Fitch's Paradox,
Stumbling Block
or Touchstone
for Knowability

Abstract:

If we want to say that all truths are knowable Fitch's Paradox leads us to conclude that all truths are known.
Is it a real philosophical problem or a mere modeling problem?
Is it possible to express the idea of knowability using modal logic?
The Knowability Principle is expressed by the formula: if φ is true then it is possible to know that φ ($\varphi \rightarrow \Diamond K\varphi$).
But what is the meaning of possibility in this context?
Given that $\Diamond$ and K are standard modal operators under what condition can they express the idea of knowability?
We will in particular examine the subjacent relations for K and $\Diamond$ in a Kripke Model.
We will define the possibility as the possibility of learning opposed to an ill-defined possibility.
Then we will show that Fitch's Paradox becomes clearer and we will examine how the Knowability Principle could be expressed in such frame.

1- For those unaware of Fitch's Paradox

Some want to claim that all truths are knowable, for various reasons.
This could be expressed by the Knowability Principle: $\varphi \rightarrow \Diamond K\varphi$.
But this is incompatible with the existence of unknown truths: $\varphi \wedge \neg K\varphi$.
It is not possible to know simultaneously that

- φ
- φ is unknown: $\neg K\varphi$.

Expressed in a formula: $\neg\Diamond K (\varphi \wedge \neg K\varphi)$.
So we must accept that not everything is knowable or that all is known.
It is a strong argument to reject the Knowability Principle and it leads to large discussions.

2- Relationship between Knowledge and Possibility

One thing that is very unclear about knowability is the relationship between knowledge and possibility.
Here, we attempt to clarify this relationship.
So we will examine the Kripke Model and the accessibility relation for K and $\Diamond$, respectively $\mathfrak{R}_K$ and $\mathfrak{R}_\Diamond$.

The meaning of these accessibility relations could be expressed as follows:

- $\mathfrak{R}_K$ relates all the worlds compatible with my present knowledge (the plausible worlds).
  Regarding what I know, these worlds are equivalent.
  ($p \in w1$; $p \notin w2$; $w1\ \mathfrak{R}_K\ w2$) : $w1 \vdash \neg Kp$
- As a first approach, let us consider $\mathfrak{R}_K$ as an equivalence relation.
- By $\mathfrak{R}_\Diamond$ we mean to express that something is possible from the point of view of knowability. But what we want to be able to learn, are only the things which are actually true.
  So $\mathfrak{R}_\Diamond$ must express the increase (or decrease) of knowledge and nothing else.
  e.g. if it rains and I don't know it, I can learn that it is raining but I cannot learn that it is sunny.
- We can consider $\mathfrak{R}_\Diamond$ as an equivalence relation.
- We can equally define $\mathfrak{R}_\Diamond$ as a partial order relation. In this case, we have 2 possibility operators :
    - $\Diamond_+$ for the increase of knowledge
    - $\Diamond_-$ to point out the previous states of knowledge

We present below a very simple model compatible with these requirements.

a) basic propositions are propositions without modal operator (K, $\Box$,$\Diamond$)
b) every basic proposition is necessary : if φ is basic then ($\varphi \rightarrow \Box\varphi$)
c) $\mathfrak{R}_K$ and $\mathfrak{R}_\Diamond$ are disjointed: (w1 $\mathfrak{R}_K$ w2) $\wedge$ (w1 $\mathfrak{R}_\Diamond$ w2) $\rightarrow$ (w1=w2)

Figure 1 - an example

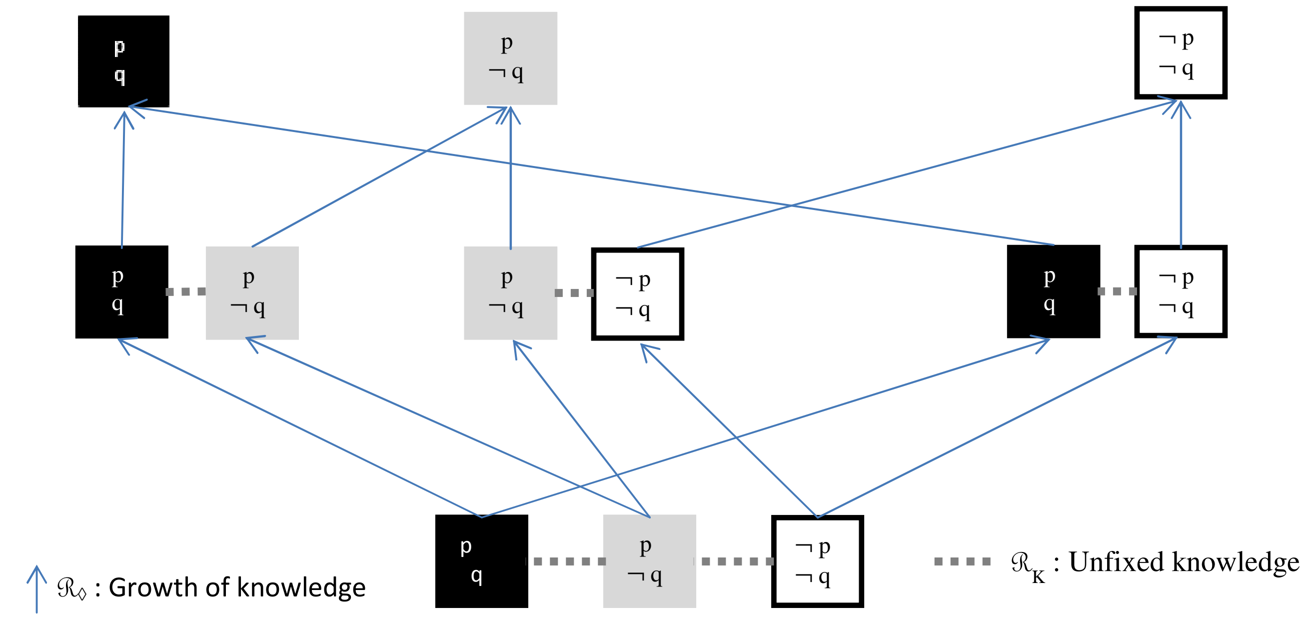
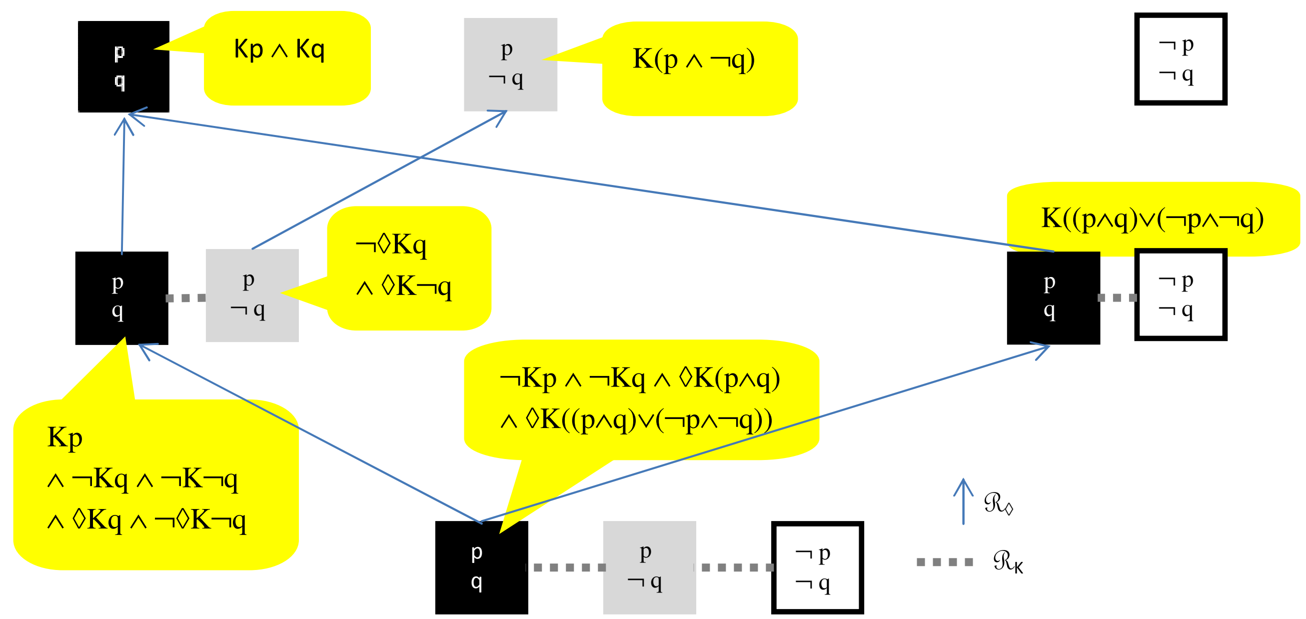


Figure 2 - a shorter example with comments



3-About Fitch's Paradox

Nevertheless, Fitch's Paradox remains.
We can never know a Moorean statement:
($\varphi \wedge \neg K\varphi$) is unknowable per se and there is no sensible way to get around this.
But now we have a solid basis to review this problem.
Once admitted that the naïve formulation of the Knowability Principle has to be adapted, we can examine the two main strategies for doing this: the reformulation strategy and the restriction strategy.

3.1 Restriction strategy

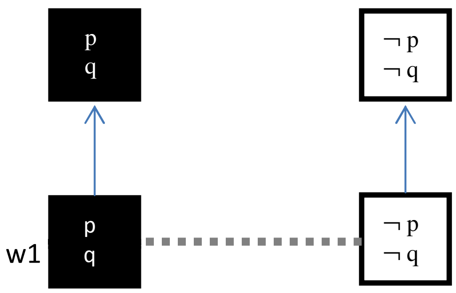
Basic Statements

In this case we keep the standard formulation of the Knowability Principle: $\varphi \rightarrow \Diamond K\varphi$
but applying it only to the basic statements.
By restricting the Knowability Principle to the basic statements, it remains philosophically debatable but it cannot anymore be rejected by using a mere logic trick.
The restricted Principle is a little weaker but it remains sufficiently strong to feed philosophical controversy on knowledge and knowability.
They are some who pretend to have refuted antirealism using the Fitch's argument. But who among them are ready to accept the Knowability Principle for all the basic statements?
If everybody accepts the **restricted Knowability Principle**, it turns out to be **a very fruitful defeat for the Knowability advocates**.

Cartesian statement

The restriction of the Knowability Principle to Cartesian statements is defended by Tennant.
A statement φ is Cartesian if and only if $K\varphi$ is not provably inconsistent.
The Cartesian restriction seems a very ad hoc restriction:
every Truth can be known except the Truth which cannot.
Furthermore we can see that **this restriction doesn't work in our model**:

Figure 3



$w1 \vdash p \wedge \neg Kq$

$\neg\Diamond K\,(p \wedge \neg Kq)$

($p \wedge \neg Kq$) is Cartesian but cannot be known.
(Van Bentham published a similar result several years ago)

Signed Strategy

We can assign a sign to each statement:

- Null if the statement has constant value. Typically, it is the case of basic statements:
  → ...FFFFFFF... →
  → ...TTTTTTT... →
- Positive if the statement can become and stay true. Typically, it is an assertion of knowledge about a basic statement (Kp):
  → ...FFFFFFF → TTTT... →
- Negative if the statement can become and stay false (e.g. ¬Kp):
  → ...TTTTT → FFFFFFF... →
- Undefined in the other cases
  ... → FFF → T → FFFFF → TTT → ...

Then every positive or null statement is knowable: if $\varphi \geq 0$ then $\varphi \rightarrow \Diamond K\varphi$.
We can then define an array for each combination (Kp, ¬p ; p∧q ; ◊p) and hence obtain a lighter restriction.
But we won't develop this here, because it's little tedious and probably not so interesting.

3.2 Reformulation strategy

A detour by Temporal Logic

We use the temporal operators below:

- **P**: it has at some time been the case that ...
- **F**: it will at some time be the case that ...
- **G**: it will always be the case that ...

The Discovery Principle is the equivalent in the Temporal Logic of the Knowability Principle.
DP: $\varphi \rightarrow FK\varphi$: if something is true it will be known that it is true.
The Discovery Principle suffers the same objection than the Knowability Principle.
Furthermore it implies, K being factive ($K\varphi \rightarrow \varphi$), that:
$\boldsymbol{\varphi \rightarrow FK\varphi \rightarrow F\varphi}$
In other words: **all truths will have to be true again.**
It is a very simple proof of the Eternal Return but we can consider this to be an unwanted effect.
So we have to reformulate the Discovery Principle.

Using the usual temporal operators, we can write 2 formulas:

1) $G\varphi \rightarrow FK\varphi$ (if something is going to stay true then it will be known in the future)
2) $\varphi \rightarrow FKP\varphi$ (if something is true then it will be known that it was true)

Moreover we can notice that the two formulas are equivalent:

(1) → (2): $\varphi \rightarrow GP\varphi \rightarrow^{(1)} FKP\varphi$
(2) → (1): $G\varphi \rightarrow^{(2)} FKPG\varphi$; $PG\varphi \rightarrow \varphi$ then $G\varphi \rightarrow FK\varphi$

We can also use an operator to mark the current instant as in this formula:
$@t\ \varphi \rightarrow FKP(t \wedge \varphi)$
This can be useful and presents no real difficulty, on condition that coherent hybrid logic is respected.
But it would require too long developments for this short article.

Comeback to KP

We can reformulate the Knowability Principle in the same way than the Discovery Principle:

$\Box_+\varphi \rightarrow \Diamond_+ K\varphi$ (if something stay true in every greater state of knowledge then it can be known)

$\varphi \rightarrow \Diamond_+ K\Diamond_-\varphi$ (if something is true it is possible to know that it was true in a previous state of knowledge)

We can see the strong link between this strategy and the restriction strategy: φ Basic implies $\Box\varphi$.
The reformulation strategy is a little more general but one may prefer the restricted formula which is simpler and clearer.

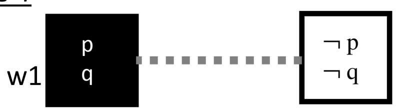
To quickly mention the recourse to hybrid logic we can notice that, unlike with an ill-defined notion of possibility, **we can refer to epistemic states which are well defined:** they are the equivalence classes defined by $\mathfrak{R}_K$.
So we can speak about the current state in our formula:

$@_{epistemic\_state\_1}\ \varphi \rightarrow \Diamond_+ K\Diamond_- (epistemic\_state\_1 \wedge \varphi)$

4- Properties of our model

<u>The **Knowability Principle** is **compatible** with our model **but** it is **not a tautology**.</u>

<u>*Figure 4*</u>



$w1 \vdash (p \wedge \neg\Diamond Kp)$

<u>The **Knowability is factive**:</u>
I can learn only the truth.
If φ is basic: $\Diamond K\varphi \rightarrow \varphi$
We must here remark that this factivity does not entail a collapse of our model.
When I am in an epistemic state, I don't know if I am in w1 or in w2,
so I don't know $\Diamond K\varphi$ more than I know φ : $\neg K\varphi \rightarrow \neg K\Diamond K\varphi$

<u>**The Omniscience Thesis is equivalent to the Knowability Principle:**</u>
The verification of the Knowability Principle (KP) is equivalent to the closure of our model by omniscient epistemic states, like in *figure 1*.
<u>Omniscient Thesis</u> (OT): <u>every world is related by $\mathfrak{R}_\Diamond$ with a world which has no $\mathfrak{R}_K$ related world</u>.

$\forall w1 , \exists w_{omniscient1} / \quad (w1\ \mathfrak{R}_{\Diamond+}\ w_{omniscient1})$
$\qquad \wedge \forall w2\ (w_{omniscient1}\ \mathfrak{R}_K\ w2) \rightarrow (w2 = w_{omniscient1})$
$\qquad \wedge \forall w3\ (w_{omniscient1}\ \mathfrak{R}_{\Diamond+}\ w3) \rightarrow (w3 = w_{omniscient1})$

OT -> KP : obvious
KP -> OT : by reductio ad absurdum
This could explain why, if not watchful, the Knowability Principle could lead us to accept the omniscience.

<u>Nevertheless this model remains **an idealisation**.</u>
I don't think that any modeling could express exactly what are Truth, Knowledge and Knowability. These concepts are not sufficiently well defined to be caught by a simple logical model. Nevertheless a logical model can help us to examine and to argue about them.

5- Conclusion

To enlighten the debate on knowability we have defined possibility as possibility to increase knowledge. This has allowed us to clarify the relationship between the relations expressing knowledge and possibility. Then we have examined the restriction strategy and the reformulation strategy. We have seen the affinity between them and also with our model.

Finally we have seen the strong bind between the Knowability Principle and the existence of Omniscient States.

Credit

This work is largely inspired from rich previous works about Fitch's Paradox, in particular:

BURGESS, John P. - Can truth out? - in [New Essays on the Knowability Paradox] - 2009

PROIETTI, Carlo - Le problème des futurs contingents et le paradoxe de Fitch.
Une étude commune de deux problèmes en logique modale - Thèse de doctorat à Paris I - 2008.

van BENTHEM, Johan - 2009 Actions that Make us Know - in [New Essays on the Knowability Paradox] - 2009.

Though I wrote this paper independently, it would seem that a very similar approach exists in Costa-Leite.