The Basics of Display Calculi

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1 Introduction

This paper gives a concise introduction to display logic as originally developed by Belnap [1].[1] Display logic concerns more than a specific formalization of a particular logic. It stands as a formal framework wherein many logics are representable. The reader might wonder why such a project is fruitful when there are well-known Hilbert-style calculi that do exactly this. The problem is that these Hilbert-style calculi represent logics in a *destructive* way. Let us explain what we mean by this through the introduction of Gentzen-style calculi. After that, the alleged destructivity of Hilbert-style calculi is addressed.

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[1] Though, the actual presentation of several aspects of the presented display calculi (most notably the display equivalences and the presentation of modalities) follow more closely [11].

Gentzen [5] is arguably the founder of two, rather powerful, formal frameworks: (1) natural deduction[2] and (2) sequent calculi. Since natural deduction formalizations of logic are not the main concern of this paper we focus strictly on sequent calculi. Let us suppose some usual propositional language in the following. In addition to formulas, Gentzen introduced *sequents*, which are syntactic objects of the following form:

$$\Gamma \Longrightarrow \Delta$$

Γ and Δ are considered to be *lists* of formulas, so that $\Gamma \Longrightarrow \Delta$ is shorthand for:

$$A_1, A_2, \ldots, A_m \Longrightarrow B_1, B_2, \ldots, B_n$$

where $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_n$ are formulas from our background propositional language. The arrow $\Longrightarrow$ occurring in the sequent is often referred to as the *sequent-arrow*. A natural interpretation of the sequent-arrow is that from the truth of $A_1, A_2, ..., A_m$ the cases $B_1, B_2, ..., B_n$ follow. An interpretation more on the semantic side is that the conjunction of $A_1, A_2, ..., A_m$ materially implies the disjunction of $B_1, B_2, ..., B_n$. Both motivations indicate that a sequent can be interpreted propositionally as:

$$A_1 \wedge A_2 \wedge ... \wedge A_m \rightarrow B_1 \vee B_2 \vee \cdots \vee B_n$$

or equivalently as:

$$\bigwedge \Gamma \rightarrow \bigvee \Delta$$

Γ is commonly coined the antecedent of a sequent and Δ the consequent or succedent of a sequent. From a classical point of view the order of the formulas occurring in both Γ and Δ, or $A_1, A_2, ..., A_m$ and $B_1, B_2, ..., B_n$, does not matter. For example, if it is possible to prove the sequent $A_{17}, A_{19} \Longrightarrow B_5$, then it should also be possible to prove the sequent $A_{19}, A_{17} \Longrightarrow B_5$. Similarly, if it is possible to prove $A_7 \Longrightarrow B_8, B_{205}$, then it should also be possible to prove $A_7 \Longrightarrow B_{205}, B_8$. This follows from our interpretation of sequents as expressing a relation of provability between antecedent formulas and succedent formulas. If B_5 follows from the derivability of A_{17} and A_{19}, then B_5 follows from the derivability of A_{19} and A_{17}. Similarly, if B_8 and B_{205} follow from the derivatibility of A_7, then B_{205} and B_8 follow from the derivability of A_7. Order does not matter. This example shows that we should be able to syntactically manipulate the order of the formulas in both the antecedent and succedent of a sequent[3]. Formally, this is expressed by a pair of structural rules, called left and right *permutation*. The intuitive understanding of a structural rule is that it allows for syntactical manipulation of Γ and Δ, where no logical operator is concerned.

Another syntactic transformation rule is *contraction*. To see the intuition behind this rule, let us give an example. If the sequent $A_{17}, A_{17} \Longrightarrow B_5$ is provable,

[2]Note that Jaskowski (1934) developed natural deduction as well, but independently of Gentzen.

[3]In classical logic such rules of syntactic manipulation are allowed, however, there are non-classical calculi which restrict these intuitive rules.

then it should also be possible to prove $A_{17} \Longrightarrow B_5$, and if a the sequent $A_7 \Longrightarrow B_8, B_8$ is provable, then it should also be possible to prove $A_7 \Longrightarrow B_8$. In other words, if B_5 follows from the derivability of A_7 and A_7, then it follows strictly from A_7 itself since the additional copy is nothing but extraneous information– similarly for the second example. Informally speaking, the contraction rules allow us to delete additional copies of a formula from either the antecedent or succedent. From a classical perspective contraction is an acceptable rule of inference.

The last pair of structural rules are called the *weakening* rules. The idea is that if a certain sequent $\Gamma \Longrightarrow \Delta$ is provable, then so are both $A, \Gamma \Longrightarrow \Delta$ and $\Gamma \Longrightarrow \Delta, A$ for any arbitrary formula A. Let us put this rule together with the former structural rules in a compact manner:[4]

Permutation (left and right):

$$\frac{\Phi, A, B, \Gamma \Longrightarrow \Delta}{\Phi, B, A, \Gamma \Longrightarrow \Delta}\ (\mathsf{Pl}) \qquad \frac{\Gamma \Longrightarrow \Delta, A, B, \Phi}{\Gamma \Longrightarrow \Delta, B, A, \Phi}\ (\mathsf{Pr})$$

Contraction (left and right):

$$\frac{A, A, \Gamma \Longrightarrow \Delta}{A, \Gamma \Longrightarrow \Delta}\ (\mathsf{Cl}) \qquad \frac{\Gamma \Longrightarrow \Delta, A, A}{\Gamma \Longrightarrow \Delta, A}\ (\mathsf{Cr})$$

Weakening (left and right):

$$\frac{\Gamma \Longrightarrow \Delta}{A, \Gamma \Longrightarrow \Delta}\ (\mathsf{Wl}) \qquad \frac{\Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, A}\ (\mathsf{Wr})$$

The validity of the structural rules follows easily from both the propositional interpretation of a sequent $\Gamma \Longrightarrow \Delta$ as $\bigwedge \Gamma \to \bigvee \Delta$ as well as from the truth table method. Under the propositional interpretation of a sequent, the left and right weakening rules are:

$$\frac{\bigwedge \Gamma \to \bigvee \Delta}{A \wedge \bigwedge \Gamma \to \bigvee \Delta}\ (\mathsf{Wl}') \qquad \frac{\bigwedge \Gamma \to \bigvee \Delta}{\bigwedge \Gamma \to \bigvee \Delta \vee A}\ (\mathsf{Wr}')$$

Thus far we have only addressed the structural aspects of sequents, *i.e.* we have only considered rules that manipulate the structure of the antecdent and consequent of a sequent. However, proof-theoretic calculi also consist of axioms and logical rules, which endow the calculus with the ability to prove logical truths. In the calculus considered here, there is exactly one axiom schema, which is the following:

$$A \Longrightarrow A$$

[4] The rule of cut, which is a structural rule as well, will be discussed below.

where A is restricted to atomic formulas. For example, an instance of the axiom schema would be $p \Longrightarrow p$. Although the axiom on its own is not very informative, the logical power of the calculus is increased with the introduction of *logical* rules of inference. This contrasts with Hilbert-style calculi where the logical power follows from the axioms. When we emphasize the logical rules as opposed to the logical axioms an interesting symmetry arises. This symmetry originates from the fact that each logical connective comes (typically) in a pair consisting[5] of a left and right introduction rule. For example, the logical rules for conjunction are:

$$\frac{A, \Gamma \Longrightarrow \Delta}{A \wedge B, \Gamma \Longrightarrow \Delta}\ (\wedge_1 \mathsf{l}) \qquad \frac{\Gamma \Longrightarrow \Delta, A \qquad \Gamma \Longrightarrow \Delta, B}{\Gamma \Longrightarrow \Delta, A \wedge B}\ (\wedge \mathsf{r})$$

$$\frac{B, \Gamma \Longrightarrow \Delta}{A \wedge B, \Gamma \Longrightarrow \Delta}\ (\wedge_2 \mathsf{l})$$

If we represent these rules via our propositional interpretation, then they can be viewed as:

$$\frac{A \wedge \bigwedge \Gamma \to \bigvee \Delta}{A \wedge B \wedge \bigwedge \Gamma \to \bigvee \Delta}\ (\wedge_1' \mathsf{l}) \qquad \frac{\bigwedge \Gamma \Longrightarrow \bigvee \Delta \vee A \qquad \bigwedge \Gamma \Longrightarrow \bigvee \Delta \vee B}{\bigwedge \Gamma \to \bigvee \Delta \vee A \wedge B}\ (\wedge' \mathsf{r})$$

$$\frac{B \wedge \bigwedge \Gamma \to \bigvee \Delta}{A \wedge B \wedge \bigwedge \Gamma \to \bigvee \Delta}\ (\wedge_2' \mathsf{l})$$

By use of truth tables the validity of these logical rules is readily verifiable. This understanding extends to the logical rules for the remaining operators as well:

$$\frac{\Gamma \Longrightarrow \Delta, A}{\neg A, \Gamma \Longrightarrow \Delta}\ (\neg \mathsf{l}) \qquad \frac{A, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta, \neg A}\ (\neg \mathsf{r})$$

$$\frac{A, \Gamma \Longrightarrow \Delta \qquad B, \Gamma \Longrightarrow \Delta}{A \vee B, \Gamma \Longrightarrow \Delta}\ (\vee \mathsf{l}) \qquad \frac{\Gamma \Longrightarrow \Delta, A}{\Gamma \Longrightarrow \Delta, A \vee B}\ (\vee_1 \mathsf{r})$$

$$\frac{\Gamma \Longrightarrow \Delta, B}{\Gamma \Longrightarrow \Delta, A \vee B}\ (\vee_2 \mathsf{l})$$

$$\frac{\Gamma \Longrightarrow \Delta, A \qquad B, \Phi \Longrightarrow \Psi}{A \to B, \Gamma, \Phi \Longrightarrow \Delta, \Psi}\ (\to \mathsf{l}) \qquad \frac{A, \Gamma \Longrightarrow \Delta, B}{\Gamma \Longrightarrow \Delta, A \to B}\ (\to \mathsf{r})$$

Given axiom instances, we can continually apply the structural rules and logical rules to deduce logical truths. The mathematical structure which results from such a process is called a *derivation*. To demonstrate how derivations are constructed, we have included a derivation of $\Longrightarrow p \vee \neg p$ below:

[5]Obviously, some logical connectives have three introduction rules, eg. $\wedge$, and $\vee$. This, however, depends heavily on the formulation of the sequent calculus.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Longrightarrow p}{\Longrightarrow p, \neg p}\,(\neg\mathsf{r})}{\Longrightarrow p, p \vee \neg p}\,(\vee_2\mathsf{r})}{\Longrightarrow p \vee \neg p, p}\,(\mathsf{Pr})}{\Longrightarrow p \vee \neg p, p \vee \neg p}\,(\vee_1\mathsf{l})}{\Longrightarrow p \vee \neg p}\,(\mathsf{Cr})$$

With the exception of right contraction (Cr) and right permutation (Pr) the derivation consists of logical rule applications which start from the axiom instance $p \Longrightarrow p$.

At this point we have introduced enough machinery to explain the destructive character of Hilbert-style calculi. Formulations of propositional logic in a Hilbert-style calculus often rely on exactly one rule of inference, namely, *modus ponens*: From A and $A \to B$ infer B. Such a rule allows us to conclude a simpler formula, namely B, from the more complex formula $A \to B$ (in conjunction with A). If one observes the logical rules presented thus far, it is easily seen that all of them build more complex formulas from simpler formulas, and that the process is never reversed. Our calculus presented thus far is purely constructive, which contrasts with Hilbert-style calculi that allow for complex formulas to be deduced from simple formulas, and vice-versa.

So far our calculus adds more and more logical complexity with each additional inference, except for in cases of contraction. In our presentation of the current proof-theoretic calculus, called LK, one crucial component is missing–a general version of modus ponens. The specific rule of modus ponens in a sequent calculus formulation is as follows:[6]

$$\frac{(\Gamma) \Longrightarrow A \qquad (\Delta) \Longrightarrow A \to B}{(\Gamma, \Delta) \Longrightarrow B}$$

while the general version in LK is the cut rule:

$$\frac{\Gamma \Longrightarrow \Delta, A \qquad A, \Phi \Longrightarrow \Psi}{\Gamma, \Phi \Longrightarrow \Delta, \Psi}\,(\mathsf{Cut})$$

It was Gentzen's ingenious insight that for every derivation of some sequent containing at least one application of cut, there is a derivation of this sequent without the use of cut. This result is Gentzen's celebrated *Hauptsatz*–also known as the *cut elimination theorem*. As stated earlier, our calculus without the cut rule is purely constructive. Although the cut rule reverses the constructive process much like *modus ponens* does in a Hilbert-style calculus, Gentzen's theorem shows that the rule is extraneous, *i.e.* its addition or removal from our calculus has no effect on what is deducible.

Although the removal of the rule has not effect on what can be deduced, it does have a practical effect on *how* something might be deduced. The length of derivations without cut are often much longer. Boolos describes the negative impact of removing the rule in his 1984 paper "Don't eliminate Cut" [2]. On the

[6] We include the parentheses around Γ and Δ since one may formulate the rule with contexts, or without contexts (so that modus ponens can only be used with theorems).

positive side, there are many insightful results connected with the cut elimination theorem. For example, it can be shown that the first-order version of the calculus presented here with Peano Arithmetic is consistent. Though a proof of this does require more advanced techniques which are omitted in the present treatment.

Take note that this paper is not written with an "expert reader" of proof theory in mind. The intended audience are those with a solid understanding of first-order logic with some interest in alternative proof calculi. This is why Gentzen's sequent calculus has been gently introduced. However, the remaining part on display logic, which is self-contained, is technically more challenging. The plan of this paper is to give an introduction to display logic in proof theory. We cover the most basic results, such as soundness, completeness, cut-elimination, and the sub-formula property, for a propositional, first-order, and modal display calculus. The calculi and theorems to follow should provide the interested reader with an introductory understanding of display logic as well as the properties of proof-theoretic calculi.

2 Propositional Display Logic

This section contains a description of the propositional display calculus D.Cp. We prove various results concerning the calculus such as completeness, soundness, the subformula property, etc. More importantly however, we give a general cut elimination procedure, and demonstrate its application to our specific display calculus. The calculus D.Cp follows from [1] and is reduced to the bare minimum, which has the benefit of least distraction.

2.1 The Calculus D.Cp

Before we define the propositional display calculus D.Cp, we need to introduce the structural connectives. The connective I is the empty structure, which is understood as $\top$ in the antecedent and as $\bot$ in succedent, though structurally the connective is meant to represent empty data. For example, when I occurs in either antecedent or succedent it represents an empty antecedent or succedent, respectively (observe the connection to a Gentzen-style calculus where an empty antecedent is interpreted as $\top$, and an empty succedent as $\bot$). The unary connective $*$ is interpreted as negation regardless of if it occurs in the antecedent or succedent. Lastly, the binary connective $\circ$ is thought of as structural addition and is interpreted as conjunction in the antecedent and disjunction in the succedent. Using these connectives we go beyond the usual formula-based calculus, and construct a calculus that includes *structures* as well:

Definition 1 (Formulas of D.Cp)**.** $A := p \mid \top \mid \bot \mid \neg A \mid A \to B \mid A \vee B \mid A \wedge B$

Definition 2 (Structures of D.Cp)**.** $X := \mathrm{I} \mid A \mid *X \mid X \circ Y$

A structure is built from formulas using the structural connectives or I. Note also that a substructure is defined to be a structure occurring in another structure, and that every structure is a substructure of itself.

Example 1. The sequent $p \circ *q \Longrightarrow p \circ r$ contains $p \circ *q$, p, $*q$, q, $p \circ r$, r as substructures.

Axiom $p \Longrightarrow p$ with p atomic.

Structural Rules[7]

$$\frac{X \Longrightarrow Y}{\mathrm{I} \circ X \Longrightarrow Y}\ (\mathrm{I}{+}) \qquad \frac{\mathrm{I} \circ X \Longrightarrow Y}{X \Longrightarrow Y}\ (\mathrm{I}{-}) \qquad \frac{\mathrm{I} \Longrightarrow Y}{X \Longrightarrow Y}\ (\mathsf{Il})$$

$$\frac{X \Longrightarrow \mathrm{I}}{X \Longrightarrow Y}\ (\mathsf{Ir}) \qquad \frac{X \circ Y \Longrightarrow Z}{Y \circ X \Longrightarrow Z}\ (\mathsf{Pl}) \qquad \frac{X \circ X \Longrightarrow Y}{X \Longrightarrow Y}\ (\mathsf{Cl})$$

$$\frac{X \circ (Y \circ Z) \Longrightarrow U}{(X \circ Y) \circ Z \Longrightarrow U}\ (\mathsf{Al}) \qquad \frac{X \Longrightarrow A \qquad A \Longrightarrow Y}{X \Longrightarrow Y}\ (\mathsf{Cut})$$

Logical Rules

$$\frac{*A \Longrightarrow X}{\neg A \Longrightarrow X}\ (\neg\mathsf{l}) \qquad \frac{X \Longrightarrow *A}{X \Longrightarrow \neg A}\ (\neg\mathsf{r}) \qquad \frac{A \circ B \Longrightarrow X}{A \wedge B \Longrightarrow X}\ (\wedge\mathsf{l})$$

$$\frac{X \Longrightarrow A \qquad Y \Longrightarrow B}{X \circ Y \Longrightarrow A \wedge B}\ (\wedge\mathsf{r}) \qquad \frac{A \Longrightarrow X \qquad B \Longrightarrow Y}{A \vee B \Longrightarrow X \circ Y}\ (\vee\mathsf{l}) \qquad \frac{X \Longrightarrow A \circ B}{X \Longrightarrow A \vee B}\ (\vee\mathsf{r})$$

$$\frac{X \Longrightarrow A \qquad B \Longrightarrow Y}{A \to B \Longrightarrow *X \circ Y}\ (\to\mathsf{l}) \qquad \frac{X \circ A \Longrightarrow B}{X \Longrightarrow A \to B}\ (\to\mathsf{r})$$

Display Equivalence Rules (DE)

$$\cfrac{\cfrac{X \circ Z \Longrightarrow Y}{X \Longrightarrow Y \circ *Z}}{Z \Longrightarrow *X \circ Y} \qquad \cfrac{\cfrac{X \Longrightarrow Y}{*Y \Longrightarrow *X}}{X \Longrightarrow * * Y} \qquad \cfrac{\cfrac{X \Longrightarrow Y \circ Z}{X \circ *Z \Longrightarrow Y}}{*Y \circ X \Longrightarrow Z}$$

The display equivalence rules, which all fall under the label (DE), are vital for the theorems given in display calculi. Examples of their use show up in almost every proof of this paper. It should be noted that the double line occurring between each sequent is meant to represent that the sequents are defined (via the display equivalence rule) to be mutually derivable from one another. For example, in the rules given directly above we may infer $X \Longrightarrow Y \circ *Z$ from $X \circ Z \Longrightarrow Y$, and vice-versa. These rules in conjunction with the other structural rules provide us with fruitful consequences that, in effect, act as additional structural rules:

Fact 1. The rules

[7] Note that we diverge from the pattern of presentation given in Belnap [1] since we place (Cut) among the structural rules. Moreover, our presentation is Non-Belnapian in the sense that we prove completeness prior to cut-elimination in each section and our structural rules are due to Wansing (See [11]). Nevertheless, we have chosen to organize the paper in this way since we believe it to ease the presentation of the content.

$$\frac{Z \Longrightarrow X \circ Y}{Z \Longrightarrow Y \circ X}\,(\mathsf{Pr}) \qquad \frac{Y \Longrightarrow X \circ X}{Y \Longrightarrow X}\,(\mathsf{Cr}) \qquad \frac{X \Longrightarrow Z}{X \circ Y \Longrightarrow Z}\,(\mathsf{Wl})$$

$$\frac{U \Longrightarrow X \circ (Y \circ Z)}{U \Longrightarrow (X \circ Y) \circ Z}\,(\mathsf{Ar}) \qquad \frac{X \Longrightarrow Z}{X \Longrightarrow Z \circ Y}\,(\mathsf{Wr})$$

are derivable.

To give the reader a feel for display logic proofs, we provide a couple examples below. First, we deduce the rule (Pr) by making use of (Pl) and the third display equivalence rule of the three given above. For our second example, we deduce axiom two of Hilbert's propositional calculus Cp. For the definition of Cp, see section 2.3 below.

Example 2. We can derive the rule (Pr) as follows:

$$\cfrac{\cfrac{\cfrac{Z \Longrightarrow X \circ Y}{Z \circ {*Y} \Longrightarrow X}\,(\mathsf{DE})}{{*Y} \circ Z \Longrightarrow X}\,(\mathsf{Pl})}{Z \Longrightarrow Y \circ X}\,(\mathsf{DE})$$

Example 3. We now derive $(\neg A \to \neg B) \to (B \to A)$ in D.Cp:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \Longrightarrow A}{{*A} \Longrightarrow {*A}}\,(\neg\mathsf{r})}{{*A} \Longrightarrow \neg A}\,(\mathsf{DE}) \quad \cfrac{\cfrac{B \Longrightarrow B}{{*B} \Longrightarrow {*B}}\,(\neg\mathsf{l})}{\neg B \Longrightarrow {*B}}\,(\mathsf{DE})}{\neg A \to \neg B \Longrightarrow {*}{*}A \circ {*}B}\,(\to\mathsf{l})}{(\neg A \to \neg B) \circ B \Longrightarrow {*}{*}A}\,(\mathsf{DE})}{(\neg A \to \neg B) \circ B \Longrightarrow A}\,(\mathsf{DE})}{\neg A \to \neg B \Longrightarrow B \to A}\,(\to\mathsf{r})}{I \circ (\neg A \to \neg B) \Longrightarrow B \to A}\,(\mathrm{I}+)}{I \Longrightarrow (\neg A \to \neg B) \to (B \to A)}\,(\to\mathsf{r})$$

Let us now emphasize a useful property characteristic of display logics: *the display property*. The intuition of the display property is that we may focus our attention on a specific structure within a display sequent and use the display equivalence rules to make the structure the entire antecedent or succedent of the sequent. This will be useful for the general cut elimination theorem given later on in this section. Before we can dig into the mechanisms of this property, there are two notions necessary to understand it, which are defined as follows:

Definition 3 (Positive and Negative Occurrence)**.** An occurrence of a substructure in a given structure is called *positive* if it is in the scope of an even number of $*$ (otherwise its coined *negative*).

Definition 4 (Antecedent and Succedent Parts)**.** In a sequent $Y \Longrightarrow Z$ an occurrence of X is an *antecedent part* if it occurs positively in the antecedent or negatively in the succedent. An occurrence that is not an antecedent part is a *succedent part*.

Working with proofs in display logic one may notice an interesting phenomenon. If we focus on any substructure occurring as an antecedent part or succedent part, then we may always *display* that substructure. For example, suppose we want to display the positive occurrence of the substructure X in the sequent $*(*X \circ Y) \circ A \Longrightarrow Z$, or the negative occurrence of the substructure X in $*X \circ B \Longrightarrow Z \circ *Y$. We could use our display equivalence rules as follows:

$$\dfrac{\dfrac{*(*X \circ Y) \circ A \Longrightarrow Z}{A \circ *Z \Longrightarrow *X \circ Y}}{X \Longrightarrow Y \circ *(A \circ *Z)} \qquad \dfrac{*X \circ B \Longrightarrow Z \circ *Y}{B \circ *(Z \circ *Y) \Longrightarrow X}$$

Notice that X occurs as an antecedent part in the first proof and as a succedent part in the second. Using our rules we were able to display X as the entire antecedent in the first case and the entire succedent in the second case. This suggests that antecedent parts and succedent parts should always be displayable in this way, and if they can be, then this means the calculus possesses the display property:

Definition 5 (Display Property)**.** A display calculus possesses the *display property* if and only if any antecedent (succedent) part X of a sequent S can be displayed as the entire antecedent (succedent) of a sequent S' which is display equivalent to S.

Our previous examples suggest that our calculus $\mathsf{D.Cp}$ has the display property. The following theorem confirms our insight:

Theorem 1 (Display Theorem)**.** *Each antecedent part X of a sequent S can be displayed as the whole antecedent of a display-equivalent sequent $X \Longrightarrow Y$ in* $\mathsf{D.Cp}$*. Likewise, each consequent part of a sequent can be displayed as the whole succedent of a display-equivalent sequent in* $\mathsf{D.Cp}$*.*

Proof. Suppose that X is an arbitrary antecedent part of a sequent S. Note that X may occur as a substructure in either the antecedent, succedent, or both. We only consider the cases where X occurs as a substructure in the antecedent since all other cases are similar. Recall that we are trying to show that every antecdent part can be displayed as the entire antecedent and every succedent part can be displayed as the entire succedent.

Let S be the sequent $\phi(X) \Longrightarrow Z$ with $\phi(X)$ a structure containing X as a substructure. We prove our theorem by induction on the structural-complexity of $\phi(X)$. For the base case, suppose that $\phi(X)$ is either the empty structure I or a formula A. Then our sequent is of the form $I \Longrightarrow Z$ or $A \Longrightarrow Z$, and so the result follows trivially. Suppose now that $\phi(X)$ is of the form $*\psi(X)$, where $\psi(X)$ is a structure containing an antecedent part X as a substructure. It is easy to see that the result follows from our rules and the inductive hypothesis (IH):

$$\dfrac{\dfrac{\dfrac{*\psi(X) \Longrightarrow Z}{*Z \Longrightarrow \psi(X)}\,(\mathsf{DE})}{\vdots}\,(\mathrm{IH})}{X \Longrightarrow Y}$$

Once we have reached line three in the proof, the inductive hypothesis (IH) guarantees that X can be displayed as an antecedent because $\psi(X)$ is of less complexity. Furthermore, we are justified in our use of double lines to signify mutual derivability, since we make use of only the (DE) rules (the inductive hypothesis assumes this). Let us now suppose that $\phi(X)$ is of the form $W \circ V$. We can assume without loss of generality that X occurs as a substructure of W, so we denote W as $\psi(X)$ for emphasis, *i.e.* $\phi(X) = \psi(X) \circ V$. Again, we can easily display X by making use of our display equivalence rules and the inductive hypothesis (IH):

$$\dfrac{\dfrac{\dfrac{\psi(X) \circ V \Longrightarrow Z}{\psi(X) \Longrightarrow Z \circ *V}\,(\mathsf{DE})}{\vdots}\,(\mathsf{IH})}{X \Longrightarrow Y}$$

Similar to the previous case, once we reach line three of the proof, $\psi(X)$ is of less complexity and the inductive hypothesis (IH) does the rest; moreover, we are justified by our use of double lines for the same reasons given previously. The remaining cases, which include the display of succedent parts, are all proved by similar argumentation.

□

The display property is used in the proof of general cut elimination given later in this section. One should note that the display property is an attribute of every display calculus given in this paper. Therefore, we will only mention the display theorem here, but the reader should keep in mind that each calculus to be presented possesses this property.

2.2 Completeness and Soundness

We now prove that the propositional display calculus D.Cp is complete and sound. In attempt to keep our paper along proof-theoretic lines, we introduce the complete and sound Hilbert calculus Cp defined below. Our completeness and soundness theorems are proven relative to the calculus Cp–allowing us to circumvent the introduction of a semantic system, while retaining the desired results.

Definition 6 (Provable in D.Cp)**.** We say that a propositional formula A is *provable in* D.Cp if and only if there is a derivation in D.Cp with the conclusion $I \Longrightarrow A$.

Definition 7 (Hilbert calculus Cp)**.** Cp is the deductive calculus consisting of the inference rule:

$$\dfrac{A \qquad A \to B}{B}\,(\mathsf{MP})$$

and the axioms (Cp1)–(Cp3):

(Cp1) $A \to (B \to A)$

(Cp2) $(\neg A \to \neg B) \to (B \to A)$

(Cp3) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

We assume the usual definitions the other Boolean connectives. To establish completeness, we show that if a formula is derivable in the Hilbert calculus Cp, then it is also derivable in D.Cp. Since Cp is complete, we know that if a formula is true, then it it is provable in Cp. By showing that anything provable in Cp is provable in D.Cp, it follows that every true formula is provable in D.Cp. Part of our completeness proof consists of showing that (MP) is an *admissible* rule of inference in D.Cp, *i.e.* the calculus proves the exact same formulas regardless of if the rule (MP) is added or omitted from the calculus. The notion of admissibility plays a large role in the theorems to come, so we provide a general definition of it here:

Definition 8 (Admissible Rule of Inference)**.** An inference rule (Inf) is admissible in a calculus S if and only if the set of formulas provable in S is equal to the set of formulas provable in S + (Inf).

The intuition behind admissibility is that the admissible rule fails to bring new deductive power to the calculus. Thus, we can always acquire a proof of a provable formula without the admissible rule. Let us now make use of this tool, and the following fact, to prove completeness:

Fact 2. For all formulas A, the sequent $X \circ A \circ X' \Longrightarrow Y \circ A \circ Y'$ is derivable in D.Cp, where X, X', Y, and Y' are arbitrary structures. Notice that the sequent $A \Longrightarrow A$ follows from this when the surrounding context of A is empty.

This fact is useful in the completeness theorem since we use $A \Longrightarrow A$ (for arbitrary A) as a starting point to show that all instances of the Hilbert axioms are provable in the calculus D.Cp.

Theorem 2 (Completeness of D.Cp)**.** *If a formula A is provable in* Cp*, then A is provable in* D.Cp.

Proof. We show that if A is derivable in Cp, then $I \Longrightarrow A$ is derivable in D.Cp. To demonstrate this, it suffices to show that the axioms (Cp1)–(Cp3) are derivable in D.Cp and that (MP) is an admissible rule of inference in D.Cp:

(Cp1) $I \Longrightarrow A \to (B \to A)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Longrightarrow A}{I \circ A \Longrightarrow A}\,(\mathrm{I}+)}{I \Longrightarrow A \circ {*A}}\,(\mathsf{DE})}{B \Longrightarrow A \circ {*A}}\,(\mathrm{I}|)}{B \circ A \Longrightarrow A}\,(\mathsf{DE})}{A \circ B \Longrightarrow A}\,(\mathsf{P})}{A \Longrightarrow B \to A}\,(\to\mathsf{r})}{I \circ A \Longrightarrow B \to A}\,(\mathrm{I}+)}{I \Longrightarrow A \to (B \to A)}\,(\to\mathsf{r})$$

(Cp2) $I \Longrightarrow (\neg A \to \neg B) \to (B \to A)$

See Example 3.

(Cp3) $I \Longrightarrow (A \to (B \to C)) \to ((A \to B) \to (A \to C))$

$$
\begin{prooftree}
\AxiomC{$A \Longrightarrow A$}
\AxiomC{$A \Longrightarrow A$}
\AxiomC{$B \Longrightarrow B$}
\AxiomC{$C \Longrightarrow C$}
\RightLabel{$(\to\mathsf{l})$}
\BinaryInfC{$B \to C \Longrightarrow *B \circ C$}
\RightLabel{$(\mathsf{DE}) \cdot 2$}
\UnaryInfC{$B \Longrightarrow C \circ *(B \to C)$}
\RightLabel{$(\to\mathsf{l})$}
\BinaryInfC{$A \to B \Longrightarrow *A \circ (C \circ *(B \to C))$}
\RightLabel{(DE)}
\UnaryInfC{$A \circ (A \to B) \Longrightarrow C \circ *(B \to C)$}
\RightLabel{(P)}
\UnaryInfC{$(A \to B) \circ A \Longrightarrow C \circ *(B \to C)$}
\RightLabel{(DE)}
\UnaryInfC{$((A \to B) \circ A) \circ (B \to C) \Longrightarrow C$}
\RightLabel{(A)}
\UnaryInfC{$(A \to B) \circ (A \circ (B \to C)) \Longrightarrow C$}
\RightLabel{$(\mathsf{DE}) \cdot 2$}
\UnaryInfC{$(B \to C) \Longrightarrow *A \circ (*(A \to B) \circ C)$}
\RightLabel{$(\to\mathsf{l})$}
\BinaryInfC{$A \to (B \to C) \Longrightarrow *A \circ (*A \circ (*(A \to B) \circ C))$}
\RightLabel{$(\mathsf{DE}) \cdot 2$}
\UnaryInfC{$A \circ (A \circ (A \to (B \to C))) \Longrightarrow *(A \to B) \circ C$}
\RightLabel{(A)}
\UnaryInfC{$(A \circ A) \circ (A \to (B \to C)) \Longrightarrow *(A \to B) \circ C$}
\RightLabel{(DE)}
\UnaryInfC{$A \circ A \Longrightarrow (*(A \to B) \circ C) \circ *(A \to (B \to C))$}
\RightLabel{(C)}
\UnaryInfC{$A \Longrightarrow (*(A \to B) \circ C) \circ *(A \to (B \to C))$}
\RightLabel{$(\mathsf{DE}) \cdot 2$}
\UnaryInfC{$(A \to B) \circ (A \circ (A \to (B \to C))) \Longrightarrow C$}
\RightLabel{(P)}
\UnaryInfC{$(A \circ (A \to (B \to C))) \circ (A \to B) \Longrightarrow C$}
\RightLabel{(A)}
\UnaryInfC{$A \circ ((A \to (B \to C)) \circ (A \to B)) \Longrightarrow C$}
\RightLabel{(P)}
\UnaryInfC{$((A \to (B \to C)) \circ (A \to B)) \circ A \Longrightarrow C$}
\RightLabel{$(\to\mathsf{r}) \cdot 2$}
\UnaryInfC{$A \to (B \to C) \Longrightarrow (A \to B) \to (A \to C)$}
\RightLabel{$(\mathrm{I}+)$}
\UnaryInfC{$I \circ (A \to (B \to C)) \Longrightarrow (A \to B) \to (A \to C)$}
\RightLabel{$(\to\mathsf{r})$}
\UnaryInfC{$I \Longrightarrow (A \to (B \to C)) \to ((A \to B) \to (A \to C))$}
\end{prooftree}
$$

(MP) The rule

$$
\frac{I \Longrightarrow A \qquad I \Longrightarrow A \to B}{I \Longrightarrow B}\ (\mathsf{MP})
$$

is admissible in D.Cp:

$$
\begin{prooftree}
\AxiomC{$I \Longrightarrow A \to B$}
\AxiomC{$I \Longrightarrow A$}
\AxiomC{$B \Longrightarrow B$}
\RightLabel{$(\to\mathsf{l})$}
\BinaryInfC{$A \to B \Longrightarrow *I \circ B$}
\RightLabel{(DE)}
\UnaryInfC{$I \circ (A \to B) \Longrightarrow B$}
\RightLabel{$(\mathrm{I}\text{-})$}
\UnaryInfC{$A \to B \Longrightarrow B$}
\RightLabel{(Cut)}
\BinaryInfC{$I \Longrightarrow B$}
\end{prooftree}
$$

Observe that this is in accordance with our definition of admissibility; the combination of inference rules in the proof, along with the axiom $B \Longrightarrow B$, produce the same conclusion as (MP) from the same premises $I \Longrightarrow A$ and $I \Longrightarrow A \to B$. Note that we also made use of the cut rule here. In the next section we show that (Cut) is another rule admissible in D.Cp. This concludes the completeness theorem.

□

We now move on to the soundness theorem, which is also proven relative to Cp. In order to complete the proof we construct a translation function $\mathfrak{I}$ that maps sequents of D.Cp to formulas of Cp. We then show that the axiom and rules of D.Cp can be mirrored in Cp with $\mathfrak{I}$, and thus, any formula provable in D.Cp is also provable in Cp. Furthermore, since every formula provable in Cp is true, it follows that any formula provable in D.Cp is true–establishing soundness. We first define $\mathfrak{I}$ and then provide a couple examples regarding translation. The soundness theorem has consistency as a corollary.

The key idea underlying our translation function is that it treats antecedent parts differently from consequent parts. Following the definition in [1], our function $\mathfrak{I}$ is defined with respect to distinct, yet interrelated, functions $\mathfrak{I}_1$ and $\mathfrak{I}_2$, where $\mathfrak{I}$ translates the entire sequent as a conditional formula with $\mathfrak{I}_1$ acting initially on the antecedent and $\mathfrak{I}_2$ acting initially on the consequent. The definitions of both are as follows:

Definition 9 (Translation function $\mathfrak{I}$)**.** We define $\mathfrak{I}(X \Longrightarrow Y)$ to be equal to $\mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)$. Moreover, let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ map from the set of structures to the set of propositional formulas such that:

$$\mathfrak{I}_1(X) = \begin{cases} A \text{ if } X = A, \\ \top \text{ if } X = I, \\ \neg\mathfrak{I}_2(Y) \text{ if } X = *Y, \\ \mathfrak{I}_1(Y) \wedge \mathfrak{I}_1(Z) \text{ if } X = (Y \circ Z) \end{cases} \qquad \mathfrak{I}_2(X) = \begin{cases} A & \text{if } X = A, \\ \bot & \text{if } X = I, \\ \neg\mathfrak{I}_1(Y) & \text{if } X = *Y, \\ \mathfrak{I}_2(Y) \vee \mathfrak{I}_2(Z) & \text{if } X = (Y \circ Z) \end{cases}$$

Fact 3. The above definition implies that $\neg\mathfrak{I}_1(X) = \mathfrak{I}_2(*X)$ and $\mathfrak{I}_1(*X) = \neg\mathfrak{I}_2(X)$. This fact will be useful below in our proof of soundness.

Example 4. Let us consider translating the sequent $**(\neg A \circ B) \Longrightarrow *(*I \circ *C)$:

$$\begin{gathered} \mathfrak{I}(**(\neg A \circ B) \Longrightarrow *(*I \circ *C)) \\ \mathfrak{I}_1(**(\neg A \circ B)) \to \mathfrak{I}_2(*(*I \circ *C)) \\ \neg\mathfrak{I}_2(*(\neg A \circ B)) \to \neg\mathfrak{I}_1(*I \circ *C) \\ \neg\neg\mathfrak{I}_1(\neg A \circ B) \to \neg(\mathfrak{I}_1(*I) \wedge \mathfrak{I}_1(*C)) \\ \neg\neg(\mathfrak{I}_1(\neg A) \wedge \mathfrak{I}_1(B)) \to \neg(\neg\mathfrak{I}_2(I) \wedge \neg\mathfrak{I}_2(C)) \\ \neg\neg(\neg A \wedge B) \to \neg(\neg\bot \wedge \neg C) \end{gathered}$$

Example 5. Let us consider translating the sequent $I \circ *(A \wedge B \circ X) \Longrightarrow Z \circ B$:

$$\begin{gathered} \mathfrak{I}(I \circ *(A \wedge B \circ X) \Longrightarrow Z \circ B) \\ \mathfrak{I}_1(I \circ *(A \wedge B \circ X)) \to \mathfrak{I}_2(Z \circ B) \\ \mathfrak{I}_1(I) \wedge \mathfrak{I}_1(*(A \wedge B \circ X)) \to \mathfrak{I}_2(Z) \vee \mathfrak{I}_2(B) \\ \top \wedge \neg\mathfrak{I}_2(A \wedge B \circ X) \to \mathfrak{I}_2(Z) \vee B \\ \top \wedge \neg(\mathfrak{I}_2(A \wedge B) \vee \mathfrak{I}_2(X)) \to \mathfrak{I}_2(Z) \vee B \\ \top \wedge \neg((A \wedge B) \vee \mathfrak{I}_2(X)) \to \mathfrak{I}_2(Z) \vee B \end{gathered}$$

Theorem 3 (Soundness Theorem for D.Cp)**.** *If a formula A is provable in* D.Cp*, then A is provable in* Cp*.*

Proof. We show that for any sequent S derivable in D.Cp, $\mathfrak{I}(S)$ is provable in Cp. In our proof we only consider a few cases since the others are carried out

similarly. It is easy to see that the translation of the D.Cp axiom is provable in Cp. If we suppose that A is an atomic formula, then $\mathfrak{I}(A \Longrightarrow A) = \mathfrak{I}_1(A) \to \mathfrak{I}_2(A) = A \to A$, which is provable in Cp. To give an idea of how the remaining translated rules are proved, we provide a few examples:

(I−)

$\mathfrak{I}(I \circ X \Longrightarrow Y)$
$\mathfrak{I}_1(I \circ X) \to \mathfrak{I}_2(Y)$
$\mathfrak{I}_1(I) \wedge \mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)$
$\top \wedge \mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)$
$\neg\top \vee \neg\mathfrak{I}_1(X) \vee \mathfrak{I}_2(Y)$
$\neg\mathfrak{I}_1(X) \vee \mathfrak{I}_2(Y)$
$\mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)$
$\mathfrak{I}(X \Longrightarrow Y)$

(Cut)

$\mathfrak{I}(X \Longrightarrow A)$
$\mathfrak{I}(A \Longrightarrow Y)$
$\mathfrak{I}_1(X) \to \mathfrak{I}_2(A)$
$\mathfrak{I}_1(A) \to \mathfrak{I}_2(Y)$
$\mathfrak{I}_1(X) \to A$
$A \to \mathfrak{I}_2(Y)$
$\mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)$
$\mathfrak{I}(X \Longrightarrow Y)$

(DE)

$\mathfrak{I}(X \Longrightarrow Y)$
$\mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)$
$\neg\mathfrak{I}_2(Y) \to \neg\mathfrak{I}_1(X)$
$\mathfrak{I}_1(*Y) \to \mathfrak{I}_2(*X)$
$\mathfrak{I}(*Y \Longrightarrow *X)$
$\mathfrak{I}_1(*Y) \to \mathfrak{I}_2(*X)$
$\neg\mathfrak{I}_2(Y) \to \neg\mathfrak{I}_1(X)$
$\mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)$
$\mathfrak{I}_1(X) \to \neg\neg\mathfrak{I}_2(Y)$
$\mathfrak{I}_1(X) \to \neg\mathfrak{I}_1(*Y)$
$\mathfrak{I}_1(X) \to \mathfrak{I}_2(**Y)$
$\mathfrak{I}(X \Longrightarrow **Y)$

(→I)

$\mathfrak{I}(X \Longrightarrow A)$
$\mathfrak{I}(B \Longrightarrow Y)$
$\mathfrak{I}_1(X) \to \mathfrak{I}_2(A)$
$\mathfrak{I}_1(B) \to \mathfrak{I}_2(Y)$
$\mathfrak{I}_1(X) \to A$
$B \to \mathfrak{I}_2(Y)$
$\neg A \to \neg\mathfrak{I}_1(X)$
$\neg A \vee B \to \neg\mathfrak{I}_1(X) \vee \mathfrak{I}_2(Y)$
$(A \to B) \to \mathfrak{I}_2(*X) \vee \mathfrak{I}_2(Y)$
$\mathfrak{I}_1(A \to B) \to \mathfrak{I}_2(*X \circ Y)$
$\mathfrak{I}(A \to B \Longrightarrow *X \circ Y)$

Continuing in this fashion is is easy to show that for every sequent S derivable in D.Cp, its interpretation $\mathfrak{I}(S)$ is provable in Cp. Therefore, for all sequents $I \Longrightarrow A$ derivable in D.Cp, $\mathfrak{I}(I \Longrightarrow A) = \top \to A$ is provable in Cp, so A is provable in Cp. □

It is true in general that consistency follows from soundness, but we will still give a demonstration of the corollary here with respect to our calculus D.Cp:

Corollary 1 (Consistency of D.Cp)**.** *There does not exist a formula A such that both A and $\neg A$ are provable in* D.Cp*.*

Proof. We prove the consistency of D.Cp by contradiction. Suppose that D.Cp is inconsistent. Then, $I \Longrightarrow A$ and $I \Longrightarrow \neg A$ are derivable in D.Cp, for some formula A. By the soundness theorem, it follows that $\mathfrak{I}(I \Longrightarrow A) = \top \to A$ and $\mathfrak{I}(I \Longrightarrow \neg A) = \top \to \neg A$ are provable in Cp. However, this contradicts the fact that Cp is consistent, so it must be the case that D.Cp is also consistent. □

2.3 Cut Elimination

A useful feature of our calculus D.Cp is that the rule

$$\frac{X \Longrightarrow A \qquad A \Longrightarrow Y}{X \Longrightarrow Y}\ (\mathsf{Cut})$$

is admissible. We have already seen that (MP) is admissible since the inference rule can be simulated with other rules of D.Cp. It was a simple procedure to show that a combination of other rules have the same effect as (MP). However, our proof for the admissibility of (Cut) requires more resources, and so, it is not as straightforward.

A significant result in [1] is that there are general conditions implying the admissibility of cut, *i.e.* if a display calculus satisfies all criteria given there, then (Cut) is an admissible rule of inference in the calculus. Any calculus which satisfies all of Belnap's desiderata will necessarily exhibit the *cut elimination property*. Our aim in the current section is to prove that these conditions do in fact imply the cut elimination property for display calculi *in general*. After securing a proof of this fact, we prove that our calculus D.Cp satisfies all conditions, from which we conclude that (Cut) is an admissible rule of inference in D.Cp.

Definition 10. (Relevant Terminology for Display Calculi) To provide the reader with some intuition, we include an example with each term defined below:

(1) An instantiation of an inference rule, where each metavariable is uniformly replaced by a concrete structure, is called an *inference*.

Example 6. Consider the inference rule

$$\frac{X \circ A \Longrightarrow B}{X \Longrightarrow A \to B}\ (\to\mathsf{r})$$

X is a structural metavariable, whereas A and B are formulaic metavariables. If we instantiate each metavariable with a concrete structure or formula, then we obtain an inference:

$$\frac{*(p \vee q) \circ \neg r \Longrightarrow s}{*(p \vee q) \Longrightarrow \neg r \to s}\ (\to\mathsf{r})$$

(2) Every structure and substructure occurring in an inference is called a *constituent* of the inference.

Example 7. If we observe the inference above, then we can see that it contains the following constituents: the premise contains $*(p \vee q) \circ \neg r$, $*(p \vee q)$, $(p \vee q)$, p, q, $\neg r$, r, and s, whereas the conclusion contains $*(p \vee q)$, $(p \vee q)$, p, q, $\neg r \to s$, $\neg r$, r, and s as constituents.

(3) A constituent is is called a *parameter*, or is said to be *parametric*, in an inference if and only if it is a substructure of a structure that was assigned to a (structural) metavariable. Intuitively, parameters are all structures, or substructures, in an inference rule, that remain unchanged when going from the premises to the conclusion.

Example 8. If we observe the inference in example 6, we see that $*(p \vee q)$, $(p \vee q)$, $\neg r$, s, $*(p \vee q)$, and $(p \vee q)$ are parametric constituents since each is a substructure of a structure assigned to a metavariable in the inference rule. From an intuitive standpoint, both $*(p \vee q) \circ \neg r$ and $\neg r \rightarrow s$ fail to be parametric because they are not preserved from premise to conclusion.

(4) A constituent is *principal* in an inference if and only if it is part of the conclusion and not parametric. Intuitively, a principal constituent is one that is introduced by the inference rule.

Example 9. Continuing with the example, we see that $\neg r \rightarrow s$ is the only principal constituent. Due to the fact that our inference introduced the formula in the conclusion, it is obvious that it cannot be parametric. This is typical of parametric constituents–notice that all of the logical rules introduce principal constituents since the rules generate higher complexity formulas.

(5) Two parameters are *congruent* if and only if they are both occurrences of the same structure and one of the following is true: (i) they were instantiated for the same structural metavariable, or (ii) they are the same substructure with the same shape and position in the structure that was instantiated for a metavariable.

Example 10. In the inference above, the occurrences of $*(p \vee q)$ and $(p \vee q)$ are the only pairs of congruent parameters. We observe that $*(p \vee q)$ was instantiated for the same structural metavariable X, and that $(p \vee q)$ is a substructure of the structure $*(p \vee q)$ instantiated for X.

Note that the definitions of constituent, parameter, principality, and congruence may vary with different display calculi as long as they comply with the conditions given in definition 11 below:

Definition 11. (Conditions (C2)–(C8)). The following general conditions guarantee cut elimination:

(C1) *Preservation of formulas:* With the exception of (Cut), each formula occurring in a premise of an inference is a subformula of some formula in the conclusion.

(C2) *Shape-alikeness of parameters:* Congruent parameters are occurrences of the same structure.

(C3) *Non-proliferation of parameters:* Each parameter is congruent to at most one constituent in the conclusion; that is, no two constituents in the conclusion are congruent to each other.

(C4) *Position-alikeness of parameters:* Congruent parameters are either all antecedent or all consequent parts in their respective sequence.

(C5) *Display of principal constituents:* If a formula is principal constituent in the conclusion of an inference, then it is either the entire antecedent or the entire consequent of the conclusion.

(C6) *Closure under substitution of consequent parts:* Each inference rule is closed under simultaneous substitution of arbitrary structures in consequent parts for congruent parameters.

(C7) *Closure under substitution of antecedent parts:* Each inference rule is closed under simultaneous substitution of arbitrary structures in antecedent parts for congruent parameters.

(C8) *Cut of matching principal constituents:* Suppose there are inferences (Inf_1) and (Inf_2) with respective conclusions $X \Longrightarrow M$ and $M \Longrightarrow Y$, where M principal in both inferences. Then, one of two things must follow: (1) $X \Longrightarrow Y$ is identical to $X \Longrightarrow M$ or $M \Longrightarrow Y$, or (2) there is a derivation of $X \Longrightarrow Y$ from the premises of (Inf_1) and (Inf_2), where (Cut) is only used on proper subformulas of M.

It should be noted that condition (C1) does not play a role in proving the admissibility of (Cut), *i.e.* conditions (C2) through (C8) are sufficient to prove the general cut elimination theorem. However, if all eight conditions do hold for a calculus, then it follows that the calculus possesses the *subformula property* – meaning that each provable sequent has a proof where every formula occurring in any step of the derivation is a subformula of a formula in the conclusion.

To observe examples of the subformula property take a look at the logical rules, (Il), and (Ir) for the calculus $\mathsf{D.Cp}$ given above. Notice that for every inference rule, any formula occurring in the premise is a subformula of some formula in the conclusion. Furthermore, if you look underneath the logical rules and at the list of display equivalence rules, you will notice that our calculus does not have a *substructure property*. For example, in the second (DE) rule $*Y \Longrightarrow *X$ is deducible from $X \Longrightarrow **Y$, and $**Y$ is not a substructure of $*Y$. We can see that structural connectives introduced in some line of a derivation may disappear later on, and so, they need not necessarily be present in the last line of the derivation.

So, although $\mathsf{D.Cp}$ possesses the subformula property, it therefore does not posses the substructure property which is an often given criticism of display calculi. If the calculus were to also possess the substructure property, then we could apply the inference rules in reverse to a given sequent, for example, and uncover a proof of the given sequent. The violation of the substructure property makes such a proof search procedure difficult, if not impractical, and is an example of one serious limitation of the display formalism.

Let us now move on to the general cut elimination theorem. We first prove that the condition (C8) implies the admissibility of principal cuts, and then demonstrate the general result for all cut formulas:

Lemma 1 (Admissibility of Principal Cuts)**.** *The condition (C8) implies that the rule (Cut) is admissible in a proof where the cut formula is principal in the premises of the final inference. In other words, if condition (C8) holds and the sequents $X \Longrightarrow M\dagger$ and $M\dagger \Longrightarrow Y$ are cut-free derivable, then the sequent $X \Longrightarrow Y$ is cut-free derivable, where $\dagger$ indicates that M is principal in the last inference of the derivation.*

Proof. Our proof is by induction on the complexity of the cut formula M. We leave it to the reader to prove the base case where M is atomic. Suppose that M is a complex formula and that X and Y are arbitrary structures. For the inductive step, we want to show that if $X \Longrightarrow M\dagger$ and $M\dagger \Longrightarrow Y$ are cut-free derivable, then $X \Longrightarrow Y$ is cut-free derivable. By the inductive hypothesis (IH) we know that for all proper subformulas M' of M and for arbitrary structures X and Y that if $X \Longrightarrow M'\dagger$ and $M'\dagger \Longrightarrow Y$ are cut-free derivable, then $X \Longrightarrow Y$ is cut-free derivable. Observe that the inductive step follows directly from condition (C8) and (IH). By (C8) we have that, given $X \Longrightarrow M\dagger$ and $M\dagger \Longrightarrow Y$, $X \Longrightarrow Y$ can be derived with the help of the cut rule restricted to proper subformulas of M. By the inductive hypothesis (IH), cuts on proper subformulas of M are admissible.

□

Theorem 4 (General Cut Elimination)**.** *If a display calculus satisfies (C2)–(C8), then the cut rule is admissible.*

Proof. Assume that conditions (C2) through (C8) hold. By lemma 1, we have that cuts on principal formulas are admissible. We now make use of lemma 1 to show that cut is admissible in general. First, we relax the requirement on the left premise and prove the following:

(1) If the sequents $X \Longrightarrow M$ and $M\dagger \Longrightarrow Y$ are cut-free derivable, then the sequent $X \Longrightarrow Y$ is cut-free derivable.

Second, we show that the right principality-condition $\dagger$ can be relaxed as well:

(2) If the sequents $X \Longrightarrow M$ and $M \Longrightarrow Y$ are cut-free derivable, then the sequent $X \Longrightarrow Y$ is cut-free derivable.

Notice that (2) is the result we are aiming to show. It says that if the premises of the cut rule are cut-free derivable, then so is the conclusion. Thus, anything provable with (Cut), can also be proven directly without the rule. It remains to show (1) and (2):

(1) Suppose $X \Longrightarrow M$ and $M\dagger \Longrightarrow Y$ are cut-free derivable. Let $\mathcal{D}$ be a derivation of $X \Longrightarrow M$. We transform $\mathcal{D}$ into a derivation of $X \Longrightarrow Y$. It is necessary to differentiate between different occurrences of the same formula within $\mathcal{D}$. To do so, we use the following definition:

Definition 12 (Congruent Parametric Ancestors)**.** For an occurrence t of a formula A in a derivation $\mathcal{D}$, define the set of *congruent parametric ancestors* $Anc(t)$ as follows: (i) t is in $Anc(t)$ and (ii) for all inferences in $\mathcal{D}$, each formula that is congruent to a member of $Anc(t)$ is also in $Anc(t)$.

By (C2), all members of $Anc(t)$ are occurrences of the same formula A. To give the reader some intuition regarding this definition, we provide an example:

Example 11. For the occurrence t of $\neg E$, the members of $Anc(t)$ are indicated in bold in the derivation below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{E \Longrightarrow A}}{*A \Longrightarrow *E}\,(\mathsf{DE})}{*A \Longrightarrow \neg \boldsymbol{E}}\,(\neg\mathsf{r})}{*\neg \boldsymbol{E} \Longrightarrow A}\,(\mathsf{DE}) \qquad \dfrac{\vdots}{B \Longrightarrow C}}{*\neg \boldsymbol{E} \circ B \Longrightarrow A \wedge C}\,(\wedge\mathsf{r})}{*\neg \boldsymbol{E} \Longrightarrow B \to (A \wedge C)}\,(\to\mathsf{r})}{*(B \to A \wedge C) \Longrightarrow \underbrace{\neg \boldsymbol{E}}_{t}}\,(\mathsf{DE})$$

The topmost member of $Anc(t)$, which occurs in the sequence $*A \Longrightarrow \neg E$, is principal in the inference ($\neg$r) leading to it. All other members of $Anc(t)$ are merely parametric in their rules of inference. This shows that $Anc(t)$ can be split in principal occurrences and parametric occurrences.

The derivation $\mathcal{D}$ can be transformed into a derivation of $X \Longrightarrow Y$ in the following way: Let t be the occurrence of M in the conclusion $X \Longrightarrow M$ of $\mathcal{D}$ and $Anc(t)$ its set of congruent parametric ancestors as defined above. Let $\mathcal{D}'$ be the result of replacing all parametric members of $Anc(t)$ in $\mathcal{D}$ with Y.

In a second step, we deal with the principal occurrences in $Anc(t)$: For each principal member u of $Anc(t)$, take the sequent S in $\mathcal{D}$ in which it occurs. By the shape-alikeness of parameters condition (C2), u is an instance of the formula M. Moreover, since u is principal and the display of principal constituents condition (C5) as well as the position-alikeness of parameters conditions (C6) and (C7) hold, we know that S is of the form $Z \Longrightarrow M\dagger$ with Z a structure and M displayed on the right and principal.

By assumption we have that from $M\dagger \Longrightarrow Y$ together with S and lemma 1, it follows that $Z \Longrightarrow Y$ is cut-free derivable. Let $\mathcal{D}''$ result from $\mathcal{D}'$ by replacing the part of the derivation leading to $Z \Longrightarrow M\dagger$ with a derivation of $Z \Longrightarrow Y$. Now $\mathcal{D}''$ is again a valid derivation, with the conclusion $X \Longrightarrow Y$.

(2) By assumption, we have that $X \Longrightarrow M$ and $M \Longrightarrow Y$ are cut-free derivable. Let $\mathcal{D}$ be a proof of $M \Longrightarrow Y$. The derivation $\mathcal{D}$ can be transformed into a derivation of $X \Longrightarrow Y$, in the same way as above by invoking (1) instead of lemma 1. □

Theorem 5 (Cut Elimination for D.Cp)**.** *The cut rule is admissible for the display calculus* D.Cp.

Proof. This is an application of theorem 4. We only need to check that condition (C8) is satisfied, since the conditions (C2)–(C7) can be verified by eye. To confirm (C8), we have to check that cuts on matching principal formulas can be replaced with a derivation that contains only cuts on subformulas of the original cut formula. This is confirmed by case distinction on the shape of the cut formula M:

(1) If M is of the form $\neg A$, then a derivation where M is princial in both premises must look as follows:

$$\dfrac{\dfrac{\dfrac{\vdots}{X \Longrightarrow *A}}{X \Longrightarrow \neg A}\,(\neg\mathsf{r}) \qquad \dfrac{\dfrac{\vdots}{*A \Longrightarrow Y}}{\neg A \Longrightarrow Y}\,(\neg\mathsf{l})}{X \Longrightarrow Y}\,(\mathsf{Cut})$$

It is easy to show that (Cut) can be moved up to a subformula:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{*A \Longrightarrow Y}}{*Y \Longrightarrow A}\,(\mathsf{DE})\cdot 2 \qquad \dfrac{\dfrac{\vdots}{X \Longrightarrow *A}}{A \Longrightarrow *X}\,(\mathsf{DE})\cdot 2}{*Y \Longrightarrow *X}\,(\mathsf{Cut})}{X \Longrightarrow Y}\,(\mathsf{DE})$$

The remaining cases are shown in a similar fashion. We first write what the proof must look like if M is principal in the premises, and then show how to move the cut upwards in the proof:

(2) Suppose that $M = A \to B$:

$$\dfrac{\dfrac{\dfrac{\vdots}{X \circ A \Longrightarrow B}}{X \Longrightarrow A \to B}\,(\to\mathsf{r}) \qquad \dfrac{\dfrac{\vdots}{Y \Longrightarrow A} \qquad \dfrac{\vdots}{B \Longrightarrow Z}}{A \to B \Longrightarrow *Y \circ Z}\,(\to\mathsf{l})}{X \Longrightarrow *Y \circ Z}\,(\mathsf{Cut})$$

$$\dfrac{\dfrac{\vdots}{Y \Longrightarrow A} \qquad \dfrac{\dfrac{\dfrac{\vdots}{X \circ A \Longrightarrow B} \qquad \dfrac{\vdots}{B \Longrightarrow Z}}{X \circ A \Longrightarrow Z}\,(\mathsf{Cut})}{A \Longrightarrow *(X \circ *Z)}\,(\mathsf{P}),\,(\mathsf{DE})\cdot 4}{Y \Longrightarrow *(X \circ *Z)}\,(\mathsf{Cut})}{X \Longrightarrow *Y \circ Z}\,(\mathsf{DE})\cdot 3$$

(3) Suppose that $M = A \vee B$:

$$\dfrac{\dfrac{\dfrac{\vdots}{X \Longrightarrow A \circ B}}{X \Longrightarrow A \vee B}\,(\vee\mathsf{r}) \qquad \dfrac{\dfrac{\vdots}{A \Longrightarrow Y} \qquad \dfrac{\vdots}{B \Longrightarrow Z}}{A \vee B \Longrightarrow Y \circ Z}\,(\vee\mathsf{l})}{X \Longrightarrow Y \circ Z}\,(\mathsf{Cut})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{X \Longrightarrow A \circ B}}{X \circ *B \Longrightarrow A}\,(\mathsf{DE}) \qquad \dfrac{\vdots}{A \Longrightarrow Y}}{X \circ *B \Longrightarrow Y}\,(\mathsf{Cut})}{*Y \circ X \Longrightarrow B}\,(\mathsf{DE})\cdot 2 \qquad \dfrac{\vdots}{B \Longrightarrow Z}}{*Y \circ X \Longrightarrow Z}\,(\mathsf{Cut})}{X \Longrightarrow Y \circ Z}\,(\mathsf{DE})$$

(4) Suppose that $M = A \wedge B$:

$$\dfrac{\dfrac{\dfrac{\vdots}{X \Longrightarrow A} \quad \dfrac{\vdots}{Y \Longrightarrow B}\,(\wedge\mathsf{r})}{X \circ Y \Longrightarrow A \wedge B} \quad \dfrac{\dfrac{\vdots}{A \circ B \Longrightarrow Z}\,(\wedge\mathsf{l})}{A \wedge B \Longrightarrow Z}}{X \circ Y \Longrightarrow Z}\,(\mathsf{Cut})$$

$$\dfrac{\dfrac{\vdots}{Y \Longrightarrow B} \quad \dfrac{\dfrac{\dfrac{\vdots}{X \Longrightarrow A} \quad \dfrac{\dfrac{\vdots}{A \circ B \Longrightarrow Z}\,(\mathsf{DE})}{A \Longrightarrow Z \circ {*}B}\,(\mathsf{Cut})}{X \Longrightarrow Z \circ {*}B}\,(\mathsf{DE}) \cdot 2}{B \Longrightarrow {*}X \circ Z}\,(\mathsf{Cut})}{\dfrac{Y \Longrightarrow {*}X \circ Z}{X \circ Y \Longrightarrow Z}\,(\mathsf{DE})}$$

□

Corollary 2 (Subformula Property of D.Cp)**.** *The display calculus* D.Cp *without* (Cut) *has the subformula property.*

Proof. This is straightforward to verify by checking each inference rule. If formulas do not get lost when going from premise to conclusion for any rule, then they do not get lost in whole derivations as well. This is the case for our calculus, and hence, D.Cp possesses the subformula property. □

3 First-Order Display Logic

In this section we extend our display calculi to included first-order formulas with quantification. After defining our first-order display calculus, we prove that our extension of D.Cp is sound, complete, and possesses the cut elimination property. The following subsection on the calculus D.QE extends the first-order calculus to one which includes equality. The properties possessed by D.QE easily follow from D.Q with the exception of the subformula property.

3.1 The Calculus D.Q

The calculus D.Q is defined as an augmentation of the calculus D.Cp. We achieve D.Q from D.Cp by allowing the instantiation of first-order formulas in the axiom $A \Longrightarrow A$ and by adding two rules of universal quantification, and two rules of existential quantification.

Each rule uses notation that we ought to clarify for the reader: the formula $A(x)$ in each rule is assumed to have at least one free occurrence of the variable x. The formulas $A(t/x)$ and $A(y/x)$ represent $A(x)$, but with t and y replacing x, respectively. Making use of this notation, the quantifier rules are as follows:

Definition 13 (Quantifier Rules)**.**

$$\frac{A(t/x) \Longrightarrow Y}{\forall x A(x) \Longrightarrow Y}\,(\forall \mathsf{l}) \qquad \frac{X \Longrightarrow A(y/x)}{X \Longrightarrow \forall x A(x)}\,(\forall \mathsf{r})$$

$$\frac{X \Longrightarrow A(t/x)}{X \Longrightarrow \exists x A(x)}\,(\exists \mathsf{r}) \qquad \frac{A(y/x) \Longrightarrow Y}{\exists x A(x) \Longrightarrow Y}\,(\exists \mathsf{l})$$

where y does not occur free in X or Y for the ($\exists$l) rule and ($\forall$r) rule. We refer to y as an *eigenvariable*.

It is important to point out the *eigenvariable restriction* imposed on the ($\exists$l) rule and ($\forall$r) rule. By "y does not occur free in X or Y" we mean that the variable y does not occur as a free variable in any of the formulas of X or Y. This condition is necessary to ensure soundness. For example, without this restriction our calculus derives invalidities:

Example 12. In the absence of the eigenvariable restrictions, the formula $\exists x A(x) \to \forall x A(x)$ is deducible:

$$\dfrac{\dfrac{\dfrac{\dfrac{A(y/x) \Longrightarrow A(y/x)}{A(y/x) \Longrightarrow \forall x A(x)}\,(\forall \mathsf{r})}{\exists x A(x) \Longrightarrow \forall x A(x)}\,(\exists \mathsf{l})}{I \circ \exists x A(x) \Longrightarrow \forall x A(x)}\,(\mathrm{I}+)}{I \Longrightarrow \exists x A(x) \to \forall x A(x)}\,(\to \mathsf{r})$$

We now prove the completeness and soundness of D.Q. Our proof demonstrates an equivalence of provability between our calculus and the sound and complete calculus Q. This strategy is also followed in the next section, and thus, we also define the calculus QE.

Definition 14 (The calculus Q and QE)**.** The calculus Q consists of the three axioms of Cp in conjunction with the following two axioms, and two inference rules:

$$(\mathsf{Q1})\ \ \forall x A(x) \to A(t/x) \qquad (\mathsf{Q2})\ \ A(t/x) \to \exists x A(x)$$

$$\frac{A \to B(y/x)}{A \to \forall x B(x)}\,(\forall \mathsf{QR}) \qquad \frac{A(y/x) \to B}{\exists x A(x) \to B}\,(\exists \mathsf{QL})$$

where y does not occur as a free variable of A in the ($\forall$QR) rule, and does not occur as a free variable of B in the ($\exists$QL) rule.

The calculus QE is defined on the basis of Q by adding the following equality axioms:

$$(\mathsf{QE1})\ \ \forall x (x = x) \qquad\qquad (\mathsf{QE2})\ \ s = t \to (P(s) \to P(t))$$

Theorem 6 (Completeness of D.Q)**.** *If a formula A is provable in* Q*, then A is provable in* D.Q*.*

Proof. In the completeness proof of D.Cp we have shown that the first three axioms of Q are provable in D.Cp. Since D.Q is an extension of D.Cp, these axioms are also provable in D.Q. We now show that the remaining two axioms and two inference rules of Q are provable in D.Q:

$$\dfrac{\dfrac{\dfrac{A(t/x) \Longrightarrow A(t/x)}{\forall x A(x) \Longrightarrow A(t/x)}\,(\forall \mathsf{l})}{I \circ \forall x A(x) \Longrightarrow A(t/x)}\,(\mathrm{I}+)}{I \Longrightarrow \forall x A(x) \to A(t/x)}\,(\to \mathsf{r}) \qquad \dfrac{\dfrac{\dfrac{A(t/x) \Longrightarrow A(t/x)}{A(t/x) \Longrightarrow \exists x A(x)}\,(\exists \mathsf{r})}{I \circ A(t/x) \Longrightarrow \exists x A(x)}\,(\mathrm{I}+)}{I \Longrightarrow A(t/x) \to \exists x A(x)}\,(\to \mathsf{r})$$

$$\dfrac{\dfrac{I \Longrightarrow A \to B(y/x) \qquad \dfrac{A \Longrightarrow A \qquad B(y/x) \Longrightarrow B(y/x)}{A \to B(y/x) \Longrightarrow *A \circ B(y/x)}\,(\to \mathrm{l})}{I \Longrightarrow *A \circ B(y/x)}\,(\mathrm{Cut})}{I \circ A \Longrightarrow B(y/x)}\,(\mathsf{DE})}{I \circ A \Longrightarrow \forall x B(x)}\,(\forall\ \mathsf{r})}{I \Longrightarrow A \to \forall x B(x)}\,(\to \mathsf{r})$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{I \Longrightarrow A(y/x) \to B \qquad \dfrac{A(y/x) \Longrightarrow A(y/x) \qquad B \Longrightarrow B}{A(y/x) \to B \Longrightarrow *A(y/x) \circ B}\,(\to \mathrm{l})}{I \Longrightarrow *A(y/x) \circ B}\,(\mathrm{Cut})}{I \circ A(y/x) \Longrightarrow B}\,(\mathsf{DE})}{A(y/x) \Longrightarrow B}\,(\mathrm{I}-)}{\exists x A(x) \Longrightarrow B}\,(\exists\ \mathrm{l})}{I \circ \exists x A(x) \Longrightarrow B}\,(\mathrm{I}+)}{I \Longrightarrow \exists x A(x) \to B}\,(\to \mathsf{r})$$

Note that in the derivation of the ($\forall$QR) rule, we assume that A does not contain a free occurrence of y, and in the derivation of the ($\exists$QL) rule, we assume that B does not contain a free occurrence of y. This completes the theorem.

□

Theorem 7 (Soundness of D.Q)**.** *If a formula A is provable in* D.Q*, then A is provable in* Q*.*

Proof. To demonstrate this theorem we need only consider the additions we made to D.Cp. Thus, we prove that our generalized axiom can be proven in Q and that our quantifier rules can be mirrored in Q. It is easy to show that the translation of the axiom is provable. If A is an atomic first-order formula, then $\mathfrak{I}(A \Longrightarrow A) = \mathfrak{I}_1(A) \to \mathfrak{I}_2(A) = A \to A$, which is provable in Q.

$$
\begin{array}{ll}
(\forall \mathsf{l}) & \begin{array}{l}
\mathfrak{I}(A(t/x) \Longrightarrow Y) \\
\mathfrak{I}_1(A(t/x)) \to \mathfrak{I}_2(Y) \\
A(t/x) \to \mathfrak{I}_2(Y) \\
\underline{\forall x A(x) \to A(t/x)} \\
\forall x A(x) \to \mathfrak{I}_2(Y) \\
\mathfrak{I}_1(\forall x A(x)) \to \mathfrak{I}_2(Y) \\
\mathfrak{I}(\forall x A(x) \Longrightarrow Y)
\end{array}
&
(\exists \mathsf{r}) \quad \begin{array}{l}
\mathfrak{I}(X \Longrightarrow A(t/x)) \\
\mathfrak{I}_1(X) \to \mathfrak{I}_2(A(t/x)) \\
\mathfrak{I}_1(X) \to A(t/x) \\
\underline{A(t/x) \to \exists x A(x)} \\
\mathfrak{I}_1(X) \to \exists x A(x) \\
\mathfrak{I}_1(X) \to \mathfrak{I}_2(\exists x A(x)) \\
\mathfrak{I}(X \Longrightarrow \exists x A(x))
\end{array}
\end{array}
$$

$$
\begin{array}{ll}
(\forall \mathsf{r}) & \begin{array}{l}
\mathfrak{I}(X \Longrightarrow A(y/x)) \\
\mathfrak{I}_1(X) \to \mathfrak{I}_2(A(y/x)) \\
\mathfrak{I}_1(X) \to A(y/x) \\
\underline{\mathfrak{I}_1(X) \to \forall x A(x)} \\
\mathfrak{I}_1(X) \to \mathfrak{I}_2(\forall x A(x)) \\
\mathfrak{I}(X \Longrightarrow \forall x A(x))
\end{array}
&
(\exists \mathsf{l}) \quad \begin{array}{l}
\mathfrak{I}(A(y/x) \Longrightarrow Y) \\
\mathfrak{I}_1(A(y/x)) \to \mathfrak{I}_2(Y) \\
A(y/x) \to \mathfrak{I}_2(Y) \\
\underline{\exists x A(x) \to \mathfrak{I}_2(Y)} \\
\mathfrak{I}_1(\exists x A(x)) \to \mathfrak{I}_2(Y) \\
\mathfrak{I}(\exists x A(x) \Longrightarrow Y)
\end{array}
\end{array}
$$

The underlined steps in each proof correspond to the quantifier axioms and rules of Q. We assume that in the $(\forall \mathsf{r})$ case, the structure X is does not contain a free occurrence of the variable y, and in the $(\exists \mathsf{l})$ case, the structure Y does not contain a free occurrence of the variable y.

□

There is a small issue regarding the cut elimination theorem for $\mathsf{D.Q}$ which must be addressed prior to the proof of the theorem. Strictly speaking, the conditions (C6) and (C7) are not satisfied:

Example 13. Suppose we take the quantifier rules:

$$
\frac{X \Longrightarrow A(y/x)}{X \Longrightarrow \forall x A(x)}\ (\forall \mathsf{r}) \qquad \frac{A(y/x) \Longrightarrow Y}{\exists x A(x) \Longrightarrow Y}\ (\exists \mathsf{l})
$$

and replace the structures X and Y by the formula $B(y)$, which contains a free occurrence of the variable y:

$$
\frac{B(y) \Longrightarrow A(y/x)}{B(y) \Longrightarrow \forall x A(x)}\ (\forall \mathsf{r}) \qquad \frac{A(y/x) \Longrightarrow B(y)}{\exists x A(x) \Longrightarrow B(y)}\ (\exists \mathsf{l})
$$

Recall that condition (C6) is satisfied when each inference rule is closed under simultaneous substitution of arbitrary structures in consequent parts for congruent parameters. Similarly, the condition (C7) is satisfied when each inference rule is closed under simultaneous substitution of arbitrary structures in antecedent parts for congruent parameters.

In the $(\exists \mathsf{l})$ rule we have substituted the structure/formula $B(y)$ in the consequent for congruent parameters, and in the $(\forall \mathsf{r})$ rule we have substituted the structure/formula $B(y)$ in the antecedent for congruent parameters. However, the result of this substitution does not produce an instance of either rule, due to the violation of the eigenvariable condition. Nevertheless, the problem can be fixed by noting the arbitrariness of the eigenvariable y. It is always possible to pick a variable z not occurring in the structure substituted for X or Y to

achieve an instance of the rule. This follows from the fact that X and Y are finite entities, and thus, can only contain a finite number of variables.

Let z be a variable not occurring in the formula $B(y)$. Then, the following are valid instances of the ($\forall$r) and ($\exists$l) rules:

$$\frac{B(y) \Longrightarrow A(z/x)}{B(y) \Longrightarrow \forall x A(x)}\ (\forall \mathsf{r}) \qquad \frac{A(z/x) \Longrightarrow B(y)}{\exists x A(x) \Longrightarrow B(y)}\ (\exists \mathsf{l})$$

So long as we assume that the necessary variable substitution take place when replacing congruent parameters with structures, conditions (C6) and (C7) will hold. Due to the fact that such a substitution is always permissible in D.Q, no problems arise from the addition of this subtle assumption.

Before proving cut elimination, it is useful to prove the substitution lemma for D.Q. The substitution lemma allows us to replace free variables occurring in a derivation with arbitrary terms. We use the notation $X(t/x)$ to denote the structure resulting from the replacement of all occurrences of x in X with the arbitrary term t, and use the notation $X[x]$ to represent that x may occur in the structure X. Note that if x does not occur in a structure X, then both $X(t/x)$ and $X[x]$ are identical to X. Also, before we proceed with the proof of the lemma, we need to define the derivation height since the argument will proceed by induction on the height of the given derivation:

Definition 15 (Derivation Height)**.** A *thread* in a derivation to be a path from the end sequent to one of the initial sequents, and the *length of the thread* is the number of sequents in the thread including the initial and end sequent. We define the *derivation height* to be the length of the maximum, or longest, thread in the derivation.

Lemma 2 (Substitution Lemma)**.** *For any sequent $X[x] \Longrightarrow Y[x]$ derivable in* D.Q *with x free, the sequent $X(t/x) \Longrightarrow Y(t/x)$ is derivable with a derivation of the same height.*

Proof. Suppose that the sequent $X[x] \Longrightarrow Y[x]$ is derivable. We show by induction on the height of the derivation of $X[x] \Longrightarrow Y[x]$ that the sequent $X(t/x) \Longrightarrow Y(t/x)$ is height-preserving derivable as well. For the base case, assume that the height of the derivation is one. Then, $X[x] \Longrightarrow Y[x]$ is an axiom instance of the form $A[x] \Longrightarrow A[x]$. Observe that $A(t/x) \Longrightarrow A(t/x)$ is an axiom as well. This proves the base case.

For the inductive step, we assume the result holds for all derivations of height n; we show that the result holds for all derivations of height $n+1$. Assume that the derivation of $X[x] \Longrightarrow Y[x]$ is of length $n+1$. We now prove the result by considering each rule that could have been used last to derive the end sequent $X[x] \Longrightarrow Y[x]$.

Suppose that the last rule used is either a structural rule or display equivalence rule:

$$\dfrac{\dfrac{\vdots}{X'[x] \Longrightarrow Y'[x]}}{X[x] \Longrightarrow Y[x]}$$

By the inductive hypothesis, we know that $X'(t/x) \Longrightarrow Y'(t/x)$ is derivable with height n, which gives us $X(t/x) \Longrightarrow Y(t/x)$ with height $n+1$ if we apply the same structural rule or display equivalence rule.

Suppose that the last rule used is a non-quantifier, logical rule. Then, our derivation is of one of two forms:

$$\dfrac{\dfrac{\vdots}{X'[x] \Longrightarrow Y'[x]}}{X[x] \Longrightarrow Y[x]} \qquad \dfrac{\dfrac{\vdots}{X'[x] \Longrightarrow Y'[x]} \quad \dfrac{\vdots}{X''[x] \Longrightarrow Y''[x]}}{X[x] \Longrightarrow Y[x]}$$

By the inductive hypothesis, we know that $X'(t/x) \Longrightarrow Y'(t/x)$ is deducible with height n in the first case, and that $X'(t/x) \Longrightarrow Y'(t/x)$ and $X''(t/x) \Longrightarrow Y''(t/x)$ are deducible with heights n and m in the second case. The result immediately follows in either case by applying the logical rule used in the original derivation.

Suppose that the last rule used is either ($\forall$l) or ($\exists$r):

$$\dfrac{\dfrac{\vdots}{A(y/x) \Longrightarrow Y[x]}}{\forall x A(x) \Longrightarrow Y[x]} \qquad \dfrac{\dfrac{\vdots}{X[x] \Longrightarrow B(y/x)}}{X[x] \Longrightarrow \exists x B(x)}$$

$$\dfrac{\dfrac{\vdots}{A[x] \Longrightarrow Y[x]}}{\forall y A[x] \Longrightarrow Y[x]} \qquad \dfrac{\dfrac{\vdots}{X[x] \Longrightarrow B[x]}}{X[x] \Longrightarrow \exists y B[x]}$$

Note that in the first two instances, the free variable x becomes bounded by the quantifier, whereas in the second two instances, another free-variable becomes bounded by the quantifiers. Since both cases can occur in our calculus D.Q, we include both for the sake of completeness. Observe that by the inductive hypothesis, $A(t/x) \Longrightarrow Y(t/x)$ and $X(t/x) \Longrightarrow B(t/x)$ are derivable with height n, and hence $\forall x A(x) \Longrightarrow Y(t/x)$ and $X(t/x) \Longrightarrow \exists x B(x)$ are derivable with height $n+1$ regarding the first two derivations. Regarding the second two instances, $\forall y A(t/x) \Longrightarrow Y(t/x)$ and $X(t/x) \Longrightarrow \exists y B(t/x)$ are derivable with height $n+1$ as well.

Suppose that the last rule used in our derivation is either ($\forall$r) or ($\exists$l):

$$\dfrac{\dfrac{\vdots}{X[x] \Longrightarrow B[x]}}{X[x] \Longrightarrow \forall y B[x]} \qquad \dfrac{\dfrac{\vdots}{A[x] \Longrightarrow Y[x]}}{\exists y A[x] \Longrightarrow Y[x]}$$

Observe that the quantifiers do not bound the variable x, but rather, bound some other variable, which we denote as z, occurring in the sequent. This follows from the assumption that the variable x occurs free in the end sequent if it occurs at all. If t is a term not containing z, then we can conclude that $\forall y A(t/x) \Longrightarrow Y(t/x)$ and $X(t/x) \Longrightarrow \exists y B(t/x)$ are derivable with height $n+1$, since $A(t/x) \Longrightarrow Y(t/x)$ and $X(t/x) \Longrightarrow B(t/x)$ are derivable with height n by the inductive hypothesis. Also, the eigenvariable condition is not violated in either case.

Note that the result still follows in the instance where t contains the eigenvariable z, however, we must do some additional work. By the inductive hypothesis, we have a derivation of $A(t/x) \Longrightarrow Y(t/x)$ and $X(t/x) \Longrightarrow B(t/x)$ with height n. Hence, we also have derivations of $A(w/z)(t/x) \Longrightarrow Y(t/x)$ and $X(t/x) \Longrightarrow B(w/z)(t/x)$ of height n where the variable w is a fresh variable not occurring in the original sequents $A(t/x) \Longrightarrow Y(t/x)$ and $X(t/x) \Longrightarrow B(t/x)$. We can now apply the rules ($\forall$r) and ($\exists$l) to their respective sequents, with w the eigenvariable, and obtain the desired results.

□

Theorem 8 (Cut Elimination for D.Q)**.** *The cut rule is admissible for the display calculus* D.Q.

Proof. Conditions (C2)–(C7) can be verified easily by looking at the rules of D.Q. The only condition that must be verified then is (C8). This condition holds for all cases of (Cut) presented in the cut elimination proof of D.Cp; however, we have two additional cases to consider: when the cut formula is principal and of the form $\exists x A(x)$, and when it is of the form $\forall x A(x)$. First, suppose we have a derivation of the following form:

$$\dfrac{\dfrac{\dfrac{\vdots}{X \Longrightarrow A(t/x)}}{X \Longrightarrow \exists x A(x)}\,(\exists\mathsf{r}) \qquad \dfrac{\dfrac{\vdots}{A(y/x) \Longrightarrow Y}}{\exists x A(x) \Longrightarrow Y}\,(\exists\mathsf{l})}{X \Longrightarrow Y}\,(\mathsf{Cut})$$

If we apply the previous lemma to the portion of the derivation down to, and including, the sequent $A(y/x) \Longrightarrow Y$, then we can cut on the proper subformula $A(t/x)$ after making a substitution:

$$\dfrac{\dfrac{\vdots}{X \Longrightarrow A(t/x)} \qquad \dfrac{\vdots}{A(t/x) \Longrightarrow Y}\,\text{Lemma 2}}{X \Longrightarrow Y}\,(\mathsf{Cut})$$

For the $\forall x A(x)$ case, suppose we have the following cut in our derivation:

$$\dfrac{\dfrac{\dfrac{\vdots}{X \Longrightarrow A(y/x)}}{X \Longrightarrow \forall x A(x)}\,(\forall\mathsf{r}) \qquad \dfrac{\dfrac{\vdots}{A(t/x) \Longrightarrow Y}}{\forall x A(x) \Longrightarrow Y}\,(\forall\mathsf{l})}{X \Longrightarrow Y}\,(\mathsf{Cut})$$

Similar to the existential case, we can move the cut upwards after applying the previous lemma, and cut on the proper subformula after making a substitution:

$$\dfrac{\dfrac{\vdots}{X \Longrightarrow A(t/x)} \qquad \dfrac{\vdots}{A(t/x) \Longrightarrow Y}\ \text{Lemma 2}}{X \Longrightarrow Y}\ (\forall \mathsf{l})$$

Both cases demonstrate that condition (C8) holds, which completes the theorem.

□

Theorem 9 (Subformula Property of D.Q)**.** *The first-order display calculus* D.Q *without* (Cut) *has the subformula property.*

Proof. We know that D.Cp has the subformula property and it is easy to check that the addition of the rules (∀l) and (∀r) preserve this property as well. These facts, in conjunction with the cut elimination theorem above, imply that D.Q has the subformula property. □

3.2 The Calculus D.QE

We extend the calculus D.Q to the calculus D.QE by adding two equality rules similar to those in [9], and allow the instantiation of equality formulas of the form $t = s$ in the axiom. The remainder of this section focuses on the various properties of D.QE.

Definition 16 (The rules (E1) and (E2))**.**

$$\dfrac{t = t \circ X \Longrightarrow Y}{I \circ X \Longrightarrow Y}\ (\mathsf{E1}) \qquad \dfrac{t = s \circ P(t) \circ P(s) \circ X \Longrightarrow Y}{t = s \circ P(t) \circ I \circ X \Longrightarrow Y}\ (\mathsf{E2})$$

Theorem 10 (Completeness of D.QE)**.** *If a formula A is provable in* QE*, then A is provable in* D.QE*.*

Proof. Since D.QE is an extension of D.Q, all of the axioms of Q are provable in D.QE. To fully demonstrate the completeness theorem then, we further show that the equality axioms of QE are provable in D.QE.

(QE1) $I \Longrightarrow \forall x(x = x)$

$$\dfrac{\dfrac{y = y \Longrightarrow y = y}{I \Longrightarrow y = y}\ (\mathsf{E1})}{I \Longrightarrow \forall x(x = x)}\ (\forall r)$$

(QE2) $I \Longrightarrow a = b \to (P(a) \to P(b))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(b) \Longrightarrow P(b)}{a = b \circ P(a) \circ P(b) \Longrightarrow P(b)}\,(\mathsf{WI})\cdot 2}{a = b \circ P(a) \circ I \Longrightarrow P(b)}\,(\mathsf{E2})}{I \circ a = b \circ P(a) \Longrightarrow P(b)}\,(\mathsf{P})\cdot 2}{I \circ a = b \Longrightarrow P(a) \to P(b)}\,(\to\mathsf{r})}{I \Longrightarrow a = b \to (P(a) \to P(b))}\,(\to\mathsf{r})$$

□

Theorem 11 (Soundness of D.QE)**.** *If a formula A is provable in* D.QE*, then A is provable in* QE*.*

Proof. Since we already proved the soundness of D.Q we need only show that the additional two rules of D.QE can be simulated in QE:

(E1)
$$\begin{array}{l}\mathfrak{I}(t = t \circ X \Longrightarrow Y)\\ \mathfrak{I}_1(t = t \circ X) \to \mathfrak{I}_2(Y)\\ t = t \wedge \mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)\\ \mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)\\ \mathfrak{I}(X \Longrightarrow Y)\end{array}$$

(E2)
$$\begin{array}{l}\mathfrak{I}(t = s \circ P(t) \circ P(s) \circ X \Longrightarrow Y)\\ \mathfrak{I}_1(t = s \circ P(t) \circ P(s) \circ X) \to \mathfrak{I}_2(Y)\\ t = s \wedge P(t) \wedge P(s) \wedge \mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)\\ t = s \wedge P(t) \wedge \mathfrak{I}_1(X) \to \mathfrak{I}_2(Y)\\ \mathfrak{I}(t = s \circ P(t) \circ X \Longrightarrow Y)\end{array}$$

□

Theorem 12 (Cut Elimination for D.QE)**.** *The cut rule is admissible for the display calculus* D.QE*.*

Proof. Since condition (C8) holds for D.Q, and the rules (E1) and (E2) do not introduce principal formulas, it also holds for D.QE. We can also verify conditions (C2)–(C7) by eye.

□

Although the calculus D.QE admits cut elimination, it does not posses the subformula property. This is due to the rule (E2) which allows for the deletion of formulas:

Theorem 13. D.QE *does not posses the subformula property*[8]*.*

Proof. Observe the following derivation of the sequent $x = y \circ y = z \circ z = w \Longrightarrow x = w$:

$$\dfrac{\dfrac{\dfrac{\dfrac{x = w \Longrightarrow x = w}{x = y \circ x = w \circ y = w \Longrightarrow x = w}\,(\mathsf{WI})\cdot 2}{x = y \circ y = w \Longrightarrow x = w}\,(\mathsf{E2})}{x = y \circ y = z \circ y = w \circ z = w \Longrightarrow x = w}\,(\mathsf{WI})\cdot 2}{x = y \circ y = z \circ z = w \Longrightarrow x = w}\,(\mathsf{E2})$$

[8] Thank you to the anonymous reviewer who pointed this out.

Notice that in the first use of (E2) we deleted $x = w$ and in the second use of (E2) we deleted $y = w$. If we let P be the property of being equal to w, then both uses of the rule are in fact instances of the rule.

What is problematic here is that the subformula property has been violated. Moreover, every derivation of the sequent $x = y \circ y = z \circ z = w \Longrightarrow x = w$ is similar to the one above in that not every formula occurring in the proof is a subformula of a formula in the conclusion. Hence, the calculus D.QE does not possess the subformula property.

□

This is the first example we have seen where our calculus fails to obtain the subformula property, despite admitting cut elimination. It therefore stands as an example that cut elimination alone is not sufficient to ensure the subformula property. One must additionally check the rules of the calculus to see if they preserve subformulas, and if this be the case, then the desirable trait follows.

4 Modal Display Logic

The general framework of display logic can easily be expanded to non-classical logics by the addition or removal of inference rules. In this section, we define the display logic D.K and then prove completeness, soundness, and that cut is admissible in the calculus. Our approach for defining the calculus consists of adding introduction rules for the necessity operator □ along with a new structural connective. By shifting the structural rules for this connective, we can represent different modal logics, though we only focus on the modal logic K here.

4.1 The Calculus D.K

Following the usual pattern, we first provide the reader with a deductively equivalent calculus that is known to be sound and complete. Let us make use of the sound and complete modal calculus K in the present section. It is defined as follows:

Definition 17 (The Calculus K)**.** The calculus K consists of the three axioms of Cp in conjunction with the following axiom and rule of inference:

$$(\mathsf{K})\ \ \Box(A \to B) \to (\Box A \to \Box B) \qquad \frac{A}{\Box A}\ (\mathsf{Nec})$$

Similar to how the calculus K is defined as an extension of the calculus Cp, our modal display calculus will be defined as an extension of D.Cp. We give the definition of our modal display calculus below:

Definition 18 (The Calculus D.K)**.** The calculus D.K is an extension of D.Cp that contains the following four additional rules of inference along with an additional structural rule and display equivalence rule:

$$\frac{A \Longrightarrow Y}{\Box A \Longrightarrow \bullet Y}\,(\Box\mathsf{l}) \qquad \frac{X \Longrightarrow \bullet A}{X \Longrightarrow \Box A}\,(\Box\mathsf{r})$$

$$\frac{* \bullet * A \Longrightarrow Y}{\Diamond A \Longrightarrow Y}\,(\Diamond\mathsf{l}) \qquad \frac{X \Longrightarrow A}{* \bullet * X \Longrightarrow \Diamond A}\,(\Diamond\mathsf{r})$$

$$\frac{\mathrm{I} \Longrightarrow Y}{\bullet \mathrm{I} \Longrightarrow Y}\,(\mathrm{I}\bullet) \qquad \frac{\bullet X \Longrightarrow Y}{X \Longrightarrow \bullet Y}\,(\bullet)$$

Notice that our logical rules allow for the introduction of the $\Box$ necessity and the $\Diamond$ possibility operators. Furthermore, the new structural connective $\bullet$ bullet has been introduced via the latter two rules. The bullet connective is used in conjunction with the two operators to deduce valid modal formulas. It is additionally used in varying modal calculi and assists in the deduction of the many modal truths relevant to those calculi.

The interpretation of the $\bullet$ changes depending on the side of the sequent where it appears. The $\bullet$ bullet on the right can be thought of as the $\Box$ necessity operator, much like how the $*$ corresponds to the $\neg$ operator. On the left side, however, the $\bullet$ bullet is interpreted as the $\blacklozenge$ backwards looking diamond. This will be important in our proof of soundness. For more information on this connective, see Wansing [11].

Let us now prove one direction of the deductive equivalence between the calculi K and $\mathsf{D.K}$. It takes additional work to demonstrate the other direction, since we have to make use of a translation function that will be defined later on. Before we prove the completeness theorem, let us show the admissibility of the medial rule (given below) that allows us to distribute the $\bullet$ out of structures on the left side of a sequent.

Lemma 3. *The following rule is admissible in* $\mathsf{D.K}$*:*

$$\frac{\bullet X \circ \bullet Y \Longrightarrow Z}{\bullet(X \circ Y) \Longrightarrow Z}\ \textit{(Medial)}$$

Proof. The strategy behind this proof is to use the display property and weakening to turn $\bullet X$ and $\bullet Y$ on the left side of the sequent into $\bullet(X \circ Y)$ and then use contraction. Since the strategy for the first part is identical for X and Y, we will show only one part of it in order to shorten the proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\bullet X \circ \bullet Y \Longrightarrow Z}{\bullet X \Longrightarrow * \bullet Y \circ Z}\,(\mathsf{DE})}{X \Longrightarrow \bullet(* \bullet Y \circ Z)}\,(\bullet)}{X \circ Y \Longrightarrow \bullet(* \bullet Y \circ Z}\,(\mathrm{Wl})}{\bullet(X \circ Y) \Longrightarrow * \bullet Y \circ Z}\,(\bullet)}{\bullet(X \circ Y) \circ \bullet Y \Longrightarrow Z}\,(\mathsf{DE})}{\bullet(X \circ Y) \circ \bullet(X \circ Y) \Longrightarrow Z}\,\cdots}{\bullet(X \circ Y) \Longrightarrow Z}\,(\mathrm{Cl})$$

□

Let us now use this fact to prove the completeness theorem, *i.e.* let us show that any formula deducible in K is also deducible in $\mathsf{D.K}$:

Theorem 14 (Completeness of D.K)**.** *If a formula A is provable in K, then A is provable in D.K.*

Proof. To prove this theorem, we will build off the completeness theorem of D.Cp, and show that the additional axioms and rules of K are deducible in D.K. We omit the $\Diamond$–rules sine they are proved similarly.

$$\dfrac{\dfrac{\mathrm{I} \Longrightarrow A}{\bullet \mathrm{I} \Longrightarrow A}\,(\mathrm{I}\bullet)}{\mathrm{I} \Longrightarrow \Box A}\,(\Box \mathsf{r}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to B \circ A \Longrightarrow B}{A \to B \Longrightarrow B \circ *A}\,(\mathsf{DE})}{\Box(A \to B) \Longrightarrow \bullet(B \circ *A)}\,(\Box \mathsf{I})}{\bullet\Box(A \to B) \Longrightarrow B \circ *A}\,(\bullet)}{A \Longrightarrow B \circ * \bullet \Box(A \to B)}\,(\mathsf{DE})}{\Box A \Longrightarrow \bullet(B \circ * \bullet \Box(A \to B))}\,(\Box \mathsf{I})}{\bullet\Box A \Longrightarrow B \circ * \bullet \Box(A \to B)}\,(\bullet)}{\bullet\Box(A \to B) \circ \bullet\Box A \Longrightarrow B}\,(\mathsf{DE})}{\bullet(\Box(A \to B) \circ \Box A) \Longrightarrow B}\,(\mathrm{Medial})}{\Box(A \to B) \circ \Box A \Longrightarrow \bullet B}\,(\bullet)}{\Box(A \to B) \circ \Box A \Longrightarrow \Box B}\,(\Box \mathsf{r})}{\Box(A \to B) \Longrightarrow \Box A \to \Box B}\,(\to \mathsf{r})}{\mathrm{I} \Longrightarrow \Box(A \to B) \to (\Box A \to \Box B)}\,(\to \mathsf{r})$$

□

Not only do we need the translation function for soundness, but we also need to incorporate the $\blacklozenge$ operator into our treatment. Intuitively, a formula of the form $\blacklozenge A$ holds at a world if and only if there a world in the past where A holds. Since our focus in this paper is proof theory, we will not discuss the semantics of the backwards looking diamond in great detail. In the current context it may be thought of as a purely syntactic entity, which is allowed to operate over our formulas. Nevertheless, the operator does have a significant and useful relation to the necessity operator:

Fact 4. The formula $\blacklozenge A \to B \leftrightarrow A \to \Box B$ is deducible in the calculus Kt, which is the calculus K extended with axioms for the $\blacklozenge$ operator. For more information on Kt see Wansing [11].

Note that this fact corresponds to the $(\bullet)$ rule and this is the reason why we interpret $\bullet$ the way that we do. Additionally, note that the calculus Kt is conservative over K, meaning that it cannot derive pure modal theorems (not containing $\blacklozenge$) that K itself cannot deduce. Therefore, if we can show that any formula deducible in D.K is deducible in Kt, then since every pure modal formula provable in Kt is provable in K, and D.K only proves pure modal formulas, we have shown that everything deducible in D.K is deducible in K. This establishes soundness as well as the deductive equivalence between D.K and K when we take the completeness theorem into account.

With this fact at our disposal, we are in a position to prove the soundness result. Let us first define the translation function that will translate sequents of D.K

into formulas of Kt. We then prove the soundness theorem on the basis of this function:

Definition 19 (Translation functions $\mathfrak{I}_1$ and $\mathfrak{I}_2$)**.** Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ map from the set of structures to the set of propositional formulas such that:

$$\mathfrak{I}_1(X) = \begin{cases} A \text{ if } X = A, \\ \top \text{ if } X = I, \\ \neg\mathfrak{I}_2(Y) \text{ if } X = *Y, \\ \mathfrak{I}_1(Y) \wedge \mathfrak{I}_1(Z) \text{ if } X = (Y \circ Z) \\ \blacklozenge\mathfrak{I}_1(Y) \text{ if } X = \bullet Y \end{cases} \quad \mathfrak{I}_2(X) = \begin{cases} A & \text{if } X = A, \\ \bot & \text{if } X = I, \\ \neg\mathfrak{I}_1(Y) & \text{if } X = *Y, \\ \mathfrak{I}_2(Y) \vee \mathfrak{I}_2(Z) & \text{if } X = (Y \circ Z) \\ \Box\mathfrak{I}_2(Y) & \text{if } X = \bullet Y \end{cases}$$

Theorem 15 (Soundness of D.K)**.** *If a formula A is provable in D.K, then A is provable in K.*

Proof. We prove the soundness theorem by showing that for any sequent S provable in D.K, $\mathfrak{I}(A)$ is provable in K. Building off the soundness theorem for D.Cp, we only have to show that the additional rules of D.K can be mirrored in Kt. Again we omit the $\Diamond$–rules.

(□l)

$\mathfrak{I}(A \Longrightarrow Y)$
$A \to \mathfrak{I}_2(Y)$
$\Box(A \to \mathfrak{I}_2(Y))$
$\Box(A \to \mathfrak{I}_2(Y)) \to (\Box A \to \Box\mathfrak{I}_2(Y))$
$\Box A \to \Box\mathfrak{I}_2(Y)$
$\mathfrak{I}_1(\Box A) \to \mathfrak{I}_2(\bullet Y)$
$\mathfrak{I}(\Box A \Longrightarrow \bullet Y)$

(□r)

$\mathfrak{I}(\bullet X \Longrightarrow A)$
$\mathfrak{I}_1(\bullet X) \to \mathfrak{I}_2(A)$
$\mathfrak{I}_1(\bullet X) \to A$
$\blacklozenge\mathfrak{I}_1(X) \to A$
$\mathfrak{I}_1(X) \to \Box A$
$\mathfrak{I}_1(X) \to \mathfrak{I}_2(\Box A)$
$\mathfrak{I}(X \Longrightarrow \Box A)$

Note that the inference from $\blacklozenge\mathfrak{I}_1(X) \to A$ to $\mathfrak{I}_1(X) \to \Box A$ in the (□r) proof follows from fact 5 above.

□

Let us now apply the general cut elimination result to our specific calculus. Just as with the previous theorems, the general cut elimination result allows for easy confirmation that our calculus possesses the cut elimination property. The proof is as follows:

Theorem 16 (Cut Elimination for D.K)**.** *The cut rule is admissible for the display calculus* D.K.

Proof. The conditions (C2)-(C7) can be verified by eye. Thus, the only condition we need to check in order to apply Theorem 4, and prove cut elimination, is condition (C8). Since all the other logical connectives remain the same, we need only consider the case where $M = \Box A$:

$$\dfrac{\dfrac{\dfrac{\vdots}{\bullet X \Longrightarrow A}}{X \Longrightarrow \Box A}\,(\Box\mathsf{r}) \qquad \dfrac{\dfrac{\vdots}{A \Longrightarrow Y}}{\Box A \Longrightarrow \bullet Y}\,(\Box\mathsf{l})}{X \Longrightarrow \bullet Y}\,(\mathsf{Cut})$$

$$\dfrac{\dfrac{\vdots}{\bullet X \Longrightarrow A} \quad \dfrac{\vdots}{A \Longrightarrow Y}\,(\mathsf{Cut})}{\dfrac{\bullet X \Longrightarrow Y}{X \Longrightarrow \bullet Y}\,(\bullet)}$$

□

Theorem 17 (Subformula Property of Modal Display Calculi)**.** *The modal display calculus D.K without (Cut) has the subformula property.*

Proof. We know that the subformula property for the calculus D.Cp holds, so all that is left is to check the additional rules of the display calculi D.K. It is easy to see that all rules retain the subformula property. □

Note that display logic and the construction of display calculi are not simply restricted to the basic modal logic K. To apply display logic to stronger modal logics than K we can add structural rules. As an example, if we add the structural rules for T, 4, and B, then this would give us the calculus D.S5:

$$\dfrac{X \Longrightarrow \bullet Y}{X \Longrightarrow Y}\,(\mathsf{T}) \qquad \dfrac{X \Longrightarrow \bullet Y}{X \Longrightarrow \bullet\bullet Y}\,(4)$$

$$\dfrac{\bullet * X \Longrightarrow Y}{\bullet X \Longrightarrow Y}\,(\mathsf{B})$$

Also, it should not come as a surprise that there is a straightforward way to deal with temporal logics (or other logics making use of backwards looking modalities) as we already have the ⧫ backwards looking diamond operator implicitly in the structural rules of our calculus. So, in order to go from D.K to the temporal modal logic D.Kt we simply need to add logical rules that make the ⧫ operator and its dual the ■ operator explicit:

$$\dfrac{X \Longrightarrow * \bullet * A}{X \Longrightarrow \blacksquare A}\,(\blacksquare\mathsf{r}) \qquad \dfrac{A \Longrightarrow Y}{\blacksquare A \Longrightarrow * \bullet * Y}\,(\blacksquare\mathsf{l})$$

$$\dfrac{X \Longrightarrow A}{\bullet X \Longrightarrow \blacklozenge A}\,(\blacklozenge\mathsf{r}) \qquad \dfrac{\bullet A \Longrightarrow Y}{\blacklozenge A \Longrightarrow Y}\,(\blacklozenge\mathsf{l})$$

5 Conclusion

The aim of this paper has been to introduce the fundamental notions of proof theory and display logic. We have provided the reader with various display calculi and shown how to prove soundness, completeness, cut-elimination, and the subformula property, among other things. Each of these properties are desirable for any deductive calculi and the reader is likely to encounter facts concerning such upon further study of proof theory.

In the last section we introduced the display calculus corresponding to the modal logic K. Display logic derives its power from the fact that logics alternative to ordinary propositional and first-order logic can be expressed display-systematically. For example, intuitionistic logic, relevance logic, the modal logic T, and the modal calculi S.4 and S.5 can all be written as display calculi.

As seen in this paper, expressing such logics as display calculi eases the proof of many systematic properties. General theorems for display calculi, such as the general cut-elimination theorem, are readily applicable to more specific display calculi. For instance, in order to demonstrate cut elimination, it is sufficient to confirm that conditions (C2) through (C8) hold for the calculus under consideration.

There still remain many questions in the realm of display logic. We have by no means investigated all significant proof-theoretic properties or calculi in this text. Interpolation, decidability, and questions regarding what logics are display-systematically formalizable are some further points of investigation for the interested reader. Detailed studies of these topics can be found within the list of references. Regardless of these further notions, we hope the reader has found value in the introductory concepts presented here.

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