Unfolding FOLDS: A foundational framework for abstract mathematical concepts

Jean-Pierre Marquis[1]
Département de philosophie
Université de Montréal
Montréal, Canada
Jean-Pierre.Marquis@umontreal.ca

[1]The technical apparatus and the formal definitions given in this paper have been presented by Michael Makkai over the last twenty years or so in various forms and in various places. The definitions are taken directly from various papers, talks and presentations given by Makkai. I will give specific references in the text. I want to thank Michael Makkai for his help. I also want to thank two anonymous referees for their comments, criticisms and suggestions. They allowed me to clarify important points and avoid some mistakes. All remaining mistakes, errors and omissions are mine.

Abstract

FOLDS, First Order Logic with Dependent Sorts, has been introduced by the logician Michael Makkai as a foundational framework to capture the abstract nature of contemporary mathematical concepts. In this paper, we present the underlying philosophical motivation of FOLDS as well as some of the salient technical features of the framework. We end by discussing what we take to be philosophically meaningful aspects of FOLDS and the accompanying framework.

1 Introduction

One of the greatest intellectual accomplishments of the 20th century is the creation and the development of a *science* of the foundations of mathematics. The roots of this development go back to the 19th century[1]. The development of first order logic together with the development of set theory lead to the constitution of a scientific foundational framework[2]. On the one hand, a precise formal system with definite properties in which axioms for various mathematical systems can be written was developed. On the other hand, a precise and mathematically defined universe of entities, e.g. a universe of pure sets, in which these axiomatic presentations can be interpreted and for which various results and theorems can be proved was constructed. The latter was conceived as the ontological component of the discipline, linked to the formal system by a rigorous semantic. These precise formal developments constituted a science in the following sense. First, both these components are taken as capturing significant properties of *already given* components of mathematical knowledge, e.g. a systematic but informal language and a system of informal mathematical concepts. Second, the precise and rigorous framework and its accompanying theoretical means allow for the rigorous and exact proofs of many important and significant results, e.g. completeness and compactness of first order logic, Gődel's incompleteness results, the consistency and the independence of the continuum hypothesis with the usual axioms of set theory, to mention but the most obvious. One should also keep in mind the creation and development of model theory and proof theory which can be seen as specific articulations and emphasis of various components of this picture.

The success of this intellectual enterprise should also be seen as giving us a set of norms for *any scientific* foundational framework[3]. Thus, *any scientific* foundational framework should be based on the following components:

[1]One could possibly go back to Leibniz to look for preliminary, informal ideas on the subject, but I believe that a reasonable demarcation point is the constitution of precise technical tools and in this regard, it is hard to deny that the 19th century provided the first real, precise and technical developments. However, the *scientific* character of the foundational enterprise has to be underlined and that feature came about only in the course of the 20th century. For a foundational framework to be considered scientific, it has to satisfy certain properties. I believe that this is a crucial component of Makkai's view on the subject.

[2]The history of the field is, as almost all histories are, quite convoluted. In particular, it is important to understand that the status that first order logic acquired is directly linked to its role in the development of axiomatic set theory. See, for instance, Moore (1980, 1987, 1988), Ferreirós (1996, 2001, 2007), Mancosu et al. (2009), Schiemer & Reck (2013).

[3]I emphasize the scientific character of the enterprise once again, for some might argue that the foundations must ultimately rest upon simple informal ideas or conceptions and that, for that reason, *formal* set theory or *formal* category theory cannot provide "real" foundations. I take it that this is or is sufficiently close to Mayberry's view as expressed in **?**. By underlining the scientific character of the discipline, I want to emphasize the fact that the methods used in the science of foundations are the same as in the other sciences. Thus, there is, as I have said, an informal or intended collection of facts that the foundational system is supposed to capture, reflect and illuminate. In that respect and as in the natural sciences, there may be a chasm between the scientific image resulting from the theoretical work and the pretheoretical ideas used and known.

1. A precise and explicit mathematical syntax, also known as a formal system or sometimes a language, $\mathcal{L}$, given by recursive rules, together with a deductive structure, also known as a logic; the latter can be given independently or it may be inherent to the formal system right from the start[4];

2. A mathematical construction or definition of a universe $\mathbb{U}$ of mathematical objects;

3. A systematic interpretation of a theory written in the language in the constructed universe such that some of the propositions of the theory, usually called "axioms", become "obviously" true under that interpretation.

As we have said, in the present state of affairs, the formal system is first order logic with the usual axioms or rules and the theory is given by the axioms of Zermelo-Fraenkel set theory. The specific universe intended is provided by the cumulative hierarchy, thus the universe of pure or regular sets[5].

It should be clear from our opening sentence that we believe that the standard framework has much to commend. Our goal in this paper is to present a different foundational framework which captures another, complementary aspect of mathematical concepts that the logician Michael Makkai has been developing. It is not that the standard framework has to be rejected. It does what it does very well. But it suffers from a conceptual limitation: it does not model an aspect of mathematical knowledge that emerged concurrently with the development of the foundational framework itself. As is often the case in the history of ideas, it is only when an alternative theoretical framework is sufficiently developed that one can clearly see a "flaw", "limitation" or an "anomaly" in the accepted theory[6].

The "anomaly" of the current set-theoretical picture has to do with the fact that it fails to capture the *abstract* character of contemporary mathematical concepts. With the advent of the abstract method and the resulting development of conceptual mathematics, a large portion of contemporary mathematics has become resolutely abstract. Of course, what the latter expression means is debatable[7]. Be that as it may, the anomaly comes from the codification of the

[4] An instance of the latter case is now given by Homotopy Type Theory. The standard presentation of first-order logic separates the purely linguistic component from the logical system.

[5] Of course, there numerous, incompatible models of the universe of set theory and it is part of the scientific foundational enterprise to build and investigate the properties of these universes to adjudicate their values and merits.

[6] Thus, instead of seeing set theory and category theory as rivals, one could draw an analogy with the case of newtonian physics and relativity theory or classical physics and quantum physics.

[7] One thing *is* sure. It certainly does not have the usual ontological meaning. Mathematicians certainly do *not* mean, when they talk about abstract algebra, that that kind of mathematics has no spatio-temporal coordinates or that it is causally inert, etc. They do not have in mind the traditional philosophical distinction between the abstract and the concrete. I, for one, am convinced that they have an epistemological distinction in mind. See, for instance, Marquis (2014). The underlying view of what it is to be abstract is here rooted in the

abstract part of contemporary mathematics into the universe of pure sets. The abstract nature of the concepts involved is simply lost by the codification into pure sets. To see this, we need to delve into the nature of abstract mathematical concepts and pure sets. Once this is done, we will sketch the basic components of the alternative foundational framework that is emerging and present some of its philosophical implications.

2 Some generalities about abstract mathematical concepts

2.1 Contrast and compare: pure sets

Let us start with a recapitulation of the universe of pure sets, for this will allow us to exhibit the main conceptual differences between the latter and the picture we are introducing. Informally, a pure set is a collection of pure sets. This apparently circular characterization can be explained analytically or synthetically. Analytically, it means that when one decomposes a pure set into its elements, one again finds pure sets and this process repeats itself uniquely until at the end of each branch of the decomposition one finds the empty set. Synthetically, it means that one starts with the empty set, which is a pure set, and constructs sets from there by various well-known (possibly infinitary) operations[8].

There is no need to present and discuss the universe of pure sets in great detail, for our point here is very simple[9]. I claim that pure sets do not encode abstract mathematical concepts properly[10]. The introduction of pure sets in the foundational landscape provides a clean and tidy house: every object has a definite structure, a definite nature and can be identified uniquely by its elements. If pure sets had physical properties, they would be prototypically concrete entities. To clarify this latter statement, consider an arbitrary pure set X and an element $x \in X$. The element x is itself a pure set and it is

way it is understood in the development of mathematics, particularly in the 20th century. It is certainly close to some views defended in the philosophical literature, for instance **?**. However, we refrain from using the term "structuralism" in this context and prefer to concentrate on the abstract character of the concepts. The term "structuralism" as it is used by most philosophers is appropriate to but a small fragment of the universe we have in mind here.

[8]Needless to say, this is very informal and one has to be careful. In the analytic description, carelessness will lead to non-well founded sets, whereas in the synthetic description, one has to make sure to have infinitary operations at hand, otherwise the universe will contain only hereditarily finite sets. The latter are certainly important, but they clearly do not provide the proper picture of the usual universe of sets.

[9]Category theorists have been critical of the underlying conception of sets provided by ZF from the 1960s onwards. These criticisms are in fact pointing in the direction we are engaging in, without emphasizing the abstract nature of the notion of sets defended. In fact, many opponents to the views articulated then thought that category theorists were simply trying to eliminate sets altogether, which lead to a profound misunderstanding between the two groups. For some of the critical arguments and alternative theory presented, see, for instance, **???**.

[10]In a sense, one could interpret Benacerraf's well-known argument as saying just that. In fact, as Makkai has already mentioned, FOLDS provides an elegant and direct solution to that problem. See, for instance, Makkai (1998, 1999).

therefore itself entirely determined by its own elements. In that sense, x has an independent identity from X. In fact, x has its own unique identity in the universe of pure sets. To illustrate this, consider a specific pure set, e.g. let X be the pure set $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ and let x be $\{\emptyset\}$. Both x and X are pure sets and $x \in X$. Although it cannot be said that pure sets are concrete in the standard ontological sense of that expression — that is they do not possess spatio-temporal coordinates nor are they causally efficacious, at least in the way molecules are — they certainly are concrete in an epistemic/semantic sense. We have a complete picture of the components and the composition of a pure set. Furthermore, all this information is contained in the notation itself. When we write down a specific pure set, e.g. $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, we can say exactly what are its components and how these are put together to yield that particular, unique pure set. Last, but not least, this also allows us to see why the axiom of extensionality holds for pure sets. The latter only confirms the fact that pure sets — in fact any system of sets satisfying the ZF axioms — should be thought of as being particulars or individuals[11]. Indeed, the axiom of extensionality constitutes the criterion of identity for ZF-sets: two sets X and Y are the same if and only if they have the same elements. The beauty of pure sets is that the criterion of identity is homogeneous or uniform: it works all the way down on the elements of pure sets, that is to determine when two given elements of X and Y are the same is still an internal matter. Let us now contrast this with the way mathematicians talk about abstract mathematical concepts.

2.2 Compare and contrast: being abstract

"Let G be an abstract group."

This is a common way of talking in contemporary mathematics, say in group theory or in representation theory. If every mathematical object is a set, then the abstract group G ought to be a set or have an underlying set. Can this underlying set be a *pure* set? Suppose it is. Then, it should be a particular pure set X_G with specific properties, e.g. having specific elements, e.g. $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \in X_G$. But the specification that the group G is an *abstract* group is given *precisely* to avoid this situation, that is, the elements and the properties of the underlying set as a specific, concrete set are completely irrelevant. What mathematicians want to say in this context is that the abstract group has an underlying *abstract* set and the properties of the latter that ought to be considered are those that a set possesses as an abstract set and nothing else.

The situation is even more strange when one considers *operations* on abstract

[11] I prefer this terminology to the term "concrete". In fact, the term "abstract" should perhaps be replaced by the term "universals" as the latter was used in philosophy. The problem is not so much with the term "abstract", but more with the term "concrete". It is more difficult to talk about concrete mathematical entities. The term "abstract" in mathematics is linked to the abstract method or to a method of abstraction and, as such, should be opposed to a method of representation. For more on the relevance of these distinctions in the present context, see Ellerman (1988), Marquis (2000).

groups[12]. Thus, let G and H be abstract groups. Now, suppose a mathematician was to present the following construction: consider the underlying sets X_G and Y_H of G and H and take their intersection $X_G \cap Y_H$. And now suppose that the mathematician suggests to work with the latter set-theoretical construction to get some results about abstract groups. Now, if the underlying sets X_G and Y_H are pure sets, then they have a well-defined and unique intersection which is a pure set. Clearly, the latter has nothing to do with the group structure of G and H. The construction presented would certainly be judged awkward and irrelevant.

When a mathematician says "let G be an abstract group", she probably has two features of G in mind: 1. The precise nature of the elements of G is left unspecified and 2. the only properties that she is interested in are those that are attributable to G as a group. More specifically, the first feature means that the elements of G can be presented in various ways and be of various types, the nature of which can vary considerably from one embodiment to the other and the properties of these elements are irrelevant. The second feature means that our mathematician knows what it is to be a group-theoretical property. A mathematician certainly learns to identify the relevant group-theoretical properties and distinguish them from, say, set-theoretical properties. In the vernacular language, a mathematician can identify a group-theoretical property when she sees one. Is there a way to formally express what the latter means? Structure-preserving maps, in this case group homomorphisms, preserve some group-theoretical properties. As is well-known, this is not enough. Some important algebraic properties are not preserved by homomorphisms[13]. The proper answer is this: the relevant properties are those that are preserved under the right criterion of identity for the entities considered. In the case of groups, this means that the relevant properties are those that are invariant under group isomorphisms[14].

Our mathematician certainly thinks that the abstract group G has an underlying *abstract* set[15]. An abstract set is basically a set whose elements have

---

[12]This example is given by Makkai at various places.

[13]For instance, an ideal of a ring is not necessarily preserved by an arbitrary ring homomorphism.

[14]This invariance property has also been associated with abstraction in modern mathematics in the literature. The first person to explicitly make the connection is, to my knowledge, Hermann Weyl in **?**. It was discussed and developed by Stephen Pollard in two articles. See **??**.

[15]The French mathematician Maurice Fréchet, who was one of the pionners of the abstract method in mathematics, described abstract sets thus:

> In modern times it has been recognized that it is possible to elaborate full mathematical theories dealing with elements of which the nature is not specified, that is with abstract elements. A collection of these elements will be called an *abstract set*. (...)
>
> It is necessary to keep in mind that these notions are *not of a metaphysical nature*; that when we speak of an abstract element we mean that the nature of this element is indifferent, but *we do not mean at all that this element is unreal.* Our theory will apply to all elements; in particular, applications of it may be made to the natural sciences. ((**?**, p. 147))

no structure. It is therefore made up of faceless, or in the words of Fréchet, abstract individuals. The *only* information we have about such elements is that it is possible to differentiate them somehow, that is, each set X comes equipped with an identity relation that allows us to tell, for two elements x and y of X, what is the truth value of the proposition "$x =_X y$". Since the elements x and y are abstract, they have no independent existence outside of X. They are given by X.

If an abstract set has abstract elements, to what extent do these elements determine the identity of that set? There is no reason to believe that they should. The identity of an abstract set should not depend on the identity of its elements. One key observation leads us towards the solution. Abstract elements together with an identity relation allows us to take the notion of *function* between abstract sets as primitive. Thus, abstract sets are taken to form a system. After all, if these abstract sets have to capture the fact that certain abstract mathematical concepts have underlying abstract sets and if these abstract mathematical concepts are naturally connected to one another or exhibit some form of conceptual dependence, then it seems reasonable to expect that some of these dependences are already noticeable at the level of sets. However, once the notion of function is available and given that there is a criterion of identity for functions, then there is a natural criterion of identity that emerges naturally, namely the notion of bijection between abstract sets. In fact, it makes perfect sense to say that two abstract sets are identical if there is a bijection between them. For, this is indeed a criterion of identity that applies to a set as a unit, independently of the nature of its elements, as it should be.

The question naturally arises at this stage as to whether one can construct a theory of abstract sets as a special case of a theory of abstract mathematical concepts. Indeed, such a theory can be developed and has been developed. I refer the reader to **?** for details[16].

The foregoing discussion contrasting pure sets with abstract sets allows us to postulates two basic principles about abstract mathematical concepts in general. First, we posit that one of the specific properties of abstract concepts is that they are known via their instances, that is something that is seen or conceived as being an instance of that concept. It is important to understand that the last claim means that the instance is given as an instance, not as something that has an independent identity. It certainly can, but in the context where the abstract concept is given, the instance is dependent on that concept and is not and cannot be considered independently of the concept. Thus, abstract concepts come *together with* and are *inseparable from* instances. They are *not*, however, *identified* with these instances. We think about the abstract concepts with the instances and never solely with the concepts. I believe that this is a fundamental cognitive aspect of abstract mathematical thinking.

Second, abstract mathematical concepts are not given individually, inde-

This is a specific quote by a mathematician that specifies that the property of being abstract is epistemological rather than ontological.

[16]The question as to which mathematical concepts ought to have underlying abstract sets can also receive a precise mathematical answer.

pendently of one another. There is a natural order, a natural organization of concepts. In fact, some concepts depend upon others, previously given, concepts. The easiest way to illustrate the idea is via the idea of dependent variable in the natural sciences. When various concepts are linked to one another by a functional relation, then certain concepts depend upon others. In these cases, a function f represents a certain dependency between concepts and the function is tied to the conceptual context. It might be mathematically simple and, in a direct sense, detachable from that context as a mathematical operation, but it appears in the conceptual framework and plays a role as such in that framework. The surprising fact is that something similar occurs in mathematics and in such a way that it ought to be captured by a foundational framework.

Although this is not quite a form of conceptual holism, it is certainly at odds with a long standing tradition in analytical philosophy, namely logical atomism[17]. It is not a form of holism simply because there is an organization which allows one to separate some parts from the others and consider certain components independently of others. However, the latter cannot be done arbitrarily nor can it be done completely, as if one would decompose a living cell down to its atoms and thus believe to completely understand what life is. These informal, general and imprecise remarks will hopefully become clearer once we will have introduced the framework more formally.

2.3 Informal remarks about the syntax and the logic for abstract concepts

Both of these basic epistemological tenets are reflected in the grammar of the theory itself. The first tenet is encoded by adopting the following convention: we write "$x : X$" to declare that x is an instance of the concept X. Since this is a declaration, the expression "$x : X$" is *not* a proposition. It cannot be true or false. We can immediately infer that the string of symbols "$\neg(x : X)$" will not be well-formed in the syntax of the system. The instance x comes with the concept X[18]. The instance x cannot be considered by itself. It is only seen as an instance of X, thus bringing to the front some of its features and pushing in the background other features. In contrast with the case of set theory, we do not have x on the one hand, and the concept X on the other and verify that indeed x has the right properties associated to X. Thus, abstract mathematics starts with concepts together with instances of the latter. This is an ontological

[17] It is certainly not a coincidence that the doctrine of logical atomism made its appearance soon after the acceptance of the doctrine of atomism by physicists. One wonders how atomism in general influenced the thinking about the foundations of mathematics and the establishment of set theory as such a foundation.

[18] Some would say that it is constructed from X. For the time being, we want to stay away from that terminology. The main point is simply that, to know X, one has to consider x and x naturally comes with X by a certain, unspecified process. The latter process can be clarified in the semantics. For instance, in a category $\mathbb{C}$, an instance $x : X$ can be given by a morphism $x : 1 \to X$, where 1 denotes the terminal object of the category, when it has one. It can also be what is called a *generalized element* $x : U \to X$ from an arbitrary object U of the category.

shift with respect to the standard set-theoretical picture. In the latter, a set is built up from its elements, whereas in the picture I am presenting, the instances are always presented as such, that is as representations of a given abstract concept[19].

The second tenet is expressed by the fact that we are introducing a language with dependent sorts[20]. These dependencies put strict constraints on the grammar of the language, as we will see.

Let us now come back to the development of abstract mathematical concepts. While a science of the foundations of mathematics was put on firm grounds, mathematics itself was undergoing profound changes. The abstract method played a key role in these changes. With its help, mathematicians started to talk and theorize about monoids, groups, rings, fields, vector spaces, topological spaces, differential manifolds, Banach spaces, Hilbert spaces, partially ordered sets, lattices, categories, homology and cohomology theories, abelian categories, triangulated categories, derived categories, monoidal categories, etc. Each one of these abstract concepts comes with its own criterion of identity, different from the criterion of identity for sets. In fact, the criterion of identity for these concepts is extracted from the concepts. It is not given *a priori*. For example, the criterion of identity for groups is the notion of group isomorphism, the criterion of identity for topological spaces is given by the notion of homeomorphism and the criterion of identity for categories is the notion of equivalence of categories. We take it that this facet of abstract mathematical concepts ought to be reflected directly in the foundational framework. In particular, the framework should not have a universal identity relation, usually denoted by the equality sign "=". The criterion of identity relevant for the entities at hand has to be determined by the abstract concepts themselves. The criterion of identity should be *derived* from the concept themselves.

Another key feature of abstract mathematical concepts is that although the concept determines the notion of identity for its instances, there can be many different identities between two given instances. Thus, there can be many different identities, that is isomorphisms, between two given groups and even between one and the same group. This might sound strange and surprising, but it is as it should be for instances of abstract concepts. For, two instances of an abstract concept are identical (or should we say "equivalent") whenever they have the same properties determined by the abstract concept they are instances of. Since these properties are preserved by any isomorphism between them, each and any one of these isomorphism constitute a way of being identical as in-

[19]Thus, in some sense, we are trying to reintroduce a certain aspect of the comprehension principle. Of course, the principle itself is not introduced, but the constraints put on the grammar are such that there are no independently given atoms from which the universe can be built. To repeat what was already said in the previous section: concepts and their instances are woven together right from the start.

[20]This is not new, nor is the previous grammatical convention with sorted variables. These features have been used by logicians and computer scientists for more than forty years now. In particular, the specific language with dependent sorts or types, as they are also known, was introduced by Martin-Löf in 1970. There are new formal elements introduced by Makkai, as we will see in the next sections.

stances of the given abstract concept. In fact, knowing ways of being identical, even self-identical, reveals a lot of information about the concept itself.

Furthermore, an instance of an abstract concept will have the properties determined by the concept, but it will also have other properties inherent to the mode of presentation used. Thus, in the context of pure sets, when groups are presented, they not only have group theoretical properties as they should, but they also have irrelevant set-theoretical properties. This may bring in a certain confusion. An adequate language for an abstract concept should allow us to write only properties that are relevant for the concept. The notion of relevance is here determined by the criterion of identity. Only the properties invariant under the given criterion of identity should be expressible in the language. More precisely, given a language $\mathcal{L}$, a signature $\mathcal{S}$ in $\mathcal{L}$ and a derived criterion of identity $x \simeq_{\mathcal{S}} y$, then if $P(x)$ and $x \simeq_{\mathcal{S}} y$, then we should have $P(y)$. It should be possible to *prove* this invariance principle in the foundational framework for abstract mathematical concepts.

These remarks indicate that the language developed will have different properties from the standard syntax of the language for set theory.

2.4 Informal remarks about the universe of abstract mathematical concepts

The universe of abstract mathematical concepts will also differ considerably from the universe of pure sets, e.g. the cumulative hierarchy. Let us emphasize immediately one point they have in common: in both cases, we deal with hierarchies. However, this common feature is in fact very superficial, for the hierarchies are deeply different, both from an ontological point of view and from an epistemological point of view. In the case of abstract mathematical concepts, the hierarchy is based on the introduction of levels of abstraction, that is systems of different kinds, irreducible to one another[21]. In the case of sets, the universe is composed of a unique kind. The hierarchy in the cumulative hierarchy is determined by the *rank* of a set, that is the least ordinal number greater than the rank of any member of the set[22].

The basic structure of the universe can be given informally as follows[23]. The first level is made up of abstract sets[24]. As we have said, each abstract set X comes with an identity relation $=_X$ for its elements. The identity criterion for sets is given by the notion of bijection between sets. The totality of sets is not a set, it is a *category*. Most of the abstract mathematical concepts introduced in the last quarter of the 19th century and the first half of the 20th century can be described and studied at this level. Thus, the category of monoids,

[21] Contemporary mathematicians commonly talk about levels of abstraction. See, for instance, **?**, for a preliminary exploration of the idea. Introducing such a hierarchy seems to be a key idea in contemporary artificial intelligence. See, for instance, **?**.

[22] Assuming the axiom of foundation, of course.

[23] A more technical description will be given in section 4.

[24] There is some fluctuation still. One could start with a different first level. But we won't get into these options here.

groups, rings, fields, topological spaces, vector spaces, etc., form categories in this sense[25].

Let us be a little bit more precise: abstract sets are connected by functions. The latter compose and satisfy the expected equalities and each abstract set has an identity function whose composition satisfies obvious equalities. Thus, the system of abstract sets is a category. Identities between sets is given by the notion of isomorphism or bijection between sets. Recall how one defines the notion of isomorphism between sets: an isomorphism between sets X and Y is a function $f : X \to Y$ such that there is a function $g : Y \to X$ satisfying $f \circ g = 1_Y$ and $g \circ f = 1_X$. Notice the presence of the identity symbol between functions. The identity symbol is in fact coherent, since we have assumed that every abstract set X possesses an internal identity relation $=_X$ and it is these relations that are at work here. In fact, we should write $f \circ g =_Y 1_Y$ and $g \circ f =_X 1_X$ to be exact. This notion of isomorphism within a category works perfectly well for the usual set-based notions: isomorphism for groups, homeomorphism for topological spaces, diffeomorphism for manifolds, etc. are the criteria of identity for these concepts. So far so good.

We now have categories. Since there is no identity relation between the objects of a category – isomorphisms as defined in a given category play that role –, an abstract category is not an abstract set. Hence a category is a new kind of object or system. Whereas sets are connected by functions, categories are connected by *functors*. When categories are defined as sets, a functor $F : \mathbf{C} \to \mathbf{D}$ is given by two functions, a function $F_O : Ob(\mathbf{C}) \to Ob(\mathbf{D})$ that sends objects of $\mathbf{C}$ to objects of $\mathbf{D}$ and a function $F_M : Mor(\mathbf{C}) \to Mor(\mathbf{D})$ sending morphisms of $\mathbf{C}$ to morphisms of $\mathbf{D}$ such that $F_M(1_X) = 1_{F_O(X)}$ for all objects X of $\mathbf{C}$ and $F_M(f \circ g) = F_M(f) \circ F_M(g)$[26]. This seems innocuous: it simply says that a functor preserves the structure of composition of morphisms of a category. Thus a functor is a structure preserving morphism between categories. However, the definition is inconsistent with the facts – I am tempted to say "the reality" – of category *theory*[27]. The inconsistency can be seen from different angles. Here is

[25]Ironically, these categories, the category of monoids, groups, rings, fields, posets, lattices, topological spaces, manifolds, vector spaces, etc., that is categories of structured sets, are called "concrete categories" in the literature! The latter term has a precise technical meaning: a category $\mathbf{C}$ is said to be *concrete* whenever there is a faithful functor to the category of sets. It can be shown that there are non-concrete categories in this sense. See, for instance, **??**.

[26]For a covariant functor.

[27]I should emphasize that this fact was not and could not be obvious to Eilenberg and Mac Lane. The reason for this is simple: one could argue that although Eilenberg and Mac Lane introduced the concepts of category, functor and natural transformation, they did not develop category *theory* in their original paper. For some of the core concepts of the theory were introduced 10 years later by Kan and Grothendieck. (For more on this, see, for instance **??**.) The inconsistency as it is presented here was observed already in the 1960s and was made explicitly by Jean Bénabou and by G. Maxwell Kelly and probably many others. In Bénabou's mind, it gave rise to the development of an idea introduced by Lawvere, namely what are called distributors and it can be shown that a locally representable distributor is an anafunctor. Kelly introduced what essentially became the notion of anafunctor in Makkai's writings, but did not develop the theory. See **??**. For the sake of completeness, I should mention that these notions are linked to the notions of profuntors and pseudofunctors in the literature. These remarks do not do justice to the history of the subject, but we will leave it

a simple one. Within a category and from the perspective of category theory, any object isomorphic to a given one does the same categorical work. In other words, in a category, mathematics is done up to isomorphism. Now, when a functor F is defined as in the foregoing definition, it assigns to a given object X of $\mathbf{C}$ a unique object $F_O(X)$ of $\mathbf{D}$. But, and this is the inconsistency introduced by the usual set theoretical definition, any object isomorphic to $F_O(X)$ would do and should do. A functor cannot identify, in the usual sense of that word, two objects X_1 and X_2 of $\mathbf{C}$, i.e. one cannot write $F_O(X_1) = F_O(X_2)$, since there are no identities between objects in a category.

Consider now the effect this fact has on the identity of categories themselves. In their original paper on categories, Eilenberg and Mac Lane treated the question of the identity of categories in a standard algebraic fashion for the period: they stipulated that two categories are identical whenever there is an isomorphism between them. An isomorphism between categories is defined in the expected way: a functor $F : \mathbf{C} \to \mathbf{D}$ is an isomorphism if there is a functor $G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F = 1_{\mathbf{C}}$ and $F \circ G = 1_{\mathbf{D}}$. These equalities hold also for the *objects* of the categories involved, e.g. $G(F(X)) = 1_{\mathbf{C}}(X)$ for all objects X of $\mathbf{C}$. Since there are no equalities between objects, the notion of isomorphism of categories is inadequate. Ironically, Eilenberg and Mac Lane's motivation for the introduction of categories was to provide the proper mathematical setting to express the notion of natural transformation, that is the appropriate notion of morphism between functors, and the latter is the key to the concept of identity for categories.

Indeed, to obtain the right notion of identity for categories, one has to replace the identities between the compositions and the identity functors by natural transformations that are isomorphisms, i.e. the foregoing equations are replaced by the existence of two natural isomorphisms $\eta : G \circ F \simeq 1_{\mathbf{C}}$ and $\mu : F \circ G \simeq 1_{\mathbf{D}}$, satisfying certain obvious conditions. Thus, when one starts with an object X, coming back in the category via the composition of the functors G and F yields an object *isomorphic* to X. Whenever these functors and natural isomorphisms exist, the categories involved are said to be *equivalent*. This is the correct notion of identity for categories.

Thus, a system of categories is a new kind of system. It consists of categories, functors *and* natural transformations. In other words, such a system is composed of objects, morphisms between objects, called 1-morphisms, and morphisms between morphisms, called 2-morphisms. Informally, it seems to be simple enough, but the complexity of the notion follows from the various ways these morphisms compose. There is an operation of composition for 1-morphisms and another one for 2-morphisms and these two operations necessarily interact and these interactions have to be coherent with one another. It is tempting to say that a system of categories has to be a set and has to be a category. From the point of view we are trying to develop, both claims are false. That is, from a specific theoretical point of view, a system of categories does *not* have an underlying abstract set. Furthermore, since a category only has (1-)morphisms and

at that.

since a system of categories has 2-morphisms that play a key role in the structure of the system, a system of categories cannot be merely a category. Thus, a system of categories is an entity of a new kind, usually called a bicategory or a weak 2-category[28]. It takes more than one page to give the formal definition of a weak 2-category. The difficulty is not conceptual, but combinatorial, so to speak. We have now introduced a new kind of entity: a weak 2-category. One can easily convince oneself that there are many weak 2-categories and that there are morphisms between them. Thus, given the latter, we have that a system of weak 2-categories is made up of objects, 1-morphisms, 2-morphisms and, these are now new, 3-morphisms between 2-morphisms. All these morphisms compose and these compositions have to respect certain "laws". Once again, they form a system and one has to find a proper criterion of identity for weak 2-categories. At this stage, the reader will not be surprised by the claim that a system of weak 2-categories cannot be a set, a category or a weak 2-category. It is a weak 3-category.

There is a clear pattern emerging. Informally, one expects this organization to go on and to consider weak n-categories which would constitute a totality of weak $(n-1)$-categories. The totality of all these would then be an ω-category and at this stage, it is possible that the latter would itself be an ω-category[29].

One of the reasons motivating the formulation of FOLDS is precisely to provide an appropriate formal language to describe this hierarchy directly and properly. As we will briefly indicate, one of its central notions can be used to give the fundamental components of the universe we are trying to grasp.

The informal picture of the universe of abstract mathematical concepts is clear enough. The precise, technical mathematical picture is also becoming clear. We now have the barebones of a formal system, a universe of mathematical objects in which the language can be interpreted systematically. Let us now put some flesh on these bones.

3 The Formal System: FOLDS

We now come to the first component of the extended science of foundations: the formal system. We will not present the system of FOLDS in all its details. We will sketch some of the main features of the language[30].

[28]There is also a notion of strict 2-category, but we will ignore the subtle differences between these notions.

[29]I should underline that there is no a priori necessity involved here. It could be that at some level, these notions stabilize into a unique notion. For instance, it is possible to show that every weak 2-category is biequivalent to a 2-category. This latter result can be interpreted as saying that the notion of weak 2-category is reducible to the notion of 2-category. However, this result does not hold for the notions of weak 3-categories and 3-categories. I will not give the definitions of all these mathematical concepts here. In fact, there are various different definitions in the literature, but their study is evolving rapidly. See **??**, for instance. I will sketch the formal approach for one of these concepts in section 4.

[30]FOLDS is a type theory and like all type theories, it tries to avoid the paradoxes not by stipulating certain axioms, but by constraining the linguistic resources right from the start. One of the underlying ideas is that the linguistic constraints should reflect a certain ontology.

3.1 FOLDS-signatures

FOLDS is an extension of first-order logic (FOL). Thus, it has the usual quantifiers and propositional logical connectives, to which we add two propositional constants $\top, \bot$. The language comes with dependent sorts (or types) and variables are sorted. Thus, the quantifiers are always bounded since the variables are sorted.

All textbooks in model theory start by defining what is called a *language*. The latter is not arbitrary: a language $\mathcal{L}$ is designed to describe what is called a mathematical *structure*. Sometimes, an author specifies a *signature* for a given type of structure, but most of the time the notions of language and signature are identified.

Recall that a first-order language $\mathcal{L}$ or a $\mathcal{L}$-signature is given by:

1. a set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$;
2. a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$;
3. a set of constant symbols $\mathcal{C}$;

where n_f and n_R are the usual functions giving the arities of the function symbols and relation symbols in both sets. Any of these sets can be empty and, it is enough, although in many cases unnatural, to work with a set of relation symbols only. This latter remark is important since this is how one can show that FOLDS is an extension of FOL.

FOLDS departs from FOL by introducing a different notion of $\mathcal{L}$-signature. This is as it should be given that the "structures" we are interested in are of different kinds than the usual structures based on abstract sets. As we have just remarked, this new notion of signature still covers the usual notion of signature. To understand the notion of FOLDS signature, observe that an n-ary relation symbol sorted as $R \subset X_1 \times ... \times X_n$ can be replaced by a new sort R together with operations $R \xrightarrow{p_i} X_i$. The latter symbol with the (sorted) operations is a one-way graph, that is a graph whose arrows are all going in the same direction. Thus, in the language of FOLDS, an n-ary relational symbol becomes a one-way graph with two levels and n operations. A FOLDS-signature is a generalization of that situation, that is, there can be (usually finitely) many levels and the morphisms between the levels compose. Instead of considering (one-way) graphs, it turns out to be simpler to use the notion of (one-way) category directly. Let us fix the terminology for the remaining sections of the paper: a *proper* morphism is a morphism different from an identity morphism.

For a short history of type theory, see **??**. When Makkai introduced FOLDS twenty years ago, Bart Jacobs' book on categorical logic had not been published. Dependent type theories were well-known to computer scientists and logicians working in theoretical computer science. Jacobs gives a presentation of first-order dependent type theory in chapter 10 of his book. His presentation and development of the framework is completely different from Makkai's development. Makkai's motivation is closer to Martin-Löf's motivation, although the latter was trying to provide a foundational framework for *constructive* mathematics, whereas I hope it is clear by now, Makkai is trying to articulate a foundational framework for *abstract* mathematics. See **?** for Jacobs presentation. For Martin-Löf's work, see **?**, **?**.

Before we give the definition of a FOLDS-signature, we need the following formal definition.

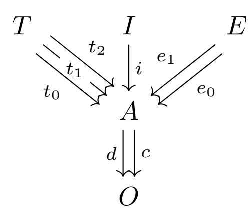
Definition 1 *A category $\mathcal{L}$ is said to be* one-way *if it satisfies the following conditions:*

1. *$\mathcal{L}$ is small;*
2. *$\mathcal{L}$ has the finite fan-out property: for any object K of $\mathcal{L}$, there are only finitely many morphisms with domain K;*
3. *$\mathcal{L}$ is reversed well-founded: there is no infinite ascending chain $\langle K_n \xrightarrow{f_n} K_{n+1} \rangle_{n \in N}$ of composable proper morphisms ($f_n \neq 1_{K_n}$).*

It follows from the definition that in any one-way category, the only morphism from an object K to itself is the identity morphism 1_K. The last condition of the definition implies that there are no cycles in a one-way category, that is there are no cycles $K_0 \to K_1 \to ... \to K_n$ of proper morphisms with $K_0 = K_n$.

Definition 2 *An $\mathcal{L}$-signature for FOLDS is a one-way category.*

The standard example of a FOLDS signature is the $\mathcal{L}_{Cat}$-signature for the concept of abstract category. Here is a representation of the one-way category with all the non-composite morphisms displayed and the identity morphisms omitted.



Since the signature is a category, the morphisms compose and there are identities between some of them. In fact, we must have the following identities:

$$\begin{array}{lll} d \circ t_0 = d \circ t_2 & d \circ i = c \circ i & d \circ e_0 = d \circ e_1 \\ c \circ t_0 = d \circ t_1 & & c \circ e_0 = c \circ e_1 \\ c \circ t_1 = c \circ t_2 & & \end{array}$$

There is a natural informal interpretation of this signature. The symbol 'O' stands for the sort of objects, 'A' for the sort of arrows, 'I' for the sort of identity arrows, 'E' for the equalities between arrows and 'T' for the commutative triangles of a category. Given this informal interpretation, the identities between the composites become obvious. The morphisms between the sorts denote the dependencies involved. Reading from bottom to top, we understand that the morphisms depend upon the objects, more precisely a morphism must be given with a domain and a codomain. Thus, in order to write down that the symbol 'f' is an arrow, we must first give '$x : O$', '$y : O$' and then write '$f : A(x, y)$'. The notation indicates that the symbol 'f' is of type 'A', for arrow, but the latter depends on the type 'O' of objects with two parameters, 'x' denoting the domain of 'f' and 'y' denoting its codomain. In fact the number of morphisms, including the composites, from a sort to another sort gives us the arity of the former sort and its dependence structure. Thus, the dependency structure yields constraints on the grammar of the language.

The definition of an $\mathcal{L}$-signature for FOLDS is deceptively simple, but the example illustrates some of the complexities that arise. There are clearly levels in a FOLDS-signature. The objects in the bottom level, which we call L_0, are not the domain of any proper morphism. When we go up, that is for levels $i > 0$, the sorts are made up of objects x for which all proper morphisms have a codomain in lower levels, e.g. in L_j, for $j < i$, and for which there is at least one proper morphism with codomain in the next lower level L_{i-1}. Thus, all proper morphisms go from a level to a lower level. We will call the objects of a $\mathcal{L}$-signature, its *kinds* and, clearly, each kind K has a level. For any morphism p, K_p will denote the codomain of p. For any kind K, the (finite) set of all proper morphisms $p : K \to K_p$ will be denoted by $K|\mathcal{L}$.

It is not obvious to see how, given a FOLDS signature $\mathcal{L}$, one writes formulas and propositions in $\mathcal{L}$. When we are given a FOL-signature, we understand how terms are constructed, how atomic formulas are constructed and how arbitrary formulas are constructed. As we have already said, the introduction of dependent sorts modifies how the grammar of the language works. Although the ensuing discussion is more technical, it is important to understand the impact this new notion of signature has on the structure of the syntax.

Let us now fix an arbitrary FOLDS-signature $\mathcal{L}$. The *sorts* and *variables* are defined recursively as follows. Let $n \in \mathbb{N}$ be a natural number and suppose that sorts of kinds of level less than n have been defined, as well as variables of such sorts.

Definition 3 *Let K be a kind of level n. A* sort *of kind K is a formal set*[31] $\langle 1, K, \langle x_p \rangle_{p \in K|\mathcal{L}} \rangle$*, which we will denote by* $K(\langle x_p \rangle_{p \in K|\mathcal{L}})$*, such that:*

1. *For every $p \in K|\mathcal{L}$, x_p is a variable of sort $K_p(\langle x_{p,q} \rangle_{q \in K_p|\mathcal{L}})$;*
2. *For every $q \in K_p|\mathcal{L}$, $x_{p,q} = x_{qp}$.*

Roughly, a sort is obtained by filling the p^{th} place of a kind K, for any p in the arity of $K|\mathcal{L}$ of K, by an appropriate variable x_p.

Definition 4 *Given a sort $X = K(\langle x_p \rangle_{p \in K|\mathcal{L}})$, a* variable *$x$ of sort X is a formal set $\langle 2, X, \alpha \rangle$, where α is a formal set called the* parameter *of x. We write $x : X$ to indicate that x is a variable of sort X.*

We see how, even at the formal, syntactical level, a variable "comes" from a sort. A variable is not merely a symbol having a specific syntactical status and function. A variable in FOLDS always carries its origin upfront.

[31] A formal set is a syntactic entity. Sets are here used to codify syntactic entities.

Definition 5 *For a sort* $X = K(\langle x_p \rangle_{p \in K|\mathcal{L}})$*, the* variables *of* X*,* $\mathrm{Var}(X)$ *is defined as* $\{x_p : p \in K|\mathcal{L}\}$*. If* $x : X$*, we write* $\mathrm{Dep}(x)$ *for* $\mathrm{Var}(X)$*.*

Looking at the signature $\mathcal{L}_{Cat}$ given above and the foregoing definitions, the reader can easily convince herself that $O|\mathcal{L} = \varnothing$, $A|\mathcal{L} = \{d, c\}$, and, for instance, $T|\mathcal{L} = \{d \circ t_0, d \circ t_1, c \circ t_2, t_0, t_1, t_2\}$. Thus, $O(*)$, where $*$ denotes the empty sequence of variables, is a sort. So is $A(x_d, x_c)$, whenever $x_d : O, x_c : O$. Similarly, given $x : O, y : O, z : O$ and $f : A(x, y), g : A(y, z), h : A(x, z)$, then $T(x, y, z, f, g, h)$ is a sort of kind T.

The reader will have noticed that sorts and variables of a given sorts are, in fact, formal sets determined by the given signature. The strategy is extended to cover the notions of a free variable and of a formula for a FOLDS signature.

Definition 6 *The* free variables *of* ϕ*,* $\mathrm{Var}(\phi)$*, is defined as follows:*

1. *Base case, already defined: for a kind* K*,* $\mathrm{Var}(K(\langle x_p \rangle_{p \in K|\mathcal{L}})) = \{x_p : p \in K|\mathcal{L}\}$*;*
2. $\mathrm{Var}(\top) =_{def} \mathrm{Var}(\bot) =_{def} \varnothing$*;*
3. *If* $\mathrm{Var}(\phi)$ *and* $\mathrm{Var}(\psi)$ *are defined, then:*
   (a) $\mathrm{Var}(\neg\phi) =_{def} \mathrm{Var}(\phi)$*, where* $\neg\phi$ *is the abbreviation of the formal set* $\langle 3, \neg, \phi \rangle$*;*
   (b) $\mathrm{Var}(\phi \to \psi) =_{def} \mathrm{Var}(\phi) \cup \mathrm{Var}(\psi)$*, where* $\phi \to \psi$ *is the abbreviation of the formal set* $\langle 3, \to, \phi, \psi \rangle$*;*
   (c) $\mathrm{Var}(\phi \wedge \psi) =_{def} \mathrm{Var}(\phi \vee \psi) =_{def} \mathrm{Var}(\phi) \cup \mathrm{Var}(\psi)$*, where* $\phi \wedge \psi$ *is the abbreviation of the formal set* $\langle 3, \wedge, \phi, \psi \rangle$ *and* $\phi \vee \psi$ *is the abbreviation of* $\langle 3, \vee, \phi, \psi \rangle$[32]*;*
4. *If* $\mathrm{Var}(\phi)$ *is defined,* $x : X$*, and there is no* $y : Y \in \mathrm{Var}(\phi)$ *such that* $x \in \mathrm{Dep}(y)$*, then*

$$\mathrm{Var}(\forall x : X.\phi) =_{def} \mathrm{Var}(\exists x : X.\phi) =_{def} (\mathrm{Var}(\phi) \cup \mathrm{Var}(\psi)) - \{x\},$$

where $\forall x : X.\phi$ *and* $\exists x : X.\phi$ *stand for* $\langle 3, \forall, \{x\}, \phi \rangle$ *and* $\langle 3, \exists, \{x\}, \phi \rangle$*.*

The formulas of the language $\mathcal{L}_{\omega,\omega}$ is the least class of formal sets containing $\top, \bot$ and such that $\mathrm{Var}(\phi)$ is defined according to definition 6, for every ϕ in the language[33]. Not surprisingly, a *sentence* in $\mathcal{L}$ is a formula ϕ with no free variables, that is, such that $\mathrm{Var}(\phi) = \varnothing$.

It is extremely important to give examples of formulas and sentences in FOLDS, for the grammar that is obtained from the foregoing definition is not immediate. Here is a formula in $\mathcal{L}_{Cat}$:

$$\exists \tau : T(x, y, z, f, g, h).\top$$

[32]Of course, the definition can be adapted for infinitary conjunctions and disjunctions in the obvious way.

[33]Once again, it is easy to give the definition for infinitary first-order languages.

with free variables $\{x, y, z, f, g, h\}$. Here is a sentence in $\mathcal{L}_{Cat}$:

$\forall x : O.\forall y : O.\forall z : O.\forall f : A(x, y).\forall g : A(y, z).\exists h : A(x, z).\exists \tau : T(x, y, z, f, g, h).\top$

This sentence says that, whenever two arrows are composable, then there is a composite arrow.

What is striking about formulas and sentences in FOLDS is the role played by the dependencies themselves. It constrains the grammar considerably. The following expression, for instance, is not a formula of the language $\mathcal{L}_{Cat}$:

$$\forall x : O.\exists \tau : T(x, y, z, f, g, h).\top$$

for to assert the existence of a triangle τ, one would have to quantify on the variables for arrows before quantifying on objects. In a sense, a FOLDS-signature provides an underlying ontology built in the language itself and such that certain expressions are excluded from the language[34]. FOLDS incorporates the idea in the syntax that the universe of mathematics is not ontologically homogeneous, uniform and isotropic. It reflects in the syntax itself the fact that mathematics is build up of entities of different kinds and that the mathematical universe is heterogenous. Of course, one quickly introduces abbreviations so that the grammar of formulas and sentences resemble more the usual grammar.

At this stage, in textbooks on model theory, authors introduce the notion of an interpretation for a language in the usual way: a domain of interpretation is fixed, constant symbols become elements of a set, functional symbols become functions and relational symbols become relations. The system used to interpret a language is called a *structure*. The same can be done for FOLDS, with the required adjustments.

Definition 7 *Given a FOLDS-signature $\mathcal{L}$, an $\mathcal{L}$-*structure *is a functor*

$$M : \mathcal{L} \to \mathbf{Set}.$$

This is once more deceptively simple, although one of the reasons is precisely because a FOLDS-signature is a category. Of course, this is a semantics valued in the category of sets. It is possible to replace the latter with a category $\mathbf{C}$, say a category with finite limits. Note that the totality of $\mathcal{L}$-structures, denoted by $Str_{\mathbf{Set}}(\mathcal{L})$, form a category. Its objects are the **Set**-valued $\mathcal{L}$-structures and the morphisms, the natural transformations between them.

We will not give the complete, general description of FOLDS semantics. It would be unnecessarily technical. We do have to add a few additional ingredients nonetheless. In particular, we need to describe more explicitly how *valuations* are defined in FOLDS, for some of the elements involved will be referred to later.

A *context* (of variables) is a finite set $\mathcal{Y}$ of variables such that, for all $y \in \mathcal{Y}$, $\mathrm{Dep}(y) \subset \mathcal{Y}$. Note that for any formula ϕ, $Var(\phi)$ is a context. Recall

[34] I am using the word "ontology" in the same way as it is used in information science. As we will see, a FOLDS-signature is intimately related to a criterion of identity, thus specifying when two objects should be seen as being the same. The signature therefore provides a clear and strong ontological constraint.

that from definition 3 above, for a variable $y \in \mathcal{Y}$, the sort of y is written $y : K_y(\langle x_{p,q}\rangle_{p\in K_y|\mathcal{L}})$. The next definition connects, so to speak, the relations of dependence correctly, thus giving the legitimate valuations of a context of variables $\mathcal{Y}$ in M.

Definition 8 *Let M be a $\mathcal{L}$-structure and K be a kind of $\mathcal{L}$.*

1.

$$M[\mathcal{Y}] = \left\{ \langle a_y\rangle_{y\in\mathcal{Y}} \in \prod_{y\in\mathcal{Y}} M(K_y) : (Mp)(a_y) = a_{x_{y,p}} \text{for all } y \in \mathcal{Y}, p \in K_y|\mathcal{L} \right\}.$$

2. The set $M[K]$ of valuations *of K in M is defined similarly:*

$$M[K] = \left\{ \langle a_p\rangle_{p\in K|\mathcal{L}} \in \prod_{p\in K|\mathcal{L}} M(K_p) : (Mq)(a_p) = a_{qp} \text{ for all } q \in K_p|\mathcal{L} \right\}.$$

The elements of the set $M[K]$ are called the *contexts* for K in M.

It is now possible to define the notion of an interpretation of a formula ϕ in a structure M in the context $\mathcal{Y}$. We will not give the complete definition here, for it is done as usual by recursion on the complexity of the formula ϕ. Here are some of the simplest cases. Let ϕ be a formula in a given $\mathcal{L}$-signature, M an $\mathcal{L}$-structure and $\mathcal{Y}$ a context such that $Var(\phi) \subset \mathcal{Y}$. The *interpretation* of ϕ in M *in the context* $\mathcal{Y}$, $M[\mathcal{Y} : \phi]$ is defined thus:

$$M[\mathcal{Y} : \top] =_{df} M[\mathcal{Y}];$$
$$M[\mathcal{Y} : \bot] =_{df} \emptyset;$$
$$\langle a_y\rangle_{y\in\mathcal{Y}} \in M[\mathcal{Y} : \psi \wedge \theta] =_{df} \langle a_y\rangle_{y\in\mathcal{Y}} \in M[\mathcal{Y} : \psi] \text{ and } \langle a_y\rangle_{y\in\mathcal{Y}} \in M[\mathcal{Y} : \theta].$$

And the other connectives are defined in a similar manner.

We refer the reader to Makkai's papers for the complete description of the semantics for FOLDS.

We have to describe and explain the notion of equivalence in FOLDS, since it is, after the notion of signature, the original element of the theory and arguably the most important.

3.2 Equivalences in FOLDS

FOLDS was designed by Makkai for the following purpose: to define languages such that all statements in such a language are invariant under the equivalence appropriate for the kind of structures described by the given language. This is feasible only if it is possible to capture basic facts about the different kinds of equivalences involved in the universe of abstract mathematical concepts. We have here a crucial and significant departure from the traditional logical analysis in which the identity relation is presented as an *a priori* and universal or global notion. In FOLDS, an adequate notion of identity is *derived* from an $\mathcal{L}$-signature. Thus, the identity relation comes *a posteriori* or, in a slightly different vocabulary, it is a local relation. This order of presentation reflects the historical order of introduction of abstract mathematical concepts. Indeed, if abstract mathematical concepts are *abstracted* from various mathematical contexts, then the identity criterion should also be abstracted from the basic properties that define these new abstract concepts and this is indeed how it happened historically[35]. What is surprising is that it is possible to find a way to extract a criterion of identity from a given signature. Here is roughly how it works.

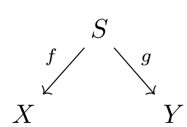
Let $\mathcal{L}$ be a given FOLDS-signature and let M and P be $\mathcal{L}$-structures and $h : P \to M$ be a natural transformation between $\mathcal{L}$-structures. For each kind $K \in \mathcal{L}$ and each $x \in P[K]$, there is an induced map

$$h_{K,x} : PK(x) \to MK(hx),$$

where hx is an abbreviation for $\langle h_{K_p} x_p \rangle_{p \in K|\mathcal{L}}$. The natural transformation is said to be *fiberwise surjective* whenever $h_{K,x}$ is surjective for all kinds $K \in \mathcal{L}$ and all $x \in P[K]$.

Definition 9 *Let $\mathcal{L}$ be a given FOLDS-signature and let M and N be $\mathcal{L}$-structures. M and N are said to be* $\mathcal{L}$-equivalent*, written $M \sim_{\mathcal{L}} N$, if there is an $\mathcal{L}$-structure P together with natural transformations $m : P \to M$ and $n : P \to N$ such that m and n are fiberwise surjective.*

This is certainly a surprising way of introducing an equivalence. It is not even clear that it *is* an equivalence relation. Let us try to unpack it. First, in any category $\mathbf{C}$, given objects X and Y of $\mathbf{C}$, a *span* on X and Y is an object S of $\mathbf{C}$ together with morphisms



Spans can be thought of as being generalizations of relations between X and Y. Indeed, a relation $R \subset X \times Y$ of X and Y is a span, with the projections $\pi_X : R \to X$ and $\pi_Y : R \to Y$. Thus, the $\mathcal{L}$-structure P together with the natural transformations m and n is a span on M and N. It is not a new notion. When the category $\mathbf{C}$ has a minimum of structure, for instance pullbacks, then

[35] This is interesting in itself and, historically, the situation is more complex than one might expect from how we learn these notions. Contemporary mathematicians are usually convinced that the criterion of identity for various abstract concepts came simultaneously with the concepts themselves. For instance, category theorists are often surprised to learn that the notion of categorical equivalence *was not* in Eilenberg and Mac Lane's original paper in category theory published in 1945. Eilenberg and Mac Lane defined the notion of *isomorphism* of categories. The correct criterion of identity for categories was introduced by Grothendieck in his paper on homological algebra published in 1957.

it is easy to show that spans compose in the obvious way. Given the latter fact, it is easy to verify that the foregoing relation is indeed an equivalence relation.

It is a remarkable fact that this definition, together with the definition of FOLDS-signature, yields the appropriate notion of identity for the concepts captured by the signature. Thus, for a signature corresponding to a classical first-order signature, the definition yields the usual notion of isomorphism of structure[36]. In the case of the signature for categories given above, one gets the notion of equivalence of categories. It is also possible to use the simplex category Δ to define a FOLDS-signature $\mathcal{L}$ such that for Kan complexes X, Y, X and Y are homotopy equivalent if and only if they are equivalent as $\mathcal{L}$-structures[37]. In other words, it is possible to give a FOLDS-signature such that the notion of equivalence for that language is equivalent to the notion of homotopy equivalence.

Moreover, it can then be *proved* that the derived notion of equivalence satisfies the following invariance principle[38]: given a FOLDS-signature $\mathcal{L}$, M and N $\mathcal{L}$-structures, for any FOLDS-formula ϕ of $\mathcal{L}$[39],

$$M \models \phi \wedge M \simeq_{\mathcal{L}} N \Longrightarrow N \models \phi.$$

Again, I want to emphasize that the notion of equivalence is not given *a priori*. The theorem has to be proved for each particular case. It basically says that, in FOLDS, it is not possible to make irrelevant claims about the entities talked about in a given language.

4 The Universe: Higher-dimensional categories

We can now move to the universe $\mathcal{U}$ of mathematical objects in which the language can be interpreted systematically. It is the universe of higher-dimensional categories. The informal target is clearly identified. Historically, bicategories or, equivalently, weak 2-categories came on the mathematical scene rather early in the development of category theory (**?**), although they appeared as a conceptual curiosity at first. It became progressively clear, mostly in the 1980s, that bicategories (weak 2-categories) and tricategories (weak 3-categories) had to be used in various contexts. Thus, the necessity of having a clear picture of the universe of higher-dimensional categories (HDC) imposed itself to the com-

[36] This is not trivial and requires some care in the statement of the theorem and its proof.

[37] The simplex category Δ can be described thus: its objects are the ordinals $[n] = \{0, 1, ..., n\}$, for all $n \in \mathbb{N}$ and the morphisms $[m] \to [n]$ are the order preserving functions. To construct the FOLDS-signature, define the category $\Delta^{\uparrow}$ with the same objects as the simplex category but restrict the morphisms to the injective functions of Δ. The FOLDS-signature is then $(\Delta^{\uparrow})^{op}$, the opposite of the previous category. It can be shown that given this notion of FOLDS-signature, an $\mathcal{L}$-structure is, in that case, a simplicial set and, in particular, a Kan complex.

[38] We give a loose and imprecise formulation of the theorem. The precise formulation would not yield more insight at this stage.

[39] The proof is by induction on the complexity of ϕ.

munity[40]. Thus, in the 1990s, category theorists started to articulate a formal, theoretical analysis of the universe and, interestingly enough, various and different formalizations appeared quickly on the scene. It brought to the fore the mathematical question of proving that these various mathematical frameworks were equivalent, a problem still evolving today. From a philosophical point of view, the question as to how the mathematical community will determine which formal framework is the most appropriate is interesting. It is a philosophical case study of the development of mathematical knowledge[41].

I will give some of the key ideas underlying the definitions of the universe of HDC. The next step is to give one precise, rigorous, technical definition of the universe, the universe $\mathbb{U}$ of section 1, simply to illustrate how it is done and what it looks like.

The basic ingredients of category theory are simple: there are categories, functors and natural transformations between functors. These are the building blocks of the theory. Thus, at first sight, one might conclude that the universe of categories is and cannot be anything else than a category, that is, the category of categories. However, as we have already said in section 2.4, it turns out that new kinds of entities are required and show up naturally in certain contexts. When categories are put up together, something else emerges and when these entities are put up together, then, again, something else emerges and so on and so forth.

At the heart of the description of higher-dimensional categories one finds certain geometric shapes which are introduced to capture how various morphisms compose. Thus, in the formal definitions, one finds a collection of basic abstract shapes that determine the underlying structure of the universe. The universe in then given by specifying certain properties of these shapes. It should be noted that Makkai has used FOLDS and the concept of FOLDS-signature to present these shapes and the universe. See **?**. We will merely sketch one way of thinking about the constructions involved.

Consider the following common situations in basic category theory. First, a functor F from a category $\mathbf{C}$ to a category $\mathbf{D}$ is represented by an arrow:

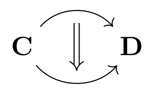
$$F : \mathbf{C} \to \mathbf{D}.$$

The situation that more or less gave rise to category theory is given by two parallel functors $F, G : \mathbf{C} \to \mathbf{D}$ with a natural transformation $\eta : F \to G$. This

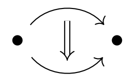
[40] I should add that many people still doubt the necessity of introducing these levels of abstraction. It is interesting to note that Lawvere introduced a universe of category as a foundations of mathematics in the early 1960s. See **?**. However, this universe is simply a category: it is the category of categories. The picture that is now emerging is considerably more intricate and radically different from the category of categories. As we have said and as Makkai has emphasized in many occasions, the bicategory of categories is *not* a category with additional structure. It does not even have an underlying category. It *is* a genuinely new kind of entity.

[41] A parallel with the development of set theory and its formalization might be interesting also, as well as a parallel with Maddy's work on the choice of axioms in set theory. See **??**. Quasicategories are now fashionable, but there are not the only concepts used. For the definition of quasicategories, see **?**. For its role in the development of higher dimensional algebra, see **??**.

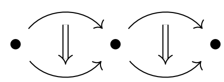
situation is sometimes depicted as follows:



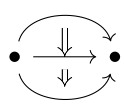
Abstracting from the specific elements, we have a geometric form composed of points, directed lines and the double line can be thought of as representing a surface that is stretched from the upper arrow to the lower one. The representation becomes:



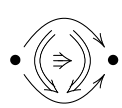
Geometrically, this can be thought of as a disk. But, the image is dynamic – stretching a surface from one directed line to the other directed line – and this aspect is crucial and has to be kept in mind when reading it and similar cases. Since functors compose and natural transformations compose, this basic geometric form can be pasted with other basic forms, yielding new complex forms. For instance, the composition of two functors with natural transformations between them yields the form:



Of course, the composition of the natural transformations together with the functors is not entirely clear from the diagram itself. Similarly, the composition of two natural transformations between three functors, the so-called vertical composition, yields another form:



It is now easy to see that these basic forms can be pasted together in various ways and the question as to how to determine which ones can be transformed into one another immediately arises[42]. And as should be clear from informal geometric knowledge, these forms are 2-dimensional. The basic 3-dimensional case can be depicted thus:



Thus, the informal description of the universe of higher-dimensional categories is understood, in the same way that the informal universe of sets is

[42] Of course, I did not write "equal" in the last sentence. This is the key element of the situation. Composing various transformations does not yield, in the general case, equalities between the morphisms. At best, there is a equivalence between them.

understood or was understood by Cantor and others before the introduction of a formal axiomatic framework with a rigorous and precise description of the cumulative hierarchy. As we have said, but should perhaps be repeated at this point, the move to FOLDS and the universe of higher dimensional categories corresponds to the move to first order logic and the universe of sets. In the latter case, the universe of sets was crystallized in the definition of the cumulative hierarchy. There are now various definitions that can be used to define a universe of higher dimensional categories, the universe $\mathbb{U}$.

To describe the universe of higher-dimensional categories, one therefore needs a language in which the basic geometric forms, also called the k-cells, are given. Once these have been described, the theory has to specify how these compose and which ones are equivalent. In the end, that is after many pages of formal definitions and clarifications, the definition of the universe itself holds in a short paragraph. To wit, here is Makkai's definition.

Definition 10 *A* multitopic ω-category *is a multitopic set S such that: For every multitope σ and every σ-shaped pasting diagram α in S, there is at least one cell* a *parallel to α such that, for $\theta = c\sigma$, the $Mlt\langle\theta\rangle$-structures $S\langle \mathrm{a} \rangle$ and $S\langle\alpha\rangle$ are $Mlt\langle\theta\rangle$-equivalent by an equivalence span that extends the identity on Mlt.*

Without the appropriate definitions and results, the definition is incomprehensible. Many notions involved have to be explained and defined. For instance, the multitopes are here the basic forms used. (See **?** for the definition of multitopes.) Makkai gives a FOLDS-signature for the multitopic ω-category. Two elements have to be mentioned. First, the required properties for a multitopic set to be a multitopic ω-category are all of the form that certain composites, defined by universal properties, are to exist. Second, the notion of FOLDS-equivalence plays a key role in the definition.

All these technicalities are neither surprising nor a problem. After all, the definition of the cumulative hierarchy underlying the set theoretical foundational framework is also technical and requires the certain sophisticated set-theoretical notions, e.g. transfinite ordinals, are understood.

5 Conclusion

Once the community will have a good grasp of the various parts constituting the new foundational space, logicians will be able to extend and develop the science of the foundations of mathematics. This is clearly Makkai's goal. Thus the point here is not to defend an ideological position regarding the nature of mathematical knowledge, for the whole project starts from the observation that abstract mathematical concepts are now part and parcel of contemporary mathematics. Nor is it to develop a formal system that will lead to automated proof checking or automated theorem proving. The goal is to extend in a specific direction the science of the foundations of mathematics, thus to obtain results about certain structures, results with an intrinsic conceptual value and that can be relevant for mathematics and for philosophy.

Why should philosophers care about FOLDS or, more generally, about a science of the foundations of mathematics[43]? Why should they care about a foundational framework for *abstract* mathematics? Most philosophers nowadays probably think that the time when they had to know and understand technical issues related to the foundations of mathematics is now behind them. Issues related to the foundations of mathematics were philosophically relevant, so the argument goes, in as much as they were related to the fact that mathematical knowledge could no longer be *rationally justified*. Once a reasonable solution to the paradoxes of set theory had been found, the technical developments leading to the creation of model theory and proof theory seemed to be of interest only to a handful of technically sensitive philosophers of logic and mathematics.

I will not try to argue for the importance of a science of the foundations of mathematics for philosophy in general. This is a topic for a whole book. I will here concentrate on a few aspects that are inherent to FOLDS.

The first element that is worth underlining is the fact that FOLDS is aimed at *abstract* mathematical concepts. This in itself is central to epistemology and ontology in general. Indeed, a better understanding of what is, how we know and how we understand abstract mathematical concepts is bound to open the way to a better understanding of abstractness in general. The latter notion is omnipresent, from aesthetic to ethics. Specific results about FOLDS could be read as exhibiting singular aspects of abstract mathematical concepts.

Second, philosophers should be interested in a language that can only express invariant properties of abstract objects. This is in itself a remarkable original feature. There is no need to underline the importance of what is called Benacerraf's problem in philosophy of mathematics and how FOLDS leads to a direct and simple solution to this problem, as Makkai has already seen himself.

Third, FOLDS and the universe of HDC force us to think differently about the notion of structure and various kinds of structuralism in mathematics and the sciences in general. For, in most cases, the debate surrounding structuralism is based on a set-theoretical notion of structure[44].

Finally, specific technical results of FOLDS will also have a philosophical impact. Certain model theoretical properties, e.g. existential closure or model completeness, have a direct philosophical interpretation which shed an important light on the nature of mathematical knowledge and mathematical understanding[45]. Similar results for theories written in FOLDS will allow a better analysis of certain aspects of the development of contemporary mathematics and its peculiar features, e.g. the use of abstract concepts.

[43]We take it for granted here that we do not have to convince logicians about the importance of FOLDS. We may be wrong about this and, if we are, it is an interesting and intriguing fact.

[44]I should hasten to add: on a concrete notion of set. Bourbaki, in his notion of structure, included the isomorphisms and thus, one could argue, on an abstract notion of set as it should. Indeed, one could use Makkai's notion of abstract set to reconstruct Bourbaki's notion of structure. But again, as I have already indicated, this is but one small fragment of the universe we are discussing here.

[45]See **?**.

In the end, FOLDS is a starting point for anyone who wants to develop a foundation for abstract mathematical concepts and, perhaps, for any abstract concepts.

References

Ellerman, D. P. (1988), 'Category theory and concrete universals', *Erkenntnis* **28**(3), 409–429.

Ferreirós, J. (1996), 'Traditional logic and the early history of sets, 1854–1908', *Arch. Hist. Exact Sci.* **50**(1), 5–71.
URL: *http://dx.doi.org/10.1007/BF00375789*

Ferreirós, J. (2001), 'The road to modern logic—an interpretation', *Bull. Symbolic Logic* **7**(4), 441–484.
URL: *http://dx.doi.org/10.2307/2687794*

Ferreirós, J. (2007), *Labyrinth of thought*, second edn, Birkhäuser Verlag, Basel. A history of set theory and its role in modern mathematics.

Makkai, M. (1998), Towards a categorical foundation of mathematics, *in* 'Logic Colloquium '95 (Haifa)', Vol. 11 of *Lecture Notes in Logic*, Springer, Berlin, pp. 153–190.
URL: *http://dx.doi.org/10.1007/978-3-662-22108-2*$_1$1

Makkai, M. (1999), On structuralism in mathematics, *in* 'Language, logic, and concepts', Bradford Book, MIT Press, Cambridge, MA, pp. 43–66.

Mancosu, P., Zach, R. & Badesa, C. (2009), The development of mathematical logic from Russell to Tarski, 1900–1935, *in* 'The development of modern logic', Oxford Univ. Press, Oxford, pp. 318–470.
URL: *http://dx.doi.org/10.1093/acprof:oso/9780195137316.003.0029*

Marquis, J.-P. (2000), 'Three kinds of universals in mathematics?', *Logical Consequence: Rival Approaches and New Studies in Exact Philosophy: Logic, Mathematics and Science* **2**, 191–212.

Marquis, J.-P. (2014), Mathematical abstraction, conceptual variation and identity, *in* P. S.-H. G. H. W. H. P. E. Bour, ed., 'Logic, Methodology and Philosophy of Science - Proceedings of the 14th International Congress (Nancy)', College Publications, London, pp. 299–322.

Moore, G. H. (1980), Beyond first-order logic: the historical interplay between mathematical logic and axiomatic set theory, *in* 'History and philosophy of logic, Vol. 1', Abacus, Tunbridge Wells, pp. 95–137.

Moore, G. H. (1987), A house divided against itself: the emergence of first-order logic as the basis for mathematics, *in* 'Studies in the history of mathematics', Vol. 26 of *MAA Stud. Math.*, Math. Assoc. America, Washington, DC, pp. 98–136.

Moore, G. H. (1988), The emergence of first-order logic, *in* 'History and philosophy of modern mathematics (Minneapolis, MN, 1985)', Minnesota Stud. Philos. Sci., XI, Univ. Minnesota Press, Minneapolis, MN, pp. 95–135.

Schiemer, G. & Reck, E. H. (2013), 'Logic in the 1930s: type theory and model theory', *Bull. Symbolic Logic* **19**(4), 433–472.
URL: *http://projecteuclid.org/euclid.bsl/1388953941*