Canonical maps

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Abstract

Categorical foundations and set-theoretical foundations are sometimes presented as alternative foundational schemes. So far, the literature has mostly focused on the weaknesses of the categorical foundations. We want here to concentrate on what we take to be one of its strengths: the explicit identification of so-called canonical maps and their role in mathematics. Canonical maps play a central role in contemporary mathematics and although some are easily defined by set-theoretical tools, they all appear systematically in a categorical framework. The key element here is the systematic nature of these maps in a categorical framework and I suggest that, from that point of view, one can see an architectonic of mathematics emerging clearly. Moreover, they force us to reconsider the nature of mathematical knowledge itself. Thus, to understand certain fundamental aspects of mathematics, category theory is necessary (at least, in the present state of mathematics).

1 Introduction

The foundational status of category theory has been challenged as soon as it has been proposed as such[1]. The literature on the subject is roughly split in two camps: those who argue against category theory by exhibiting some of its shortcomings and those who argue that it does not fall prey to these shortcomings[2]. Detractors argue that it supposedly falls short of some basic desiderata that any foundational framework ought to satisfy: either logical, epistemological, ontological or psychological. To put it bluntly, it is sometimes claimed that

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[1] The first publication presenting the category of categories as a foundational framework is Lawvere's [23]. As far as I know, the first printed reaction against such an enterprise is Kreisel's appendix in [26].

[2] Here are some of the standard references on the topic: [10, 3, 16, 2, 36, 37, 22, 42, 41, 24, 38, 43, 21, 28, 33].

category theory fails to rest on simple notions, or its objects are too complicated, or it presupposes more basic concepts, or it can only be understood after some prior notion has been assimilated and, of course, these disjunctions are not exclusive. In this paper, I want to reverse this perspective completely. I want to present what I take to be one of the main virtues of category theory as a foundational framework. More precisely, I claim that only a categorical framework, or a framework that encodes the same features than the ones I am going to present here, can capture an essential and fundamental aspect of contemporary mathematics, namely the existence of canonical maps and their pivotal role in the practice and development of mathematics, or should I say their pivotal role in mathematics, period. Moreover, not only can this be seen as a virtue of category theory, but it should also be thought as constituting a drawback of purely extensional set theories, e.g. ZF(C). I should emphasize immediately that I am *not* arguing against set theory in general, but only a specific formalization of it. In fact, sets still play a crucial role in a categorical foundational framework[3]. To be entirely clear: I will not be arguing in favor of a *particular* categorical framework in this paper. The point I will be trying to make is simply that any foundational framework that is written in the language of category theory *necessarily* exhibits essential conceptual features of mathematical practice.

From a philosophical and a practical point of view, set theory has the advantage of offering an ontological unification: all mathematical objects can be defined or ought to be defined as sets. Thus, number systems, functions, relations, geometrical spaces, topological spaces, Banach spaces, groups, rings, fields, categories, etc., can all be defined as sets. This is very well-known. The epistemological aspects of mathematics are supposed to be taken care of by the logical machinery. This also provides a form of unification[4]. I want to underline that the analysis of mathematics via first-order logic and set theory, which used to be called metamathematics, offers clear and important payoffs. As is also very well-known, category theory itself challenges the ontological and methodological unifications provided by set theory and first-order logic. It is not so much the notion of large categories which is the problem, but rather the inescapable usage of functor categories and functors between them that raises the issue. There *are* various technical solutions to the problem and we won't discuss them here[5]. The philosophical limitations of the purely extensional set-theoretical framework are too familiar for us to mention them here[6].

[3] How this can be is explained in [33],[28]. See also [45].

[4] Category theory also provides a form of unification, in fact many forms of unification. The most obvious and in some sense superficial is the fact that almost any kind of mathematical structure together with their morphisms form a category. Then, there are deeper forms of unification, forms that are revealed after serious mathematical work has been done. To mention but one example, the notion of Grothendieck topos provides a deep unification between the continuous and the discrete, a unification which lead to the development of arithmetic geometry. Our main claim in this paper is based on a different form of unity. The various forms of unification inherent to category theory and their philosophical advantages will be explored elsewhere.

[5] There is a vast literature on the subject of large categories and set-theoretical foundations for the latter. See, for instance, [11, 12, 4].

[6] See, for instance, [29, 35, 27, 34]

One of the advantages of category theory is that it puts morphisms on a par with objects, both ontologically and epistemologically. As such, it does not seem to differ that much from one formulation of set theory. After all, Von Neumann has given an axiomatic set theory based on the notion of mapping. (See [46].) What does a categorical framework add to the foundational picture, apart from organizing mathematics differently?

Let me start with a loose and informal sketch. Suppose you have laid out in front of view a network of objects with multiple arrows between them. At first sight, the whole thing looks like a messy graph. The fact that stands out from the development of mathematics from the last fifty years or so is that if this graph represents a (potentially large) portion of mathematics, then not all arrows in the graph have the same role nor the same status. When we use a road map or any geographical map, there are conventions underlying the representation that allow us to see immediately which roads are highways or which portions of the map are major rivers, mountains, etc. The latter representations work because the language used for the construction of these maps contain a code that captures these elements. The point I want to make here is extremely simple: category theory, and not just its language, provides us with the proper code to represent the map of mathematical concepts. To use another metaphor: when the graph is illuminated with the proper lighting, some mappings stand out as having singular properties. It is as if one would lit the given network with a blacklight which make some of the mappings become apparent through fluorescence. The interesting thing is that this can now be reflected in the foundational domain and thus acquire a philosophical interpretation. I claim that the illuminated mappings occupy a privileged position: they provide, to use yet another metaphor, the basic routes along which concepts can be moved around. They constitute the highway system of mathematical concepts. This is already significant, but that is not all. Perhaps even more important is the fact that these basic, even elementary, roads open up the way for other conceptually important roads that are in some sense built upon them. Thus, not only do they provide the roads, they also provide the frame upon which the other concepts are constructed or erected. These morphisms are usually called "canonical maps". Once the latter have been identified or recognized for what they are, morphisms that are not canonical but that are important acquire a new meaning too. To use an analogy here, one could say that in the same way that symmetries are fundamental in the sciences, broken symmetries are just as important as long as one has understood the importance of symmetry in the first place. From a set theoretical point of view, canonical maps are merely maps like all the others. They can be defined as sets and they are not highlighted in any special manner. From a categorical point of view, the situation is quite different. The language and the concepts of category theory provide the blacklight. These maps show up as having a special character. They are singled out as being significant. Some mathematicians have decided to use them as much as possible, once they realized the role they played in the architecture of mathematics. Thus, the very practice of mathematics is modified by the conscious recognition of these maps and their status. And their status is not innocuous. Fundamental mathematical theorems rest on the existence and the properties of these morphisms. It then seems reasonable to have a way to reflect this fundamental character in a foundational framework and that is precisely what a categorical framework does.

Here is another metaphor: among all the morphisms, some are given right from the start. Mathematics is, in some sense, born with them. They come with the objects, or rather, the objects come with them. For, in some sense, they are inseparable from the objects themselves and it is hard to say which is which, that is, whether we are dealing with the objects or we are dealing with the morphisms. In fact, there is no conundrum: we are in fact dealing with both. However, there are *other* morphisms between objects. The latter are defined and used for specific purposes whereas canonical maps are constitutive of the framework itself. Thus, canonical morphisms provide the blueprint of mathematics. It is perhaps not totally inappropriate to develop an analogy with genetics. It is unquestionable that genes play a key role in the development of an organism. But as is well known, the environment plays a key role too. It is a different role and the important point is that the same genetic pool can yield organisms with significantly different phenotypes in different environments. There is subtle form of dualism at work here and we propose that the same kind of subtle dualism is revealed by the presence and the role of canonical morphisms. One of the main points to be made is that canonical morphisms are built-in a given framework. They are not an option. And once they are seen as such, mathematics develops around them in a completely organic manner. One of the main functions of category *theory*is precisely to reveal these maps and their properties. In turn, the nature and the role of these maps allow mathematicians to explain various mathematical facts.

2 Canonical maps: the folklore

In the mathematical vernacular, a canonical map f is usually said to be a map defined without any arbitrary decision. What can this possibly mean? It suggests that the definition of the map is somehow read off directly from what is given in a specific context. This is of course very rough, imprecise and approximate. We should be careful and beware of the folklore. The ability to read off directly from a context is far from being a clear and transparent notion. There are, in all cases, too much presupposed knowledge and abilities involved in the act of *reading off directly* from something.

The following operational characterization is also part of the folklore: a map f is canonical whenever any two mathematicians having to define f give the same definition. This is another attempt at capturing the idea that it is made without any arbitrary choice[7].

[7] I should emphasize immediately that this has nothing to do with the axiom of choice at this point. I should also emphasize that I am focusing on canonical *maps*, but that I am not talking about morphisms of symplectic manifolds which are also called "canonical transformations". In the latter case, the terminology refers to a completely different concept,

Another, slightly different, informal characterization says that a map f is canonical if the definition of f depends exclusively on structural elements of the context in which the definition is made. As to how one determines what is a structural component or that only structural components are used in a definition, this is left unspecified. I suppose that it is assumed that professional mathematicians are taken to be able to identify the structural components of a situation.

Categories were introduced for the sole purpose of clarifying the informal notion of being 'natural' for a map. In that respect, it was entirely successful. Every mathematician now knows what it means for a map to be a natural transformation. What was an informal and vague term became a purely technical term that captured in a satisfactory manner an important and general mathematical notion. This usage, in turn, helps us see that the usage of 'natural' does not coincide with the usage of 'canonical':

> Occasionally, we use the equal sign to mean a "canonical" isomorphism, perhaps not, strictly speaking, an equality. The word "canonical" is often used for the concept for which the word "natural" was used before category theory gave the word a precise meaning. That is, "canonical" certainly means natural when the latter has meaning, but it means more: that which might be termed "God-given." We shall make no attempt to define that concept precisely. (Thanks to Dennis Sullivan for a theological discussion in 1969.) [6, vii]

Bredon makes two considerably different claims in the preceding quote. First, canonical maps should be natural, when the latter has meaning. Although I am not quite sure I understand this first claim, I believe it can be given a very plausible reading. I take it to mean that whenever it is meaningful for a canonical map to be a natural transformation, then it is one. More precisely, this says that if a given canonical map can be put in the right commutative diagram, then it should turn out to be a natural transformation in that context. If this interpretation is correct, then it is indeed a very reasonable claim. However, since we do not have a definition of what it is to be canonical for a map, I am not quite sure what to make of this claim. I submit that it should be taken as a desiderata for any characterization of the notion of being canonical for a map. Second, canonical maps might be termed "God-given". Of course, there is certainly a form of humor at work here. There is a reference to the tables given by God and it is well-known that the term 'canon' was also used to designate mathematical tables[8]. For the term "canonical" literally means "required by

since a symplectomorphism is a transformation of canonical *coordinates*. Finally, I am not considering other usages of the word 'canonical' in mathematics and logic, for instance when one talks about canonical models. I will say more about the historical usage of the terms "canon" and "canonical" involved in §3 and §4.

[8]For instance, Napier's book of logarithmic tables was called *Mirifici Logarithmorum Canonis Descriptio* in latin. I thank Daniel Otero for this remark. Another example is provided by Jacobi's book entitled *Canon Arithmeticus*, which is a book of tables of numbers n and i The first such tables were published by Gauss in his *Disquisitiones*.

canon law" or coming from a sacred text. We would therefore be in front of a piece of mathematics that comes from "The Book". Of course, there is a wink towards Kronecker's infamous statements about the natural numbers and Erdös' ideas that some proofs are taken directly from God's book of mathematical proofs (See, for instance, [1]). There is nonetheless room for interpretation. Here are a few obvious options: canonical maps are inherent to mathematics; they are completely independent of human choices and representations; we discover them. These are all possible interpretations. I will come back to this very important dimension at the end of the paper. I first have to establish that some maps, that mathematicians dub 'canonical', have a very special property that separates them from other maps. Furthermore, we have to try to see what it is about these maps that one is inclined to qualify them as being God-given.

One of the claims I want to make here is that some of these canonical maps arise spontaneously when mathematics is developed in a categorical framework. They are inevitable. They are omnipresent. And they are significant. As the foregoing quote indicates, even in the context of category theory, canonical maps are not necessarily the same as natural transformations. Some natural transformations are not canonical. In fact, one often sees the expression 'a canonical natural isomorphism' in the literature, emphasizing the fact that the natural isomorphism *is* canonical. Not all canonical maps are natural transformations. Some functors are canonical. Furthermore, there may be canonical maps that appear elsewhere or escape the categorical framework. This latter possibility does not affect my claim. The claim that all canonical maps, informally understood, are captured by the categorical language is a thesis akin to the Church-Turing Thesis and I do not want to make it here.

The explicit presence of canonical maps is certainly one of the reasons why one would want to develop mathematics in a categorical framework and, in fact, one of the main reasons why one would want to develop a foundational framework in categorical terms. The purpose of this paper is to explore the nature and status of these canonical maps and, from there, establish their significance in mathematics in general. Once the latter is granted, one can use these facts to argue that a foundational framework ought to reflect the presence and the role of these canonical maps. We will try to provide some of the arguments at the end of this paper. Thus, our plan here is simple: we will first sketch how, from the historical point of view, the expression 'canonical map' arose naturally in a categorical framework and was used extensively by certain mathematicians (but not all!), then give some non-trivial examples of these maps and their roles in certain proofs, move to their presence in a foundational framework and end with some speculation on the links between canonical maps and mathematical cognition.

3 Examples

Before going any further, I will give three simple examples of maps that are considered to be canonical by mathematicians. Other examples will be given in a latter section. I simply want to provide four rudimentary illustrations immediately to set up the stage.

What seems to be the very first map to have been called 'canonical' is the usual quotient map. This construction appears everywhere in mathematics, from topology to algebra. In the set theoretical framework, one starts with a set X and an equivalence relation $R \subset X \times X$ on X. An equivalence set is defined as usual to be $[x] =_{df} \{y : R(x, y)\}$ and we can now consider the set of equivalence sets $X/R =_{df} \{[x] : x \in X\}$. There is then a special map, called the quotient map, $\pi : X \to X/R$ defined by $\pi(x) = [x]$. This map is always called canonical. One could say that it is the canonical canonical map. In a way, if you follow every step of the construction, the language itself tells you how to define it. There are other maps from a set to a quotient of that set by an equivalence relation, but π is, in some sense, given by the construction itself. There is no choice being made. The notation itself takes care of it. Whenever there is more data involved, e.g. when X is a topological space or a group, the quotient map has the 'right' properties, i.e. it is continuous, a homomorphism, etc. A very well-known special case of this construction is the canonical map $\mathbb{N} \rightarrowtail \mathbb{Z}$ defined by $n \to [(n, 0)]$ with the usual equivalence relation.

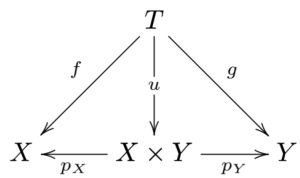
Although the preceding example is given in set-theoretical terms, it turns out to be but one more example of what we get when we move to the categorical context. As I said, category theory shows immediately what the foregoing map has in common with other cases of canonical maps. I assume the reader knows the basic constructions of category theory, e.g. products, pullbacks, equalizers, terminal object and limits in general, their dual as well as the notions of functors, natural transformations and natural isomorphisms.

Let C be a category with pullbacks and binary products. Consider the following pullback diagram:

$$\begin{array}{ccc} P & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[m]{} & Z \end{array}$$

Then, it is easy to show that there is a canonical *monomorphism* $h : P \rightarrowtail X \times Y$, where $X \times Y$ is the product of X and Y. Category theory tells us why. The product $X \times Y$ comes with two maps and a universal property. This is how the notion of product is *defined* in category theory. The two maps that are inherent to a product are the so-called *projections* $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$. They are also called the *canonical projections* in the literature. Thus, when described in category-theoretical terms, the notion of a product automatically comes with canonical maps.

The universal property is expressed as follows: given an object T *together with* two maps $f : T \to X$ and $g : T \to Y$, then there is *a unique* map $u : T \to X \times Y$ such that $p_X \circ u = f$ and $p_Y \circ u = g$. In turn, these data can be represented by the following commutative diagram:



With these information at hand, the reader can now see why and how the foregoing map h automatically shows up in the commutative square above. The fact that h is a monomorphism also follows directly from the data. The map h and the fact that it is a monomorphism are in a precise sense given. There is absolutely no choice made. If this is a pullback square and if the category C has binary products, then h exists and it is uniquely defined by the data. Of course, as usual, there are many other monomorphisms in general between these objects, but the map h depends entirely on specific features of the situation[9].

The quotient presented above can be given a similar presentation. It becomes a special case of what is called a *coequalizer* in category theory. (See .)

The third example is slightly more involved and corresponds to familiar mathematical situations. Let C by a category with finite products, finite coproducts and a null object 0, which amounts to the existence of a null morphism $0 : X \to Y$ between any two objects of C, defined by the composition of morphisms $X \to 0 \to Y$. Notice that the null morphism is canonical, but that is not the example we want to give. For, in this case there is a canonical map

$$X_1 \sqcup \cdots \sqcup X_n \to X_1 \times \cdots \times X_n.$$

Again, this map comes for free. It is there as soon as the category C has the right categorical properties. It can then be used to prove other properties, construct other morphisms, perhaps specific to a particular category, e.g. the category of abelian groups.

Let me finish with a slightly different example, namely the example of a canonical functor. Let (X, x_0) be a pointed topological space, namely a topological space X with a specified point x_0 and let (S^1, p) denote the unit circle with the usual topology with the point p be the north pole. Then, there is a functor $\pi_1 : \mathbf{Top} \to \mathbf{Grp}$ from the category of topological spaces and continuous functions to the category of groups with group homomorphisms defined by $(X, x_0) \mapsto Hom((S^1, p), (X, x_0))$ which is the fundamental group of the pointed space. It follows directly from general facts that it *is* a functor. The fact that it is a group also follows from underlying general facts.

These maps appear for purely general reasons in the context of category theory. It is tempting to say that they appear for structural reasons. But since we do not have a general theory of structures and it is hard to pin point exactly what is meant by 'structural', I will not use this terminology for the

[9] An interesting variation on this situation is this: suppose there is a monomorphism $Z \rightarrowtail T$ and that, as above, P is the pullback of $X \xrightarrow{e} Z \leftarrow^{m} Y$ and that $X \times_T Y$ is the obvious other pullback, then it is easy to show that there is a canonical isomorphism $P \to X \times_T Y$.

moment. Although it is certainly not obvious from this very short discussion, the reader has to take our word for it when we claim that these maps are literally everywhere in mathematics. As soon as someone needs to make various constructions in a mathematical context, these maps show up when the proper theoretical tools are used adequately. They are built-in the constructions in the categorical language.

4 Canonical maps: brief historical remarks

Canonical maps permeate category theory. However, the terminology does not originate with the field and it took some time before it became accepted and commonly used[10]. In the categorical context, one observes that the term was not used by the American school of category theorist in the 1950s and the early 1960s and that it clearly was under the influence of the French school, more specifically Grothendieck's school who used category theory to develop new foundations of algebraic geometry that the terminology acquired a clear status. Here is a sketch of this development. A full story will have to be told elsewhere.

Although Eilenberg and Mac Lane's goal was to give a precise general theory of natural equivalences – or as we now say "natural isomorphisms" – and natural transformations, along the way they stumbled upon examples of canonical maps, but did not give them that name, nor did they underline their presence. For some natural isomorphisms are certainly canonical maps. The very first example given to motivate their whole paper about general equivalences is a canonical map: it is the canonical natural isomorphism between a finite-dimensional vector space V and its dual V^{**}. Interestingly enough, Eilenberg and Mac Lane do emphasize what could be taken as being one difference between a natural transformation and a canonical map. The first part expresses what we have presented as being now part of the folklore:

> For the iterated conjugate space $T(T(L))$, on the other hand, it is well known that one can exhibit an isomorphism between L and $T(T(L))$ *without* using any special basis in L. [9, 232]

This is a well-known fact, but extremely important conceptually. There *is* an isomorphism between a finite-dimensional vector space V and its dual space V^*, but in order to define it, one has to fix a basis in V. In fact, there are many of them. But when one moves to the double dual V^{**}, the isomorphism is defined independently of the chosen basis. This is a remarkable fact: one does not have

[10] As we have already alluded to in preceding footnotes, the term "canonical" has been used in different contexts in mathematics, for instance to circumscribe certain matrix forms in the 19th century or curves in geometry also at the same time. As far as I can tell, the term "canonical matrix" was introduced by Sylvester in the mid 19th century, following apparently a suggestion made by Hermite. As I have indicated, the terminology was used earlier in various contexts. It would be interesting to explore its history and usage by different mathematicians and philosophers, for instance in Descartes and Leibniz.

to *choose* a particular basis. The map is given by the construction of V^{**} (their $T(T(L))$) and its links to V. The foregoing passage then shifts to the specific property of a natural transformation:

> This exhibition of the isomorphism $L \cong T(T(L))$ is "natural" in that it is given *simultaneously* for *all* finite-dimensional vector spaces L. [9, 232]

And there lies one difference between the two concepts. What the technical notion of natural transformation captures is the fact that the transformation is given *uniformly* for *all* the objects of the category and it is a map between functors. Naturality is a global phenomenon, so to speak, whereas canonicity need not be in general in that sense. Furthermore, a natural transformation is not necessarily canonical[11]. This is, however, a nice example of a canonical map that satisfied the desiderata mentioned in section 2: it is a canonical map that can be put into a commutative diagram expressing the property of being a natural transformation and it is one.

Various constructions presented in their 1942 paper and in their 1945 paper include what they call "induced maps" and the latter are canonical.

The terminology simply does not show up in the American school in the 1950's and early 1960's. The word appears in Mac Lane's paper on duality for groups published in 1950 to designate the map from a group to its quotient by a normal subgroup and the decomposition of a group homomorphism into an epimorphism and a monomorphism. There is nothing in Kan's paper on adjoint functors in 1958 and the accompanying paper. There is no occurrence of the word in Eckmann and Hilton's paper in 1962 and no occurrence of the term in Freyd's book *Abelian Categories* published in 1964. In Mac Lane's paper on categorical algebra, published in 1965, one finds 21 occurrences of the word 'canonical'. They are not all used to denote maps and sometimes it designate the same map, but it is nonetheless a sign that the vocabulary was getting established in the english speaking community. It is somewhat surprising to observe that the terminology is not used in Freyd's *Aspects of Topoi* in 1972.

Things change radically when one moves to the French school of category theory, in particular Grothendieck's school of algebraic geometry.

The origin of the terminology clearly goes back to Bourbaki. It was probably intended whimsically at first, typical of Bourbaki's humor. As far as I can tell, it appears in Weil's version of Bourbaki's book on set theory, where one finds the notion of a "canonical correspondence" between two sets: given two sets X and Y, there is a canonical map $f : X \times Y \longrightarrow Y \times X$. In his thesis published in 1951, Serre uses the term 24 times, not only to denote the usual quotient map, but many isomorphisms as well.

Given that Grothendieck was a member of Bourbaki in the 1950s and given the influence of Serre on Grothendieck, it is no surprise to find Grothendieck use

[11] There is at least one paper whose title is revealing: "Non-canonical isomorphisms". One has to be careful, for in this case, the author, the category theorist Steve Lack equates being canonical in a categorical framework with being a universal morphism. See [20]

the term extensively in his work. Indeed, the terminology of canonical maps is omnipresent in Grothendieck's writings. It is already present in his first writings in functional analysis. It is however less predominant than in his subsequent writings developed in a categorical context. Those who have read the *Éléments de géométrie algébrique* will have seen the word 'canonical' sometimes more than ten times in a single page[12]! The omnipresence of the terminology in Grothendieck's work and thereafter is certainly a significant fact. Grothendieck systematically search for and uses canonical maps in his work. But it is not so much the terminology that is interesting. It is what it reveals about a certain class of morphisms and their role in the mathematical landscape.

5 Canonical maps: taking stock

Our claim is that category theory allows us to identify certain morphisms as being canonical. As we saw above, when one introduces certain constructions in category theory, the constructed objects *always* come equipped with canonical maps. The general notions involved are those of limits and colimits of a diagram in a category. I will give the general definition of a limit of a diagram J in a category $\mathcal{C}$. This will allows us to circumscribe more precisely what we want to include in our characterization of canonical maps.

Let J be a small category and $\mathcal{C}$ be a category. A *diagram* of shape J in a category $\mathcal{C}$ is a functor $\delta : J \to \mathcal{C}$[13]. Informally, a diagram of shape J can be pictured as being composed of objects $\delta(X)$ of $\mathcal{C}$ together with morphisms $\delta(X) \to \delta(Y)$ between these objects. A *cone over the diagram* is an object c of $\mathcal{C}$ together with morphisms $p_X : c \to \delta(X)$ for each X in J such that for each morphism $i : X \to Y$ in J, $\delta(i) \circ p_X = p_Y$.

A *limit*, denoted by $\varprojlim J$ or simply by $\lim J$, for the diagram J is a *universal cone*. Thus, it is a cone over the given diagram. As such, and this point is fundamental for us, it comes with morphisms $\pi_X : \varprojlim J \to \delta(X)$. Second, a limit satisfies the following universal property: given any cone c over the diagram, there is a unique morphism $u : c \to \varprojlim J$ such that $\pi_X \circ u = p_X$ for all objects X in J.

We should note that a limit for a diagram is unique up to a unique isomorphism. This means that, given a diagram of shape J in $\mathcal{C}$, it both L_1 and L_2 are limits for this diagram, then there is a unique isomorphism between them. This fact follows immediately from the definition of a limit.

The notion of a *colimit* is the dual of the notion of a limit. Thus, everything we will say about limits and their canonical morphisms applies to colimits and their canonical morphisms.

[12]It would be interesting to explore the number of times the expression appears in the works of various mathematicians, especially those who use category theory. For instance, it appears often in Jacob Lurie's work, no less than 749 times in his book *Higher Algebra* and 139 times in his book *Higher Topos Theory*.

[13]I am being a bit loose here. The informed reader knows how to fix the definition. To the uninformed reader, the differences are uninformative.

We are now in a position to circumscribe more precisely what we want to include in the notion of canonical morphisms or maps.

1. Morphisms that are part of the data of a limit are canonical morphisms; for instance the projection morphisms that are part of the notion of a product;
2. The unique morphism from a cone to a limit determined by a universal property is a canonical morphism;
3. In particular, the unique isomorphism that arise between two candidates for a limit is a canonical morphism.

Of course, this is not a definition of the notion of canonical morphism. I will not even attempt to give such a definition. I do not need to. My point is simply this: no matter how one defines the notion of canonical morphism, the proposed definition will have to include these cases as generic instances.

One important remark about my presentation via the notion of limit. There are well-known alternatives. Grothendieck used the notion of representable functors to capture these cases systematically[14]. We could have used the notion of adjoint functors. The fact that these different approaches are all equivalent is another important result of category theory.

We are now ready to turn to more philosophical issues.

6 Canonical maps, category theory and the architectonic of mathematics

> By an **architectonic** I understand the art of systems. Since systematic unity is that which makes ordinary cognition into science, i.e., makes a system out of a mere aggregate of it, architectonic is the doctrine of that which is scientific in our cognition in general, and therefore necessarily belongs to the doctrine of method. (Kant, *Critique of Pure Reason*, A 832/B 860, translation Guyer & Wood)

By using the term "architectonic", I want to emphasize the fact that mathematical concepts as developed in a categorical framework form a system whose nature is different from the systems we are accustomed to. And I claim that this new architectonic has philosophical traction. I won't be able to develop these claims as well as they should in such a short paper. Needless to say, many concepts and claims have to be clarified and a lot of background material has to be assimilated[15]. The challenge is, at the very least, threefold: 1. to explicate precisely the *kind* of system I have in mind; 2. to say how that kind differs from other kinds of systems; 3. what makes that particular kind of system

[14] This explains in part why the Yoneda lemma and the Yoneda embedding are central in his work.

[15] For some of the background material, I refer the reader to [19, 31, 32].

philosophically relevant. Given that the type of architectonic I am presenting depends upon the presence of canonical morphisms, I will call it "the Canonical Architectonic" from now on. I will briefly develop these three challenges in turn.

6.1 The Canonical Architectonic: what kind of underlying principles?

Let me start with what most people have in mind when they think about category theory and the system of mathematical concepts. It is well-known that mathematical concepts, like the notions of monoids, groups, rings, fields, topological spaces, metric spaces, vector spaces, partial orders, etc., all form categories. Thus, one has the category of groups and group homomorphisms, the category of topological spaces and continuous maps, the category sets and functions, etc. This is the usual understanding of what mathematicians have in mind when they think about the organizational power of category theory. Mathematics is thus presented as a system of *concrete* categories with functors between them (and natural transformations between the latter)[16]. For most mathematicians and logicians, that is all there is to it. It is not that this picture is false nor that it is useless to mathematicians. It does some work, but it is philosophically completely uninteresting and it is mathematically and from a foundational point of view radically incomplete. The unifying principle underlying this view of category theory in mathematics and its foundations is simply the fact that these notions and their morphisms satisfy the axioms of a category and, by doing so, it allows for a certain organisation of mathematics. Furthermore, this organization is seen to be at a very high, algebraic level, useful for the mathematician doing algebraic geometry, algebraic topology or what have you, but completely useless for foundational purposes and philosophical analyses[17]. The standpoint in this case is that category theory and its concepts are *applied* to various situations in fruitful ways. These successes are seen, so to speak, after the fact and it is as if one would not have good theoretical reasons to develop mathematics in categories in the first place. This situation is general in philosophy of mathematics and was already put forward by Ken Manders more than twenty-five years ago.

> ...suppose we could attribute "fruitfulness-engendering" structural properties to ways of treating a subject matter, ..., instead of judging the outcome of attempts to treat the subject after the fact. Then there could be reasons for success and failure: x succeeds, because x has α and having α is inherently fruitful. Then there would be a normative fact of the matter, with theoretical rather than just practical standing, whether a treatment was correct, preferable, or

[16] A *concrete* category is usually thought of as being a category of structured sets of some sort. A more technical definition was provided by Freyd in [14, 15] where a concrete category is a category that has a full and faithful functor into the category of sets. Freyd even gives necessary and sufficient conditions for a category to be concrete.

[17] And we are back to the opening paragraph of this paper...

> appropriate, based on intrinsic merits, and regardless of attempts to carry it out. ([30, p. 554])

The problem with this view of category theory as applied mathematics is that it misses crucial elements of the picture, in particular the principles underlying the new unity. In accordance with Kant's idea, the use of the term "architectonic" is justified inasmuch as the system is developed according to certain principles and, in some sense, in a coherent or cohesive fashion[18]. The latter requirement, interpreted appropriately, is central to our claim.

In the foregoing picture, categories are introduced *after* some given structures are provided by other means, usually by an axiomatic presentation in first-order logic interpreted in the universe of sets. In the picture I am proposing, the underlying principles *precede* particular structures and specific categories[19]. Again, I am not arguing in favor of one of singular, complete, technical development of the category of categories, but rather I am saying that the presence of canonical morphisms ought to be included in any such picture, since they already provide the underlying new principle[20].

Thus, canonical morphisms provide the architectonic of mathematics[21].

> Human reason is by nature architectonic, i.e., it considers all cognitions as belonging to a possible system, and hence it permits only such principles as at least do not render an intended cognition incapable of standing together with others in some system or other. (Kant, *Critique of Pure Reason*, A474, translation Guyer & Wood)

Canonical morphisms are not *merely* morphisms that connect various "objects" together. In the three types that we have identified in this paper, the nature of the connections is revealing. The first group of morphisms, those that are constitutive of the notions of limits or colimits already give us an underlying network between concepts of a certain sort. Thus, by stipulating that a certain category has limits or colimits of a certain kind, one immediately specify an underling system of concepts that determine a global network.

[18]Kant would say *a priori*. This is precisely what we want to claim here, but the term *a priori* is philosophically so charged that I hesitate to use it explicitly.

[19]It is in this sense that the underlying principles are *a priori*. They do not depend on any previous formal system. They might be explained by informal concepts and pictures, as I am trying to do here.

[20]I will nonetheless say some things about such technical developments in the conclusion of this paper.

[21]Bourbaki explicitly wrote about the *architecture* of mathematics and talked about three mother structures: algebraic structures, order structures and topological structures. See [5]. Bourbaki's paper is fascinating and it constitutes a clear and concise presentation of what mathematicians understood and, to a large extent, still understand by the axiomatic method and how it unifies mathematics. It should be clear that I am going in a different direction altogether. I am not claiming that there are "mother structures". I am claiming that there is a global architectonic of mathematical concepts, a global conceptuel unity revealed by canonical morphisms. The term has also been used by Balzer and Moulines in the title of their book about the structuralist program in philosophy of science. See . Although there might be links to their program, I am not making a structuralist claim in this paper.

It is, however, the second kind of canonical morphisms that is even more important. The universal property satisfied by limits and colimits is attached to a universal and canonical morphism. This says that limits and colimits occupy a privileged position in the given system of concepts. By stipulating that one is working in a category with these limits or colimits, one automatically knows that there are routes in the network that have a special status. By travelling through these routes, one obtains important and often crucial information.

Often in mathematics, one has to prove that two independently given constructions are in fact isomorphic or equivalent in an appropriate sense. When the constructions are done in a categorical framework, they often come with canonical maps and the isomorphism or the equivalence that one is searching for can sometimes fall automatically from these canonical maps. In other cases, the canonical maps provide the starting steps in the construction and the required isomorphism or equivalence amounts to putting together the given canonical maps together with specific maps inherent to the situation at hand. This is just not handy and useful for a mathematician. One uses all the given resources at hand, all the underlying capacities of the situation.

The last type of canonical morphisms simply reinforces the second point. The unique isomorphism between two limits for the same diagram of a certain shape can be thought of as an identity. This last fact ought to be interpreted as a special case of Leibniz's Principle of identity: if $P(X)$ and $X \simeq Y$, then $P(Y)$, where X and Y are objects in a given category $\mathcal{C}$ and $P(-)$ is a property of the theory[22].

The canonical architectonic arises from the intrinsic properties of the system. Clearly, it is an *abstract* system and I claim that it is precisely this abstract character that makes it philosophically interesting *and* problematic.

6.2 The Canonical Architectonic: how does it differ from other systems?

> Mathematics constitutes an organic whole of concepts, methods and theories crossing each other and interdependent in such a way that any attempt to isolate one branch, even for the sake of providing a foundation for the whole body, would be in vain; nor would it make sense to break this organic solidarity by looking for an external foundation (be it natural science or logic or some psychological reality). (Cavaillès quoted in [8, p. 104])

Cavaillès underlines the *organic* whole of concepts, methods and theories and their interdependence. One could argue that Kant had a similar view of scientific knowledge when he decided to characterized it as being architectonic. As we have already indicated, it is precisely this specific aspect of the canonical

[22] We are in this particular case dealing with a *unique* isomorphism. It should be noted that the principle ought to apply to cases where there are many isomorphisms or equivalences between given objects. This is a topic in itself that will certainly be addressed in this book by other authors.

architectonic that I want to put forward. Some would like to talk about its holistic nature. I prefer to avoid this terminology altogether. It is systematic and the system forms a whole in the same way that an organism form a whole.

As I have already indicated in the opening section of this paper, set theory and first-order logic, taken together, already provide a unity to mathematical knowledge. What are the underlying principles of this unity? There is no need to elaborate this picture extensively. The axioms of set theory, say ZFC, when interpreted in the cumulative hierarchy, are the fundamental principles and are supposed to be self-evident, at least with respect to the cumulative hierarchy[23]. However, the system is in this case purely deductive. Logic provides the links between various mathematical propositions. The epistemological standpoint is the usual foundationalist position. The gain is a fine-grained look at justification of propositions. On the practical side, axioms are given for various structures. They are interpreted in the universe of sets. But the latter does not reflect the various relationships between the concepts, e.g. the numerous links between topological spaces, groups, more generally modules.

> Conceptual unification, understandability, clarity, even length of proofs, fall outside the narrow justificational concern of foundational epistemology. That concern, we should now see with the benefit of a hundred years' hindsight, fails to capture the intellectual enterprise of mathematics. Any dispassionate look at mathematical sciences should teach that the "mathematical way of knowing" seeks as much to render things understandable as it does to establish theorems or avoid error. The process of establishing deductive relations is subsidiary to the larger goal of rendering understandable. ([30, p. 562])

To put it bluntly: the universe of sets is conceptually flat, but combinatorially rich. I propose we look at the dual picture.

In the picture proposed here, mathematics is developed within a universe of categories. The usual mathematical concepts, e.g. groups, topological spaces, etc., can be defined in that universe[24]. Canonical morphisms are built-in that universe and can be used to develop various theories and connections between them. This is what is usually and unfortunately called "pure category theory". In the latter, one finds theorems about additive categories, abelian categories, triangulated categories, exact categories, regular categories, toposes, monoidal categories, braided categories, Quillen model categories, Waldhausen categories, etc., to mention but the most well-known.

Results in these "pure" categories are seldom by themselves equivalent to what mathematicians are directly interested in, although, in some cases, they are[25]. However, one can then bring in what was seen as the "applied" dimension of categories, e.g. more specific structures or data, etc. required by the problem

[23] The standard presentation of this position can be found in [44]. Historically, the situation is considerably more delicate and intricate. See, for instance, [13, 39]

[24] See, for instance, [40].

[25] For instance, many duality theorems find their natural expressions at this level.

or theory at hand. Thus, the applied dimension finds its full expression when the built-in morphisms have been identified and used.

We end up with a picture of mathematics that is not quite standard: it is a new form of dualism. There is the canonical part of mathematics which seems entirely written in the concepts themselves and various contraptions that we invent, manipulate for more specific needs and purposes. From the point of view of mathematicians, it is a methodological dualism and it has been developed and propounded by category theorists from the 1960s onwards. The strategy is to used the canonical morphism to obtain as much information as possible at the most abstract level and, then, add the specific, particular or concrete ingredients of the situation one is interested in to prove the wanted results.

Even though this is a form of dualism, it is not a form of platonism in the standard sense of that expression.

6.3 The Canonical Architectonic: Why is it philosophically relevant?

The philosophical content of the presence of canonical morphisms is multifaceted and deep.

Firstly, to work at the level of canonical morphisms is to work at an abstract level. One uses only the properties of canonical morphisms to obtain various results. For instance, when a mathematician obtains a result in a topos using purely topos theoretical properties, it is certainly not yet a result in, say, algebraic geometry. One could wonder what is the gain to work this way. For one thing, it is similar to the gain that was associated with the axiomatic method and that led Bourbaki to talk about the architecture of mathematics. Whereas Bourbaki thought that there were three fundamental mathematical structures, I see a global organization of mathematical concepts deriving from the presence of canonical morphisms. The analogy with an organism seems to be apt and adequate. The unity of an organism depends upon the presence of various organs that are organized into a system, that is, relationships between these organs. Categories defined via the existence of canonical morphisms roughly correspond to these organs and results obtained from properties of these canonical morphisms correspond to the relationships. An architectonic provides us with a plan of mathematical knowledge and an understanding of its fundamental organization.

Secondly, results proved in categories defined via canonical morphisms are valid in all applications where these morphisms exist. This provides a vast unification of mathematical concepts and results. Associated with this new form of unification, one finds also a simplification of various results. This might appear paradoxical, since category theory is usually seen as being difficult and complicated. To wit:

> Any relation in the language of derived categories and functors gives rise to assertions formulated in the more traditional language of cohomology groups, filtrations, spectral sequences.... Of course, these

> can frequently be proved without explicitly mentioning derived categories so that we may wonder why we should make the effort of using this more abstract language. The answer is that the simplicity of the phenomena, hidden by the notation in the old language, is clearly apparent in the new one. The example of the Kenneth relations ... serves to illustrate this point... ([18, p. 672])

Or, again:

> The first part of the paper, on which everything else depends, may perhaps look a little frightening because of the abstract language that it uses throughout. This is unfortunate, but there is no way out. It is not the purpose of the abstract language to strive for great generality. The purpose is rather to simplify proofs, and indeed to make some proofs understandable at all. The reader is invited to run the following test: take theorem 2.2.1 (this is about the worst case), translate the complete proof into not using the abstract language, and then try to communicate it to somebody else. ([47, p. 318])

Notice the explicit claim made by Waldhausen that the usage of the categorical language is not to strive for great generality. One too often reads that this is why some mathematicians use category theory[26]. These two quotes put to the fore two very different reasons: simplicity and understandability. In both cases, canonical morphism play a key role.

Thirdly, one of the indications of the fundamental nature of canonical morphisms is the fact that basic logical rules are captured in a categorical framework by canonical morphisms.

Let us come back to Bredon's quote. It is now common in textbooks on algebraic geometry to replace *canonical* isomorphisms by a sign of equality. This might seem innocuous, but I believe that it indicates an important cognitive aspect of the situation.

Canonical morphisms are interesting only when they arise systematically from a theoretical framework. Otherwise, they would not deserve to be called architectonic. Once they are seen as such, the cognitive gain is substantial. The introduction of the arrow notation $X \to Y$ acquires all its power once the presence of canonical morphisms has been recognized and is used explicitly.

7 Conclusion

It is not possible to end without pointing towards higher-dimensional category theory. As I am writing this, there is still no adopted definition of higher-dimensional categories. There is, however, a very good theory of quasi-categories,

[26]This was mentioned again and again in the homages to Grothendieck after his death in November 2014. I think that these quotes reflect more adequately Grothendieck's motivations.

also known as $(\infty, 1)$-categories[27]. There is no need to go in the technical details of what these objects are. A fundamental point can nonetheless be made already. The *language* and the *core* concepts of category theory can be used *as they are* for quasi-categories. The computations are far from trivial, but one can show that the main concepts and theorems of category theory still hold for quasi-categories. Thus, and this is the moral I want to make, the notion of canonical morphism is directly lifted to this new context.

It is clear that, no matter what the universe of categories will look like, if it is taken as a foundational framework, then canonical morphisms will play a central role. And our point will still be valid.

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[27] See, for instance, [17, 25, 7]. A full theory of higher-dimensional categories will be a theory of (∞, ∞)-categories. The particular feature of $(\infty, 1)$-categories is that morphisms of dimension strictly greater than 1 are invertible.

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