# Finite axiomatizability of logics of distributive lattices with negation

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## Abstract

This paper focuses on order-preserving logics defined from varieties of distributive lattices with negation, and in particular on the problem of whether these can be axiomatized by means Hilbert-style calculi that are finite (i.e. that contain a finite number of finitary rule schemata). On the negative side, we provide a syntactic condition on the equational presentation of a variety that entails failure of finite axiomatizability for the corresponding logic. An application of this result is that the logic of all distributive lattices with negation is not finitely axiomatizable; we likewise establish that the order-preserving logic of the variety of all Ockham algebras is also not finitely axiomatizable. On the positive side, we show that an arbitrary subvariety of semi-De Morgan algebras is axiomatized by a finite number of equations if and only if the corresponding order-preserving logic is axiomatized by a finite Hilbert-style calculus. This equivalence also holds for every subvariety of a Berman variety of Ockham algebras. We obtain, as a corollary, a new proof that the implication-free fragment of intuitionistic logic is finitely axiomatizable, as well as a new corresponding Hilbert-style calculus. Our proofs are constructive in that they allow us to effectively convert an equational presentation of a variety of algebras into a Hilbert-style calculus for the corresponding order-preserving logic, and vice versa. We also consider the assertional logics associated to the above-mentioned varieties, showing in particular that the assertional logics of finitely axiomatizable subvarieties of semi-De Morgan algebras are finitely axiomatizable as well.

*Keywords*: Finite axiomatizability, lattices with negation, Ockham algebras, semi-De Morgan algebras, Berman varieties, pseudo-complemented lattices.

## 1 Introduction

In the present paper we study logics associated to subvarieties of the class $\mathbb{DN}$ of *distributive lattices with negation* (Definition 2.2) considered, for instance, in the papers [12, 13]. $\mathbb{DN}$ is a variety that includes many well-known classes of algebras of non-classical logics, such as (semi-)De Morgan algebras, Stone algebras, pseudo-complemented distributive lattices and Ockham algebras; thus $\mathbb{DN}$ provides a common semantical framework for the study of the corresponding logics.

We will be mostly concerned with the order-preserving logics associated to the above-mentioned varieties, focusing in particular on the issue of whether they can be axiomatized or not by means of a Hilbert-style calculus consisting of finitely many rule schemata; if this is the case, the logic will be called *finitely based*.

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On the side of negative results, we are going to show that the order-preserving logic associated to the variety $\mathbb{DN}$ is not finitely based; the same holds for the order-preserving logic of all Ockham algebras (Definition 2.3). Indeed, we will give a syntactic criterion regarding the equations that axiomatize (relatively to $\mathbb{DN}$) a variety $\mathbb{V} \subseteq \mathbb{DN}$ implying that the same holds for the corresponding logic. On the positive side, we will show how to obtain a finite Hilbert-style calculus that is complete with respect to the logic of semi-De Morgan algebras, entailing that the latter is finitely based. The same techniques will allow us to obtain finite calculi for the logics associated to so-called Berman varieties of Ockham algebras [6]. As a corollary of our results, we will also obtain a finite axiomatization for the logic of pseudo-complemented distributive lattices (i.e. the implication-free fragment of intuitionistic logic) alternative to the one introduced in [31].

Our proof strategies are discussed in more detail in Sections 3 and 4, but we give here an introductory account on the finite axiomatizability problem for order-preserving logics and the difficulties one faces. First of all, let us clarify the meaning of the terms 'order-preserving logic' and 'finite Hilbert-style calculus'.

Let $\mathbb{K}$ be a class (say, a variety) of algebras such that each algebra $\mathbf{A} \in \mathbb{K}$ has a bounded lattice reduct $\langle A; \wedge, \vee, \bot, \top \rangle$. One of the standard ways of associating a (finitary) Tarskian logic $\vdash^{\leq}_{\mathbb{K}}$ to $\mathbb{K}$ is the following. One lets $\emptyset \vdash^{\leq}_{\mathbb{K}} \varphi$ if and only if the equation $\varphi \approx \top$ is valid in $\mathbb{K}$ and, for all $\Gamma \cup \{\varphi\} \subseteq Fm$ such that $\Gamma \neq \emptyset$, one lets $\Gamma \vdash^{\leq}_{\mathbb{K}} \varphi$ iff there is a natural number $n$ and formulas $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\mathbb{K}$ validates the equation:

$$\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \varphi \approx \gamma_1 \wedge \ldots \wedge \gamma_n.$$

Thus $\vdash^{\leq}_{\mathbb{K}}$ is by definition a finitary logic, called the *order-preserving logic* of the class $\mathbb{K}$. Note that $\vdash^{\leq}_{\mathbb{K}}$ coincides with the logic defined by the class of matrices

$$\{\langle \mathbf{A}, F \rangle : \mathbf{A} \in \mathbb{K},\ F \subseteq A \text{ is a non-empty lattice filter of } \mathbf{A}\}.$$

Other logics may of course be defined from $\mathbb{K}$, for instance, the class of matrices $\{\langle \mathbf{A}, \{\top\} \rangle : \mathbf{A} \in \mathbb{K}\}$ also determines a (stronger) logic associated to $\mathbb{K}$. Following [19], we call this the $\top$-*assertional logic* of $\mathbb{K}$ (denoted $\vdash^{\top}_{\mathbb{K}}$) and will be considered in Section 6.

By a *Hilbert-style calculus* we mean a logical calculus whose every rule schema is a pair $\frac{\Gamma}{\varphi}$ where $\Gamma$ is a (possibly empty) set of formulas and $\varphi$ is a formula; a rule of type $\frac{\emptyset}{\varphi}$ is usually called an *axiom (schema)*. We say that a Hilbert-style calculus is *finite* when it consists of finitely many rule schemata (each schema having only finitely many premisses). Following [29, p. 607], we call a logic that can be axiomatized by a finite Hilbert-style calculus *finitely based*. In [16, Sec. 2.1] the authors introduce a finite calculus for the order-preserving logic of the variety $\mathbb{SDM}$ of semi-De Morgan algebras (Definition 2.3). This, however, is not a Hilbert-style calculus *stricto sensu*, because it involves meta-rule schemata such as the following: from $\langle \varphi, \psi \rangle$ infer $\langle \sim \psi, \sim \varphi \rangle$. The 'axioms' of the calculus introduced in [16], on the other hand, are examples of what are usually called (single-premiss) Hilbert-style rules. Finite Hilbert-style calculi for the order-preserving logics of De Morgan algebras ($\mathbb{DM}$) and pseudo-complemented distributive lattices ($\mathbb{PL}$) can be found in the papers [14, 31]. We note in this respect that $\vdash^{\top}_{\mathbb{PL}} = \vdash^{\leq}_{\mathbb{PL}}$, while $\vdash^{\top}_{\mathbb{DM}}$ is strictly stronger than $\vdash^{\leq}_{\mathbb{DM}}$, which is the well-known Belnap–Dunn logic [4, 5]. $\vdash^{\top}_{\mathbb{DM}}$ is the *Exactly True Logic* introduced and axiomatized by means of a Hilbert-style calculus in [26]; see also [1, 32] .

A closer look at the order-preserving logic $\vdash^{\leq}_{\mathbb{SDM}}$ associated to semi-De Morgan algebras explains the choice of a hybrid calculus in [16], as well as the challenge one faces when trying to axiomatize $\vdash^{\leq}_{\mathbb{K}}$ (for $\mathbb{K} \subseteq \mathbb{DN}$) by means of a calculus that is Hilbert-style in the strict sense. In fact, the consequence relation of each order-preserving logic $\vdash^{\leq}_{\mathbb{K}}$ corresponds to the lattice order on $\mathbb{K}$, in the sense that one has $\varphi \vdash^{\leq}_{\mathbb{K}} \psi$ if and only if the inequality $\varphi \leq \psi$ (taking the latter as a shorthand for the equation $\varphi \wedge \psi \approx \varphi$) is valid in $\mathbb{K}$. Such a partial order relation on each $\mathbf{A} \in \mathbb{K}$ enjoys certain (meta-) properties that need to be mirrored by the logical calculus. Indeed, every order-preserving logic $\vdash^{\leq}_{\mathbb{K}}$ is self-extensional (see Section 3); moreover, observe that, if $\mathbb{K} \vDash \varphi \leq \psi$, then $\mathbb{K} \vDash {\sim}\psi \leq {\sim}\varphi$, but also $\mathbb{K} \vDash \varphi \vee \gamma \leq \psi \vee \gamma$ for every $\gamma \in Fm$, and so on.

In [16], the above meta-properties are imposed by adding suitable meta-rule schemata such as the one mentioned earlier (from $\langle \varphi, \psi \rangle$ infer $\langle {\sim}\psi, {\sim}\varphi \rangle$). As is well known, pure Hilbert-style calculi (*stricto sensu*) lack the expressive power needed to directly impose such (meta-)properties, which is one of the reasons of interest in more expressive (e.g. Gentzen-style) calculi. Hilbert-style rules, however, are a most useful tool in characterizing the sets of formulas that are closed with respect to the derivability relation of a given logic; thus the first step in the study of algebraic models of a logic is in most cases a proof of completeness with respect to some Hilbert-style calculus (see e.g. [14, p. 414]). Indeed, the whole theory of the algebraization of logics pioneered by H. Rasiowa [28] and perfected by W. Blok and D. Pigozzi [7] relies on the strong formal analogies that exist between the equational consequence of abstract algebras and deductive systems presented via Hilbert-style calculi.

A first approach to the above-mentioned axiomatizability problem suggests the following strategy. Take a finite set of Hilbert-style rule schemata $\mathcal{R} \subset \vdash^{\leq}_{\mathbb{K}}$ and recursively close it as follows: whenever $\langle \varphi, \psi \rangle \in \mathcal{R}$, add to $\mathcal{R}$ also $\langle {\sim}\psi, {\sim}\varphi \rangle$, $\langle \varphi \vee \gamma, \psi \vee \gamma \rangle$, etc. As Theorem 3.3 shows, such a process may indeed allow us to show that the derivability relation $\vdash_{\mathcal{R}}$ thus obtained coincides with $\vdash^{\leq}_{\mathbb{K}}$. The non-trivial question is whether some *finite* subset $\mathcal{R}_0 \subseteq \mathcal{R}$ will also suffice or not. The main result of the present paper consists in providing a sufficient condition for the negative result to hold as well as a few conditions that are sufficient for ensuring a positive answer. As we shall see, the answer relies crucially on the possibility to add certain rule schemata (i.e. on whether the corresponding inequalities hold in $\mathbb{K}$).

We note for the algebraic logician that the logics considered in the present paper are not algebraizable in the sense of Blok and Pigozzi, and indeed they are easily shown to be non-protoalgebraic either (see e.g. [15] for the relevant definitions). This is one of the challenges of our study, for one cannot rely on the existence of the translations between equations and formulas that are provided by the general theory of algebraizable logics. Thus, in this setting, there is no standard recipe for obtaining a Hilbert-style axiomatization of a given logic from an equational presentation of the corresponding class of algebras. Also, no isomorphism is readily available between (say) the lattice of subquasivarieties of $\mathbb{DN}$ and the lattice of finitary extensions of $\vdash^{\leq}_{\mathbb{DN}}$ (but see Theorem 2.7 in Section 2).

The paper is organized as follows. Section 2 collects the fundamental definitions on algebras and logics, as well as a few useful lemmas. In Section 3 we give a recipe for obtaining a (potentially infinite) Hilbert-style axiomatization for the logic $\vdash^{\leq}_{\mathbb{K}}$ for each class $\mathbb{K} \subseteq \mathbb{DN}$. We investigate conditions entailing that the above-mentioned axiomatization must be infinite, and in particular we show that $\vdash^{\leq}_{\mathbb{DN}}$ is not finitely based; the same holds for the logic $\vdash^{\leq}_{\mathbb{O}}$ of the variety of all Ockham algebras (Definition 2.3). By contrast, we show in Section 4 that, for an arbitrary variety $\mathbb{K} \subseteq \mathbb{SDM}$, where $\mathbb{SDM}$ is the class of semi-De Morgan algebras (Definition 2.3), the logic $\vdash^{\leq}_{\mathbb{K}}$ is finitely based if and only if $\mathbb{K}$ is axiomatized by a finite number of equations (in particular, $\vdash^{\leq}_{\mathbb{SDM}}$ is itself finitely based). In Section 5 we adapt our proof techniques to show that, unlike the whole variety $\mathbb{O}$ of Ockham algebras, every Berman subvariety $\mathbb{O}^m_n \subseteq \mathbb{O}$ determines a logic $\vdash^{\leq}_{\mathbb{O}^m_n}$ that is finitely based. In Section 6 we briefly consider $\top$-assertional logics associated to varieties of distributive lattices with negation, showing in particular that $\vdash^{\top}_{\mathbb{SDM}}$ is finitely based. Lastly, Section 7 contains some concluding remarks and suggestions for further research.

## 2 Algebraic and Logical Preliminaries

### *2.1 Algebras*

We adopt the standard conventions and notation of modern universal algebra, for which we refer the reader to [8]. All algebras considered in the present paper are bounded (distributive) lattices (Definition 2.1) enriched with a unary *negation* operation $\sim$ on which different requirements will be imposed, giving rise to the various classes of algebras of interest. The algebraic (as well as the logical) language $\{\wedge, \vee, \sim, \bot, \top\}$, consisting of a conjunction (interpreted as the lattice meet on algebras), a disjunction (the join), a negation and truth constants (the top and bottom of the lattice) will stay fixed throughout the paper. We shall denote by **Fm** the algebra of formulas over this language, freely generated by a denumerable set of variables *Var* (denoted $x, y, z$ etc.; we shall instead use $p, q, r$ etc. for propositional variables appearing further on in the definition and axiomatization of logics), and by *Fm* the universe of this algebra. We shall mostly be interested in equational classes of algebras (since Tarski's classic characterization [8, Thm. II.9.5], also known as *varieties*). An *equation* is a pair of formulas $\langle\varphi, \psi\rangle \in Fm \times Fm$, and every set $\mathsf{E} \subseteq Fm \times Fm$ of equations determines a variety which will be denoted by $\mathbb{V}_{\mathsf{E}}$.

Definition 2.1 ([8]).
A *bounded distributive lattice* is an algebra $\mathbf{A} = \langle A; \wedge, \vee, \bot, \top\rangle$ of type $\langle 2, 2, 0, 0\rangle$ such that the following equations are satisfied:

(L1) $x \vee y = y \vee x \qquad x \wedge y = y \wedge x.$
(L2) $x \vee (y \vee z) = (x \vee y) \vee z \qquad x \wedge (y \wedge z) = (x \wedge y) \wedge z.$
(L3) $x \vee x = x \qquad x \wedge x = x.$
(L4) $x \vee (x \wedge y) = x \qquad x \wedge (x \vee y) = x.$
(L5) $x \wedge \bot = \bot \qquad x \vee \top = \top.$
(L6) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$

Definition 2.2 ([12, 13]).
A *distributive lattice with negation* is an algebra $\mathbf{A} = \langle A; \wedge, \vee, \sim, \bot, \top\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that $\langle A; \wedge, \vee, \bot, \top\rangle$ is a bounded distributive lattice (Definition 2.1) and the following equations are satisfied:

(N1) $\sim\bot = \top.$
(N2) $\sim(x \vee y) = \sim x \wedge \sim y.$

We shall denote by $\mathbb{DN}$ the variety of distributive lattices with negation, and by $\mathsf{DN}$ the set of equations axiomatizing this class according to Definition 2.2.

The choice of the class of distributive lattices with negation as our base variety is due to the following reasons. On the one hand, $\mathbb{DN}$ is sufficiently general to include many algebras of non-classical logics that interest us, in particular pseudo-complemented distributive lattices and semi-De Morgan algebras (our original case study). On the other hand, the two items of Definition 2.2 are some minimal equational requirements ensuring that the connective $\sim$ indeed behaves like a negation (in particular, $\sim$ is order-reversing); also, the theory of $\mathbb{DN}$ is sufficiently well developed to allow us to rely on a few algebraic lemmas. Besides $\mathbb{DN}$, we shall be mainly working with the subvarieties introduced below.

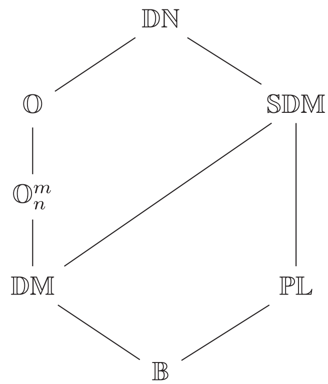


FIGURE 1. Varieties of distributive lattices with negation, ordered by inclusion.

DEFINITION 2.3 ([33]).
A distributive lattice with negation $\mathbf{A} = \langle A; \wedge, \vee, \sim, \bot, \top \rangle$ is:

- a *semi-De Morgan algebra*, if $\mathbf{A}$ satisfies the following equations:

(SDM1) $\sim\top = \bot$.
(SDM2) $\sim\sim(x \wedge y) = \sim\sim x \wedge \sim\sim y$.
(SDM3) $\sim x = \sim\sim\sim x$.

- a *De Morgan algebra*, if $\mathbf{A}$ is a semi-De Morgan algebra satisfying:

  (DM) $\sim\sim x = x$.

- a *pseudo-complemented distributive lattice* (*p-lattice*, for short), if $\mathbf{A}$ is a semi-De Morgan algebra satisfying:

  (PL) $x \wedge \sim(x \wedge y) = x \wedge \sim y$.

- an *Ockham algebra*, if $\mathbf{A}$ satisfies (SDM1) plus the following equation:

  (O) $\sim(x \wedge y) = \sim x \vee \sim y$.

We shall denote by $\mathbb{SDM}$ the variety of distributive lattices with negation, and by $\mathsf{SDM}$ the set of equations axiomatizing this class according to Definition 2.3.

We shall also be interested in the so-called *Berman varieties* of Ockham algebras [6], defined via the following terms. Let $\sim^0 x := x$ and $\sim^{n+1} x := \sim\sim^n x$. For $m \geq 1$ and $n \geq 0$, the variety $\mathbb{O}_n^m$ is defined as the subclass of those Ockham algebras that satisfy the equation $\sim^{2m+n} x = \sim^n x$. The class of Boolean algebras, viewed as a subvariety of $\mathbb{DN}$, will be denoted by $\mathbb{B}$; also recall from the preceding Section that $\mathbb{SDM}$, $\mathbb{DM}$ and $\mathbb{PL}$ denote, respectively, the variety of semi-De Morgan algebras, De Morgan algebras and $p$-lattices. The following inclusions (all proper) hold among the above-defined varieties: $\mathbb{B} \subseteq \mathbb{DM} \subseteq \mathbb{SDM} \subseteq \mathbb{DN}$, $\mathbb{B} \subseteq \mathbb{DM} \subseteq \mathbb{O}_n^m \subseteq \mathbb{O} \subseteq \mathbb{DN}$ and $\mathbb{B} \subseteq \mathbb{PL} \subseteq \mathbb{SDM} \subseteq \mathbb{DN}$.

Since their introduction about three decades ago [33], semi-De Morgan algebras have been studied especially in the setting of universal algebra [25] and duality theory [12, 13, 18]. On the other hand, a logic associated to semi-De Morgan algebras (here denoted $\vdash^{\leq}_{\mathbb{SDM}}$) has been first considered only in the recent paper [16]. Having been introduced in the late 1970s, Ockham lattices are slightly older than semi-De Morgan algebras; logics associated to (Berman subvarieties of) Ockham lattices are considered in [21, 22].

De Morgan algebras (i.e. involutive semi-De Morgan algebras) are worth mentioning in the present context especially because of their logical interpretation. In fact, since the 1970s with the seminal papers by N. Belnap [4, 5], the variety $\mathbb{DM}$ has been associated to and studied as the standard semantics of the Belnap–Dunn four-valued logic (see e.g. [14]). Indeed, the consequence relation $\vdash^{\leq}_{\mathbb{DM}}$ is precisely the Belnap–Dunn logic (on the other hand, $\vdash^{\top}_{\mathbb{DM}}$ is strictly stronger than $\vdash^{\leq}_{\mathbb{DM}}$). Sub(quasi)varieties of $\mathbb{DM}$ have also been studied from a logical standpoint in the more recent papers [1, 27, 32]. From a technical point of view, we shall also be interested in exploiting the structural relation between semi-De Morgan and De Morgan algebras stated in Lemma 2.5.

The study of $p$-lattices can be traced back to the 1920s with V. Glivenko's classical work on intuitionistic logic. From a logical point of view, the importance of $p$-lattices stems from their relation with intuitionistic logic. In fact, it is well known that $p$-lattices are precisely the implication-free subreducts of Heyting algebras: in logical terms, this entails that the logic $\vdash^{\leq}_{\mathbb{PL}}$, or equivalently $\vdash^{\top}_{\mathbb{PL}}$ (both defined as in Section 1), captures the implication-free fragment of intuitionistic logic.

We end the section with a few algebraic lemmas that will be used to make sure that certain rules are sound with respect to particular subclasses of $\mathbb{DN}$.

Lemma 2.4
Let $\mathbf{A}$ be a semi-De Morgan algebra and $a, b, c \in A$. Then,

(i) $\sim(a \wedge b) = \sim(\sim\sim a \wedge b) = \sim(a \wedge \sim\sim b) = \sim(\sim\sim a \wedge \sim\sim b)$.
(ii) $\sim(\sim(\sim a \wedge b) \wedge c) \leq \sim(a \wedge c)$.

Proof. (i). See [13,Lemma 1.1].

(ii). Let $a, b, c \in A$. Observe that, by the preceding item, $\sim(a \wedge b) = \sim(\sim\sim a \wedge b)$. Since $\sim$ is order-reversing, from $\sim a \wedge b \leq \sim a$ we have $\sim\sim a \wedge c \leq \sim(\sim a \wedge b) \wedge c$ and $\sim(\sim(\sim a \wedge b) \wedge c) \leq \sim(\sim\sim a \wedge c) = \sim(a \wedge c)$. ☐

Let $\mathbf{A} = \langle A; \wedge, \vee, \sim, 0, 1\rangle$ be a semi-De Morgan algebra. Defining $A^* := \{\sim a : a \in A\}$ and $a \vee^* b := \sim\sim(a \vee b)$ for all $a, b \in A^*$, we consider the algebra $\mathbf{A}^* = \langle A^*; \wedge, \vee^*, \sim, 0, 1\rangle$. It is easy to show that $A^*$ is indeed closed under the operations $\{\wedge, \vee^*, \sim, 0, 1\}$. Moreover, we have the following result, which may be viewed as a generalization of Glivenko's theorem relating Heyting and Boolean algebras.

Lemma 2.5 ([33], Thm. 2.4).
If $\mathbf{A}$ is a semi-De Morgan algebra, then $\mathbf{A}^*$ is a De Morgan algebra.

The preceding lemma is interesting for us because of the following logical consequence. Let $\varphi$ be a formula in the language of semi-De Morgan logic. Define the formula $\varphi^*$ recursively as follows:

$$\varphi^* := \begin{cases} \sim\sim\varphi & \text{if } \varphi \in \mathit{Var} \cup \{\top\} \\ \sim\varphi_1^* & \text{if } \varphi = \sim\varphi_1 \\ \varphi_1^* \wedge \varphi_2^* & \text{if } \varphi = \varphi_1 \wedge \varphi_2 \\ \sim\sim(\varphi_1^* \vee \varphi_2^*) & \text{if } \varphi = \varphi_1 \vee \varphi_2. \end{cases}$$

Lemma 2.6
Let $\langle \varphi, \psi\rangle$ be a rule that is sound w.r.t. $\vdash^{\leq}_{\mathbb{DM}}$ (i.e. the Belnap–Dunn logic). Then $\langle \varphi^*, \psi^*\rangle$ is sound w.r.t. $\vdash^{\leq}_{\mathbb{SDM}}$.

PROOF. By contraposition, assume $\langle \varphi^*, \psi^* \rangle$ is not sound in $\vdash^{\leq}_{\mathbb{SDM}}$. Then there is a semi-De Morgan algebra $\mathbf{A}$ that witnesses the failure of the inequality $\varphi^* \leq \psi^*$. It is then easy to check that $\mathbf{A}^*$ (which is a De Morgan algebra, by Lemma 2.5) witnesses the failure of $\varphi \leq \psi$, contradicting the assumption that $\langle \varphi, \psi \rangle$ is sound w.r.t. the Belnap–Dunn logic. □

*2.2 Logics*

Here, a *logic* is a structural (Tarskian) consequence relation on $\mathbf{Fm}$, i.e. a subset of $\mathcal{P}(Fm) \times Fm$. Logics will be denoted by $\vdash$ with suitable subscripts, regardless of the way (syntactical or semantical) they are defined. A logic can, for instance, be defined through a *logical matrix*, i.e. a pair $\mathbb{M} = \langle \mathbf{A}, D \rangle$ where $\mathbf{A}$ is an algebra and $D \subseteq A$ a set of designated elements. One sets $\Gamma \vdash_{\mathbb{M}} \varphi$ iff for every homomorphism $h \colon \mathbf{Fm} \to \mathbf{A}$, we have $h(\varphi) \in D$ whenever $h(\Gamma) \subseteq D$. Similarly, a class of logical matrices defines a logic by considering the intersection of the logics defined by each member of the class. Another way is by considering a class of partially ordered algebras $\mathbb{K}$, giving rise to the order-preserving logic $\vdash^{\leq}_{\mathbb{K}}$ defined in the Introduction. Indeed, for a class $\mathbb{K}$ of lattice-ordered algebras, $\vdash^{\leq}_{\mathbb{K}}$ is the logic defined by the class of all matrices $\langle \mathbf{A}, D \rangle$ such that $\mathbf{A} \in \mathbb{K}$ and $D$ is a lattice filter of $\mathbf{A}$.

We shall also be interested in logics defined through Hilbert-style calculi consisting of a finite or denumerable set of rule schemata. By a *Hilbert-style rule* we mean a pair $\langle \Gamma, \psi \rangle$, usually denoted $\frac{\Gamma}{\psi}$, where $\Gamma \cup \psi \subseteq Fm$. When $\Gamma$ is a singleton (say, $\Gamma = \{\varphi\}$ for some $\varphi \in Fm$), we speak of a *formula-to-formula* rule, usually written $\frac{\varphi}{\psi}$. We shall write $\frac{\varphi}{\overline{\psi}}$ to denote the 'bidirectional rule', which is really just an abbreviation for the pair of formula-to-formula rules $\{\frac{\varphi}{\psi}, \frac{\psi}{\varphi}\}$. Every set $\mathcal{R}$ of Hilbert-style rules determines a logic $\vdash_{\mathcal{R}}$ in the standard way, and we write $\Gamma \vdash_{\mathcal{R}} \varphi$ whenever there is a Hilbert-style derivation of $\varphi$ from $\Gamma$ that uses rules in $\mathcal{R}$.

Below we state formally a result that will be central to our study of the relation between order-preserving logics and varieties of distributive lattices with negation.

Recall that a logic is said to be *non-pseudo-axiomatic* if the set of its theorems is the set of formulas that are derivable from every formula [19, p. 78]. Every order-preserving logic $\vdash^{\leq}_{\mathbb{K}}$ considered in the present paper is non-pseudo-axiomatic. Moreover, since all algebras in $\mathbb{K} \subseteq \mathbb{DN}$ have a lattice reduct, $\vdash^{\leq}_{\mathbb{K}}$ is *semilattice-based* relative to $\wedge$ and $\mathbb{K}$ [19, p.76]. Therefore, we can apply [19, Thm. 3.7] to obtain the following.

THEOREM 2.7
There is a dual isomorphism between the class of all subvarieties of $\mathbb{DN}$, ordered by inclusion, and the class of logics $\vdash^{\leq}_{\mathbb{K}}$ for $\mathbb{K} \subseteq \mathbb{DN}$, ordered by extension. The isomorphism is given by $\mathbb{K} \mapsto \vdash^{\leq}_{\mathbb{K}}$.

In the present paper, we will study the problem of obtaining, from a basis $\mathsf{E}$ for the equational theory of $\mathbb{K} \subseteq \mathbb{DN}$, a set of rules that are a basis for the logic $\vdash^{\leq}_{\mathbb{K}}$; in particular, we shall be interested in conditions ensuring that such a basis may be given by a finite set of rule schemata.

## 3 The Order-Preserving Logic of $\mathbb{DN}$

In this section we introduce an infinite Hilbert-style calculus for the order-preserving logic of the variety $\mathbb{DN}$. Our calculus is obtained by translating the set $\mathsf{DN}$ of equations that axiomatize $\mathbb{DN}$ into a set $\mathcal{R}^{\mathsf{DN}}$ of bidirectional rules, which we then suitably enlarge in order to ensure that the corresponding inter-derivability relation is a congruence of $\mathbf{Fm}$. After showing that the denumerable set $\mathcal{R}_\omega$ of rules thus obtained axiomatizes $\vdash^{\leq}_{\mathbb{DN}}$ (Corollary 3.4), we will proceed to show that $\mathcal{R}_\omega$ cannot be replaced by any finite set. This is the main result of this section: *the order-preserving* *logic of* $\mathbb{DN}$ *is not finitely based* (Theorem 3.8). We note that most of the results that we proceed to prove below also hold for classes of algebras more general than $\mathbb{DN}$, and thus for logics weaker than $\vdash^{\leq}_{\mathbb{DN}}$ (for instance, Lemma 3.1 only relies on having the set of commutativity rules $\mathcal{R}_{\mathsf{C}}$ defined below, etc.). In view of future research, this suggests the project of applying our techniques to more general logics/classes of algebras.

Given a set of equations $\mathsf{E} := \{\varphi_i = \psi_i : i \in I\} \subseteq Fm \times Fm$, we define the following set of bidirectional rules:

$$\mathcal{R}^{\mathsf{E}} := \left\{ \frac{\varphi_i}{\psi_i} : i \in I \right\}.$$

Note that every rule in $\mathcal{R}^{\mathsf{E}}$ is formula-to-formula.

Following standard notation, we shall use the letters $p, q, r$ etc. to denote logical variables, rather than $x, y, z$ etc. which we have reserved for algebraic terms and equations. Thus, for instance, the equations (L1) in Definition 2.1 give us the rules $\frac{p \vee q}{q \vee p}$ and $\frac{p \wedge q}{q \wedge p}$, and so on.

Given a set of rules $\mathcal{R}$, we say that $Q \subseteq Var$ is *fresh* for $\mathcal{R}$ if none of the variables in $Q$ occurs in $\mathcal{R}$. If the cardinality of $Var \setminus Q$ is greater or equal to the one of the set of variables occurring in $\mathcal{R}$, we can rename the variables in $\mathcal{R}$ and obtain an equivalent set of rules for which $Q$ is fresh. Hence, if $Var \setminus Q$ is infinite, we can always assume $Q$ is fresh for $\mathcal{R}$. Since $Var \cup Q$ and $Var$ are both denumerable, we can fix a bijection $\mathsf{b} : Var \to Var \setminus Q$ and the set $\{r^{\sigma} : r \in \mathcal{R}\}$ with $\sigma(p) = \mathsf{b}(p)$. Clearly, $\mathcal{R}^{\sigma}$ axiomatizes the same logic as $\mathcal{R}$. The key point here is that we make $Q$ fresh without collapsing any two distinct variables occurring in $\mathcal{R}$.

Given a set $\mathcal{R} \subseteq Fm \times Fm$ of formula-to-formula rules, let $\{q_i : i < \omega\}$ be a set of fresh variables and define

$$\begin{aligned}
\mathcal{R}_0 &:= \mathcal{R} \\
\mathcal{R}_{n+1} &:= \left\{ \frac{\varphi \vee q_n}{\psi \vee q_n} : \frac{\varphi}{\psi} \in \mathcal{R}_n \right\} \cup \left\{ \frac{\varphi \wedge q_n}{\psi \wedge q_n} : \frac{\varphi}{\psi} \in \mathcal{R}_n \right\} \cup \left\{ \frac{\sim \psi}{\sim \varphi} : \frac{\varphi}{\psi} \in \mathcal{R}_n \right\} \\
\mathcal{R}_{\omega} &:= \bigcup_{n<\omega} \mathcal{R}_n.
\end{aligned}$$

Let us also fix the set $\mathcal{R}_{\mathsf{C}} = \{\frac{p \wedge q}{q \wedge p}, \frac{p \vee q}{q \vee p}\}$ and $\mathcal{R}_{\mathsf{F}} = \{\frac{}{\top}, \frac{p\,,\,q}{p \wedge q}, \frac{p}{p \vee q}\}$. As the notation suggests, the set $\mathcal{R}_{\mathsf{C}}$ ensures that the conjunction and disjunction are commutative, while the rules in $\mathcal{R}_{\mathsf{F}}$ say that the designated elements are (non-empty) lattice filters of the algebraic models of the logic.[1]

Recall that a logic $\vdash$ is said to be *self-extensional* if the inter-derivability relation $\dashv\vdash$ is a congruence of the formula algebra $\mathbf{Fm}$. Every order-preserving logic $\vdash^{\leq}_{\mathbb{K}}$ is obviously self-extensional: thus one needs to ensure that the syntactic counterpart of $\vdash^{\leq}_{\mathbb{K}}$ also enjoys this property.

LEMMA 3.1
Let $\mathcal{R} \subseteq Fm \times Fm$ be a set of formula-to-formula rules such that $\mathcal{R}_{\mathsf{C}} \subseteq \mathcal{R}$. Then the inter-derivability relation $\dashv\vdash_{\mathcal{R}_{\omega}}$ is a congruence of $\mathbf{Fm}$.

PROOF. A $\mathcal{R}_{\omega}$-derivation of $\varphi \vdash_{\mathcal{R}_{\omega}} \psi$ consists in the application of rules $\frac{\gamma_i}{\delta_i} \in \mathcal{R}_{\omega}$ and substitutions $\sigma_i : P \to Fm$ for $0 \leq i \leq n$, such that $\gamma_1^{\sigma_1} = \varphi$, $\gamma_{j+1}^{\sigma_{j+1}} = \delta_j^{\sigma_j}$ for $1 \leq j \leq n-1$, and $\delta_n^{\sigma_n} = \psi$. Since for every $0 \leq i \leq n$, $\frac{\gamma_i \vee q_i}{\delta_i \vee q_i} \in \mathcal{R}_{\omega}$ with $q_i$ not occurring in $\gamma_i$ nor $\delta_i$, we obtain that by applying rules

[1] Observe that $\mathcal{R}_{\mathsf{C}} \subseteq \mathcal{R}^{\mathsf{DN}}$, so we will not need to worry about adding $\mathcal{R}_{\mathsf{C}}$ when dealing with $\vdash^{\leq}_{\mathbb{DN}}$ and stronger logics.

$\frac{\gamma_i \vee q_i}{\delta_i \vee q_i}$ and substitution $\sigma_i'$ with $\sigma_i'(q_i) = \gamma$ and $\sigma_i'(p) = \sigma_i(p)$ for $p \neq q_i$, we obtain a derivation of $\varphi \vee \gamma \vdash_{\mathcal{R}_\omega} \psi \vee \gamma$. Similarly, from the fact that $\frac{\gamma_i \wedge q_i}{\delta_i \wedge q_i}, \frac{\sim\delta_i}{\sim\gamma_i} \in \mathcal{R}_\omega$ we obtain that $\varphi \wedge \gamma \vdash_{\mathcal{R}_\omega} \psi \wedge \gamma$ and $\sim \psi \vdash_{\mathcal{R}_\omega} \sim \varphi$.

Hence, if $\varphi_i \dashv\vdash_{\mathcal{R}_\omega} \psi_i$ we have $\varphi_1 \vee \varphi_2 \dashv\vdash_{\mathcal{R}_\omega} \psi_1 \vee \psi_2, \varphi_1 \wedge \varphi_2 \dashv\vdash_{\mathcal{R}_\omega} \psi_1 \wedge \psi_2$ and $\sim \psi_i \dashv\vdash_{\mathcal{R}_\omega} \sim \varphi_i$ for $i = 1, 2$. From $\varphi_1 \dashv\vdash_{\mathcal{R}_\omega} \psi_1$ we obtain $\varphi_1 \vee \varphi_2 \dashv\vdash_{\mathcal{R}_\omega} \psi_1 \vee \varphi_2$. And from $\varphi_2 \dashv\vdash_{\mathcal{R}_\omega} \psi_2$ and $\frac{p \vee q}{q \vee p} \in \mathcal{R}_{\mathsf{C}}$, we obtain $\psi_1 \vee \varphi_2 \dashv\vdash_{\mathcal{R}_\omega} \psi_1 \vee \psi_2$. Thus by transitivity we conclude that $\varphi_1 \vee \varphi_2 \dashv\vdash_{\mathcal{R}_\omega} \psi_1 \vee \psi_2$. The remaining cases are analogous. $\square$

The following lemma is an immediate consequence of the definition of $\mathcal{R}^{\mathsf{E}}$.

LEMMA 3.2
Let $\vdash$ be a logic over $\mathbf{Fm}$, and let $\mathsf{E} \subseteq Fm \times Fm$ a set of equations. If $\mathcal{R}^{\mathsf{E}} \subseteq \vdash$ and $\dashv\vdash$ is a congruence of $\mathbf{Fm}$, then the quotient $\mathbf{Fm}/\dashv\vdash$ satisfies all the equations in $\mathsf{E}$. In particular, if $\mathsf{DN} \subseteq \mathsf{E}$, then $\mathbf{Fm}/\dashv\vdash$ is a distributive lattice with negation (Definition 2.2) with the order given by $\vdash$, that is $[\varphi] \leq [\psi]$ whenever $\varphi \vdash \psi$.

Given a set of equations $\mathsf{E} \subseteq Fm \times Fm$, we denote by $\mathbb{V}_{\mathsf{E}}$ the variety axiomatized by $\mathsf{E}$.

THEOREM 3.3
Let $\mathsf{E} \subseteq Fm \times Fm$ be a set of equations such that $\mathsf{DN} \subseteq \mathsf{E}$. Then $\mathcal{R}^{\mathsf{E}}_\omega \cup \mathcal{R}_{\mathsf{F}}$ axiomatizes $\vdash^{\leq}_{\mathbb{V}_{\mathsf{E}}}$.

PROOF. Let $\vdash := \vdash_{\mathcal{R}^{\mathsf{E}}_\omega \cup \mathcal{R}_{\mathsf{F}}}$. It is clear that $\vdash \subseteq \vdash^{\leq}_{\mathbb{V}_{\mathsf{E}}}$. To prove completeness, assume $\Gamma \nvdash \varphi$ for some $\Gamma \cup \{\varphi\} \subseteq Fm$. By Lemma 3.1 and the fact that $R_{\mathsf{C}} \subseteq R^{\mathsf{DN}} \subseteq R^{\mathsf{E}}$, the relation $\dashv\vdash_{R^{\mathsf{E}}_\omega}$ is a congruence of $\mathbf{Fm}$, which in this proof we denote by $\equiv$. Let $\Gamma^{\vdash} = \{\varphi : \Gamma \vdash \varphi\}$ and $F := \Gamma^{\vdash}/{\equiv}$. Consider the matrix $\langle \mathbf{Fm}/{\equiv}, F\rangle$. Observe that $\vdash_{R^{\mathsf{E}}_\omega} \subseteq \vdash$ implies that $\Gamma^{\vdash}$ is compatible with $\equiv$ (i.e. $\psi \in \Gamma^{\vdash}$ and $\psi \equiv \psi'$ entail $\psi' \in \Gamma^{\vdash}$). It follows from $R^{\mathsf{E}} \subseteq \vdash$ and Lemma 3.2 that $\mathbf{Fm}/{\equiv}$ is in $\mathbb{V}_{\mathsf{E}}$. In particular, $\mathbf{Fm}/{\equiv}$ is a lattice. Thus, to show that $F$ is a non-empty lattice filter, it suffices to use the rules in $\mathcal{R}_{\mathsf{F}}$. To conclude the proof, observe that the canonical homomorphism $\pi : Fm \to Fm/{\equiv}$ is a valuation that satisfies all formulas in $\Gamma$ but not $\varphi$. $\square$

COROLLARY 3.4
$\vdash_{\mathcal{R}^{\mathsf{DN}}_\omega \cup \mathcal{R}_{\mathsf{F}}} = \vdash^{\leq}_{\mathbb{DN}}$.

With regards to the problem of axiomatization, a crucial difference between $\vdash^{\leq}_{\mathbb{DN}}$ and the logic $\vdash^{\leq}_{\mathbb{SDM}}$ of semi-De Morgan algebras considered in the next section is the fact that $\vdash^{\leq}_{\mathbb{SDM}}$ validates certain rules such as

$$\frac{\sim\sim(p \wedge q)}{\sim\sim p \wedge \sim\sim q}$$

that allow us to 'push' negations inside formulas, thus being able to work with formulas having a certain shape (a normal form as it were). Since in the axiomatization for $\vdash^{\leq}_{\mathbb{DN}}$ we cannot make use of a similar rule, it is important to have at our disposal a way of measuring the depth of nested negations in each formula. To this end we introduce the following notions.

DEFINITION 3.5
Recall that an *atomic formula* is a propositional variable or a constant belonging to our language ($\perp$ or $\top$). The $\sim$-*depth* of an occurrence of an atomic formula $\varphi$ in $\psi$ is the number of $\sim$-headed subformulas of $\psi$ with that occurrence of $\varphi$. In other words, we consider the tree representation

of $\psi$ and a leaf labelled $\varphi$ (i.e. a selected occurrence) and count the number of $\sim$-labelled nodes that are ancestors of that leaf. The $\sim$-depth of a formula $\psi$ is the maximum $\sim$-depth of the atomic subformulas of $\psi$. The $\sim$-depth of a set of rules $\mathcal{R}$ is the maximum $\sim$-depth among the formulas in $\mathcal{R}$. We say that a rule $\mathsf{r}$ is *$\sim$-balanced* if all occurrences of all variables in $\mathsf{r}$ have the same $\sim$-depth. We say that a set of rules $\mathcal{R}$ is *$\sim$-balanced* if every rule $\mathsf{r} \in \mathcal{R}$ is $\sim$-balanced.



We shall now focus on invariants of logics axiomatized by $\sim$-balanced rules having $\sim$-depth $k < \omega$. This will allow us to single out certain non-finitely based logics extending $\vdash^{\leq}_{\mathbb{DN}}$. To this end, we shall also need the following functions.

Given that the set of propositional variables *Var* and the language *Fm* are both denumerable we can fix a bijection $\mathsf{b} : Fm \to Var$. We let, for all $k < \omega$ and for all $\varphi, \psi \in Fm$, $f_k : Fm \to Fm$ be given by:

$$\begin{aligned}
f_k(\top) &:= \top \\
f_k(\bot) &:= \bot \\
f_k(p) &:= \mathsf{b}(p) \\
f_k(\varphi \wedge \psi) &:= f_k(\varphi) \wedge f_k(\psi) \\
f_k(\varphi \vee \psi) &:= f_k(\varphi) \vee f_k(\psi) \\
f_k(\sim \varphi) &:= \begin{cases} \mathsf{b}(\sim \varphi) & \text{if } k = 0 \\ \sim f_{k-1}(\varphi) & \text{if } k > 0 \end{cases}
\end{aligned}$$

Essentially, $f_k$ replaces in $\varphi$ every subformula $\psi$ occurring at $\sim$-depth $k$ whose main connective is $\sim$ by the variable $b(\psi)$. It also renames the variables $p$ by $\mathsf{b}(p)$ in order to avoid conflicts.[2] We extend $f_k$ to sets of formulas, rules and sets of rules in the expected way: $f_k(\Gamma) = \{f_k(\varphi) : \varphi \in \Gamma\}$, $f_k(\frac{\Gamma}{\varphi}) = \frac{f_k(\Gamma)}{f_k(\varphi)}$ and $f_k(\mathcal{R}) = \{f_k(\mathsf{r}) : \mathsf{r} \in \mathcal{R}\}$.

LEMMA 3.6
Let $\mathcal{R}$ be a set of rules that is $\sim$-balanced and has $\sim$-depth $k$. Then $\Gamma \vdash_{\mathcal{R}} \varphi$ implies $f_n(\Gamma) \vdash_{\mathcal{R}} f_n(\varphi)$ for every $n > k$.

PROOF. Since $n > k$, for each rule $\frac{\Delta}{\psi} \in \mathcal{R}$, we have $f_n(\Delta) = \Delta$ and $f_n(\psi) = \psi$. Further, for every substitution $\sigma$ we have $f_n(\Delta^{\sigma}) = f_n(\Delta)^{\sigma'} = \Delta^{\sigma'}$ and $f_n(\psi^{\sigma}) = f_n(\psi)^{\sigma'} = \psi^{\sigma'}$, where $\sigma'(p) = f_{n-j}(\sigma(p))$ and $j$ is the $\sim$-depth of $p$ in $\frac{\Delta}{\psi}$ (note that $\sigma'$ is well defined exactly because $\mathcal{R}$ is $\sim$-balanced). As in Lemma 3.1, we can transform every $\mathcal{R}$-derivation of $\varphi$ from $\Gamma$ into a derivation of $f_n(\varphi)$ from $f_n(\Gamma)$ with the same number of steps. Simply, in each step, $i$, where the rule $\frac{\Delta_i}{\psi_i}$ and substitution $\sigma_i$ were used, the rule $\frac{f_n(\Delta_i)}{f_n(\psi_i)}$ and substitution $\sigma'_i$ are now used. □

LEMMA 3.7
Let $\mathcal{R} \subseteq Fm \times Fm$ be $\sim$-balanced and having $\sim$-depth $k$. If $f_{n+k}(\mathcal{R}_n) \not\subseteq \vdash_{\mathcal{R}_\omega \cup \mathcal{R}_{\mathsf{F}}}$ for every $n < \omega$, then the logic $\vdash_{\mathcal{R}_\omega \cup \mathcal{R}_{\mathsf{F}}}$ is not finitely based.

PROOF. Let $\vdash_n = \vdash_{\mathcal{R}_n \cup \mathcal{R}_{\mathsf{F}}}$ and $\vdash_\omega = \vdash_{\mathcal{R}_\omega \cup \mathcal{R}_{\mathsf{F}}}$. Clearly, $\mathcal{R}_\omega = \bigcup_{n<\omega} \mathcal{R}_n$. Note that every axiomatization of a finitely axiomatizable finitary logic must contain a finite subset that already axiomatizes

[2]Note that the proof only requires that $\mathsf{b}$ injects *Var* together with negated formulas into *Var*.

the logic. Thus, it is enough to show that $\vdash_n \subsetneq \vdash_{n+1}$. It is clear that $\mathcal{R}_n \cup R_{\mathsf{F}}$ is $\sim$-balanced and with $\sim$-depth $n+k$. Hence, by Lemma 3.6, $\Gamma \vdash_n \varphi$ iff $f_{n+k}(\Gamma) \vdash_n f_{n+k}(\varphi)$. Thus, from $f_{n+k}(\mathcal{R}_{n+1}) \nsubseteq \vdash_\omega$ and $\vdash_n \subseteq \vdash_\omega$ we conclude that $\mathcal{R}_{n+1} \subseteq \vdash_{n+1}$ but $\mathcal{R}_{n+1} \nsubseteq \vdash_n$, as was required to prove. □

THEOREM 3.8
The logic $\vdash^{\leq}_{\mathbb{DN}}$ of distributive lattices with negation is not finitely based.

PROOF. Recall that $\vdash_{\mathcal{R}^{\mathsf{DN}}_\omega \cup \mathcal{R}_{\mathsf{F}}} = \vdash^{\leq}_{\mathbb{DN}}$ by Corollary 3.4. Then, the result follows directly from Lemma 3.7. Indeed, $\mathcal{R}^{\mathsf{DN}}$ is $\sim$-balanced and has $\sim$-depth 1. Moreover, for every $n < \omega$, we have $\mathsf{r} = \frac{\sim^{n+1}(p \wedge p))}{\sim^{n+1} p} \in \mathcal{R}^{\mathsf{DN}}_n$ and

$$f_n(\mathsf{r}) = \frac{f_n(\sim^{n+1}(p \wedge p))}{f_n(\sim^{n+1} p)}) = \frac{\sim^n (\mathsf{b}(\sim (p \wedge p)))}{\sim^n (\mathsf{b}(\sim p))} = \frac{\sim^n (q_{\sim(p \wedge p)})}{\sim^n (q_{\sim p})} \notin \vdash^{\leq}_{\mathbb{DN}} .$$

□

Note that the result of Theorem 3.8 holds for every strengthening of $\vdash^{\leq}_{\mathbb{DN}}$ to which Lemma 3.7 applies. In particular, let $\mathsf{E}$ be a set of equations such that $\mathcal{R}^{\mathsf{E}}$ is $\sim$-balanced and has finite $\sim$-depth. Then, for Lemma 3.7 to apply, it suffices to have $\mathbb{B} \subseteq \mathbb{V}_{\mathsf{E}} \subseteq \mathbb{DN}$. For instance, denoting by $\vdash^{\leq}_{\mathbb{O}}$ the order-preserving logic of the variety of Ockham algebras (Definition 2.3), it suffices to check that the rule $\frac{\sim(p \wedge q)}{\sim p \vee \sim q}$ is $\sim$-balanced to conclude that $\vdash^{\leq}_{\mathbb{O}}$ is not finitely based. A similar argument shows that, letting $\mathbb{K} \subseteq \mathbb{DN}$ be the variety of distributive lattices with negation axiomatized (relatively to $\mathbb{DN}$) by equations (SDM1) and (SDM2) from Definition 2.3, we have that $\vdash^{\leq}_{\mathbb{K}}$ is not finitely based.

## 4 The Logics of Semi-De Morgan Algebras and of $p$-Lattices

In this section we show that, unlike $\vdash^{\leq}_{\mathbb{DN}}$ and $\vdash^{\leq}_{\mathbb{O}}$, the order-preserving logic of semi-De Morgan algebras $\vdash^{\leq}_{\mathbb{SDM}}$ is finitely based. In fact, we are going to establish a more general result, namely that every order-preserving logic extending semi-De Morgan logic is going to be finitely based as long as the corresponding variety is (Theorem 4.5).

Let us fix throughout this section a set of equations $\mathsf{E} \supseteq \mathsf{SDM}$. By Theorem 3.3, we know that $\vdash^{\leq}_{\mathsf{E}}$ is axiomatized by $\mathcal{R}^{\mathsf{E}}_\omega \cup \mathcal{R}_{\mathsf{F}}$. The task is now to try and single out a *finite* set of rule schemata that allows us to generate the same inferences; but before we come to this, let us consider an interesting (infinite) subset of $\mathcal{R}^{\mathsf{E}}_\omega$, $(\mathcal{R}^{\mathsf{E}})_{\omega,g}$.

Let $\mathcal{R} \subseteq Fm \times Fm$ be a set of rules, and let $(\{q\} \cup \{q_i : i < \omega\})$ be a fresh set of variables. Given a formula $\gamma$, let $g_0(\gamma) = \gamma \wedge q_0$ and $g_{n+1}(\gamma) = \sim g_n(\gamma) \wedge q_n$. Given a rule $\frac{\varphi}{\psi}$ and $n < \omega$, let

$$\mathsf{r}^{\varphi,\psi}_n = \begin{cases} \frac{g_n(\varphi) \vee q}{g_n(\psi) \vee q} & \text{if } n = 2k \\ \frac{g_n(\psi) \vee q}{g_n(\varphi) \vee q} & \text{if } n = 2k+1 \end{cases}$$

For $n < \omega$, let $\mathcal{R}_{n,g} = \{\mathsf{r}^{\varphi,\psi}_n : \frac{\varphi}{\psi} \in \mathcal{R}\}$, $\mathcal{R}_{\leq n,g} = \mathcal{R} \cup \bigcup_{k \leq n} \mathcal{R}_{k,g}$ and $\mathcal{R}_{\omega,g} = \bigcup_{n<\omega} \mathcal{R}_{\leq n,g}$.

EXAMPLE 4.1
Given a rule $\frac{\varphi}{\psi}$, we have

$$\frac{(\varphi \wedge q_0) \vee q}{(\psi \wedge q_0) \vee q}\,\mathsf{r}_0^{\varphi,\psi} \qquad \frac{(\sim (\psi \wedge q_0) \wedge q_1) \vee q}{(\sim (\varphi \wedge q_0) \wedge q_1) \vee q}\,\mathsf{r}_1^{\varphi,\psi} \qquad \frac{(\sim (\sim (\varphi \wedge q_0) \wedge q_1) \wedge q_2) \vee q}{(\sim (\sim (\psi \wedge q_0) \wedge q_1) \wedge q_2) \vee q}\,\mathsf{r}_2^{\varphi,\psi}$$

The general pattern is

$$\frac{(\sim \ldots (\sim (\gamma^{\mathsf{up}_n} \wedge q_0) \wedge q_1) \ldots \wedge q_n) \vee q}{(\sim \ldots (\sim (\gamma^{\mathsf{dn}_n} \wedge q_0) \wedge q_1) \ldots \wedge q_n) \vee q}\,\mathsf{r}_n^{\varphi,\psi}$$

with

$$\gamma^{\mathsf{up}_n} = \begin{cases} \varphi & \text{for even } n \\ \psi & \text{for odd } n \end{cases} \quad \text{and} \quad \gamma^{\mathsf{dn}_n} = \begin{cases} \psi & \text{for even } n \\ \varphi & \text{for odd } n. \end{cases}$$

Clearly, $\mathcal{R}_{\omega,g} \subseteq \mathcal{R}_{\omega}$. Let us fix the set $\mathcal{R}_{\bullet}$ consisting of the following rules:

$$\frac{p}{p\vee\bot}\mathsf{r}_{\bot}^{\vee} \qquad \frac{p\vee r}{(p\wedge\top)\vee r}\mathsf{r}_{\top}^{\wedge} \qquad \frac{\sim(p\wedge\top)\vee r}{\sim p\vee r}\mathsf{r}_{\top}^{\sim}$$

$$\frac{(p\vee q)\wedge r}{(p\wedge r)\vee(q\wedge r)}\mathsf{r}_{\mathsf{dist}}^{\vee\wedge} \qquad \frac{((p_1\wedge p_2)\wedge p_3)\vee q}{(p_1\wedge(p_2\wedge p_3))\vee q}\mathsf{r}_{\mathsf{ass}_{\wedge}^{\vee}} \qquad \frac{\sim(p\vee q)}{\sim p\wedge\sim q}\mathsf{r}_{\mathsf{dm}_{\vee}^{\sim}}$$

The next lemma entails that, if the set of rules $\mathcal{R}_{\bullet}$ is sound with respect to $\vdash_{\mathcal{R}_{\omega}}$, then we only need to join it with the above-defined set $\mathcal{R}_{\omega,g}$ to obtain a complete (albeit still infinite) axiomatization of $\vdash_{\mathcal{R}_{\omega}}$.

LEMMA 4.1
If $\mathcal{R}_{\bullet} \subseteq \vdash_{\mathcal{R}_{\omega}}$ then $\vdash_{\mathcal{R}_{\omega,g}\cup\mathcal{R}_{\bullet}} = \vdash_{\mathcal{R}_{\omega}}$.

PROOF. Let $\vdash = \vdash_{\mathcal{R}_{\omega,g}}$. Since $\mathcal{R} \subseteq \mathcal{R}_{\omega,g} \subseteq \vdash_{\mathcal{R}_{\omega}}$, it is enough to show that if $\varphi \vdash \psi$, given a fresh variable $q$, we have:

(i) $\varphi \vee q \vdash \psi \vee q$ (ii) $\varphi \wedge q \vdash \psi \wedge q$ (iii) $\sim \psi \vdash \sim \varphi$.

The proof is by induction on the length of the derivation showing that $\varphi \vdash \psi$. In the base case we have simply $\varphi = \psi$, in which cases (i), (ii) and (iii) follow immediately. For the inductive step, assume $\varphi, \gamma_1, \ldots, \gamma_k, \psi$ is an $\mathcal{R}_{\omega,g}$-derivation and by induction hypothesis we have that $\varphi\vee q \vdash \gamma_k\vee q$, $\varphi \wedge q \vdash \gamma_k \wedge q$ and $\sim \gamma_k \vdash\sim \varphi$. To conclude the proof, we consider in each of the cases how to complete the derivations depending on the last rule that was used. By structurality, it is enough to show that for each rule $\frac{\varphi}{\psi} \in \mathcal{R}_{\omega,g}$ we have that (i)–(iii) hold.

Concerning the rules $\frac{\varphi}{\psi} \in \mathcal{R}$, we have

(i) $\varphi \vee q \vdash_{\mathsf{r}_{\top}^{\wedge}} (\varphi \wedge \top) \vee q \vdash_{\mathsf{r}_0^{\varphi,\psi}} (\psi \wedge \top) \vee q \vdash_{\mathsf{r}_{\top}^{\wedge}} \psi \vee q$

(ii) $\varphi \wedge q \vdash_{\mathsf{r}_{\bot}^{\vee}} (\varphi \wedge q) \vee \bot \vdash_{\mathsf{r}_0^{\varphi,\psi}} (\psi \wedge q) \vee \bot \vdash_{\mathsf{r}_{\bot}^{\vee}} \psi \wedge q$

(iii) $\sim \psi \vdash_{\mathsf{r}_{\top}^{\sim}} \sim (\psi \wedge \top) \vee \bot \vdash_{\mathsf{r}_1^{\varphi,\psi}} \sim (\varphi \wedge \top) \vee \bot \vdash_{\mathsf{r}_{\top}^{\sim}} \sim \varphi \vee \bot \vdash_{\mathsf{r}_{\bot}^{\vee}} \sim \varphi$.

Now, for each $\frac{\varphi}{\psi} \in \mathcal{R}$ and $j < \omega$, consider $\mathsf{r}_j^{\varphi,\psi} = \frac{g_j(\varphi)\vee q'}{g_j(\psi)\vee q'}$. We have

(i) $(g_j(\varphi) \vee q') \vee q \vdash_{\mathsf{r}^{\vee}_{\mathsf{ass}}} g_j(\varphi) \vee (q' \vee q) \vdash_{\mathsf{r}_j^{\varphi,\psi}} g_j(\psi) \vee (q' \vee q) \vdash_{\mathsf{r}^{\vee}_{\mathsf{ass}}} (g_j(\psi) \vee q') \vee q$

(ii) For $j > 0$ (the case $j = 0$ is analogous)

$$\begin{aligned}
(g_j(\varphi) \vee q') \wedge q &= ((\sim g_{j-1}(\varphi) \wedge q_j) \vee q') \wedge q \\
&\vdash_{\mathsf{r}^{\vee\wedge}_{\mathsf{dist}}} ((\sim g_{j-1}(\varphi) \wedge q_j) \wedge q) \vee (q' \wedge q) \\
&\vdash_{\mathsf{r}_{\mathsf{ass}^{\vee}_{\wedge}}} ((\sim g_{j-1}(\varphi) \wedge (q_j \wedge q)) \vee (q' \wedge q) \\
&\vdash_{\mathsf{r}_j^{\varphi,\psi}} ((\sim g_{j-1}(\psi) \wedge (q_j \wedge q)) \vee (q' \wedge q) \\
&\vdash_{\mathsf{r}_{\mathsf{ass}^{\vee}_{\wedge}}} ((\sim g_{j-1}(\psi) \wedge q_j) \wedge q) \vee (q' \wedge q) \\
&\vdash_{\mathsf{r}^{\vee\wedge}_{\mathsf{dist}}} (g_j(\psi) \vee q') \wedge q
\end{aligned}$$

(iii)
$$\begin{aligned}
\sim (g_j(\psi) \vee q) &\vdash_{\mathsf{r}_{\mathsf{dm}^{\sim}_{\vee}}} \sim g_j(\psi) \wedge \sim q \\
&\vdash_{\mathsf{r}^{\vee}_{\bot}} (\sim g_j(\psi) \wedge \sim q) \vee \bot \\
&\vdash_{\mathsf{r}_{j+1}^{\varphi,\psi}} (\sim g_j(\varphi) \wedge \sim q) \vee \bot \\
&\vdash_{r^{\vee}_{\bot}} \sim (g_j(\varphi) \vee q) \\
&\vdash_{\mathsf{r}_{\mathsf{dm}^{\sim}_{\vee}}} \sim (g_j(\varphi) \vee q)
\end{aligned}$$

□

Since $\mathcal{R}_\bullet \subseteq \vdash^{\leq}_{\mathbb{DN}}$, by Lemma 4.1, we have that $\mathcal{R}^{\mathsf{DN}}_{\omega,g} \cup \mathcal{R}_\bullet \cup \mathcal{R}_{\mathsf{F}}$ provides an alternative (but still infinite) Hilbert-style calculus for $\vdash^{\leq}_{\mathbb{DN}}$.

In order to obtain a finite axiomatization of $\vdash^{\leq}_{\mathbb{SDM}}$, let us fix the set $\mathcal{S}_\bullet$ consisting of the following rules ( notice that they are not $\sim$-balanced):

$$\frac{\sim\sim(p \wedge q) \vee r}{\sim\sim p \vee r}\ \mathsf{r}_{\wedge} \qquad \frac{\sim(\sim\sim p \wedge q)}{\sim(p \wedge q)}\ \mathsf{r}_{\sim} \qquad \frac{\sim(\sim p_1 \wedge p_2) \qquad \sim(\sim(p_3 \wedge p_4) \wedge p_2)}{\sim(\sim(p_1 \wedge p_4) \wedge p_2)}\ \mathsf{r}^{\sim}_{\wedge}$$

The next proposition shows that if the rules in $\mathcal{S}_\bullet$ are $\vdash_{\mathcal{R}_\omega}$-sound, then we can replace the infinite set of rules $\mathcal{R}_{\omega,g}$ by the following finite one: $\mathcal{R}_+ := \mathcal{R}_{\leq 2,g} \cup \mathcal{S}_\bullet$.

PROPOSITION 4.2
If $\mathcal{S}_\bullet \subseteq \vdash_{\mathcal{R}_{\omega,g}}$, then $\vdash_{\mathcal{R}_+} = \vdash_{\mathcal{R}_{\omega,g}}$.

PROOF. We just need to show that $\mathsf{r}_n^{\varphi,\psi} \in \vdash_{\mathcal{R}_+}$ for $n > 2$. Since $\mathsf{r}_1^{\varphi,\psi}, \mathsf{r}_2^{\varphi,\psi} \in \mathcal{R}_+$, it suffices to show that we can derive $\mathcal{R}^{\varphi,\psi}_{2n+3}, \mathcal{R}^{\varphi,\psi}_{2n+4}$ using the rules in $\mathcal{R}_{n+} := \mathcal{R}_+ \cup \{\mathsf{r}^{\varphi,\psi}_{2n+1}, \mathsf{r}^{\varphi,\psi}_{2n+2}\}$.

That is, we need to show that, for every $\frac{\varphi}{\psi} \in \mathcal{R}$ and $n < \omega$,

(i) $g_{2n+3}(\psi) \vee r \vdash_{\mathcal{R}_{+n}} g_{2n+3}(\varphi) \vee r$.
We have

$$\begin{aligned}
\gamma_0 &= g_{2n+3}(\psi) \vee r \\
&= \sim (g_{2n+2}(\psi) \wedge q_{2n+3}) \vee r \\
&= \sim (\sim (g_{2n+1}(\psi) \wedge q_{2n+2}) \wedge q_{2n+3}) \vee r \\
&= \sim (\sim (\sim (g_{2n}(\psi) \wedge q_{2n+1}) \wedge q_{2n+2}) \wedge q_{2n+3}) \vee r \\
&\vdash_{\mathsf{r}_\wedge} \sim (\sim\sim (g_{2n}(A) \wedge q_{2n+1}) \wedge q_{2n+3}) \vee r \\
&\vdash_{\mathsf{r}_\sim} \sim (g_{2n}(\psi) \wedge (q_{2n+1} \wedge q_{2n+3})) \vee r \\
&\vdash_{\mathsf{r}^{\varphi,\psi}_{2n+1}} \sim (g_{2n}(\varphi) \wedge (q_{2n+1} \wedge q_{2n+3})) \vee r \\
&\vdash_{\mathsf{r}_\sim} \sim (\sim\sim (g_{2n}(\varphi) \wedge q_{2n+1}) \wedge q_{2n+3}) = \gamma_1
\end{aligned}$$

Further,

$$\gamma_0, \gamma_1 \vdash_{\mathsf{r}^\sim_\wedge} \sim (\sim (\sim (g_{2n}(\varphi) \wedge q_{2n+1}) \wedge q_{2n+2}) \wedge q_{2n+3}) \vee r = g_{2n+3}(\varphi) \vee r.$$

Hence, $g_{2n+3}(\psi) \vee r \vdash_{\mathcal{R}_{n+}} g_{2n+3}(\varphi) \vee r$.

(ii) $g_{2n+4}(\varphi) \vee r \vdash_{\mathcal{R}_{+n}} g_{2n+4}(\psi) \vee r$.
We have

$$\begin{aligned}
\gamma_0 &= g_{2n+4}(\varphi) \vee r \\
&= \sim (g_{2n+3}(\varphi) \wedge q_{2n+4}) \vee r \\
&= \sim (\sim (g_{2n+2}(\varphi) \wedge q_{2n+3}) \wedge q_{2n+4}) \vee r \\
&= \sim (\sim (\sim (g_{2n+1}(\varphi) \wedge q_{2n+2}) \wedge q_{2n+3}) \wedge q_{2n+4}) \vee r \\
&\vdash_{\mathsf{r}_\wedge} \sim (\sim\sim (g_{2n+1}(\varphi) \wedge q_{2n+2}) \wedge q_{2n+4}) \vee r \\
&\vdash_{\mathsf{r}_\sim} \sim (g_{2n+1}(\psi) \wedge (q_{2n+2} \wedge q_{2n+4})) \vee r = \varphi_2 \\
&\vdash_{\mathsf{r}^{\varphi,\psi}_{2n+2}} \sim (g_{2n+1}(\psi) \wedge (q_{2n+2} \wedge q_{2n+4})) \vee r \\
&\vdash_{\mathsf{r}_\sim} \sim (\sim\sim (g_{2n+1}(\psi) \wedge q_{2n+2}) \wedge q_{2n+4}) = \gamma_1
\end{aligned}$$

Further,

$$\gamma_0, \gamma_1 \vdash_{\mathsf{r}^\sim_\wedge} \sim (\sim (\sim (g_{2n+1}(\psi) \wedge q_{2n+2}) \wedge q_{2n+3}) \wedge q_{2n+4}) \vee r = g_{2n+4}(\psi) \vee r.$$

Hence, $g_{2n+4}(\varphi) \vee r \vdash_{\mathcal{R}_{n+}} g_{2n+4}(\psi) \vee r$.

□

Since we are assuming $\mathsf{E} \supseteq \mathsf{SDM}$, it is clear that the rules $\mathsf{r}_\wedge$ and $\mathsf{r}_\vee$ are sound for $\vdash^\leq_{\mathsf{E}}$. In the next lemma we show that the same is true for $\mathsf{r}^\sim_\wedge$.

LEMMA 4.3
The rule $\mathsf{r}^\sim_\wedge$ is sound in $\vdash^\leq_{\mathbb{SDM}}$ (hence also in $\vdash^\leq_{\mathsf{E}}$).

PROOF. Using the semi-De Morgan equations (SDM1)–(SDM3), it is easy to show that the rule $\sim(\sim p_1 \wedge p_2) \wedge \sim(\sim(p_3 \wedge p_4) \wedge p_2) \vdash \sim(\sim(p_1 \wedge p_4) \wedge p_2)$ is sound in $\vdash^\leq_{\mathbb{SDM}}$ if and only if

$({\sim}{\sim}p_1 \vee^* {\sim}p_2) \wedge (({\sim}{\sim}p_3 \wedge {\sim}{\sim}p_4) \vee^* {\sim}p_2) \vdash ({\sim}{\sim}p_1 \wedge {\sim}{\sim}p_4) \vee^* {\sim}{\sim}p_2$ is sound in $\vdash^{\leq}_{\mathbb{SDM}}$. Letting $\varphi := (p_1 \vee {\sim}p_2) \wedge ((p_3 \wedge p_4) \vee {\sim}p_2)$ and $\psi := (p_1 \wedge p_4) \vee p_2$, the rule $\varphi \vdash \psi$ is easily seen to be sound in $\vdash^{\leq}_{\mathbb{DM}}$. Moreover, $\varphi^* = ({\sim}{\sim}p_1 \vee^* {\sim}p_2) \wedge (({\sim}{\sim}p_3 \wedge {\sim}{\sim}p_4) \vee^* {\sim}p_2)$ and $\psi^* = ({\sim}{\sim}p_1 \wedge {\sim}{\sim}p_4) \vee^* {\sim}{\sim}p_2$. The soundness of $\varphi^* \vdash \psi^*$ w.r.t. $\vdash^{\leq}_{\mathbb{SDM}}$ then follows from Lemma 2.6. □

We thus arrive at our main result regarding logics stronger than $\vdash^{\leq}_{\mathbb{SDM}}$, which gives us an effective way to convert an equational presentation $\mathsf{E} \supseteq \mathsf{SDM}$ of a subvariety of semi-De Morgan algebras into a Hilbert-style axiomatization of $\vdash^{\leq}_{\mathsf{E}}$. Recall that $\mathcal{R}^{\mathsf{E}}_{+} := \mathcal{R}^{\mathsf{E}}_{\leq 2,g} \cup \mathcal{S}_{\bullet}$.

THEOREM 4.4
If $\mathsf{E} \supseteq \mathsf{SDM}$ then $\mathcal{R}^{\mathsf{E}}_{+} \cup \mathcal{R}_{\bullet} \cup \mathcal{R}_{\mathsf{F}}$ axiomatizes $\vdash^{\leq}_{\mathbb{V}_{\mathsf{E}}}$.

PROOF. Since $\mathsf{DN} \subseteq \mathsf{SDM} \subseteq \mathsf{E}$, we know by Theorem 3.3 that $\mathcal{R}^{\mathsf{E}}_{\omega} \cup \mathcal{R}_{\mathsf{F}}$ axiomatizes $\vdash^{\leq}_{\mathbb{V}_{\mathsf{E}}}$. From $\mathcal{R}_{\bullet} \subseteq \vdash^{\leq}_{\mathbb{SDM}} \subseteq \vdash^{\leq}_{\mathbb{V}_{\mathsf{E}}}$ we obtain by Lemma 4.1 that $\mathcal{R}^{\mathsf{E}}_{\omega,g} \cup \mathcal{R}_{\bullet}$ axiomatizes $\vdash^{\leq}_{\mathbb{V}_{\mathsf{E}}}$. Moreover, $\mathcal{S}_{\bullet} \subseteq \vdash^{\leq}_{\mathbb{SDM}} \subseteq \vdash^{\leq}_{\mathbb{V}_{\mathsf{E}}}$ (Lemma 4.3 deals with the less obvious case). Hence, by Proposition 4.2 we conclude that $\mathcal{R}^{\mathsf{E}}_{+} \cup \mathcal{R}_{\bullet} \cup \mathcal{R}_{\mathsf{F}}$ axiomatizes $\vdash^{\leq}_{\mathbb{V}_{\mathsf{E}}}$. □

EXAMPLE 4.6
By Theorem 4.4, the set $\mathcal{R}^{\mathsf{SDM}}_{+} \cup \mathcal{R}_{\bullet} \cup \mathcal{R}_{\mathsf{F}} = \mathcal{R}^{\mathsf{SDM}}_{\leq 2,g} \cup \mathcal{R}_{\bullet} \cup \mathcal{S}_{\bullet} \cup \mathcal{R}_{\mathsf{F}}$ axiomatizes $\vdash^{\leq}_{\mathbb{SDM}}$. Since $\mathcal{R}^{\mathsf{SDM}}_{\leq 2,g} = \mathcal{R}^{\mathsf{SDM}} \cup \mathcal{R}^{\mathsf{SDM}}_{0,g} \cup \mathcal{R}^{\mathsf{SDM}}_{1,g} \cup \mathcal{R}^{\mathsf{SDM}}_{2,g}$, the axiomatization thus obtained consists of $4 \times |\mathsf{SDM}| + |\mathcal{R}_{\bullet} \cup \mathcal{S}_{\bullet} \cup \mathcal{R}_{\mathsf{F}}| = (4 \times 16) + 11 = 75$ rules, many of which are bidirectional. However, it is not hard to see that $\mathcal{R} \subseteq \vdash_{\mathcal{R}^{\mathsf{SDM}}_{0,g} \cup \mathcal{R}_{\bullet}}$, which allows one to reduce the number of rules to $(3 \times |\mathsf{SDM}|) + 11 = 59$. Further simplifications are of course possible, and in particular cases one may obtain a much more compact axiomatization. A certain amount of redundancy in the set of rules obtained is the price we have to pay for the generality and modularity of our approach. Regarding the latter aspect, observe for instance that the variety of pseudo-complemented distributive lattices $\mathbb{PL}$ is axiomatized, relatively to $\mathbb{SDM}$, by adding the equation $x \wedge {\sim}(x \wedge y) = x \wedge {\sim}y$. Adding the rule $\frac{p \wedge {\sim}(p \wedge q)}{p \wedge {\sim}q}\ \mathsf{r_P}$ is not sufficient, for we also need to ensure that the resulting logic is self-extensional. To achieve this, by Theorem 4.4, it is enough to add the three rules in $\{r_P\}_{\leq 2,g} = \{\mathsf{r}^{\varphi,\psi}_i : 0 \leq i \leq 2\}$, where $\varphi = p \wedge {\sim}(p \wedge q)$ and $\psi = p \wedge {\sim}q$. We then have that $\vdash^{\leq}_{\mathbb{PL}}$ is axiomatized over $\vdash^{\leq}_{\mathbb{SDM}}$ by $\{\mathsf{r}^{\varphi,\psi}_0, \mathsf{r}^{\varphi,\psi}_1, \mathsf{r}^{\varphi,\psi}_2\}$.

Theorem 4.4 also provides a means to obtain (alternative) finite axiomatizations of other order-preserving logics above $\vdash^{\leq}_{\mathbb{SDM}}$. In particular, we can obtain a finite axiomatization of the logic of $p$-lattices (Definition 2.3), i.e. the implication-free fragment of intuitionistic logic, that is alternative to the one introduced in [31]. We provide a general formulation of this observation below in Theorem 4.5.

THEOREM 4.5
Let $\mathbb{V} \subseteq \mathbb{SDM}$ be a variety. The following are equivalent:

(i) $\mathbb{V}$ is axiomatized by a finite set of equations.
(ii) $\vdash^{\leq}_{\mathbb{V}}$ is axiomatized by a finite set of finitary rule schemata.

PROOF. That (i) implies (ii) follows directly from Theorem 4.4.

For the other direction, assume (ii) holds, so $\vdash^{\leq}_{\mathbb{V}}$ is axiomatized by a finite set $\mathcal{R}$ of finitary rule schemata. Given a rule $\mathsf{r} = \frac{\Gamma}{\varphi}$, let $\mathsf{E}(\mathsf{r})$ be the equation $\bigwedge \Gamma \wedge \varphi = \bigwedge \Gamma$. Let $\mathsf{E}_{\mathcal{R}} := \mathsf{SDM} \cup \{\mathsf{E}(\mathsf{r}) : \mathsf{r} \in \mathcal{R}\}$. Observe that the set $\mathsf{E}_{\mathcal{R}}$ is finite, and let $\mathbb{V}'$ be the variety defined by the equations $\mathsf{E}_{\mathcal{R}}$. We claim that $\mathbb{V}' = \mathbb{V}$. Indeed, it is clear that $\mathbb{V} \subseteq \mathbb{V}' \subseteq \mathbb{SDM}$ and therefore $\vdash^{\leq}_{\mathbb{SDM}} \subseteq \vdash^{\leq}_{\mathbb{V}'} \subseteq \vdash^{\leq}_{\mathbb{V}}$. For the other direction, we start by observing that for each $\frac{\varphi}{\psi} \in \mathcal{R}$ we have $\frac{\bigwedge \Gamma \wedge \varphi}{\bigwedge \Gamma} \in \mathcal{R}^{\mathsf{E}_{\mathcal{R}}}$. This, together with the fact that $\frac{p\,,\,q}{p \wedge q}, \frac{p \wedge q}{q} \in \vdash^{\leq}_{\mathbb{SDM}} \subseteq \vdash^{\leq}_{\mathbb{V}'}$, implies that $\mathcal{R} \subseteq \vdash^{\leq}_{\mathbb{V}'}$. Hence, $\vdash^{\leq}_{\mathbb{V}} \subseteq \vdash^{\leq}_{\mathbb{V}'}$. From, $\vdash^{\leq}_{\mathbb{V}'} = \vdash^{\leq}_{\mathbb{V}}$ and Theorem 2.7 we conclude that $\mathbb{V} = \mathbb{V}'$. □

## 5 Order-Preserving Logics of Berman Varieties

We have shown in Section 4 how to obtain a finite axiomatization of the order-preserving logic $\vdash^{\leq}_{\mathbb{K}}$ with $\mathbb{K} \subseteq \mathbb{SDM}$. Now, suppose $\mathbb{K} \subseteq \mathbb{O}$ is a variety of Ockham algebras. As observed earlier, $\vdash^{\leq}_{\mathbb{O}}$ is not finitely based. However, if we restrict our attention to a Berman variety $\mathbb{O}^m_n$ of Ockham algebras, then we can adapt the technique employed in the preceding section to obtain a finite Hilbert-style axiomatization for $\vdash^{\leq}_{\mathbb{O}^m_n}$ (an infinite one being directly given by Theorem 3.3).

From now on, let us fix a variety $\mathbb{O}^m_n$, with $m, n < \omega$, and let $\mathsf{E}^m_n$ be the equations axiomatizing $\mathbb{O}^m_n$. Let $k < \omega$ and let $t$ be a fresh variable. Given a rule $\frac{\varphi}{\psi}$, define $\mathsf{s}^{\varphi,\psi}_{2k} := \frac{\sim^{2k}(\varphi) \vee t}{\sim^{2k}(\psi) \vee t}$ and $\mathsf{s}^{\varphi,\psi}_{2k+1} := \frac{\sim^{2k+1}(\psi) \vee t}{\sim^{2k+1}(\varphi) \vee t}$. Letting

$$\frac{p \wedge q}{p}\,\mathsf{r}^1_{\wedge} \qquad \frac{p \wedge q}{q}\,\mathsf{r}^2_{\wedge} \qquad \frac{p\,,\,q}{p \wedge q}\,\mathsf{r}^{\mathsf{in}}_{\wedge}$$

define $\mathcal{R}^{mn}_{\wedge} := \mathcal{R}^{\mathsf{E}^m_n} \cup \{\mathsf{r}^1_{\wedge}, \mathsf{r}^2_{\wedge}\}$, and $\mathcal{O}^m_n := \mathcal{R}^{mn}_{\wedge} \cup \{\mathsf{s}_i(\mathsf{r}) : i \leq 2m+n, \mathsf{r} \in \mathcal{R}^{mn}_{\wedge}\} \cup \{\mathsf{r}^{\mathsf{in}}_{\wedge}\}$.

LEMMA 5.1
The relation $\dashv\vdash_{\mathcal{O}^m_n}$ is a congruence of $\mathbf{Fm}$.

PROOF. The key difference with the cases considered in the previous Sections is that $\frac{\sim(p \wedge q)}{\sim p \vee \sim q}\,\mathsf{r}^{\wedge\vee}_{\mathsf{dm}} \in \mathcal{R}^{mn}_{\wedge}$, but recall that the following rules are also in $\mathcal{R}^{mn}_{\wedge}$:

$$\frac{(p \wedge q) \wedge r}{p \wedge (q \wedge r)}\,\mathsf{r}^{\wedge}_{\mathsf{ass}} \qquad \frac{(p \vee q) \vee r}{p \vee (q \vee r)}\,\mathsf{r}^{\vee}_{\mathsf{ass}} \qquad \frac{\sim^{2m+n} p}{\sim^n p}\,\mathsf{r}^m_n \qquad \frac{p \vee (q \wedge r)}{(p \vee q) \wedge (p \vee r)}\,\mathsf{r}^{\vee\wedge}_{\mathsf{dist}}$$

In the presence of $\mathsf{r}^{\wedge\vee}_{\mathsf{dm}}$, we can show directly that, if $\Gamma \vdash \varphi$, then

$$\text{(i) } \Gamma \vee u \vdash \varphi \vee u \qquad \text{(ii) } \Gamma \wedge u \vdash \varphi \wedge u \qquad \text{(iii) } \sim \varphi \vdash \bigvee_{\gamma \in \Gamma} \sim \gamma$$

where $u$ is a fresh variable, $\Gamma \vee u := \{\gamma \vee u : \gamma \in \Gamma\}$ and $\Gamma \wedge u := \{\gamma \wedge u : \gamma \in \Gamma\}$. Once more it is enough to show that (i)–(iii) are satisfied when $\mathsf{r} = \frac{\Gamma}{\varphi} \in \mathcal{O}^m_n$.

For $\mathsf{r} = \frac{p\,,\,q}{p \wedge q}$, we have

(i) $p \vee u, q \vee u \vdash_{\mathsf{r}} (p \vee u) \wedge (q \vee u) \vdash_{\mathsf{r}^{\vee\wedge}_{\mathsf{dist}}} (p \wedge q) \vee u$
(ii) $p \wedge u, q \wedge u \vdash_{\mathsf{r}} (p \wedge u) \wedge (q \wedge u) \vdash_{\mathsf{r}^{\wedge}_{\mathsf{ass}}} (p \wedge q) \wedge u$
(iii) $\sim (p \wedge q) \vdash_{\mathsf{r}^{\wedge\vee}_{\mathsf{dm}}} \sim p \vee \sim q$.

For $\mathsf{r} = \frac{\varphi}{\psi} \in \mathcal{R}^{mn}_{\wedge}$, we have

(i) $\varphi \vee u \vdash_{\mathsf{s}_0^{\varphi,\psi}} \psi \vee u$

(ii) $\varphi \wedge u \vdash_{\mathsf{r}^{j}_{\wedge}} \varphi, u \vdash_{\mathsf{r}} \psi, u \vdash_{\mathsf{r}^{\mathsf{in}}_{\wedge}} \psi \wedge u$

(iii) $\sim \psi \vdash_{\mathsf{r}^{\vee}_{\perp}} \sim \psi \vee \perp \vdash_{\mathsf{s}_1^{\varphi,\psi}} \sim \varphi \vee \perp \vdash_{\mathsf{r}^{\vee}_{\perp}} \sim \varphi$.

For $\mathsf{s}_k^{\varphi,\psi} \in \{\mathsf{s}_i(\mathsf{r}) : i \leq 2m+n, \mathsf{r} \in \mathcal{R}^{mn}_{\wedge}\}$, we let $\gamma_k^{\mathsf{up}} = \varphi$ and $\gamma_k^{\mathsf{dn}} = \psi$ if $k$ is odd, and $\gamma_k^{\mathsf{up}} = \varphi$ and $\gamma_k^{\mathsf{dn}} = \psi$ if $k$ is even. We can write $\mathsf{s}_k^{\varphi,\psi} = \frac{\sim^k(\gamma_k^{\mathsf{up}})\vee q}{\sim^k(\gamma_k^{\mathsf{dn}})\vee q}$, as in Example 4.1.

(i) $(\sim^k (\gamma_k^{\mathsf{up}}) \vee q) \vee r \vdash_{\mathsf{r}^{\vee}_{\mathsf{ass}}} \sim^k (\gamma_k^{\mathsf{up}}) \vee (q \vee r) \vdash_{\mathsf{s}_k^{\varphi,\psi}} \sim^k (\gamma_k^{\mathsf{dn}}) \vee (q \vee r) \vdash (\sim^k (\gamma_k^{\mathsf{dn}}) \vee q) \vee r$

(ii) $(\sim^k (\gamma_k^{\mathsf{up}}) \vee q) \wedge r \vdash_{\mathsf{r}^{j}_{\wedge}} (\sim^k (\gamma_k^{\mathsf{up}}) \vee q), r \vdash_{\mathsf{s}_k^{\varphi,\psi}} (\sim^k (\gamma_k^{\mathsf{dn}}) \vee q), r \vdash (\sim^k (\gamma_k^{\mathsf{dn}}) \vee q) \wedge r$

(iii) To show that $\sim (\sim^k (\gamma_k^{\mathsf{dn}}) \vee q) \vdash \sim (\sim^k (\gamma_n^{\mathsf{up}}) \vee q)$ we must consider two cases. If $k+1 \leq 2m+n$, then

$$\sim (\sim^k (\gamma_k^{\mathsf{dn}}) \vee q) \vdash_{\mathsf{r}^{\wedge\vee}_{\mathsf{dm}}} \sim^{k+1} (\gamma_k^{\mathsf{dn}}) \vee \sim q \vdash_{\mathsf{s}_{k+1}^{\varphi,\psi}} \sim^{k+1} (\gamma_k^{\mathsf{up}}) \vee \sim q \vdash_{\mathsf{r}^{\wedge\vee}_{\mathsf{dm}}} \sim (\sim^k (\gamma_k^{\mathsf{dn}}) \vee q).$$

Otherwise, let $k+1 = n + (i2m + j)$ for $i > 0$ and $0 \leq j < 2m$ (and thus $n+j < 2m+n$). We have

$$\begin{aligned} \sim (\sim^k (\gamma_k^{\mathsf{dn}}) \vee q) &\vdash_{\mathsf{r}^{\wedge\vee}_{\mathsf{dm}}} \sim^{k+1} (\gamma_k^{\mathsf{dn}}) \wedge \sim q \\ &\vdash_{\mathsf{r}^{m}_{n}} \sim^{n+j} (\gamma_k^{\mathsf{dn}}) \wedge \sim q \\ &\vdash_{\mathsf{s}_{k+1}^{\varphi,\psi}} \sim^{n+j} (\gamma_k^{\mathsf{up}}) \wedge \sim q \\ &\vdash_{\mathsf{r}^{m}_{n}} \sim^{k+1} (\gamma_k^{\mathsf{up}}) \wedge \sim q \\ &\vdash_{\mathsf{r}^{\wedge\vee}_{\mathsf{dm}}} \sim (\sim^k (\gamma_k^{\mathsf{dn}}) \vee q) \end{aligned}$$

□

THEOREM 5.2
The set of rules $\mathcal{O}^m_n \cup \{\frac{}{\top}\}$ axiomatizes $\vdash^{\leq}_{\mathbb{O}^m_n}$

PROOF. It is clear that $\mathcal{O}^m_n \cup \{\frac{}{\top}\} \subseteq \vdash^{\leq}_{\mathbb{O}^m_n}$. Completeness follows by a similar reasoning, as in Theorem 3.3 from Lemma 5.1. □

We note that, given $\mathsf{E}$ such that $\mathbb{V}^{\mathsf{E}} \subseteq \mathbb{O}^m_n$, it is easy to see that $\vdash^{\leq}_{\mathbb{V}^{\mathsf{E}}}$ is axiomatized, relatively to $\vdash^{\leq}_{\mathbb{O}^m_n}$, by the set $\mathcal{R}^{\mathsf{E}} \cup \{\mathsf{s}_i(\mathsf{r}) : i \leq 2m+n, \mathsf{r} \in \mathcal{R}^{\mathsf{E}}\}$. Hence, if $\mathsf{E}$ is finite then $\vdash^{\leq}_{\mathbb{V}^{\mathsf{E}}}$ is finitely based. In fact, one can easily adapt the argument of Theorem 4.5 to obtain the following:

COROLLARY 5.3
Let $\mathbb{V} \subseteq \mathbb{O}^m_n$ be a variety. The following are equivalent:

(i) $\mathbb{V}$ is axiomatized by a finite set of equations.

(ii) $\vdash^{\leq}_{\mathbb{V}}$ is axiomatized by a finite set of finitary rule schemata.

## 6 On $\top$-Assertional Logics

As mentioned earlier, another logic (alternative to $\vdash^{\leq}_{\mathbb{K}}$) canonically associated to a given class $\mathbb{K}$ of algebras having a constant $\top$ is the so-called $\top$-assertional logic $\vdash^{\top}_{\mathbb{K}}$ determined by the class of all matrices $\{\langle \mathbf{A}, \{\top\}\rangle : \mathbf{A} \in \mathbb{K}\}$. By definition, $\vdash^{\top}_{\mathbb{K}}$ is stronger than $\vdash^{\leq}_{\mathbb{K}}$, but it is well known that $\vdash^{\top}_{\mathbb{K}} = \vdash^{\leq}_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{B}$ or $\mathbb{K} = \mathbb{PL}$. On the other hand, it is easy to check that $\vdash^{\top}_{\mathbb{DN}} \neq \vdash^{\leq}_{\mathbb{DN}}$. For this, it suffices to observe that the rule

$$\frac{p \wedge \sim p}{\sim q}\ \mathsf{r_{wxc}}$$

is sound w.r.t. $\vdash^{\top}_{\mathbb{DN}}$ but not w.r.t. $\vdash^{\leq}_{\mathbb{DN}}$. The same example witnesses $\vdash^{\top}_{\mathbb{SDM}} \neq \vdash^{\leq}_{\mathbb{SDM}}$ and $\vdash^{\top}_{\mathbb{O}} \neq \vdash^{\leq}_{\mathbb{O}}$.

In this section we take a closer look at the assertional logic $\vdash^{\top}_{\mathbb{SDM}}$ from an algebraic logic point of view. This perspective will allow us to obtain further information on the poset of finitary self-extensional extensions of $\vdash^{\leq}_{\mathbb{SDM}}$, as well as to provide a Hilbert-style calculus for $\vdash^{\top}_{\mathbb{SDM}}$. For all unexplained terminology used in this section, we refer the reader to [15].

As mentioned in the Introduction, all logics considered in this paper are non-protoalgebraic. We state this formally below.

THEOREM 6.1
Let $\mathbb{K} \subseteq \mathbb{DN}$. If $\mathbb{PL} \subseteq \mathbb{K}$ or $\mathbb{DM} \subseteq \mathbb{K}$, then $\vdash^{\top}_{\mathbb{K}}$ (and, a fortiori, $\vdash^{\leq}_{\mathbb{K}}$) is not protoalgebraic.

PROOF. Observe that both $\vdash^{\top}_{\mathbb{PL}}$ and $\vdash^{\top}_{\mathbb{DM}}$ are non-protoalgebraic. The former was remarked in [30, p. 320], while the latter is proved in [1, Thm. 5.1]. The result then follows from the observation that the property of being protoalgebraic is preserved by extensions. (Indeed, we notice that [1, Thm. 5.1] even entails that $\vdash^{\top}_{\mathbb{K}}$ is not protoalgebraic for every $\mathbb{K}$ with $\mathbb{B} \subsetneq \mathbb{K} \subseteq \mathbb{DM}$.) ☐

We next provide a better description of reduced matrix models of $\vdash^{\leq}_{\mathbb{SDM}}$. Recall that a matrix $\mathbb{M}$ is a *model* of a logic $\vdash$ when $\vdash \subseteq \vdash_{\mathbb{M}}$. The *Leibniz congruence* $\boldsymbol{\Omega}_{\mathbf{A}}(D)$ of a matrix $\mathbb{M} = \langle \mathbf{A}, D\rangle$ is the largest congruence of $\mathbf{A}$ that is compatible with $D$ in the following sense: for all $a, b \in A$, if $a \in D$ and $\langle a, b\rangle \in \boldsymbol{\Omega}_{\mathbf{A}}(D)$, then $b \in D$. A matrix $\mathbb{M} = \langle \mathbf{A}, D\rangle$ is *reduced* when $\boldsymbol{\Omega}_{\mathbf{A}}(D)$ is the identity relation.

PROPOSITION 6.2
Let $\mathbb{M} = \langle \mathbf{A}, D\rangle$ be a model of $\vdash^{\leq}_{\mathbb{SDM}}$ with $\mathbf{A} \in \mathbb{SDM}$, and let $a, b \in A$. Then $\langle a, b\rangle \in \boldsymbol{\Omega}_{\mathbf{A}}(D)$ if and only if, for all $c_1, c_2, c_3 \in A$, the following conditions hold:

(i) $a \vee c_1 \in D$ iff $b \vee c_1 \in D$,
(ii) $\sim(a \wedge c_2) \vee c_1 \in D$ iff $\sim(b \wedge c_2) \vee c_1 \in D$,
(iii) $\sim(\sim(a \wedge c_3) \wedge c_2) \vee c_1 \in D$ iff $\sim(\sim(b \wedge c_3) \wedge c_2) \vee c_1 \in D$.

PROOF. Let $\theta$ be the relation defined by items (i)–(iii). Let us check that $\theta$ is compatible with the algebraic operations of $\mathbf{A}$.

($\sim$). Assume $\langle a, b\rangle \in \theta$. That $\sim a \vee c_1 \in D$ iff $\sim b \vee c_1 \in D$ follows from (ii): observe that, taking $c_2 = \top$, we have $\sim a \vee c_1 = \sim(a \wedge \top) \vee c_1$ and $\sim b \vee c_1 = \sim(b \wedge \top) \vee c_1$. A similar reasoning, taking $c_3 = \top$ in (iii), shows that $\sim(\sim a \wedge c_2) \vee c_1 \in D$ iff $\sim(\sim b \wedge c_2) \vee c_1 \in D$. Now, assume $\sim(\sim(\sim a \wedge c_3) \wedge c_2) \vee c_1 \in D$. By Lemma 2.4.ii, we have $\sim(\sim(\sim a \wedge c_3) \wedge c_2) \vee c_1 \leq \sim(\sim\sim a \wedge c_2) \vee c_1 = \sim(a \wedge c_2) \vee c_1$. Hence, $\sim(a \wedge c_2) \vee c_1 \in D$, and we can apply (ii) to obtain $\sim(b \wedge c_2) \vee c_1 = \sim(\sim\sim b \wedge c_2) \vee c_1 \in D$. Thus we have $(\sim(\sim\sim b \wedge c_2) \vee c_1) \wedge (\sim(\sim(\sim a \wedge c_3) \wedge$

$c_2) \vee c_1) \in D$, because $D$ is closed under $\wedge$. Then, taking $p_1 = \sim b$, $p_2 = c_2$, $q = c_1$, $p_3 = \sim a$, $p_4 = c_3$ in $\mathsf{r}^{\sim}_{\wedge}$, we have $\sim(\sim(\sim b \wedge c_3) \wedge c_2) \vee c_1 \in D$.

To check that $\theta$ is compatible with the binary operations, assume $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in \theta$. Relying on completeness (Theorem 4.4), we can use any logical rule $\frac{\varphi}{\psi}$ such that $\varphi \leq \psi$ is an inequality valid in $\mathbb{SDM}$. In particular, in the proof below, by (e.g.) 'commutativity' for $\wedge$ we shall refer not only to the rule $\frac{p\wedge q}{q\wedge p}$, but also $\frac{\sim(p\wedge q)}{\sim(q\wedge p)}$, $\frac{\sim(p\wedge q)\vee r}{\sim(q\wedge p)\vee r}$, etc. In the computations that follow, we shall skip the steps that follow trivially (by symmetry) from the preceding ones; the dots $(\ldots)$ will be used to indicate the passages that have been omitted.

($\wedge$). We have:

(i) $(a_1 \wedge a_2) \vee c_1 \in D$

$$
\begin{aligned}
&\text{iff } (a_1 \vee c_1) \wedge (a_2 \vee c_1) \in D && \text{by distributivity}\\
&\text{iff } a_1 \vee c_1, a_2 \vee c_1 \in D && \text{by } \frac{p \wedge q}{p}\\
&\text{iff } b_1 \vee c_1, b_2 \vee c_1 \in D && \text{by (i)}\\
(\ldots)\ &\text{iff } (b_1 \wedge b_2) \vee c_1 \in D.
\end{aligned}
$$

(ii) $\sim((a_1 \wedge a_2) \wedge c_2) \vee c_1 \in D$

$$
\begin{aligned}
&\text{iff } \sim(a_1 \wedge (a_2 \wedge c_2)) \vee c_1 \in D && \text{by } \wedge\text{-associativity}\\
&\text{iff } \sim(b_1 \wedge (a_2 \wedge c_2)) \vee c_1 \in D && \text{by (ii)}\\
&\text{iff } \sim((b_1 \wedge a_2) \wedge c_2) \vee c_1 \in D && \text{by } \wedge\text{-associativity}\\
&\text{iff } \sim((a_2 \wedge b_1) \wedge c_2) \vee c_1 \in D && \text{by } \wedge\text{-commutativity}\\
&\text{iff } \sim(a_2 \wedge (b_1 \wedge c_2)) \vee c_1 \in D && \text{by } \wedge\text{-associativity}\\
&\text{iff } \sim(b_2 \wedge (b_1 \wedge c_2)) \vee c_1 \in D && \text{by (ii)}\\
(\ldots)\ &\text{iff } \sim((b_1 \wedge b_2) \wedge c_2) \vee c_1 \in D.
\end{aligned}
$$

(iii) $\sim(\sim((a_1 \wedge a_2) \wedge c_3) \wedge c_2) \vee c_1 \in D$

$$
\begin{aligned}
&\text{iff } \sim(\sim((a_1 \wedge (a_2 \wedge c_3)) \wedge c_2) \vee c_1 \in D && \text{by } \wedge\text{-associativity}\\
&\text{iff } \sim(\sim((b_1 \wedge (a_2 \wedge c_3)) \wedge c_2) \vee c_1 \in D && \text{by (iii)}\\
(\ldots)\ &\text{iff } \sim(\sim((a_2 \wedge (b_1 \wedge c_3)) \wedge c_2) \vee c_1 \in D\\
&\text{iff } \sim(\sim((b_2 \wedge (b_1 \wedge c_3)) \wedge c_2) \vee c_1 \in D && \text{by (iii)}\\
(\ldots)\ &\text{iff } \sim(\sim((b_1 \wedge b_2) \wedge c_3) \wedge c_2) \vee c_1 \in D.
\end{aligned}
$$

($\vee$). We have

(i) $(a_1 \vee a_2) \vee c_1 \in D$

$$
\begin{array}{lr}
\text{iff } a_1 \vee (a_2 \vee c_1) \in D & \text{by } \vee\text{-associativity} \\
\text{iff } b_1 \vee (a_2 \vee c_1) \in D & \text{by (i)} \\
\text{iff } (b_1 \vee a_2) \vee c_1 \in D & \text{by } \vee\text{-associativity} \\
\text{iff } (a_2 \vee b_1) \vee c_1 \in D & \text{by } \vee\text{-commutativity} \\
\text{iff } a_2 \vee (b_1 \vee c_1) \in D & \text{by } \vee\text{-associativity} \\
\text{iff } b_2 \vee (b_1 \vee c_1) \in D & \text{by (i)} \\
\text{iff } (b_1 \vee b_2) \vee c_1 \in D & \text{by } \vee\text{-associativity.}
\end{array}
$$

(ii) $\sim((a_1 \vee a_2) \wedge c_2) \vee c_1 \in D$

$$
\begin{array}{lr}
\text{iff } \sim((a_1 \wedge c_2) \vee (a_2 \wedge c_2)) \vee c_1 \in D & \text{by distributivity} \\
\text{iff } (\sim(a_1 \wedge c_2) \wedge \sim(a_2 \wedge c_2)) \vee c_1 \in D & \text{by (SDM1)} \\
\text{iff } (\sim(a_1 \wedge c_2) \vee c_1) \wedge (\sim(a_2 \wedge c_2) \vee c_1) \in D & \text{by distributivity} \\
\text{iff } \sim(a_1 \wedge c_2) \vee c_1, \sim(a_2 \wedge c_2) \vee c_1 \in D & \text{by } \dfrac{p \wedge q}{p} \\
\text{iff } \sim(b_1 \wedge c_2) \vee c_1, \sim(b_2 \wedge c_2) \vee c_1 \in D & \text{by (ii)} \\
(\ldots) \text{ iff } \sim((b_1 \vee b_2) \wedge c_2) \vee c_1 \in D. &
\end{array}
$$

(iii) $\sim(\sim((a_1 \vee a_2) \wedge c_3) \wedge c_2) \vee c_1 \in D$

$$
\begin{array}{lr}
\text{iff } \sim(\sim((a_1 \wedge c_3) \vee (a_2 \wedge c_3)) \wedge c_2) \vee c_1 \in D & \text{by distributivity} \\
\text{iff } \sim((\sim(a_1 \wedge c_3) \wedge \sim(a_2 \wedge c_3)) \wedge c_2) \vee c_1 \in D & \text{by (SDM1)} \\
\text{iff } \sim(\sim(a_1 \wedge c_3) \wedge (\sim(a_2 \wedge c_3) \wedge c_2)) \vee c_1 \in D & \text{by } \wedge\text{-associativity} \\
\text{iff } \sim(\sim(b_1 \wedge c_3) \wedge (\sim(a_2 \wedge c_3) \wedge c_2)) \vee c_1 \in D & \text{by (iii)} \\
\text{iff } \sim((\sim(b_1 \wedge c_3) \wedge (\sim(a_2 \wedge c_3)) \wedge c_2) \vee c_1 \in D & \text{by } \wedge\text{-associativity} \\
\text{iff } \sim(((\sim(a_2 \wedge c_3) \wedge \sim(b_1 \wedge c_3)) \wedge c_2) \vee c_1 \in D & \text{by } \wedge\text{-commutativity} \\
\text{iff } \sim(((\sim(b_2 \wedge c_3) \wedge \sim(b_1 \wedge c_3)) \wedge c_2) \vee c_1 \in D & \text{by (iii)} \\
\text{iff } \sim(((\sim(b_1 \wedge c_3) \wedge \sim(b_2 \wedge c_3)) \wedge c_2) \vee c_1 \in D & \text{by } \wedge\text{-commutativity} \\
(\ldots) \text{ iff } \sim(\sim((b_1 \vee b_2) \wedge c_3) \wedge c_2) \vee c_1 \in D. &
\end{array}
$$

Hence, $\theta$ is a congruence of **A**. Also, $\theta$ is obviously compatible with $D$. Indeed, if $a \in D$ and $\langle a, b\rangle \in \theta$, then we can use $\frac{p}{p \vee q}$ to conclude $a \vee b \in D$. Then we have $b \vee b \in D$ by (i), which gives us $b \in D$ using the rule of $\vee$-idempotency. Lastly, if $\theta'$ is a congruence of **A** that is compatible with $D$, then it is easy to show that $\theta' \subseteq \theta$. Indeed, if $\langle a, b\rangle \in \theta'$, then we also have, for instance, $\langle a \wedge c_2, b \wedge c_2\rangle, \langle \sim(a \wedge c_2), \sim(b \wedge c_2)\rangle, \langle \sim(a \wedge c_2) \vee c_1, \sim(b \wedge c_2) \vee c_1\rangle \in \theta'$ and so on. Thus, assuming $\sim(a \wedge c_2) \vee c_1 \in D$, we have $\sim(b \wedge c_2) \vee c_1 \in D$ because $\theta'$ is compatible with $D$. Hence, $\langle a, b\rangle \in \theta$. Thus, $\theta$ is the largest congruence compatible with $D$, as required. □

The following auxiliary result is well known to hold for semilattice-based logics (see e.g. [1, Thm. 2.13.iii]; for a definition of the classes $\mathsf{Alg}^*(\vdash)$ and $\mathsf{Alg}(\vdash)$, see [15]).

PROPOSITION 6.3
$\mathsf{Alg}(\vdash^{\leq}_{\mathbb{SDM}}) = \mathbb{SDM}$.

Table 1 introduces the two extra rules that will permit us to axiomatize $\vdash^{\top}_{\mathbb{SDM}}$. Observe that $\mathsf{r_{WP}}$ is a weaker form of the pseudo-complement rule $\mathsf{r_P}$ introduced in Example 4.6. Note also that none of the rules in $\mathcal{R}_\top$ corresponds to an (in)equality: their role is to ensure that reduced models satisfy $F = \{\top\}$, rather than to restrict the underlying class of algebras.

TABLE 1. The set of rules $\mathcal{R}_\top$.

$$\frac{p \wedge (\sim(p \wedge q) \vee r)}{\sim q \vee r}\ \mathsf{r_{WP}} \qquad \frac{p \wedge (\sim(\sim q \wedge r) \vee s)}{\sim(\sim(p \wedge q) \wedge r) \vee s}\ \mathsf{r_Q}$$

LEMMA 6.4
Let $\langle \mathbf{A}, F\rangle$ be a reduced matrix for the strengthening of $\vdash^{\leq}_{\mathbb{SDM}}$ with $\mathcal{R}_\top$. Then $F = \{\top\}$.

PROOF. By Proposition 6.3 (and the well-known fact that $\mathsf{Alg}^*(\vdash) \subseteq \mathsf{Alg}(\vdash)$ holds for any logic $\vdash$ [15, Thm. 2.23]), we have that every reduced matrix for $\vdash^{\leq}_{\mathbb{SDM}}$ is of the form $\langle \mathbf{A}, F\rangle$ with $\mathbf{A} \in \mathbb{SDM}$ and $F$ a lattice filter [19, Lemma 3.8].

Suppose, by way of contradiction, that there is $a \in F$ such that $a \neq \top$. Then $\langle a, \top\rangle \notin \boldsymbol{\Omega}_{\mathbf{A}}(F)$. This means that there are $c_1, c_2, c_3 \in A$ such that at least one of the three items of Proposition 6.2 fails. Clearly, item (i) cannot fail, because $a, \top \in F$. Thus, suppose item (ii) fails. Then there are $c_1, c_2 \in A$ such that $\sim(a \wedge c_2) \vee c_1 \in F$ and $\sim(\top \wedge c_2) \vee c_1 = \sim c_2 \vee c_1 \notin F$. But, since $a \in F$, the latter cannot happen because of the rule $\mathsf{r_{WP}}$. Now, assume item (iii) fails. Then there are $c_1, c_2, c_3 \in A$ such that $\sim(\sim(\top \wedge c_3) \wedge c_2) \vee c_1 = \sim(\sim c_3 \wedge c_2) \vee c_1 \in F$ but $\sim(\sim(a \wedge c_3) \wedge c_2) \vee c_1 \notin F$. But since $a \in F$, this cannot happen because of rule $\mathsf{r_Q}$. □

THEOREM 6.5
For every $\mathbb{K} \subseteq \mathbb{SDM}$, the logic $\vdash^{\top}_{\mathbb{K}}$ is axiomatized, relatively to $\vdash^{\leq}_{\mathbb{K}}$, by $\mathcal{R}_\top$.

PROOF. Soundness is clear. For completeness, assume $\Gamma \nvdash \varphi$ where $\vdash$ is the strengthening of $\vdash^{\leq}_{\mathbb{K}}$ with $\mathsf{r_{WP}}$ and $\mathsf{r_Q}$. Then there is a reduced matrix model $\langle \mathbf{A}, F\rangle$ of $\vdash$ witnessing this. Moreover, $\mathbf{A} \in \mathsf{Alg}(\vdash^{\leq}_{\mathbb{K}}) = \mathbb{V}(\mathbb{K}) \subseteq \mathbb{SDM}$ (cf. Proposition 6.3). So we can invoke Lemma 6.4 to obtain $F = \{\top\}$. Hence, $\Gamma \nvdash^{\top}_{\mathbb{K}} \varphi$, as required. □

Taking into account Theorem 4.4, the preceding Theorem immediately gives us the following.

COROLLARY 6.6
For every $\mathbb{K} \subseteq \mathbb{SDM}$, if $\vdash^{\leq}_{\mathbb{K}}$ is finitely based, then so is $\vdash^{\top}_{\mathbb{K}}$.

## 7 Concluding Remarks

The present paper has been a contribution to improving our current understanding of the expressivity of Hilbert-style calculi. As observed earlier, Gentzen calculi allow one to impose directly the meta-properties needed to ensure that the inter-derivability relation is a congruence of the formula algebra. By contrast, we have shown that under certain conditions this is beyond what Hilbert-style calculi can capture finitely. Our main results are displayed in Table 2 below.

TABLE 2. Finite axiomatizability results.

| toprule Conditions on $\mathsf{E}$ | $\vdash^{\leq}_{\mathbb{V}_{\mathsf{E}}}$ | $\vdash^{\top}_{\mathbb{V}_{\mathsf{E}}}$ | Examples |
|---|---|---|---|
| $\sim$-balanced, $\mathbb{V}_{\mathsf{E}} \subseteq \mathbb{DN}$ and $\sim^k p \nvdash^{\leq}_{\mathbb{V}_{\mathsf{E}}} \sim^k q$ | **N** | ? | $\mathbb{DN}$, $\mathbb{O}$ |
| finite and $\mathbb{V}_{\mathsf{E}} \subseteq \mathbb{SDM}$ | **Y** | **Y** | $\mathbb{SDM}$, $\mathbb{PL}$ |
| finite and $\mathbb{V}_{\mathsf{E}} \subseteq \mathbb{O}^m_n$ | **Y** | ? | $\mathbb{O}^m_n$, $\mathbb{DM}$ |

On the front of positive results, we have identified certain subvarieties of $\mathbb{DN}$ for which Hilbert-style calculi are indeed able to reflect finitely the effect of imposing extra equations on the algebras. The well-known result that finitely-generated varieties of lattices are finitely based [8, Cor. V.4.18] implies that our methods may be successfully applied to every finite-valued order-preserving logic that extends $\vdash^{\leq}_{\mathbb{SDM}}$. We believe it would be interesting to take a closer look at the conditions that characterize this divide.

Yet another approach to the axiomatization of logics is provided by multiple-conclusion calculi, which are still less expressive than Gentzen-style calculi. These are an extension of traditional (single-conclusion) Hilbert-style calculi where rules may have non-singleton sets of conclusions (which are read disjunctively). With multiple-conclusion calculi one gains a considerably greater expressive power without expanding the signature with metalinguistic symbols as happens with Gentzen-style systems. For instance, it is known that every finite-valued logic is finitely axiomatizable by multiple-conclusion calculi, and desirable proof-theoretical properties (e.g. analiticity, effective proof search) are more easily established for the latter than for the their single-conclusion counterparts (see e.g. [23, 24, 34]). We speculate whether the logics we have shown to be non-finitely based (by means of single-conclusion Hilbert-style calculi) might be axiomatizable by means of a finite multiple-conclusion calculus (as happens, for instance, with the logic defined by Wroński's three-element matrix: see [24, 35]).

A related question is whether logics of distributive lattices with negation that are not given by any finite set of finite matrices may admit some finite non-deterministic partial matrix semantics (see [2, 3, 9–11]).

A last research direction worth mentioning is the study of logics defined from classes of distributive lattices with negation through different choices of the designated elements. As we have seen earlier, one such choice yields $\top$-assertional logics associated to subvarieties of $\mathbb{DN}$. In this respect, we speculate whether the finite axiomatizability result obtained in Section 6 for $\vdash^{\top}_{\mathbb{SDM}}$ might be extended to other logics (e.g. $\vdash^{\top}_{\mathbb{DN}}, \vdash^{\top}_{\mathbb{O}}, \vdash^{\top}_{\mathbb{O}^m_n}$).

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