The Psychology of The Two Envelope Problem

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This article concerns the psychology of the paradoxical Two Envelope Problem. The goal is to find instructive variants of the envelope switching problem that are capable of clear-cut resolution, while still retaining paradoxical features. By relocating the original problem into different contexts involving commutes and playing cards the reader is presented with a succession of resolved paradoxes that reduce the confusion arising from the parent paradox. The goal is to reduce confusion by understanding how we sometimes misread mathematical statements; or, to completely avoid confusion, either by reforming language, or adopting an unambiguous notation for switching problems. This article also suggests that an illusion close in character to the figure/ground illusion hampers our understanding of switching problems in general and helps account for the intense confusion that switching problems sometimes generate.

This article concerns the psychology of the paradoxical Two Envelope Problem [1–11]. It opens with the introduction of a simple related problem — The Two Commuters Problem in Sec. I — with the idea that a linguistic analysis of this basic problem can shed light on the psychology of the well-known Two Envelope Problem, which will be discussed in Sec. III. The general goal is to find variants of the envelope switching problem that are capable of clear-cut resolution, while still retaining paradoxical features. By relocating the original envelope problem into different contexts involving commutes (in Sec. I), a finite number of playing cards (in Secs. IV, V, VIII, IX, and X), and a random walk (in Sec. VI), the reader is presented with a succession of resolved paradoxes that reduce the confusion arising from the parent paradox.

The author has kept in mind that where paradoxes are concerned it is not enough to point out the right answer, one also must provide an explanation of how other seemingly plausible answers are logically — or psychologically — flawed. For this reason a Socratic dialogue is used in Sec. V to explore an enigmatic switching problem from multiple perspectives: first reducing paradoxical confusion, then reviving it, then reducing it, thereby clarifying some basic (but subtle) points of switching problems. The goal is to reduce confusion through an understanding of how we misread mathematical statements, or to avoid confusion altogether through reform of language. This dialogue leads to some suggestions concerning the vocabulary of switching, and a compact notation for unambiguously specifying some switching problems; see Sec. VII.

In Sec. II it is suggested that an illusion close in character to the figure/ground illusion [12] hampers our understanding of switching paradoxes in general; this reflects this article's emphasis on psychological issues. In suggesting a role for the figure/ground illusion this article supports Raymond Smullyan's contention that The Two Envelope Problem has paradoxical features that do not concern probability [6]; see Sec. XI.

I. THE TWO COMMUTERS PROBLEM

A. Problem Statement

Every day Alice and Bob commute separately to work, where on alternate days Bob commutes first twice as far as Alice, and then half as far as Alice, keeping up this alternating pattern over the course of a leap year. The question is, who commutes the longer yearly distance?

B. Analysis

To better understand this problem we need to restate it mathematically. But what exactly do we know? We appear to know that Alice always commutes X, and that Bob commutes $2X$, $X/2$, $2X$, $X/2$, ..., etc. on alternate days throughout the year. It follows that over the 366 days of a leap year Bob commutes daily $(2X + X/2)/2$ on average, whereas Alice commutes X on average. If $X \neq 0$ then over the course of a year Bob has the longer commute.

Is this translation of the problem into mathematics correct? Not necessarily.

To see why, assume that on the first day Alice travels four miles and Bob eight. What must happen on the second day to fulfill the Problem Statement, if, say, only one of them alters their commute? The answer is: Either Bob must alter his commute to two miles (the obvious answer) *or Alice must alter her commute to 16.* Although the original wording appears to say that Alice has a fixed commute, this is largely the result of a quirk of language. In English we sometimes have an asymmetry in words that is not reflected in the situation described:

> Shoe Salesman: *Madame, your left foot is smaller than your right foot.*
> Shoe Salesman: *Madame, your right foot is larger than your left foot.*

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(Observe that in going from the first sentence to the second the swaps *left* $\leftrightarrow$ *right* and *smaller* $\leftrightarrow$ *larger* are made.) In fact a close reading of the Problem Statement reveals nothing that prohibits Alice's commute from changing.

To amplify this point, consider the following two sentences, the first of which is taken verbatim from the Problem Statement, and the second of which follows from making the swaps *Alice* $\leftrightarrow$ *Bob* and *twice* $\leftrightarrow$ *half* on the first:

- "Every day *Alice* and *Bob* commute separately to work, where on alternate days *Bob* commutes first *twice* as far as *Alice*, and then *half* as far as *Alice*, keeping up this alternating pattern over the course of a leap year."
- "Every day *Bob* and *Alice* commute separately to work, where on alternate days *Alice* commutes first *half* as far as *Bob*, and then *twice* as far as *Bob*, keeping up this alternating pattern over the course of a leap year."

However different these statements may at first appear, a case can be made that they are logically equivalent. To the author they both convey the same literal meaning, either

$$\frac{\text{Bob's commute}}{\text{Alice's commute}} = 2^{(-1)^{N-1}} \tag{1.1}$$

or, equivalently,

$$\frac{\text{Bob's commute}}{\text{Alice's commute}} = (\text{day \# is odd}) \ ? \ 2 \ : \ \frac{1}{2} \quad , \tag{1.2}$$

where N equals the day number 1 to 366. In the same way, the two renditions of the shoe salesman's excuse for ill-fitting shoes convey the same literal meaning, however they may differ subjectively.

One final point: Suppose we learn that on the first day

$$\frac{\text{Bob's commute}}{\text{Alice's commute}} = \frac{4.628 \text{ miles}}{2.314 \text{ miles}} \quad . \tag{1.3}$$

What exactly have we added to our knowledge? Absolutely nothing. The above equation merely states that on the first day Bob travels twice as far as Alice, not that Bob travels 4.628 miles.

C. Reanalysis

So what do we actually know about Alice's and Bob's commutes from the Problem Statement? Very little! It might be that on the first day Alice travels one mile and Bob two miles, and then they swap these commutes day by day. Perhaps Bob travels four miles every day, and Alice travels two and eight miles on alternate days. Perhaps Alice and Bob travel one and two miles the first day, respectively; four and two the second day, respectively; four and eight the third day, respectively; and so on. Perhaps their commutes are planned as in these examples, but are altered by a common multiplier that depends on how much it rains.

What is peculiar is that the original Problem Statement seems so definite and constraining while actually saying so little. A cabdriver charging by the mile and trying to maximize his income by transporting just one of them back and forth from work cannot tell from the original description who best to contract his yearly services to. This is because *all* of the above algorithms yield commutes consistent with the Problem Statement. But it is the exact *algorithm* used to generate these commutes that the cabdriver needs to know to determine whose yearly commute is longer — not merely the *feature* that "Bob commutes first twice as far as Alice, and then half as far as Alice, keeping up this alternating pattern over the course of a leap year," which fails to tell him the very thing he needs to know: who has the longer yearly commute.

D. Feature Versus Algorithm

In this light reconsider this earlier inference:

> *"We appear to know that Alice always commutes X, and that Bob commutes $2X$, $X/2$, $2X$, $X/2$, ... , etc. on alternate days throughout the year."*

This statement appears to be wrong because Alice's commute need not be fixed. But underlying this is a subtle question. What exactly is meant by a statement such as:

> *Today Alice commutes X and Bob $2X$ or $X/2$?*

If the reader thinks it concerns mere *feature* (i.e. an aspect of what we observe) then it could be thought to mean the same as

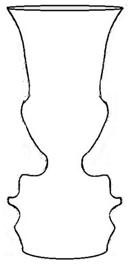
$$\frac{\text{Alice's commute}}{\text{Bob's commute}} = 2^{\pm 1} \quad . \tag{1.4}$$

But if the reader thinks it concerns algorithm (i.e. *origination*) then he is justified in imagining someone first choosing Alice's commute X, and then flipping a coin to determine whether Bob commute is $2X$ or $X/2$. Judging the meaning of the above sentence engages taste, bias, preference, and expectation, which is why such ambiguity should be avoided whenever possible.

II. THE FIGURE/GROUND ILLUSION

Why do people sometimes badly misjudge The Two Commuters Problem Statement? The author suspects it has something to do with the figure/ground illusion, also known as figure/ground reversal [12].

Firstly, note that figure/ground organization involves seeing at least one solid-looking and well-defined object (the *figure*) standing out against a fuzzy and formless background (the *ground*). Secondly, note that in the figure/ground illusion a person sees an image having details that alternately appear as figure and ground. Perhaps the most famous such illusion involves a vase and two faces:



When the vase appears as figure, the faces are unseen as part of the ground. And when the faces appear as figure, it is the vase that is unseen as part of the ground. For this illusion there is a tendency to perceive either the vase, or the faces, but not both simultaneously. Essentially, the two interpretations compete for attention with only one winning at any given time.

For The Two Commuters Problem we are inclined to see Alice as having a fixed commute (which is to say, as being the figure), and Bob as having a fuzzy commute (which is to say, as being the ground). And just as a "hidden" figure may sometimes have to be pointed out to us in some illusions, such as:

so the hidden possibility of Bob having a fixed commute may have to be pointed out to us. And, lastly, although the prototypical figure/ground illusion concerns visual stimuli, the author can see nothing that prevents it from also applying to The Two Commuters Problem in particular, or switching problems generally.

III. THE TWO ENVELOPE PROBLEM

The above commuting problem and figure/ground illusion will now be used to try to shed light on the much discussed Two Envelope Problem [1–11], which is the focus of this section.

A. Problem Statement

A Player is shown two envelopes, one of which is known to contain twice as much money as the other. The Player is invited to choose one envelope, open it, and then decide whether to switch envelopes.

B. Analysis

Employing the above figure/ground analogy we see that once The Player takes one envelope and it occludes the other envelope (presumably lying on a table) the taken envelope becomes the figure. The Player then automatically sees the occluded envelope as ground, and as having a variable or fuzzy amount. (That is to say, he thinks the occluded envelope contains an algorithmically-determined amount that is either $2X$ or $X/2$, whereas he holds X.) Hence, he wrongly concludes that it must make sense to switch, as he has more to gain $2X - X$ than lose $X - X/2$. This belief — *that switching must make great sense* — can be very strong, just as the figure/ground illusion can be very strong, and this mindset is only reinforced after he actually *opens* the envelope he holds and identifies X. Of course, we know from The Two Commuters Problem that the above reasoning is false: The player knowing that an amount is either twice or half as large as another in no way guarantees that it was not fixed.

C. A Psychologically Different Version of The Two Envelope Problem

It is illuminating to imagine a psychologically different version of The Two Envelope Problem, in which The Player grabs the envelope he does *not* want, so that the envelope he has chosen becomes part of the occluded ground. It is the author's expectation that The Player will then think of this occluded envelope as having a variable amount, despite its being the chosen envelope. In these reversed circumstances The Player may well be *dis*-inclined to switch away from his first choice.

IV. THE DAISY CHAINED CARDS PROBLEM

We will apply the above insights to The Daisy Chained Cards Problem, which is a finitely-chained variant of an infinitely-chained card problem first described by the mathematician J. E. Littlewood in 1953, who in turn credits physicist Erwin Schrödinger [2, 5].

A. Problem Statement

A deck of six playing cards is manufactured with faces as follows:

$$\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \begin{bmatrix} 2 & \\ & 3 \end{bmatrix}, \begin{bmatrix} 3 & \\ & 4 \end{bmatrix}, \begin{bmatrix} 4 & \\ & 5 \end{bmatrix}, \begin{bmatrix} 5 & \\ & 6 \end{bmatrix}, \begin{bmatrix} 6 & \\ & 7 \end{bmatrix}$$

A Player examines how the cards are numbered and notices that the cards are linked — daisy chained — by pairs of numbers. A Dealer shuffles the cards. She then places them facedown. The Player picks a card and peeks at a corner. The Player knows he can get 2^N units of cash for the number N he has chosen and that he is entitled to switch to the other numbered corner. When (if ever) should he switch if he wants to maximize his return over time?

B. Analysis

- The Player does not switch on 7 because he knows it is the maximum.
- He also knows that 6 occurs on two cards, and that his potential gain of $2^{6+1} - 2^6$ in switching is greater than his potential loss of $2^6 - 2^{6-1}$, so he switches on 6.
- Unfortunately there is a downside to switching on both 6's: Inevitably he will swap one of these 6's for a 5. No matter. By switching on both 5's he "gets back" the 6.
- Unfortunately there is a downside to switching on both 5's: Inevitably he will swap one of these 5's for a 4. No matter. By switching on both 4's he "gets back" the 5.
- Unfortunately there is a downside to switching on both 4's: Inevitably he will swap one of these 4's for a 3. No matter. By switching on both 3's he "gets back" the 4.
- Unfortunately there is a downside to switching on both 3's: Inevitably he will swap one of these 3's for a 2. No matter. By switching on both 2's he "gets back" the 3.
- Unfortunately there is a downside to switching on both 2's: Inevitably he will swap one of these 2's for a 1. No matter. By switching on 1 he "gets back" the 2.

So he switches for 1 to 6 and holds on 7.

V. A DIALOGUE ON THE ABOVE STRATEGY

It is easiest to pose and resolve the paradoxes associated with the above game in the form of a dialogue between Simplicio, Sagredo, and Salviati, the interlocutors — in increasing order of intelligence — of Galileo's *Two New Sciences* [13]:

> Sagredo: *I have an issue with the strategy just described. Let's assume that the cards are given letters* **A** *to* **F** *from the leftmost card to the rightmost. Let's assume I pick* **C**. *I peek at its upper left corner. No matter what I see I switch to the lower right corner. If I peek at the lower right corner first, I switch to the upper left corner. Is this not true?*
>
> Simplicio: *I am thinking like you.*
>
> Sagredo: *So this switching accomplishes nothing. Not switching on* **C** *would leave the results the same?*
>
> Simplicio: *Again, I am thinking like you.*
>
> Sagredo: *And so the solution as it is constructed is absurd. I am constrained by the strategy to do something that I need not do. Were I to do the opposite with* **C** *and never switch, the results would remain the same. And this is true for* **A**, **B**, **D**, *and* **E** *as well.*
>
> Simplicio: *I can see no error in your logic, Sagredo.* (he turns to Salviati) *Salviati, what do you make of it?*
>
> Salviati: *Let me first pose a question. Sagredo, how would you treat* **E**?
>
> Sagredo: *The same as* **C**. *I first look at one corner — No matter what I see, I switch to the other.*
>
> Salviati: *And how would you treat* **F**?
>
> Sagredo: **F** *is a special case. When I see* 7, *I must stay, as* 7 *is the maximum. On* 6 *I must switch in order to maximize my reward . . . it gives me a shot at getting an extra* 7 *that I cannot pass up.*
>
> Salviati: *And how do you know when you are holding* **E**?
>
> Sagredo: *How do I know? From its numbers. Small numbers tell me that I am dealing with cards that in reality do not require switching. When I see* 3, *I am dealing with* **B** *or* **C**. *In both cases I can violate the strategy and not switch without affecting the result.*
>
> Salviati: *But when you see* 6 *might not the card be* **E** *or* **F**?
>
> Sagredo: *Yes. So?*
>
> Salviati: *Well, you're telling me you can violate the strategy with impunity and never switch on* **E**, *and yet when you see* 6 *you have no way of distinguishing* **E** *from* **F**.

Sagredo: *Okay. So you've got me! I must switch when I see* 6*, so as to get a shot at an extra* 7*; and this means that I must sometimes switch on* **E**. *I concede your point. But for* **A**, **B**, **C**, *and* **D** *I can still violate the strategy and never switch without affecting the result.*

Salviati: *And how do you* know *when you are holding* **D***?*

Salviati: *When I see* 4 *or* 5.

Salviati: *But when you see* 5*, might not the card be* **E***?*

Sagredo: *So?*

Salviati: *Well, you're now trying to convince me that you can violate the strategy with impunity and never switch on* **D**. *But in practice this requires that you never switch on* 5. *But when you have* $\mathbf{E}_5$*, switching on* 5 *is beneficial, just as when you have* $\mathbf{F}_6$ *switching on* 6 *is beneficial.*

Sagredo: *I concede I want to maximize my* 6*'s and* 7*'s.*

Salviati: *Then you must always switch on* 6 *and* 5. *You switch on* 6 *to get a shot at an extra* 7*, and you switch on* 5 *to get a shot at an extra* 6.

Sagredo: *Agreed. You've caught me again! I must always switch on* 5 *to get a shot at* 6*, which means I must sometimes switch on* **D**. *I can see where this is leading.* (pauses) *I must switch all the way down to* **A**.

Salviati: *Now consider this: If you always switch on* 5 *and* 6 *then you always switch on* **E**. *But always switching on a given card is the very thing you and Simplicio said was absurd in the first place.*

Sagredo: *So I did.*

Simplicio: *And I.*

Salviati: *Do you still think it's absurd?*

Sagredo: *It still doesn't seem quite right, but I can't put my finger on where I'm going wrong.*

Simplicio: *I am still inclined to believe that never switching on* **A**, **B**, **C**, **D**, **E** *is as good a strategy as always switching — and for just the reason Sagredo gave at the start of today's discussion: The results of the game are the same either way, which is all that matters. This is a settled point among us, is it not?*

Salviati: *Not so fast, Simplicio! It's a confusing aspect of this game that you think you choose cards and corners but actually you only make decisions about numbers. Neither the cards nor corners have letters* **A** *to* **F** *that inspire decisions.*

Sagredo: *They have no letters, as you say, though we sometimes talk about "choosing cards" as if we were picking cards with labeled backs.*

Salviati: *Well then, let's for a moment assume that the backs of the cards were actually labeled with Sagredo's letters. Then you could follow a strategy of never switching on* **A**, **B**, **C**, **D**, **E***, while still getting the same reward as always switching — in this Simplicio is right! But remove these letters and you cannot distinguish* $\mathbf{F}_6$ *from* $\mathbf{E}_6$. *And so in trying to convert* $\mathbf{F}_6$ *into an extra* 7 *you will unintentionally switch on* $\mathbf{E}_6$. *Then your only way to restore the* status quo ante *is to always switch on the entire group* **A**, **B**, **C**, **D**, **E***, which you can only accomplish by switching on* 1 *to* 6.

Sagredo: *Finally I see your point! To always — or never — switch on* **A**, **B**, **C**, **D**, *and* **E** *is equally profitable. But in the absence of letters my attempt to achieve an extra* 7 *when I switch on* 6 *leads me to sometimes switch from* $\mathbf{E}_6$ *to* $\mathbf{E}_5$ *— for a large loss. The only remedy for this is to always switch on the entire group* **A**, **B**, **C**, **D**, **E**. *This, as you say, restores the* status quo ante. *And if I switch on* 1 *to* 6 *then I switch on this entire group.*

Salviati: *And you will still convert your* 6 *to* 7. *Your reasoning is the key to understanding The Player's strategy.*

Simplicio: *I still do not understand the mystery of the cards that are always switched.*

Salviati: *First imagine that you always switch on* 3. *Then, to your advantage, you eliminate all* 3's *from the outcome, while adding a* 2 *and a* 4.

Simplicio: *Clearly, I benefit by switching on* 3.

Salviati: *Now imagine that you always switch on* 3 *and* 4. *You then add a* 2 *and a* 5*, while eliminating a* 3 *and* 4.

Simplicio: *I benefit even more by switching on* 3 *and* 4.

Salviati: *And the* 3 *and* 4 *that remain occur when the card that luck assigns to you is* **C** *and you perform what amounts to self-canceling swaps.*

Simplicio: *So that makes always switching on* **C** *just a "side-effect" of always switching on* 3 *and* 4*? And I never actually know that I am switching on* **C** *until after the switch is completed?*

Salviati: *Right. And it follows in turn that always switching on* 3, 4, *and* 5 *leads to always switching on* **C** *and* **D**. *The end result is one more* 2 *and one more* 6 . . .

Sagredo: . . . *and one less* 3 *and one less* 5 . . .

Salviati: *. . . where the numbers made more frequent are added at the edges, while the frequency of* 4 *does not change.*

Sagredo: *So always switching on* 2 *to* 6 *leads to always switching on* **B** *to* **E** *with an end result of one more* 1 *and one more* 7*?*

Salviati: *Yes, and one less* 2 *and one less* 6.

Simplicio: *And if we add "switching on* 1*" we then get optimal results?*

Salviati: *Yes. By always switching on* 1 *to* 6 *we always switch on* **A** *to* **E** *and on* $\mathbf{F}_6$. *And this switching on* $\mathbf{F}_6$ *provides a gain of* $2^7 - 2^6$ *that we wouldn't receive by always holding.*

Simplicio: *Finally I understand!*

Sagredo: *Perhaps this would be a good place to stop, as we have come full circle?*

Simplicio: *But I still don't understand why, when I choose a* 3 *. . .*

Salviati: *I must stop you, Simplicio! You are going to tie yourself up in logical knots if you persist in regarding the random assignments as choices. The only choice you ever make is whether to switch. The so-called choice of card or corner is based on no information at all and is no choice at all. Treat your initial number as a random assignment! And be sure to talk of numbers not cards!*

Simplicio: *Okay, then . . . Let's assume I am randomly assigned* 3 *and I switch and get* 4. *But if I had been assigned* 4 *to begin with, I would automatically be switching back to* 3*, which shows switching to be pointless.* (long pause) *You're going to say that had I been assigned* 4 *my switching instead might produce* 5. *What I formerly took to be automatic — that I would automatically switch back to* 3 *— is obviously not automatic at all. In switching from* 4 *I might get* 3*, but* 5 *is just as likely. By speaking in terms of "assigned numbers" and ignoring what card I hold, the paradox concerning "pointless" switching between* $\mathbf{C}_3$ *and* $\mathbf{C}_4$ *never arises in the first place. After all, I only learn what card I hold after I switch, so the card I hold cannot inform any decision I make.*

Salviati: *As I said earlier, "It's a confusing aspect of this game that you think you choose cards and corners but actually you only make decisions about numbers. Neither the cards nor corners have letters* **A** *to* **F** *that inspire decisions." If you keep this in mind and learn to distinguish an assignment by fate from a true choice you will avoid much confusion.*

Sagredo: *There was a question that I was going to ask that I am now going to try out in the new vocabulary. It is this: What happens if the number of cards* Z *grows really large to* $Z > 1\,000\,000$*? Am I not almost always switching? Let's say luck assigns me a number* L *where* $1 < L < Z$. *I switch and get* $L \pm 1$. *But had luck given me* $L \pm 1$ *from the start, switching would give me either* $L - 1 \pm 1$ *or* $L + 1 \pm 1$. (pauses) *I no longer see a paradox!*

Salviati: *It is enough to know that when your reward is determined by* 2^N*, then* $N = L \pm 1$ *(if equiprobable) will pay more on average than* $N = L$*, for whatever* L *luck has assigned to you. But with* Z *cards expect to play about* Z *times before pulling ahead with this strategy.*

Sagredo: *I suggest we adjourn for today. My mind reels with this talk of strategies where cards are almost always rotated. We have much to think about.*

VI. THE RANDOM WALK PROBLEM

To elaborate upon the point that closed the dialogue, imagine that you have $\$X$, and you get to receive back either that amount, or double-or-half that, or double-or-half that again, and so on, optionally, until you decide to quit. How would you decide when to stop invoking the double-or-half option?

As it turns out, you should just keep on switching until your random walk produces your desired reward.

Now consider that the Random Walk Problem differs from The Daisy Chained Cards Problem in that the first lets you "go double-or-half" as many times as you want, whereas the second lets you go double-or-half at most once. What the two games have in common, however, is that for both:

- The likelihood of coming out ahead for each single act of switching is an even bet.
- If your number is not maximal you always switch in order to introduce *variability* into that number — *irrespective of what that number happens to be at the time.*

The more variability you can introduce, the greater your average reward. As it is, wholesale switching is forbidden in The Daisy Chained Cards Problem, but by switching whenever your number is less than the maximum you can still "do retail" what The Random Walk Problem lets you "do wholesale."

VII. STANDARDIZING SWITCHING PROBLEMS

A. Vocabulary of The Daisy Chained Cards Problem

It is instructive to consider the three so-called "choices" The Player makes in The Daisy Chained Cards Problem:

- When he "chooses" a card he uses no information and acquires no information.
- When he "chooses" a corner he uses no information and acquires some information.
- When he "chooses" whether to switch he uses some information to decide whether to acquire other information.

All three of these diverse acts can be thought of as "choices," but doing so invites insidious confusion for the reasons outlined above. The first and second *non*-choices are just so much hocus-pocus. They are equivalent to a magician misdirecting the focus of his audience by calling attention to something of no importance (the Player chooses a *card* and peeks at a *corner*), while the matter of central importance but possessing less theatrics (a *number* is randomly assigned) is missed. If the reader limits the use of the word "choice" to the act of using some information to decide whether to acquire other information he is proof against such misdirection and avoids much of this confusion.

B. A Standardized Form for Switching Problems

To better understand switching problems the author recommends they be restated in this standard form (if possible):

- Before play, The Player *examines* the distribution of numbers and builds a s*witching table* of possible switches (or, alternately, is simply *shown* the switching table). For The Daisy Chained Cards Problem this switching table is:

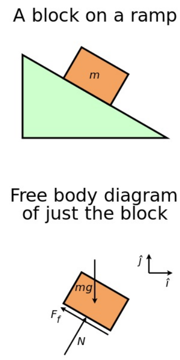
$$1 \longleftrightarrow 2$$
$$2 \longleftrightarrow 3$$
$$3 \longleftrightarrow 4$$
$$4 \longleftrightarrow 5$$
$$5 \longleftrightarrow 6$$
$$6 \longleftrightarrow 7$$

- *Luck randomly assigns* a number N from this table to The Player, where those numbers that appear more often are proportionately more likely to be assigned. (Above, the numbers 2 to 6 are each twice as likely as either 1 or 7.) Such an assignment produces a *reduced switching table* in which the *assigned number* is underlined. Here, 3 is assigned:

$$2 \longleftrightarrow \underline{3}$$
$$\underline{3} \longleftrightarrow 4$$

- The Player *chooses* whether to switch based on the possibilities listed in the reduced switching table. (Above, The Player is assigned 3 and knows he can switch to either 2 or 4, which are equiprobable.)

The above standardization prevents much of the confusion that tends to infect the discussion. There are no extraneous cards chosen or corners lifted in the final analysis; instead only essential information is abstracted. This accomplishes much the same thing that a Free Body Diagram does in physics:



In such a diagram the extraneous environment is omitted and we are left with only a free body and the forces acting on it.

C. Reframing The Daisy Chained Cards Problem

The Daisy Chained Cards Problem can be put in this standardized form.

A Player is shown this switching table:

$$1 \longleftrightarrow 2$$
$$2 \longleftrightarrow 3$$
$$3 \longleftrightarrow 4$$
$$4 \longleftrightarrow 5$$
$$5 \longleftrightarrow 6$$
$$6 \longleftrightarrow 7$$

Luck randomly assigns him a number N from the table. He knows he can get 2^N units of cash for the number N and that he is entitled to switch. When (if ever) should he switch if he wants to maximize his return over time?

The Player should switch on 1 to 6 and hold on 7.

D. The Figure/Ground Illusion and The Daisy Chained Cards Problem

With regard to the figure/ground illusion, when The Player sees a 1 he knows the other number is a 2 — so both 1 and 2 are figure. When The Player sees a 7 he knows the other number is a 6 — so both 6 and 7 are likewise figure. But when The Player sees a 2 to 6, the number he sees is always figure and the numbers he might switch to are always ground. That the numbers 2 to 6 can sometimes appear as figure and sometimes ground cannot help but cause confusion of the type illustrated by the dialogue. But this is the inevitable consequence of the cards being daisy chained.

E. Reframing The Random Walk Problem

The Random Walk Problem can be put in standardized form.

A Player is shown this infinitely large switching table:

$$\begin{array}{c} \vdots \\ -3 \longleftrightarrow -2 \\ -2 \longleftrightarrow -1 \\ -1 \longleftrightarrow -0 \\ +0 \longleftrightarrow +1 \\ +1 \longleftrightarrow +2 \\ +2 \longleftrightarrow +3 \\ \vdots \end{array}$$

He is assigned the number $N = 0$. He knows he can get 2^N units of cash for the number N, but he is entitled to evaluate N and optionally switch, iteratively, until he is satisfied with the amount he can get. What strategy should he follow?

The Player should just keep switching until a random walk takes him to the reward of 2^N units that he desires.

F. How Should You Think of the Card You Are Holding?

It is instructive to consider what it means to "hold a card." Consider this passage, written by mathematician Robert A. Rosenbaum [14]. It concerns not numbered playing cards, but baseball umpire Bill "The Old Arbitrator" Klem [15] calling balls and strikes:

> *Three umpires were discussing their own attitudes and virtues. The first umpire remarked, "Some is balls and some is strikes, and I calls 'em as I see 'em." The second, a bit more self-assured, said, "Some is balls and some is strikes, and I calls 'em as they are." Then up spoke Bill Klem, the dean of them all: "Some is balls and some is strikes, but 'til I calls 'em, they ain't* nothing.*" The first of these umpires, wrestling with reality to the extent of his own poor ability, is clearly an engineer type. The second, describing the universe in sure, bold strokes, is a close relative of a happy physicist — of the nineteenth century. But Klem takes his place with the* other *mathematical immortals — Bolyai, Lobachevsky, Riemann — who have invented their own worlds.*

Perhaps the best thing you can do when you hold an unidentified card from a deck is imagine it as "one of the deck," or, if you know one corner, as "one of several." Perhaps the worst thing you can do is treat the unidentified card you hold as a single card whose identity you happen not to know. The snare of regarding a *selected but-unlooked-at card* as somehow special is the subject of the next section.

VIII. THE MARKED CARDS PROBLEM

We will now consider The Marked Cards Problem.

A. Problem Statement

A marked deck of six playing cards is manufactured with faces as follows:

$$\begin{bmatrix} 1 & \\ & \rotatebox{180}{2} \end{bmatrix}, \begin{bmatrix} 2 & \\ & \rotatebox{180}{3} \end{bmatrix}, \begin{bmatrix} 3 & \\ & \rotatebox{180}{4} \end{bmatrix}, \begin{bmatrix} 4 & \\ & \rotatebox{180}{5} \end{bmatrix}, \begin{bmatrix} 5 & \\ & \rotatebox{180}{6} \end{bmatrix}, \begin{bmatrix} 6 & \\ & \rotatebox{180}{1} \end{bmatrix}$$

A Player examines how the cards are numbered. A Dealer shuffles the cards. She then places them facedown. The Player picks one card and secretly peeks at one numbered corner and sees a 3. He knows he can get 2^N units of cash for the number N he has chosen and that he is entitled to switch to the other numbered corner. Because the deck is marked The Dealer is able to see that the card chosen has a 3 and 4, but she does not know the number secretly chosen. Should The Player switch? What does The Dealer think The Player should do?

B. Analysis

The Player knows only the number chosen. He has narrowed down the *card* chosen to either of two.

The Dealer knows only the card chosen. She has narrowed down the *number* chosen to either of two.

For The Player the reduced switching table is:

$$\begin{array}{c} 2 \longleftrightarrow \underline{3} \\ \underline{3} \longleftrightarrow 4 \end{array}$$

and it makes sense to switch away from 3 (underlined) given the reward of 2^N.

For The Dealer the switching table is:

$$3 \longleftrightarrow 4$$

and she can see no advantage in The Player switching, because he has as much to lose as gain, the difference: $2^4 - 2^3$. Note that she does not know The Player's number, which is why no number is underlined.

Now assume The Player writes his chosen number on a chit of paper and hands it facedown to The Dealer. If she were to conclude that The Player's choice has changed into a single number, which she happens not to know, but which is on the chit she now *holds*, she would be making the same error as The Player who, *upon choosing a card and peeking at its corner*, imagines it represents a single switching table row that he happens not to know. It may be harder to see the absurdity of The Player's reasoning than The Dealer's, in part because the card he holds has two numbers. Nevertheless, The Player has no more right to think that what he holds represents a single table row, than The Dealer has to think that what she holds represents a single number. For Player and Dealer the card and number are, respectively, "one of two" until they are revealed.

The Player may be tempted to suppose:

> *I know the number under one of this card's corners. Had I initially chosen this card's other corner, then automatically switching would merely take me be back to what I have now... which would be pointless.*

But The Player who thinks this way is acting like he knows what "this card" is, which only The Dealer knows. Having made this mistake, The Player reaches the same conclusion as The Dealer — *there is as much to lose as gain* — but without The Dealer's justification.

Is The Player's reasoning totally wrong? Actually, no — it is insidiously wrong, which is far worse. By falsely imagining that he has pinned down the card he holds to just one, and therefore that he has only one corner/number to switch to, The Player is already seriously prejudiced against the benefits of switching, without even knowing the origin of his prejudice.

IX. THE TWIN'S SWITCHING PROBLEM

We will now consider The Twin's Switching Problem.

A. Problem Statement

A deck of six playing cards is manufactured with faces as follows:

$$\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \begin{bmatrix} 2 & \\ & 3 \end{bmatrix}, \begin{bmatrix} 3 & \\ & 4 \end{bmatrix}, \begin{bmatrix} 4 & \\ & 5 \end{bmatrix}, \begin{bmatrix} 5 & \\ & 6 \end{bmatrix}, \begin{bmatrix} 6 & \\ & 7 \end{bmatrix}$$

Twins, Viola and Sebastian, examine how the cards are numbered. A Dealer shuffles the cards and then places them facedown. Viola and Sebastian together settle on a single card. Viola peeks at one corner and Sebastian the other. Each can get 2^N units of cash for the number N they now have, but they are each independently entitled to switch to the card's other number, and there is no rule against them winding up with the same number. After peeking they may not communicate until after they have both decided what to do. So each must make their decision without knowing the other's number, or even whether the other has decided to switch. To maximize their joint return over time what strategy should they follow?

B. Paradox

The situation is symmetric. Sebastian reasons:

> *Let's assume we both follow the usual strategy and switch on* 1 *to* 6 *and hold on* 7*. Then, for* 1 *to* 6 *any gain I get by switching is Viola's loss, and vice versa. So the usual strategy is pointless.*

Is he right?

C. Analysis

No. Sebastian's claim that with the usual strategy "any gain I get by switching is Viola's loss" is obviously false when he has a 6 and Viola has a 7; then Sebastian gains when he switches from 6 to a 7, while Viola, holding with 7, does not lose. Moreover, by always switching on 1 to 6 they can gain this 7 with no offsetting loss, a point touched on in the dialogue:

> Sagredo: *Finally I see your point! To always — or never — switch on* **A**, **B**, **C**, **D**, *and* **E** *is equally profitable. But in the absence of letters my attempt to achieve an extra* 7 *when I switch on* 6 *leads me to sometimes switch from* $\mathbf{E}_6$ *to* $\mathbf{E}_5$ — *for a large loss. The only remedy for this is to always switch on the entire group* **A**, **B**, **C**, **D**, **E**. *This, as you say, restores the* status quo ante. *And, if I switch on* 1 *to* 6*, then I switch on this entire group.*

In settling on the above strategy Viola may reason as follows:

> *When I get a* 5*, I know that you, Sebastian, could have a* 6*. In trying to convert such a* 6 *into a* 7 *you would wind up with a* 5*, like me. Disaster! We both need to switch on* 1 *to* 6 *every time. This will allow us to convert our* 6*'s into* 7*'s, while preventing all such disasters at the smallest cost.*

In summary, following the usual strategy allows Viola and Sebastian to each convert a 6 to a 7 at no cost compared against the do-nothing-strategy. Of course, the do-nothing-strategy misses the opportunity to convert a 1 into a 2 at no risk. So, in the final accounting, the usual strategy allows Viola and Sebastian to each convert a 6 to a 7 at a cost of each converting a 2 to a 1. It involves a lot of switching, but it is the optimal plan.

It is instructive to consider what happens as the number of cards K grows ever larger. Then the situations in which Viola's strategy improves upon the above *do-almost-nothing-strategy* of "holding except on 1" are increasingly rare. Moreover, their persistent switching will — at least to some onlookers — appear to be ever more pointless: You can imagine the twins almost always rotating their jointly chosen card 180°.

On the surface the twin's strategy could not be more different from the do-almost-nothing-strategy, given that their strategy involves almost always switching, whereas the do-almost-nothing-strategy involves almost never switching. But it is Viola and Sebastian's fate to have their fortunes yoked in such a way that these strategies produce only two situations that are *economically* different. If there are K cards these two situations allow the twins to gain $2^{K+1} - 2^K$ at a cost of $2^2 - 2^1$. As K grows larger these two situations will become increasingly rare, and it may take Viola and Sebastian K or more plays before they find themselves pulling ahead of the benchmark do-almost-nothing-strategy.

D. What Should the Twins Do If They Do Not Know the Number of Cards?

It is convenient at this point to touch on the issue of what the twins should do if they know that the switching table takes the form:

$$\begin{gathered} 1 \longleftrightarrow 2 \\ 2 \longleftrightarrow 3 \\ 3 \longleftrightarrow 4 \\ \vdots \end{gathered}$$

where they do not know the number of rows/cards. A plausible strategy is for them to both switch on numbers that are less than "the largest seen so far," a strategy that would likely converge on the usual strategy within K hands if there are K cards.

That said, if the numbers in their switching table each occurred only once, and were instead *interleaved*, as here:

$$\begin{gathered} 1 \longleftrightarrow 2 \\ 3 \longleftrightarrow 4 \\ 5 \longleftrightarrow 6 \\ \vdots \end{gathered}$$

this would change everything. This is the subject of the next section.

X. THE INTERLEAVED CARDS PROBLEM

We will now consider The Interleaved Cards Problem.

A. Problem Statement

A deck of six playing cards is manufactured with faces as follows:

$$\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}, \begin{bmatrix} 3 & \\ & 4 \end{bmatrix}, \begin{bmatrix} 5 & \\ & 6 \end{bmatrix}, \begin{bmatrix} 7 & \\ & 8 \end{bmatrix}, \begin{bmatrix} 9 & \\ & 10 \end{bmatrix}, \begin{bmatrix} 11 & \\ & 12 \end{bmatrix}$$

A Player examines how the cards are numbered and notices that the cards are now not daisy chained by pairs of numbers — instead they are interleaved. One of the twelve numbers N is randomly assigned to him. He knows he can get 2^N units of cash for the number N and that he is entitled to switch to a different number on the same card. When (if ever) should he switch if he wants to maximize his return over time?

B. Reframing The Interleaved Cards Problem

The Interleaved Cards Problem can be put in standardized form.

A Player is shown this switching table:

$$\begin{gathered} 1 \longleftrightarrow 2 \\ 3 \longleftrightarrow 4 \\ 5 \longleftrightarrow 6 \\ 7 \longleftrightarrow 8 \\ 9 \longleftrightarrow 10 \\ 11 \longleftrightarrow 12 \end{gathered}$$

Luck randomly assigns him a number N from the table. He knows he can get 2^N units of cash for the number N and that he is entitled to switch. When (if ever) should he switch if he wants to maximize his return over time?

As with The Daisy Chained Cards Problem, The Player is given a chance to double or halve his reward, but now the numbers are no longer daisy chained. Instead, every number occurs just once, a key difference that makes it pointless to always switch.

But, in actual fact, The Player knows all he needs to know to make perfect choices every time, should he desire. In the absence of daisy chaining all he needs to do is identify the one row singled out by his number, and switch if his alternative number is higher than what he already has. This is a result of how the numbers in the switching table are "married" to each other. In the language of figure and ground this makes all twelve numbers figure.

Two observations:

- Always switching on *any odd number to any greater even number* is pointless.
- Always switching on *any even number to any greater odd number* is beneficial.

C. Daisy Chaining and The Two Envelope Problem

It appears to the author that The Two Envelope Problem with an assumption of at least some daisy chaining is closer to what people experience when they make decisions in everyday life, than is The Two Envelope Problem with universal interleaving, which appears to be more a product of purely mathematical thinking. Why should two dollars be paired with four dollars, but *never* with one dollar? Life is not like that. Be that as it may, we will now consider a problem where, depending on whose point of view is adopted, there either is — or is not — daisy chaining.

XI. THE TWO ENVELOPE PROBLEM OF RAYMOND SMULLYAN

With regard to The Two Envelope Problem, logician Raymond Smullyan observed [6]:

> *I suspect that probability is not the heart of the matter, and I have thought of a new version of the paradox which doesn't involve probability at all.*

A. Smullyan's Two Contradictory Propositions

He goes on to reduce the paradox to "two contradictory propositions":

> Proposition 1. *The amount that you will gain, if you do gain, is greater than the amount you will lose, if you do lose.*
> Proposition 2. *The amounts are the same.*

He then "proves" each in turn. The reader is asked:

> *Which of the two propositions is the correct one? They obviously can't both be right!*

B. Reframing Smullyan's Propositions

Smullyan's Propositions admit of more than one specific interpretation. For the sake of argument we will assume that Smullyan's Two Propositions can be reframed as:

$$2 \longleftrightarrow \underline{3}$$
$$\underline{3} \longleftrightarrow 4$$

and

$$3 \longleftrightarrow 4$$

with a reward of 2^N for Bob and Alice who have, respectively, the above switching tables. It follows that:

- Bob is assigned a 3. He has more to gain $2^4 - 2^3$ than lose $2^3 - 2^2$ by switching.
- Alice has as much to lose as gain by switching: the difference $2^4 - 2^3$. She has not been assigned a number.

C. Analysis

The reader will recognize that this is no more than The Marked Cards Problem, discussed in Sec. VIII, with:

- Bob in the role of Player.
- Alice in the role of Dealer.
- Bob deciding whether to switch from 3.
- Alice sharing Bob's reward.

Note that, originally, Alice sharing Bob's reward was not part of The Marked Cards Problem. It is added now to complete the problem.

D. The Dealer/Player Illusion

In the language of the figure/ground illusion:

- When we see things from Bob's perspective we see 3 as figure, and 2 and 4 as ground.
- When we see things from Alice's perspective we see 3 and 4 as ground.

In this way Smullyan's Propositions 1 and 2 are both accurate descriptions of The Marked Cards Problem, *but from Bob's and Alice's points of view, respectively*:

- From Bob's point of view he actually does have *something* to gain on average by his switching.
- From Alice's point of view she actually does have *nothing* to gain on average by Bob's switching.

And their respective switching tables are actually accurate representations of their situations *for each of them.*

It follows that for the omniscient reader who can see this situation either way, but who cannot adopt both perspectives simultaneously, the above *Dealer/Player Illusion* might paradoxically oscillate in the mind in much the same way this figure/ground illusion oscillates:

E. Summary

Smullyan's version of the Two Envelope Paradox is especially instructive because it directs attention toward the difference between Bob's and Alice's points of view. Smullyan's discounting the importance of probability is also insightful, as paradoxical aspects of the problem occur even when probabilities are deliberately minimized (as in The Two Commuters Problem). The author found Raymond Smullyan's version of The Two Envelope Paradox an excellent starting point for researching this article, therefore it is especially satisfying that it provides an excellent stopping point as well.

XII. CONCLUSION

This article will conclude by considering a few practicalities that have somehow not made it into the previous discussions.

A. The Two Envelope Problem Reconsidered

It is a simple matter to translate The Marked Cards Problem into a version of The Two Envelope Problem. A presenter simply picks a card, fills two envelopes according to the two numbers on the card, and then places these before The Player. However, the original Two Envelope Problem differs from The Marked Cards Problem in that no switching table is available. Are rewards/numbers daisy chained as they are in The Marked Cards Problem? Or interleaved as in The Interleaved Cards Problem? We do not know. Do the rewards/numbers have an upper or lower bound? Again we do not know. And, as The Marked Cards Problem shows, there are ample sources of confusion even when we have answers to these questions. Lacking answers, we face ambiguities that are impossible to resolve.

B. The Two Envelopes in Real Life

And what about switching envelopes in real life? If someone actually presents you with two envelopes, what should you do in the absence of a switching table? In this case you must consider how many times the offer will be repeated, your needs, and you must infer what you can about the switching table from the presenter of the envelopes. You also must keep in mind that large rewards are proportionately less likely.

So assume that you are given a one-time choice between two envelopes where one contains twice the other. You open one envelope and see so much money. Perhaps, given your limited needs, you should keep that amount? Or, perhaps, this amount is of little use to you just now, but twice that would satisfy your needs, and so you decide to gamble? Or, perhaps, from the limited means, or stinginess, of the presenter you know that the other envelope is unlikely to contain twice the amount you now hold? All of these questions can arise in making your decision.

C. The Probability Teacher Versus the Billionaire

Now imagine another one-time offer. You are shown two envelopes and are told that one contains one dollar, and the other either one million dollars or a tiny speck of copper worth 1/10 000¢. You are also told that you will only get to see the contents of the unchosen envelope if you actually switch. Luck assigns you the envelope with the dollar, so you have this reduced switching table:

$$\begin{aligned} 1/10\,000¢ &\Longleftrightarrow \underline{\$1} \\ \underline{\$1} &\Longleftrightarrow \$1\,000\,000 \end{aligned}$$

with the double arrows signifying that in this case the rows are not necessarily equiprobable. Should you switch? If the person filling the envelopes is the teacher of your probability course I would suggest you pocket the dollar (unless you *want* that speck of copper). But what if the person behind the offer is an elderly billionaire with a mischievous grin? If only for your peace of mind I would suggest you open the other envelope. And who knows?

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