\Phi : \mathcal{B}(X) \to \mathbb{C}$ linear and continuous in τ such that $\Phi(Y) = \{0\}$ and $\Phi(T) = 1$. The converse being immediate by the continuity of the linear functional Φ. The next theorem provides a natural way to construct linear functionals on $\mathcal{B}(X)$, which annihilate the finite rank operators, but not the identity operator.

THEOREM 4.3 *Let X be a separable Banach space and $\{x_1, x_2, \ldots, x_n, \ldots\}$ a countable set of unit vectors in X such that their linear span D is dense in X. Let $\alpha_n \in X^*$ have the property that $\alpha_n(x_j) = \delta_{nj}$ ($\{x_n, \alpha_n\}$ is usually called a biorthogonal system), and suppose that $\sup\limits_n \|\alpha_n\| < \infty$. Let $\mathcal{L}$ be defined by*

$$\mathcal{L} : \mathcal{B}(X) \to \ell_\infty \qquad \mathcal{L}(T) = \{\alpha_n(Tx_n)\}_{n=1}^{\infty}.$$

Then
(i) $\mathcal{L}$ is continuous in the operator norm topology,
(ii) $\mathcal{L}(\mathcal{F}(X)) \subset c_0$, and $\mathcal{L}(I) = (1, 1, \ldots)$, where I is the identity operator in $\mathcal{B}(X)$,
(iii) Consequently, given a $\gamma \in (\ell_\infty)^*$, such that $\gamma(c_0) = \{0\}$, and $\gamma(1, 1, \ldots) = 1$ the functional $\beta \in (\mathcal{B}(X))^*$ defined by $\beta(T) = \gamma(\mathcal{L}(T))$, satisfies $\beta(I) = 1$ and $\beta(\mathcal{F}(X)) = 0$.

Proof:
(i) To see that $\mathcal{L}$ is continuous in the operator norm topology, let $T \in \mathcal{B}(X)$. Since $\|x_n\| = 1$ and $\sup\limits_n \|\alpha_n\|$ are bounded we have

$$\|\mathcal{L}(T)\|_\infty = \sup_n |\alpha_n(Tx_n)| \leqslant \sup_n \|\alpha_n\| \|T\|.$$

(ii) Any operator A in $\mathcal{A}(D)$ is a finite sum of rank one operators that is

$$A(y) = \sum_{j=1}^{N} \Phi_j(y) x_j = \sum_{j=1}^{N} T_j(y),$$

where T_j is a rank one operator in $\mathcal{A}(D)$ given by $T_j(y) = \Phi_j(y)x_j$. By Theorem 4.2 $\mathcal{A}(D)$ is dense in $\mathcal{F}(X)$ in the operator norm topology. By (i) $\mathcal{L}$ is a bounded linear operator on $\mathcal{B}(X)$. Thus, since c_0 is a closed subspace of ℓ_∞ it suffices to prove that for a rank one operator $T \in \mathcal{A}(D)$ we have $\mathcal{L}(T) \in c_0$. If T is a rank one operator in $\mathcal{A}(D)$ and $y \in X$ then

$$Ty = \Phi(y)x \qquad x \in D \qquad \Phi \in X^*.$$

By the definition of the operator $\mathcal{L}$ and $Tx_n = \Phi(x_n)x$ we have

$$\mathcal{L}(T) = \{\alpha_n(Tx_n)\}_{n=1}^{\infty} = \{\Phi(x_n)\alpha_n(x)\}_{n=1}^{\infty} \in \ell^\infty. \tag{4.2}$$

Since D is the linear span of $\{x_n\}$, x can be written as

$$x = \sum_{n=1}^{N} z_n x_n \qquad z_n \in \mathbb{C}$$

Since Φ is bounded and $\alpha_n(x_k) = 0$ for all $n > N$ by (4.2) it is clear that $\mathcal{L}(T) \in c_0$ consequently by taking linear combinations and by the reasoning in the beginning of the proof $\mathcal{L}(\mathcal{A}(D))$ and $\mathcal{L}(\mathcal{F}(X)) \subset c_0$. It is clear from the definition of $\mathcal{L}$ that $\mathcal{L}(I) = (1, 1, \ldots)$.

(iii) Given a $\gamma \in (\ell_\infty)^*$, such that $\gamma(c_0) = \{0\}$, $\gamma(1, 1, 1, ...) = 1$ let $\beta \in (\mathcal{B}(X))^*$ defined as $\beta = \gamma \circ \mathcal{L}$, and apply the definition of β to obtain

$$\beta(T) = \gamma(\mathcal{L}(T)) = \gamma\Big(\{\alpha_n(Tx_n)\}_{n=1}^\infty\Big).$$

Since $c_0 \subseteq \mathrm{Ker}(\gamma)$ and $\mathcal{L}(\mathcal{F}(X) \subset c_0$ it follows that $\beta(\mathcal{F}(X)) = 0$. Similarly, one can easily see that

$$\beta(I) = \gamma(\mathcal{L}(I)) = \gamma\{\alpha_n(x_n)\}_{n=1}^\infty = \gamma(1, 1, 1, ...) = 1.$$

□

The most common examples of functionals γ as considered in the above theorem are the so-called Banach limits. Banach limits are special extensions of the notion of limit to sequences which do not converge. The existence of these functionals is a direct application of the Hahn-Banach theorem. There are several different ways of defining such functionals, which will be illustrated below.

EXAMPLE 4.4 If $x \in c$, where $c \subset \ell_\infty$ is the subspace of all convergent sequences in ℓ_∞ then we can define the functional

$$\gamma : c \to \mathbb{C} \text{ by } \gamma(x) = \lim x_n.$$

The functional γ is well defined for all elements $x \in c$ and satisfies $|\gamma(x)| \leqslant \|x\|$. By the Hahn-Banach theorem, γ can be extended to a linear functional $\widehat{\gamma}$ on the whole of ℓ_∞ with $\|\gamma\| = \|\widehat{\gamma}\| = 1$. Also, it is immediate that for any sequence $x_0 \in c_0$, $\lim(x_0) = \gamma(c_0) = 0$ thus $c_0 \subset \mathrm{Ker}(\widehat{\gamma})$ (see [C], page 82).

EXAMPLE 4.5 Another way to define a Banach limits is to consider

$$\gamma_n(x) = \frac{x_1 + \ldots + x_n}{n}$$

and let

$$s = \{x \in \ell_\infty : \lim \gamma_n(x) = \gamma(x) \text{ exists}\}.$$

If p is seminorm on s defined as

$$p(x) = \limsup_{n\to\infty} |\gamma_n(x)| \qquad \text{we have} \qquad |\gamma(x)| \leqslant p(x)$$

and by the Hahn-Banach theorem we can extend the functional γ from s to the whole of ℓ_∞ (see [R], page 85).

EXAMPLE 4.6 The third way to define a Banach limit is important for our construction of a Banach space without the approximation property. Let $\{M_k\}$ be a family of subsets of the natural numbers such that $\lim\limits_{k\to\infty} |M_k| = \infty$ with $|M_k|$ denoting the cardinality of the set M_k. Define for each $x = (x_1, x_2, \ldots) \in \ell_\infty$

$$\gamma_k(x) = \frac{1}{|M_k|} \sum_{j\in M_k} x_j.$$

As above, let $s = \{x \in \ell_\infty : \lim_{k\to\infty} \gamma_k(x) = \gamma(x) \text{ exists}\}$ and apply the Hahn-Banach theorem to obtain a Banach limit γ on ℓ_∞. By Theorem 4.3, γ induces a nonzero functional β on $\mathcal{B}(X)$ and the difficulty consists in showing that this functional is continuous with respect to the topology τ. To this end, we will approximate β by the functionals β_k induced by γ_k in a similar way. Recall that if X is a Banach space and $\{x_n, \alpha_n\}$ is the biorthogonal system where $\{x_n\} \subset X$, $\{\alpha_n\} \subset X^*$ are as in Theorem 4.3 the functionals $\beta_k = \gamma_k \circ \mathcal{L}$ have the form

$$\beta_k(T) = \frac{1}{|M_k|} \sum_{n \in M_k} \alpha_n(Tx_n). \tag{4.3}$$

The values on the right hand side of (4.3) are usually called the average trace of the operator T over the set $\{x_j\}_{j \in M_k}$ where the subset M_k is a finite non-empty subset of the index set $\mathbb{N}$. As pointed out here the functionals β_k play an essential role for our purposes.

4.3 The Banach Space X

For $n = 0, 1, 2, \ldots$ let G_n be the commutative groups with $3 \cdot 2^n$ elements and let $G = \bigcup_{n=0}^{\infty} G_n$, be the disjoint union of the groups G_n. Thus

$$G = \{g_1^0, g_2^0, g_3^0, g_1^1, \ldots, g_6^1, \ldots, g_1^n, \ldots, g_{3\cdot 2^n}^n, \ldots\}.$$

Let $\ell_\infty(G)$ be the Banach space of all functions $x : G \to \mathbb{C}$ such that

$$\|x\|_{\ell_\infty(G)} = \|\{x(g)\}\|_\infty = \sup_{g \in G} |x(g)| < \infty \tag{4.4}$$

and give $\ell_\infty(G)$ the structure of a linear space in the usual way by defining pointwise addition and multiplication by scalars. We want now to define a special subspace of $\ell_\infty(G)$.

For each n, let H_n be the set of characters of G_n, that is, the set of homomorphisms x of G_n into S^1

$$H_n = \{x : G_n \to S^1\},$$

where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. By Theorem A.2 H_n is a commutative group. For $g \in G_n$ and for every $x \in H_n$ we define the inverse $x^{-1}(g) = x(g^{-1}) = \overline{x(g)}$, where $\overline{x(g)}$ is the complex conjugate of $x(g)$. Since G_n is abelian we have by Theorem A.3 that $|H_n| = |G_n| = 3 \cdot 2^n$ for each $n \in \mathbb{N}$. We will use both these facts in the following proofs (see also Appendix A).

We can partition the elements of H_n in two disjoint subsets W_n^+ and W_n^- with cardinality 2^n and 2^{n+1} respectively such that $W_n^+ \cup W_n^- = H_n$. For each n let us denote by $\sigma_1^n, \ldots, \sigma_{2^n}^n$ the elements of W_n^+ and by $\tau_1^n, \ldots, \tau_{2^{n+1}}^n$ the elements W_n^- respectively. Now we will define the set of functions, which span the space X. For each $n = 0, 1, 2, \ldots$ and each $j = 1, \ldots, 2^n$ we define $x_j^n : G \to S^1$ by

$$x_j^n(g) = \begin{cases} \tau_j^{n-1}(g) & \text{if } \; g \in G_{n-1} \qquad (n \geqslant 1) \\ \varepsilon_j^n \sigma_j^n(g) & \text{if } \; g \in G_n \\ 0 & \text{if } \; g \in G \setminus (G_{n-1} \cup G_n), \end{cases} \tag{4.5}$$

where $\varepsilon_j^n = \pm 1$ will be chosen below. We shall consider the closed subspace X of $\ell_\infty(G)$ defined by

$$X = \overline{\bigvee \{x_j^n, n \geqslant 0, j = 1, 2, \ldots, 2^n\}}, \tag{4.6}$$

where the x_j^n's are the functions in (4.5). It is important to notice that the space X will depend on the way we partition the elements of each group H_n as well as on the choices of ε_j^n. Note that there are 2^n distinct σ_j^n's and 2^n different τ_j^{n-1}'s and all are used in the definition of x_j^n. These x_j^n will play a role similar to the vectors x_n defined in Theorem 4.3. Also by (4.5) $|x_j^n(g)| = 1$ if $g \in G_{n-1} \cup G_n$ and that $x_j^n(g) \neq 0$ only for $g \in G_{n-1} \cup G_n$. Naturally, the subsets where the $x_j^n \neq 0$ cannot be pairwise disjoint, since if they were we could form a basis for X. We will keep the notations H_n, σ_j^n, τ_j^n and ε_j^n in what follows.

PROPOSITION 4.7 *Let X be the space defined in (4.6). Then X is a separable Banach space.*

Proof:
An element x in the linear span of $\{x_j^n\}$ can be written as

$$x(g) = \sum_{n=1}^{M} \sum_{j=1}^{2^n} z_j^n x_j^n(g) \qquad z_j^n \in \mathbb{C}, g \in G \qquad \text{where } M \text{ is a positive integer.} \tag{4.7}$$

Since the linear span of $\{x_j^n\}$ is dense in X and each $z_j^n \in \mathbb{C}$ can be approximated by a corresponding complex number r_j^n with rational real and imaginary parts arbitrarily near z_j^n then X is separable. Since X is a closed subspace of a Banach space it also a Banach space. □

We have defined the Banach space X and by using the basic ideas from Section 4.2 we will prove in several steps the following result

THEOREM 4.8 *Let X be the separable complex Banach space defined above. Then there exists a choice of ε_j^n in (4.5) and partitions W_n^+ and W_n^- of H_n such that X does not have the approximation property.*

4.4 The biorthogonal functionals $\alpha_j^n \in X^*$

It is important to notice at this point that the characters of each group G_n are mutually orthogonal by Theorem A.5. Since the groups G_n are pairwise disjoint, it follows from (4.5) that $\{x_j^n\}$ is an orthogonal system in $\ell_2(G)$ with the scalar product defined by

$$(x, z) = \sum_{g \in G} x(g)\overline{z(g)}. \tag{4.8}$$

for $x, z \in X$.

Now for $n \geqslant 0$ and $1 \leqslant j \leqslant 2^n$ we define for each $x \in X$ the functionals

$$\alpha_j^n(x) = \frac{1}{3 \cdot 2^n} \sum_{g \in G_n} \varepsilon_j^n \sigma_j^n(g^{-1}) x(g) = \frac{1}{3 \cdot 2^n} \sum_{g \in G_n} \varepsilon_j^n \overline{\sigma_j^n(g)} x(g). \tag{4.9}$$

PROPOSITION 4.9 *The functionals $\alpha_j^n : X \to \mathbb{C}$ are bounded linear functionals on X with $\|\alpha_j^n\| \leqslant 1$.*

Proof:
Let x be an element of X. Since $|\varepsilon_j^n| = 1$ and $|\sigma_j^n(g)| = 1$ for all $g \in G_n$ we have

$$|\alpha_j^n(x)| \leqslant \frac{1}{3 \cdot 2^n} \sum_{g \in G} |x(g)| \leqslant \sup_{g \in G} |x(g)| = \|x\|_\infty$$

for any $x \in X$, and the proof is complete. □

PROPOSITION 4.10 *(i) For each $n \geqslant 0$ and $j = 1, \ldots, 2^n$ the system $\{x_j^n, \alpha_j^n\}$, is a biorthogonal system or equivalently $\alpha_j^n(x_i^m) = \delta_{ij}\delta_{mn}$.*
(ii) If $n \geqslant 1$, $1 \leqslant j \leqslant 2^n$ α_j^n satisfies the following identity

$$\alpha_j^n(x) = \frac{1}{3 \cdot 2^{n-1}} \sum_{g \in G_{n-1}} \tau_j^{n-1}(g^{-1})x(g) = \frac{1}{3 \cdot 2^{n-1}} \sum_{g \in G_{n-1}} \overline{\tau_j^{n-1}(g)}x(g). \quad (4.10)$$

Proof:
(i) By construction we have that $|G_n| = 3 \cdot 2^n$ and by (4.5) we have for $g \in G_n$, $x_j^n(g) = \varepsilon_j^n \sigma_j^n(g)$, with $|\sigma_j^n(g)| = 1$ and $(\varepsilon_j^n)^2 = 1$. Apply (4.9) to $x = x_j^n$ to obtain

$$\alpha_j^n(x_j^n) = \frac{1}{3 \cdot 2^n} \sum_{g \in G_n} \varepsilon_j^n \overline{\sigma_j^n(g)} x_j^n(g) = \frac{1}{3 \cdot 2^n} \sum_{g \in G_n} (\varepsilon_j^n)^2 |\sigma_j^n(g)|^2 = 1. \quad (4.11)$$

By (4.9) we have for an arbitrary function x_i^m

$$\alpha_j^n(x_i^m) = \frac{1}{3 \cdot 2^n} \varepsilon_j^n \sum_{g \in G_n} \overline{\sigma_j^n(g)} x_i^m(g). \quad (4.12)$$

We distinguish the following cases.
Case 1: If $n \notin \{m, m-1\}$, then $G_n \cap (G_{m-1} \cup G_m) = \varnothing$, by the definition of x_j^n, $x_i^m(g) = 0$ for $g \notin G_{m-1} \cup G_m$ we infer that $\alpha_j^n(x_i^m) = 0$.
Case 2: If $n = m-1$, then by (4.5) $x_i^m(g) = \tau_i^{m-1}(g) = \tau_i^n(g)$ for $g \in G_{m-1} = G_n$, from (4.12) and the orthogonality of the characters of G_n we infer:

$$\alpha_j^n(x_i^m) = \alpha_j^n(x_j^{n+1}) = \frac{1}{3 \cdot 2^n} \varepsilon_j^n \sum_{g \in G_n} \overline{\sigma_j^n(g)} \tau_i^n(g) = 0.$$

Case 3: If $n = m$, then since $x_i^m(g) = \varepsilon_i^m \sigma_i^m(g) = \varepsilon_i^n \sigma_i^n(g)$ for $g \in G_m = G_n$ from (4.12) and the orthogonality of the characters of G_n we obtain

$$\alpha_j^n(x_i^m) = \frac{1}{3 \cdot 2^n} \varepsilon_j^n \varepsilon_i^n \sum_{g \in G_n} \sigma_i^n(g) \overline{\sigma_j^n(g)} = 0 \qquad \text{for } i \neq j. \quad (4.13)$$

This proves (i).
(ii) Let us denote the right hand side of (4.10) by $\widetilde{\alpha}_j^n(x_j^n)$ and calculate its value for $x = x_j^n$ and $g \in G_{n-1}$. Note that $x_j^n(g) = \tau_j^{n-1}(g)$. Then

$$\widetilde{\alpha}_j^n(x_j^n) = \frac{1}{3 \cdot 2^{n-1}} \sum_{g \in G_{n-1}} \overline{\tau_j^{n-1}(g)} x_j^n(g) = \frac{1}{3 \cdot 2^{n-1}} \sum_{g \in G_{n-1}} |\tau_j^{n-1}(g)|^2 = 1.$$

so for $x = x_j^n$ we have $\alpha_j^n(x_j^n) = \widetilde{\alpha}_j^n(x_j^n) = 1$. Moreover if $x = x_i^m$ with i, m arbitrary

$$\widetilde{\alpha}_j^n(x_i^m) = \frac{1}{3 \cdot 2^{k-1}} \sum_{g \in G_{k-1}} x_i^m(g)\overline{\tau_j^{n-1}(g)}, \tag{4.14}$$

then by a similar computation we obtain again that $\widetilde{\alpha}_j^n(x_i^m) = 0$ for all $n \neq m$ and all i, j as well as for $n = m$ and $i \neq j$. This proves our assertion above that $\alpha_j^n(x_i^m) = \widetilde{\alpha}_j^n(x_i^m) = \delta_{ij}\delta_{mn}$. Hence by linearity and continuity $\alpha_j^n = \widetilde{\alpha}_j^n$. □

4.5 The functionals β_n on $\mathcal{B}(X)$

For each $n \geqslant 0$ we define the subspace X_n of X as the linear span of $\{x_j^n\}_{j=1}^{2^n}$ by

$$X_n = \bigvee \{x_1^n, \ldots, x_{2^n}^n\} \qquad \dim X_n \leqslant 2^n$$

For $n \geqslant 0$ we will define the average trace functionals $\beta_n \in (\mathcal{B}(X))^*$ by

$$\beta_n(T) = \frac{1}{2^n} \sum_{j=1}^{2^n} \alpha_j^n(Tx_j^n) \qquad x_j^n \in X, \tag{4.15}$$

where the α_j^n's are defined by (4.9) or equivalently by (4.10). Since $\{\alpha_j^n, x_j^n\}$ is a biorthogonal system we can use Theorem 4.3 and the observation following it to see that each β_n is induced by a functional γ_n on ℓ_∞ of the following form

$$\beta_n(T) = \gamma_n(\mathcal{L}(T)) = \gamma_n\{\alpha_j^n(Tx_j^n)\}_{n=1}^{\infty} \qquad x_j^n \in X.$$

Clearly and in accordance with notations used in the previous and in this section $\beta_n = \gamma_n \circ \mathcal{L}$, where the γ_n is a functional on ℓ_∞ of the type described in the Example 4.6, with $|M_n| = |X_n| = 2^n$. These functionals can be easily computed explicitly using an enumeration of the index set $\{(j,n), n \geqslant 0, 1 \leqslant j \leqslant 2^n\}$. Recall that our goal is to construct a functional β which is τ-continuous and induced by a Banach limit. In other words, β needs to be approximated by the functionals β_n in an appropriate way. To this end we need a good estimate of the consecutive difference $\beta_{n+1} - \beta_n$. Technically speaking, this is the heart of the proof and we will need several technical lemmas gathered in the next section in order to produce the desired estimate.

Using (4.9) for α_j^n and the linearity of the operator T we obtain

$$\beta_n(T) = \frac{1}{2^n} \sum_{j=1}^{2^n} \alpha_j^n(Tx_j^n) = \frac{1}{2^n} \sum_{j=1}^{2^n} \frac{1}{3 \cdot 2^n} \sum_{g \in G_n} \varepsilon_j^n \sigma_j^n(g^{-1})(Tx_j^n)(g) =$$

$$\frac{1}{3 \cdot 2^{2n}} \sum_{g \in G_n} T\Bigg(\sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) x_j^n\Bigg)(g) \qquad (T \in \mathcal{B}(X), n = 0, 1, 2, \ldots). \tag{4.16}$$

By applying (4.10) we obtain

$$\beta_{n+1}(T) = \frac{1}{2^{n+1}} \sum_{j=1}^{2^{n+1}} \alpha_j^{n+1}(Tx_j^{n+1}) = \frac{1}{2^{n+1}} \sum_{j=1}^{2^{n+1}} \frac{1}{3 \cdot 2^n} \sum_{g \in G_n} \tau_j^n(g^{-1})(Tx_j^{n+1})(g) =$$

$$\frac{1}{3 \cdot 2^{2n+1}} \sum_{g \in G_n} T\left(\sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}) x_j^{n+1} \right)(g) \qquad (T \in \mathcal{B}(X), n = 0, 1, 2, \ldots). \tag{4.17}$$

Hence by (4.16) and (4.17) for each $n = 0, 1, 2, \ldots$ we have

$$\beta_{n+1}(T) - \beta_n(T) = \frac{1}{3 \cdot 2^n} \sum_{g \in G_n} T(\phi_g^n), \tag{4.18}$$

where, for the sake of simplicity, we have denoted

$$\phi_g^n = \frac{1}{2^{n+1}} \sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}) x_j^{n+1} - \frac{1}{2^n} \sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) x_j^n \in X, g \in G_n. \tag{4.19}$$

So each ϕ_g^n is the difference of an element of X_{n+1} and one of X_n and in particular elements of the Banach space X. Use (4.18) and recall that the group G_n has $3 \cdot 2^n$ elements to obtain the following inequality

$$|\beta_{n+1}(T) - \beta_n(T)| = \left| \frac{1}{3 \cdot 2^n} \sum_{g \in G_n} T(\phi_g^n) \right| \leqslant \sup_{g \in G_n} \|T\phi_g^n\|, \qquad g \in G_n \tag{4.20}$$

In the next Proposition using the definition of x_j^n given by (4.5) we will calculate $\phi_g^n(h)$ for $g \in G_n$ and for $h \in G$.

PROPOSITION 4.11 *Let $n \geqslant 0$ and $g \in G_n$. Then for $h \in G = \bigcup_{n=0}^{\infty} G_n$ we have*

$$\phi_g^n(h) = \begin{cases} (i) & \dfrac{1}{2^{n+1}} \displaystyle\sum_{j=1}^{2^{n+1}} \varepsilon_j^{n+1} \tau_j^n(g^{-1}) \sigma_j^{n+1}(h) & (h \in G_{n+1}) \\ (ii) & \dfrac{1}{2^n} \left(\dfrac{1}{2} \displaystyle\sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}h) - \sum_{j=1}^{2^n} \sigma_j^n(g^{-1}h) \right) & (h \in G_n) \\ (iii) & -\dfrac{1}{2^n} \displaystyle\sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) \tau_j^{n-1}(h) & (h \in G_{n-1}, n > 0) \\ (iv) & 0 & \text{otherwise} \end{cases} \tag{4.21}$$

(v) In particular $|\phi_g^{n-1}(h)| = |\phi_{h^{-1}}^n(g^{-1})|$ for $g \in G_{n-1}, h \in G_n$.

Proof:
(i) We start by considering an $h \in G_{n+1}$. From (4.5) we have

$$x_j^{n+1}(h) = \begin{cases} \tau_j^n(h) & h \in G_n \\ \varepsilon_j^{n+1} \sigma_j^{n+1}(h) & h \in G_{n+1} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x_j^n(h) = \begin{cases} \tau_j^{n-1}(h) & h \in G_{n-1} \\ \varepsilon_j^n \sigma_j^n(h) & h \in G_n \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for any $g \in G_n$

$$\phi_g^n(h) = \frac{1}{2^{n+1}} \sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}) x_j^{n+1}(h) - \frac{1}{2^n} \sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) x_j^n(h) = \frac{1}{2^{n+1}} \sum_{j=1}^{2^{n+1}} \varepsilon_j^{n+1} \tau_j^n(g^{-1}) \sigma_j^{n+1}(h),$$

where we have used the fact that $x_j^n(h) = 0$ for $h \in G_{n+1}$.
(ii) for $h \in G_n$ we have

$$\begin{aligned} \phi_g^n(h) &= \frac{1}{2^{n+1}} \sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}) x_j^{n+1}(h) - \frac{1}{2^n} \sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) x_j^n(h) \\ &= \frac{1}{2^{n+1}} \sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}) \tau_j^n(h) - \frac{1}{2^n} \sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) \sigma_j^n(h) \\ &= \frac{1}{2^n} \left(\frac{1}{2} \sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}h) - \sum_{j=1}^{2^n} \sigma_j^n(g^{-1}h) \right). \end{aligned}$$

(iii) For $h \in G_{n-1}, n \geqslant 1$ $x_j^n(h) = \tau_j^{n-1}(h)$ and the first sum in (4.19) is identically zero. Thus we get

$$\phi_g^n(h) = -\frac{1}{2^n} \sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) x_j^n(h) = -\frac{1}{2^n} \sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) \tau_j^{n-1}(h)$$

(iv) Clear.
(v) The verification of this equality is straightforward and has been omitted. □

4.6 Probability and partition lemmas

We start by proving two inequalities which we will need for the proof of Lemma 4.13

LEMMA 4.12 *For $t \in \mathbb{R}$ we have:*

$$\cosh t \leqslant e^{t^2}, \tag{4.22}$$

and

$$\frac{1}{3} e^{2t} + \frac{2}{3} e^{-t} \leqslant e^{2t^2}. \tag{4.23}$$

Proof:
Note that

$$\cosh t = \frac{e^t + e^{-t}}{2} = \frac{1}{2}(1 + t + \frac{t^2}{2} + \dots) + \frac{1}{2}(1 - t + \frac{t^2}{2} - \dots) =$$
$$\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \leqslant \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} = e^{t^2},$$

which proves (4.22).
Since $2t \leqslant 2t^2$ and $-t \leqslant 2t^2$ for $t \geqslant 1$, the following inequality

$$\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \leqslant e^{2t^2} \qquad t \geqslant 1$$

holds.
For $t < 1$, we have that

$$e^{2t} + 2e^{-t} = \sum_{n=0}^{\infty} \frac{(2t)^n + 2(-t)^n}{n!} =$$
$$\sum_{n=0}^{\infty} \frac{(2t)^{2n} + 2(-t)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(2t)^{2n+1} + 2(-t)^{2n+1}}{(2n+1)!} =$$
$$3 + 3t^2 + t^3 + \sum_{n=2}^{\infty} \left(\frac{2^{2n} + 2}{(2n)!} + \frac{2^{2n+1} - 2}{(2n+1)!}t \right) t^{2n}$$

Now, since $t < 1$, for the factors inside brackets the following inequalities hold

$$\frac{2^{2n} + 2}{(2n)!} + \frac{2^{2n+1} - 2}{(2n+1)!}t < \frac{2^{2n+1}}{(2n)!} + \frac{2^{2n+1}}{(2n+1)!} =$$
$$\frac{2^{2n+1}(2n+1) + 2^{2n+1}}{(2n+1)!} = \frac{2^{2n+1}(2n+2)}{(2n+1)!} <$$
$$\frac{3.2^{2n}(2n+2)}{(2n+1)!} = \frac{3.2^n}{n!} . \frac{4^n}{(n+2)\dots(2n+1)} < \frac{3 \cdot 2^n}{n!}$$

for all $n \geqslant 2$.
Considering that for $t \leqslant 1$ the inequality $t^3 \leqslant 0 \leqslant 3t^2$ is valid and is equivalent to $3t^2 + t^3 \leqslant 6t^2$ we get that

$$e^{2t} + 2e^{-t} \leqslant 3 + (3t^2 + t^3) + 3\sum_{n=2}^{\infty} \frac{2^n}{n!} t^{2n}$$
$$\leqslant 3 + 6t^2 + 3\sum_{n=2}^{\infty} \frac{2^n}{n!} t^{2n} = \sum_{n=0}^{\infty} \frac{2^n}{n!} t^{2n} = 3e^{2t^2}$$

which proves (4.23). □

The following Lemma was originally proved by Kahane in [Ka]. It will be used to show that there are choices of ε_j^n and suitable partitions of each character group H_n in such a way that we will be able to get good estimates for the differences $|\beta_{n+1} - \beta_n|$.

LEMMA 4.13 *(a) Let $\alpha_1, \ldots, \alpha_N$ be complex numbers and let $\varepsilon_1, \ldots, \varepsilon_N$ be chosen independently at random, taking the values $+1$ or -1, each with probability $\frac{1}{2}$. Then*

$$P\left[\left|\sum_{j=1}^{N} \alpha_j \varepsilon_j\right| > 3\sqrt{3}\left(\sum_{j=1}^{N} |\alpha_j|^2 \log N\right)^{\frac{1}{2}}\right] < \frac{3\sqrt{3}}{N^3}. \tag{4.24}$$

(b) Let $\alpha_1, \ldots, \alpha_N$ be complex numbers and let $\rho_1, \ldots, \rho_N$ be chosen independently at random, taking the values $+2$ or -1, each with probability $\frac{1}{3}$ and $\frac{2}{3}$ respectively. Then

$$P\left[\left|\sum_{j=1}^{N} \alpha_j \rho_j\right| > 3\sqrt{3}\left(\sum_{j=1}^{N} |\alpha_j|^2 \log N\right)^{\frac{1}{2}}\right] < \frac{3\sqrt{3}}{N^3}. \tag{4.25}$$

Proof:
(a) In order to prove the above lemma, part (a), assume first that the α_j's are real and $\sum_{j=1}^{N} |\alpha_j|^2 = 1$. Let (Ω, P) be the probability space, X a (measurable) random variable from Ω taking the values $+1$ and -1, and let $\mathcal{E}(X)$ be the expectation of the random variable X. Then

$$\mathcal{E}(X) = P(X = 1) + P(X = -1) = 0.$$

Since X is measurable so is e^X. Using the inequality

$$e^{|t|} \leqslant e^t + e^{-t}, \qquad t \in \mathbb{R},$$

we have for all $t \in \mathbb{R}$ and for $\lambda > 0$,

$$\mathcal{E}\left(e^{\lambda\left|\sum_{j=1}^{N} \alpha_j \varepsilon_j\right|}\right) \leqslant \mathcal{E}\left(e^{\lambda \sum_{j=1}^{N} \alpha_j \varepsilon_j}\right) + \mathcal{E}\left(e^{-\lambda \sum_{j=1}^{N} \alpha_j \varepsilon_j}\right) =$$
$$\prod_{j=1}^{N} \mathcal{E}(e^{\lambda \alpha_j \varepsilon_j}) + \prod_{j=1}^{N} \mathcal{E}(e^{-\lambda \alpha_j \varepsilon_j}) = 2\prod_{j=1}^{N}\left(\frac{1}{2}e^{\lambda \alpha_j} + \frac{1}{2}e^{-\lambda \alpha_j}\right). \tag{4.26}$$

Using the independence of the random variables $e^{\lambda \alpha_j \varepsilon_j}$. Note that

$$\prod_{j=1}^{N}\left(\frac{1}{2}e^{\lambda \alpha_j} + \frac{1}{2}e^{-\lambda \alpha_j}\right) = \prod_{j=1}^{N} \cosh \lambda \alpha_j \leqslant \prod_{j=1}^{N} e^{(\lambda \alpha_j)^2}$$

in view of (4.22). Since $\sum_{j=1}^{N} |\alpha_j|^2 = 1$, we get

$$\mathcal{E}(e^{\lambda\left|\sum_{j=1}^{N} \alpha_j \varepsilon_j\right|}) \leqslant 2\prod_{j=1}^{N} e^{(\lambda \alpha_j)^2} = 2e^{\lambda^2 \sum_{j=1}^{N} |\alpha_j|^2} = 2e^{\lambda^2}.$$

Consequently,

$$\mathcal{E}(e^{\lambda\left|\sum_{j=1}^{N} \alpha_j \varepsilon_j\right| - \lambda^2 - 6\log N}) = e^{(-\lambda^2 - 6\log N)}\mathcal{E}(e^{\lambda\left|\sum_{j=1}^{N} \alpha_j \varepsilon_j\right|} \leqslant$$
$$e^{(-\lambda^2 - 3\log N)} \cdot 2e^{\lambda^2} \leqslant 2e^{-6logN} = \frac{2}{N^3}.$$

With $\lambda = \sqrt{3\log N}$, we obtain

$$\mathcal{E}(e^{\sqrt{3\log N}|\sum_{j=1}^N \alpha_j\varepsilon_j| - 9\log N}) \leqslant \frac{2}{N^3} \tag{4.27}$$

We shall now use the observation that for any random variable X and any $\lambda > 0$ we have $P(X > 0) \leqslant \mathcal{E}(e^{\lambda X})$. Applying this inequality to $X = |\sum_{j=1}^N \alpha_j\varepsilon_j| - 3\sqrt{3\log N}$ and $\lambda = \sqrt{3\log N}$, we get that

$$P\left[\left|\sum_{j=1}^N \alpha_j\varepsilon_j\right| - 3\sqrt{3\log N} > 0\right] \leqslant \tag{4.28}$$
$$\mathcal{E}(e^{\sqrt{3\log N}|\sum_{j=1}^N \alpha_j\varepsilon_j| - 9\log N}) \leqslant \frac{2}{N^3} < \frac{3\sqrt{3}}{N^3},$$

which proves the lemma for $\sum_{j=1}^N |\alpha_j|^2 = 1$, $\alpha_j \in \mathbb{R}$.
Now for any real $\alpha_1, \ldots, \alpha_N$ not all zero we have

$$P\left[\left|\frac{\sum_{j=1}^N \alpha_j}{\sqrt{\sum_{k=1}^N |\alpha_k|^2}}\varepsilon_j\right| > 3\sqrt{3\log N}\right] =$$

$$P\left[\left|\sum_{j=1}^N \alpha_j\varepsilon_j\right| > 3\sqrt{3\log N}\left(\sum_{k=1}^N |\alpha_k|^2\right)^{\frac{1}{2}}\right]$$

which proves that the lemma holds for any real $\alpha_1, \ldots, \alpha_N$.
Assume that the α_j's are complex numbers , say $\alpha_j = s_j + it_j$, $(j = 1, \ldots, N)$. Hence $|\alpha_j|^2 = |s_j|^2 + |t_j|^2$ $(j = 1, \ldots, N)$ and

$$\left|\sum_{j=1}^N \alpha_j\varepsilon_j\right|^2 = \left|\sum_{j=1}^N s_j\varepsilon_j\right|^2 + \left|\sum_{j=1}^N t_j\varepsilon_j\right|^2$$

Let

$$B_1 = \left\{\omega \in \Omega : |\sum_{j=1}^N s_j\varepsilon_j(\omega)|^2 > 27\sum_{j=1}^N |s_j|^2 \log N\right\},$$
$$B_2 = \left\{\omega \in \Omega : |\sum_{j=1}^N t_j\varepsilon_j(\omega)|^2 > 27\sum_{j=1}^N |t_j|^2 \log N\right\},$$
$$B = \left\{\omega \in \Omega : |\sum_{j=1}^N \alpha_j\varepsilon_j(\omega)|^2 > 27\sum_{j=1}^N |\alpha_j|^2 \log N\right\}.$$

Then $B \subseteq B_1 \cup B_2$, which implies by the subadditivity of the probability measure

P that $P(B) \leqslant P(B_1) + P(B_2)$. Thus by inequality (4.28) above, we have

$$P\left[\left|\sum_{j=1}^{N}\alpha_j\varepsilon_j\right|^2 > 27\sum_{k=1}^{N}|\alpha_j|^2\log N\right] \leqslant P\left[|\sum_{j=1}^{N}s_j\varepsilon_j|^2 > 27\sum_{k=1}^{N}|s_j|^2\log N\right] +$$

$$P\left[|\sum_{j=1}^{N}t_j\varepsilon_j|^2 > 27\sum_{k=1}^{N}|t_j|^2\log N\right] \leqslant \frac{2}{N^3} + \frac{2}{N^3} \leqslant \frac{3\sqrt{3}}{N^3}.$$

This proves part (a) of the lemma for all complex numbers $\alpha_1, \ldots, \alpha_N$.
(b) Replacing the ε_j by ρ_j, the first part of the proof is similar to part (a) until (4.26) which becomes

$$\mathcal{E}\big(e^{\lambda|\sum_{j=1}^{N}\alpha_j\rho_j|}\big) \leqslant \prod_{j=1}^{N}\mathcal{E}(e^{\lambda\alpha_j\rho_j}) + \prod_{j=1}^{N}\mathcal{E}(e^{-\lambda\alpha_j\rho_j}) =$$

$$\prod_{j=1}^{N}(\frac{1}{3}e^{2\lambda\alpha_j} + \frac{2}{3}e^{-\lambda\alpha_j}) + \prod_{j=1}^{N}(\frac{1}{3}e^{-2\lambda\alpha_j} + \frac{2}{3}e^{\lambda\alpha_j}).$$

Assuming as in (a) that all $\alpha_j \in \mathbb{R}$ with $\sum_{j=1}^{N}|\alpha_j|^2 = 1$, and using (4.23) we obtain

$$\mathcal{E}(e^{\lambda|\sum_{j=1}^{N}\alpha_j\rho_j|}) \leqslant \prod_{j=1}^{N}e^{2(\lambda\alpha_j)^2} + \prod_{j=1}^{N}e^{2(\lambda\alpha_j)^2} = 2e^{\lambda^2\sum_{j=1}^{N}\alpha_j^2} = 2e^{2\lambda^2}.$$

Consequently,

$$\mathcal{E}\big(e^{\lambda|\sum_{j=1}^{N}\alpha_j\rho_j|-2\lambda^2-3\log N}\big) = e^{-2\lambda^2-3\log N}\mathcal{E}\big(e^{\lambda|\sum_{j=1}^{N}\alpha_j\rho_j|}\big) \leqslant 2e^{-3logN} = \frac{2}{N^3},$$

whence, putting $\lambda = \sqrt{3\log N}$, we obtain

$$\mathcal{E}(e^{\sqrt{3\log N}|\sum_{j=1}^{N}\alpha_j\rho_j|-9\log N}) \leqslant \frac{2}{N^3}$$

This completes the proof for $\sum_{j=1}^{N}|\alpha_j|^2 = 1$. The rest of the proof, i.e. generalization to all real and complex numbers is done as in the proof of part (a), thus completing the proof of (b). $\square$

LEMMA 4.14 *For each $n \in \mathbb{N}$ let G_n be an abelian group with $3 \cdot 2^n$ elements and let H_n be the set of all characters of G_n. Then for each n there exist two disjoint subsets W_n^+, W_n^- of H_n with cardinalities satisfying*

$$|W_n^+| = 2^n, |W_n^-| = 2^{n+1} = 2 \cdot 2^n, \tag{4.29}$$

(hence $W_n^+ \cup W_n^- = H_n$), such that

$$\left\|2\sum_{w\in W_n^+}w - \sum_{w\in W_n^-}w\right\|_\infty \leqslant 36(n+1)^{\frac{1}{2}}2^{\frac{n}{2}}. \tag{4.30}$$

Proof:
Let $H_n = \{w_1, \ldots, w_{3\cdot 2^n}\}$ be the group of all characters of G_n. We claim that there exist number $\rho_1, \ldots, \rho_{3\cdot 2^n}$ with $\rho_j = 2$ or $\rho_j = -1$, ($j = 1, \ldots, 3\cdot 2^n$), such that

$$\left|\sum_{j=1}^{3\cdot 2^n} \rho_j w_j(g)\right| \leqslant 36(n+1)^{\frac{1}{2}} 2^{\frac{n}{2}}, \qquad g \in G_n. \tag{4.31}$$

To prove this let $\rho_1, \ldots, \rho_{3\cdot 2^n}$ be chosen independently at random, taking the values $\frac{1}{3}$ and $\frac{2}{3}$ respectively. For any fixed $g \in G_n$ we have by (4.25) with $N = 3\cdot 2^n$ and $\alpha_j = w_j(g)$ (hence $|\alpha_j| = |w_j(g)| = 1$ for $j = 1, \ldots, 3\cdot 2^n$)

$$P\left[\left|\sum_{j=1}^{3\cdot 2^n} w_j(g)\rho_j\right| > 3\sqrt{3}\left(\sum_{j=1}^{3\cdot 2^n} \log 3\cdot 2^n\right)^{\frac{1}{2}}\right] < \frac{3\sqrt{3}}{27\cdot 2^{3n}}$$

so that

$$P\left[\left|\sum_{j=1}^{3\cdot 2^n} w_j(g)\rho_j\right| > 3\sqrt{3}\left(\sum_{j=1}^{3\cdot 2^n} \log 3\cdot 2^n\right)^{\frac{1}{2}} \text{ for some } g \in G_n\right] \leqslant$$

$$\sum_{g\in G_n} P\left[\left|\sum_{j=1}^{3\cdot 2^n} w_j(g)\rho_j\right| > 3\sqrt{3}\left(\sum_{j=1}^{3\cdot 2^n} \log 3\cdot 2^n\right)^{\frac{1}{2}}\right] < \frac{3\cdot 2^n}{27.2^{3n}} 3\sqrt{3} < 1.$$

Thus

$$P\left[\left|\sum_{j=1}^{3\cdot 2^n} w_j(g)\rho_j\right| \leqslant 3\sqrt{3}\left(\sum_{j=1}^{3\cdot 2^n} \log 3\cdot 2^n\right)^{\frac{1}{2}} \text{ for all } g \in G_n\right] > 0.$$

Consequently there exists a choice of numbers $\rho_1, \ldots, \rho_{3\cdot 2^n}$ with $\rho_j = 2$ or $\rho_j = -1, (j = 1, \ldots, 3\cdot 2^n)$, such that

$$\begin{aligned}\Big|\sum_{j=1}^{3\cdot 2^n} w_j(g)\rho_j\Big| &\leqslant 3\sqrt{3}\Big(\sum_{j=1}^{3\cdot 2^n} \log 3\cdot 2^n\Big)^{\frac{1}{2}} = \\ 9\sqrt{2^n(\log 3 + n\log 2)} &\leqslant 18\cdot 2^{\frac{n}{2}}(n+1)^{\frac{1}{2}} \qquad g \in G_n,\end{aligned} \tag{4.32}$$

since $\log 3 + n\log 2 \leqslant 4(n+1)$. This proves (4.31).
Now we shall show that by changing some of the ρ_j's (so as to be still either 2 or -1) and increasing the constant 18 appearing in (4.32), we can obtain both (4.30) and

$$\sum_{j=1}^{3\cdot 2^n} \rho_j = 0. \tag{4.33}$$

This would complete the proof since among the $\rho_1, \ldots, \rho_{3\cdot 2^n}$ there must be exactly 2^n which are equal to 2 and exactly $2\cdot 2^n = 2^{n+1}$ which are equal to -1.

Applying (4.32) to the unit $g = e \in G_n$, since $w_j(e) = 1$ for all $j = 1, \dots, 3 \cdot 2^n$, we obtain

$$\left|\sum_{j=1}^{3\cdot 2^n} \rho_j\right| = \left|\sum_{j=1}^{3\cdot 2^n} \rho_j w_j(e)\right| \leqslant 18 \cdot 2^{\frac{n}{2}} (n+1)^{\frac{1}{2}}. \tag{4.34}$$

Let us denote by m and k respectively the number of those ρ_j's which are equal to 2, respectively equal to -1, then $\sum_{j=1}^{3\cdot 2^n} \rho_j = 2m - k$ and $k = 3 \cdot 2^n - m$, whence

$$\sum_{j=1}^{3\cdot 2^n} \rho_j = 2m - k = 3m - 3 \cdot 2^n = 3(m - 2^n). \tag{4.35}$$

If $m = 2^n$, we are done since $3(m-2^n) = 0$ implies that $|W_n^+| = 2^n = m, |W_n^-| = 2^{n+1}$, thus

$$\left|\sum_{j=1}^{3\cdot 2^n} \rho_j\right| = \left|\sum_{j=1}^{3\cdot 2^n} \rho_j w_j(e)\right| = \left\|2 \sum_{w \in W_n^+} w - \sum_{w \in W_n^-} w\right\|_\infty \leqslant 18(n+1)^{\frac{1}{2}} 2^{\frac{n}{2}},$$

and we get (4.30).
Assume that $m > 2^n$. Select any set S of indices such that $|S| = m - 2^n$ and $\rho_j = 2$ for all $j \in S$ and put

$$\tilde{\rho_j} = \begin{cases} 1 - \rho_j = -1 & \text{for} \quad j \in S \\ \rho_j & \text{for} \quad j \in \{1, \dots, 3 \cdot 2^n\} \setminus S \end{cases} \tag{4.36}$$

Then using the definition of $\tilde{\rho_j}$, the fact that $|w_j(g)| = 1$, from (4.34) and (4.35) we obtain

$$\left|\sum_{j=1}^{3\cdot 2^n} \rho_j w_j(g) - \sum_{j=1}^{3\cdot 2^n} \tilde{\rho_j} w_j(g)\right| = \left|\sum_{j\in S} \rho_j w_j(g) - \sum_{j \in S} \tilde{\rho_j} w_j(g)\right|$$

since they are only different on S, and then

$$\left|\sum_{j\in S} (\rho_j - (1 - \rho_j)) w_j(g)\right| = \left|\sum_{j \in S} (2\rho_j - 1) w_j(g)\right| = \left|\sum_{j\in S} 3 w_j(g)\right| \leqslant \tag{4.37}$$

$$3|S| = 3(m - 2^n) = \left|\sum_{j=1}^{3\cdot 2^n} \rho_j\right| \leqslant 18(n+1)^{\frac{1}{2}} 2^{\frac{n}{2}}. \tag{4.38}$$

By (4.32), we deduce that

$$\left|\sum_{j=1}^{3\cdot 2^n} \tilde{\rho_j} w_j(g)\right| \leqslant 36(n+1)^{\frac{1}{2}} 2^{\frac{n}{2}}, \qquad g \in G_n.$$

On the other hand by the definition of $\tilde{\rho_j}$ and (4.35), we have

$$\sum_{j=1}^{3\cdot 2^n} \rho_j - \sum_{j=1}^{3\cdot 2^n} \tilde{\rho_j} = \sum_{j \in S} (2\rho_j - 1) = \sum_{j \in S} 3 =$$

$$3|S| = 3(m - 2^n) = \sum_{j=1}^{3\cdot 2^n} \rho_j$$

so that

$$\sum_{j=1}^{3\cdot 2^n} \tilde{\rho}_j = 0$$

and hence we have (4.33) for ρ_j replaced by $\tilde{\rho}_j$. Finally in the case when $m - 2^n < 0$ the argument is similar: we choose a set of indexes T such that $|T| = 2^n - m$ and choose $\rho_j = -1$ for $\rho_j \in T$ and then replace $2^n - m$ numbers ρ_j's which are -1 by $\tilde{\rho}_j = 2$. This completes the proof of the Lemma 4.14. $\square$

4.7 The τ-continuity of the functional β

Let W_n^+ and W_n^- be the partition of H_n given by Lemma 4.14. As before, for each n let us denote by $\sigma_1^n, \dots, \sigma_{2^n}^n$, and $\tau_1^n, \dots, \tau_{2^{n+1}}^n$ the elements of W_n^+ and W_n^- respectively. Thus, with $A_1 \geqslant 36$ we can write (4.30) in the form

$$\left| 2\sum_{j=1}^{2^n} \sigma_j^n(g) - \sum_{j=1}^{2^{n+1}} \tau_j^n(g) \right| \leqslant A_1 (n+1)^{\frac{1}{2}} 2^{\frac{n}{2}}, \tag{4.39}$$

Recall also the functions ϕ_g^n defined by (4.19)

$$\phi_g^n = \frac{1}{2^{n+1}} \sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}) x_j^{n+1} - \frac{1}{2^n} \sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) x_j^n \in X, g \in G_n.$$

PROPOSITION 4.15 *Let ϕ_g^n be as above. Then there exists choice of $\varepsilon_j^n = \pm 1$ such that*

$$\|\phi_g^n\| \leqslant A_2 \frac{(n+1)^{\frac{1}{2}}}{2^{\frac{n}{2}}},$$

where A_2 is an absolute constant.

Proof:
Recall that for $g \in G_n$, $n \geqslant 0$ we have by (4.4)

$$\|\phi_g^n\| = \sup_{h \in G} |\phi_g^n(h)|.$$

We will consider different cases.
Case 1: Suppose that $g, h \in G_n$ and apply Lemma 4.14 to obtain

$$|\phi_g^n(h)| = \left| \frac{1}{2^n} \left(\frac{1}{2} \sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}h) - \sum_{j=1}^{2^n} \sigma_j^n(g^{-1}h) \right) \right| =$$

$$\left| \frac{1}{2^{n+1}} \left(\sum_{j=1}^{2^{n+1}} \tau_j^n(g^{-1}h) - 2\sum_{j=1}^{2^n} \sigma_j^n(g^{-1}h) \right) \right| \leqslant$$

$$\frac{2^{\frac{n}{2}}}{2^{n+1}} A_1 (n+1)^{\frac{1}{2}} \leqslant A_1 \frac{(n+1)^{\frac{1}{2}}}{2^{\frac{n}{2}}}$$

Case 2: Recall first that by Proposition 4.11 $|\phi_g^{n-1}(h)| = |\phi_{h^{-1}}^n(g^{-1})|$ for $g \in$

$G_{n-1}, h \in G_n$. Thus for any fixed $g \in G_{n-1}$ and $h \in G_n$, apply Lemma 4.13 to obtain

$$P\left[\left|\sum_{j=1}^{N} \alpha_j \varepsilon_j\right| > A_1 \left(\sum_{j=1}^{N} |\alpha_j|^2 \log N\right)^{\frac{1}{2}}\right] < \frac{A_1}{N^3}.$$

with $N = 2^n$ and $\alpha_j = \tau_j^n(g^{-1})\sigma_j^n(h)$ (thus $\left|\alpha_j^n\right| = 1$ for $j = 1, 2, \ldots, 2^n$),

$$P_1(g,h) = P\left[\left|\sum_{j=1}^{2^n} \varepsilon_j^n \tau_j^{n-1}(g^{-1})\sigma_j^n(h)\right| > A_1 \left(\sum_{j=1}^{2^n} \log 2^n\right)^{\frac{1}{2}}\right] < \frac{A_1}{2^{3n}},$$

and hence, considering $\overline{\alpha_j}$, for any fixed $g \in G_n$ and $h \in G_{n-1}$ we get

$$P_2(g,h) = P\left[\left|\sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1})\tau_j^{n-1}(h)\right| > A_1 \left(\sum_{j=1}^{2^n} \log 2^n\right)^{\frac{1}{2}}\right] < \frac{A_1}{2^{3n}},$$

whence since $|G_{n-1}| = 3.2^{n-1}$ and $|G_n| = 3.2^n$ we obtain for n large enough:

$$\begin{aligned}
&P\Bigg\{\text{either} \left|\sum_{j=1}^{2^n} \varepsilon_j^n \tau_j^{n-1}(g^{-1})\sigma_j^n(h)\right| > A_1\Big(\sum_{j=1}^{2^n} \log 2^n\Big)^{\frac{1}{2}} \text{ for some } g \in G_{n-1}, h \in G_n \\
&\text{or} \left|\sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1})\tau_j^{n-1}(h)\right| > A_1\Big(\sum_{j=1}^{2^n} \log 2^n\Big)^{\frac{1}{2}}, \text{for some } g \in G_n, h \in G_{n-1}\Bigg\} = \\
&\sum_{g\in G_{n-1}} \sum_{h \in G_n} P_1(g,h) + \sum_{g \in G_n} \sum_{h \in G_{n-1}} P_2(g,h) < \\
&3 \cdot 2^{n-1} \sum_{h\in G_n} P_1(g,h) + 3 \cdot 2^n \sum_{h \in G_{n-1}} P_2(g,h) < 3 \cdot 2^{n-1} \frac{A_1}{2^{3n}} + 3 \cdot 2^n \frac{A_1}{2^{3n}} < 1,
\end{aligned}$$

that is for n large enough we have,

$$\begin{aligned}
&P\Bigg\{\left|\sum_{j=1}^{2^n} \varepsilon_j^n \tau_j^{n-1}(g^{-1})\sigma_j^n(h)\right| \leqslant A_1\Big(\sum_{j=1}^{2^n} \log 2^n\Big)^{\frac{1}{2}} \text{ for all } g \in G_{n-1}, h \in G_n \text{ and} \\
&\quad \left|\sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1})\tau_j^{n-1}(h)\right| > A_1\Big(\sum_{j=1}^{2^n} \log 2^n\Big)^{\frac{1}{2}} \text{ for all } g \in G_n, h \in G_{n-1}\Bigg\} > 0
\end{aligned}$$

Consequently, for n large enough, say $n > n_0$, there exist signs $\varepsilon_j^n = \pm 1$, $(j = 1, 2, \ldots, 2^n)$ such that

$$\left| \sum_{j=1}^{2^n} \varepsilon_j^n \tau_j^{n-1}(g^{-1}) \sigma_j^n(h) \right| \leqslant A_1 \left(\sum_{j=1}^{2^n} n \log 2^n \right)^{\frac{1}{2}} =$$

$$A_1(2^n n \log 2)^{\frac{1}{2}} = A_2 2^{\frac{n}{2}} n^{\frac{1}{2}} \qquad (g \in G_{n-1}, h \in G_n),$$

$$\left| \sum_{j=1}^{2^n} \varepsilon_j^n \sigma_j^n(g^{-1}) \tau_j^{n-1}(h) \right| \leqslant A_1 \left(\sum_{j=1}^{2^n} \log 2^n \right)^{\frac{1}{2}} =$$

$$A_1(2^n n \log 2)^{\frac{1}{2}} \leqslant A_2 2^{\frac{n}{2}} (n+1)^{\frac{1}{2}} \qquad (g \in G_n, h \in G_{n-1})$$

where $A_2 = (2 \log 2)^{\frac{1}{2}} A_1$. But since each G_n is a finite set, the same inequalities hold for $n = 0, 1, 2, \ldots, n_0$ by increasing A_2 if necessary. By the definition of ϕ_g^n's given in (4.21) above and by (4.7) we can infer that for any $g \in G_n$, and $n = 0, 1, 2, \ldots$ we have

$$|\phi_g^n(h)| \leqslant \frac{1}{2^n} A_3 (n+1)^{\frac{1}{2}} 2^{\frac{n}{2}} = A_3 \frac{(n+1)^{\frac{1}{2}}}{2^{\frac{n}{2}}}$$

Thus, there is a choice of $\varepsilon = \pm 1$ such that

$$\sup_{h \in G} |\phi_g^n(h)| = \|\phi_g^n\| \leqslant A_3 \frac{(n+1)^{\frac{1}{2}}}{2^{\frac{n}{2}}}$$

where A_3 is an absolute constant. □

Consider the following subset $K \subset X$:

$$K = \{x_1^0\} \cup \{(n+1)^2 \phi_g^n : g \in G_n, n \geqslant 0\} \subset X. \tag{4.40}$$

We want to show that K is relatively compact. We apply Theorem 2.11. Since

$$\|\phi_g^n\| \leqslant A_3 \frac{(n+1)^{\frac{1}{2}}}{2^{\frac{n}{2}}}, \text{ then } (n+1)^2 \|\phi_g^n\| \leqslant A_3 \frac{(n+1)^{\frac{5}{2}}}{2^{\frac{n}{2}}} \to 0 \text{ as } n \to \infty,$$

which means that K is relatively compact.

PROPOSITION 4.16 *With the above partition and choices of $\varepsilon_j^n = \pm 1$ let*

$$\beta_n(T) = \frac{1}{2^n} \sum_{j=1}^{2^n} \alpha_j^n(T x_j^n), \qquad x_j^n \in X$$

then

$$\beta(T) = \lim_{n \to \infty} \beta_n(T). \tag{4.41}$$

exists for every $T \in \mathcal{B}(X)$ and defines a τ-continuous linear functional on $\mathcal{B}(X)$.

Proof:
By (4.20) we have the following inequality

$$|\beta_{n+1}(T) - \beta_n(T)| \leqslant \sup_{g \in G_n} \|T \phi_g^n\|.$$

Thus, for an element $x \in K$ as in (4.40),

$$\sup_{g \in G_n} (n+1)^2 \|T\phi_g^n\| \leqslant \sup_{x \in K} \|Tx\|$$

or

$$\sup_{g \in G_n} \|T\phi_g^n\| \leqslant \sup_{x \in K} \frac{\|Tx\|}{(n+1)^2}.$$

We can thus conclude that

$$|\beta_{n+1}(T) - \beta_n(T)| \leqslant \sup_{x \in K} \frac{\|Tx\|}{(n+1)^2}$$

for all $T \in \mathcal{B}(X)$. From here we will deduce that

(i) $\lim_{n \to \infty} \beta_n(T)$ exists, and
(ii) β is τ-continuous.

To see (i) observe that since $\sum_{n=1}^{\infty} \big[\beta_{n+1}(T) - \beta_n(T)\big]$ is absolutely convergent the sequence

$$\beta_N(T) = \beta_0(T) + \sum_{n=1}^{N-1} \big[\beta_{n+1}(T) - \beta_n(T)\big] \tag{4.42}$$

is convergent that is the limit $\beta(T)$ exists.

To prove (ii) we first observe that

$$|\beta_0(T)| = |\alpha_1^0(Tx_1^0)| = \left|\frac{1}{3}\varepsilon_1^0\sigma_1^0(g^{-1})Tx_1^0(g)\right| = \frac{1}{3}\|Tx_1^0\| \leqslant \frac{1}{3}\sup_{x \in K}\|Tx\|.$$

Hence, for all $T \in \mathcal{B}(X)$ by (4.42)

$$|\beta(T)| = \left|\beta_0(T) + \sum_{n=1}^{\infty} \big(\beta_{n+1}(T) - \beta_n(T)\big)\right| \leqslant \tag{4.43}$$

$$|\beta^0(T)| + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \sup_{x \in K}\|Tx\| \leqslant 3 \sup_{x \in K}\|Tx\| \tag{4.44}$$

Thus by Theorem 2.19, β is τ-continuous and the proof is complete. □

Now the argument following Theorem 4.3 (see Example 4.6) shows that β is a τ-continuous functional with $\beta(I) = 1$ and $\beta(T) = 0$ for all $T \in \mathcal{F}(X)$. Consequently the identity operator cannot be uniformly approximated by finite rank operators in the topology τ, that means that the separable Banach space X does not have the approximation property. This concludes the proof of Theorem 4.8.

Remark For convenience the above construction was carried out using complex scalars. One can see that X also fails to have the approximation property has a real Banach space. Using the fact that if T is a real-linear operator then

$$x \to \frac{1}{2}\big(Tx - iT(ix)\big)$$

is complex-linear.

Appendix A

The characters of a group

In this appendix we will introduce some notions about the characters of a group which are used in Chapter 3.

DEFINITION A.1 *Let G be a group (not necessarily abelian) and let $H = Hom(G, S^1)$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the set of homomorphisms from the group G to the multiplicative group S^1 considered along with pointwise multiplication*

$$(\chi_1 * \chi_2)(g) = \chi_1(g)\chi_2(g), \qquad \chi_j \in H, g \in G. \tag{A.1}$$

H is called the character group of G over $\mathbb{C}$.

THEOREM A.2 *The set H of all characters of G is an abelian group called the character group of G. H has as the identity element the constant function $e : G \to S^1$, defined by $e(g) = 1$ for $g \in G$. And for each χ the inverse $\chi^{-1} = \bar{\chi}$, where $\bar{\chi}$ is the complex conjugate of χ.*

THEOREM A.3 *Let G be an abelian group with character group H. Then the cardinalities of the groups G and H are equal and the two groups are isomorphic.*

For the proofs of the above two theorems see [K], pages 438-439.

DEFINITION A.4 *Let G be a finite group and define on its characters the following natural scalar product*

$$(x, z) = \frac{1}{|G|} \sum_{g \in G} x(g)\overline{z(g)}. \tag{A.2}$$

THEOREM A.5 *Let x and z be characters of G. Then $(x, z) = 0$ if $x \neq z$ and $(x, z) = 1$ if and only if $x = z$. Thus the characters of a group form an orthogonal set.*

Proof:
For $h \in G$ we have

$$(x, z) = \sum_{g \in G} x(g)\overline{z(g)} = \sum_{g \in G} x(hg)\overline{z(hg)} = x(h)\overline{z(h)} \sum_{g \in G} x(g)\overline{z(g)}$$

so that $\displaystyle\sum_{g \in G} x(g)\overline{z(g)}(1 - x(h)\overline{z(h)}) = 0$

This implies that either

$$\sum_{g \in G} x(g)\overline{z(g)} = 0$$

or for all $h \in G$ $x(h)\overline{z(h)} = 1$ i.e. $x = z$. $\square$

EXAMPLE A.6 Let $m > 1, m \in \mathbb{N}$, and consider the finite cyclic group $\mathbb{Z}_m$, which we represent as $\{0, 1, \ldots, m-1\}$ with addition modulo m. The mapping $k \to e^{(\frac{2\pi i k}{m})}$ is a character of $\mathbb{Z}_m$. Using Theorem A.3 we see that every character of $\mathbb{Z}_m$ has the form $k \to e^{(\frac{2\pi i k l}{m})}$, where l is a fixed integer such that $0 \leqslant l \leqslant m-1$.

Appendix B

Axioms of Probability

We present some basic concepts from probability theory. Let X be an arbitrary set. A collection $\mathcal{A}$ of subsets of X is a σ-algebra on X if
a) $X \in \mathcal{A}$,
b) for each set A that belongs to $\mathcal{A}$, the set A^c belongs to $\mathcal{A}$,
c) for each infinite sequence $\{A_i\} \subset \mathcal{A}$, the set $\bigcup_{i=1}^{\infty} A_i$ belongs to $\mathcal{A}$,
Thus a σ-algebra on X is a family of subsets of X that contains X and is closed under complementation, under the formation of countable unions, and under the formation of countable intersections.

DEFINITION B.1 *A probability measure defined on σ-algebra $\mathcal{A}$ of Ω is a function $P : \mathcal{A} \to [0,1]$ that satisfies:*
(a) $P(\emptyset) = 0$ and $P(\Omega) = 1$
(b) For every countable sequence $\{A_n\}_{n \geqslant 1}$ of elements of $\mathcal{A}$, pairwise disjoint, (i.e. $A_n \cap A_m = \emptyset$ if $n \neq m$) one has

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} P(A_n)$$

Axiom (b) above is called countable additivity.

DEFINITION B.2 *(a) Two events A and B (an event is a set in $\mathcal{A}$) are independent if*

$$P(A \cap B) = P(A)P(B).$$

(b) A (possibly infinite) collection of events $(A_i)_{i \in I}$ is an independent collection if for every finite subset J of I, one has

$$P\Big(\bigcap_{i \in J} A_i\Big) = \prod_{i \in J} P(A_i).$$

DEFINITION B.3 *A complex random variable is a measurable function X from a probability space $(\Omega, \mathcal{A}, P)$ into $\mathbb{C}$, that is $X^{-1}(G) \in \mathcal{A}$ whenever G is open in $\mathbb{C}$.*

A random variable X represents an unknown quantity that varies with the outcome of a random event. Before the random event, we know which values X could possible assume, but we do not know which one it will take until the random event happens. A random variable X is often thought of as representing a numerical quantity connected to the outcome of a random experiment. If one repeats the experiment, one will see values of X corresponding to the frequencies of the outcomes they represent. These frequencies are modelled by the probability measure. Thus in the long run, one would expect X to be the weighted average of its values, with weights corresponding to the probability of attaining those values. This motivates the name and definition of the expectation of X.

DEFINITION B.4 *Let X be a real-valued random variable on a countable probability space $\Omega = \{\omega_1, \omega_2, \ldots\}$. The expectation of X, denoted by $\mathcal{E}(X)$, is defined as*

$$\mathcal{E}(X) = \sum_i X(\omega_i)p_i, \text{ where } p_i = P(\omega_i),$$

provided the series is absolutely convergent or $X \geqslant 0$.

THEOREM B.5 *The expectation of a random variable is a linear operator.*

THEOREM B.6 *If X and Y are two independent real-valued random variables, then*

$$\mathcal{E}(XY) = \mathcal{E}(X)\mathcal{E}(Y),$$

or more generally if X_j's are a family of random independent variables such that each X_j has a finite expectation, then

$$\mathcal{E}\Big(\prod_{j=1}^{N} X_j\Big) = \prod_{j=1}^{N} \mathcal{E}(X_j).$$

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