Aggregation for potentially infinite populations without continuity or completeness*

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Abstract

We present an abstract social aggregation theorem. Society, and each individual, has a preorder that may be interpreted as expressing values or beliefs. The preorders are allowed to violate both completeness and continuity, and the population is allowed to be infinite. The preorders are only assumed to be represented by functions with values in partially ordered vector spaces, and whose product has convex range. This includes all preorders that satisfy strong independence. Any Pareto indifferent social preorder is then shown to be represented by a linear transformation of the representations of the individual preorders. Further Pareto conditions on the social preorder correspond to positivity conditions on the transformation. When all the Pareto conditions hold and the population is finite, the social preorder is represented by a sum of individual preorder representations. We provide two applications. The first yields an extremely general version of Harsanyi's social aggregation theorem. The second generalizes a classic result about linear opinion pooling.

Keywords. Social aggregation; discontinuous preferences and comparative likelihood relations; incomplete preferences and comparative likelihood relations; infinite populations; Harsanyi's social aggregation theorem; linear opinion pooling; partially ordered vector spaces.

JEL Classification. D60, D63, D70, D81, D83.

1 Introduction

Individuals may have incomplete values and beliefs.[1] They may be undecided which of two goods is preferable, or which of two events is more likely. The former is reflected, for example, in the literature developing multi-utility theory, and the latter in work on representing beliefs by sets of probability measures, and on decision theories that feature such sets. Individuals may also have discontinuous values and beliefs.[2] For example, they may see some goods as being infinitely more valuable than others (or, equivalently, see some goods as only having significance as tiebreakers). In addition, allowing individuals to regard some events as infinitesimally likely

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[1] The literature on the topics of this paragraph is vast. References that are directly relevant to our approach are given in section 2.1.

[2] The applications we discuss are either decision theoretic (values) or to do with comparative likelihood (beliefs). Each subject has its own family of continuity axioms, but we rarely need to specify these, and we rely on context to determine which family we are discussing.

provides a solution to a number of problems to do with conditional probability, decision theory, game theory and conditional preference.

A standard question is how to aggregate the values or beliefs of individuals to form a collective view. Here we also wish to allow the population to be infinite. One rationale is that even in a finite society, a decision maker having incomplete information about the values and beliefs of members of society may wish to model each individual as an infinite set of types.[3] But the most obvious rationale comes from the much discussed problem of intergenerational equity. Here, the possibility that society will extend indefinitely into the future requires an infinite number of people. It is also commonly modelled by an infinite sequence of generations, each one with a social utility function; in such a model, the generations play the role of 'individuals' whose interests are to be aggregated.[4]

Section 2 presents a family of abstract aggregation theorems. Each of possibly infinitely many individuals, and society, is equipped with a preorder on a given set. We assume that these preorders are represented by functions with values in partially ordered vector spaces, and whose product has convex range. The use of partially ordered vector spaces is explained and motivated in section 2.1; axiomatizations of such representations are given in sections 3.1 and 3.2.2. Roughly speaking, our main result shows that Pareto indifference holds if and only if the social preorder can be represented by a linear combination of the representations of the individual preorders. Further Pareto conditions correspond to positivity conditions on the linear mapping.

Section 3 illustrates the interest of the results with two applications, corresponding respectively to the aggregation of values and of beliefs. The first yields an extremely general version of the celebrated social aggregation theorem of Harsanyi (1955) that assumes only the central expected utility axiom of strong independence. The other shows that aggregate beliefs are given by linear pooling. To reiterate, unlike other results in the literature, these results hold without any completeness or continuity assumptions, and allow for an infinite population.

We end in section 4 with a discussion of related literature, but for now we acknowledge Mongin (1995) and De Meyer and Mongin (1995) for drawing attention to formal similarities between preference aggregation and opinion pooling, and emphasizing the usefulness of the convex range assumption we use below.

The proofs of our main theorems rely only on concepts from basic linear algebra, rather than any theorems from convex or functional analysis. All proofs are in the Appendix.

2 Main results

2.1 Representations in partially ordered vector spaces

We are going to consider representations of preorders with values in partially ordered vector spaces. We first recall the basic definitions, and give some examples illustrating this type of representation.

Recall that a *preorder* $\succsim$ is a binary relation that is reflexive and transitive; we write $\sim$ and $\succ$ for the symmetric and asymmetric parts of $\succsim$ respectively. Since preorders can be incomplete, we write $x \curlywedge y$ if neither $x \succsim y$ nor $y \succsim x$.

Let $(X, \succsim)$ and $(X', \succsim')$ be preordered sets. A function $f\colon X \to X'$ is *increasing* if $x \succsim y \Rightarrow f(x) \succsim' f(y)$; *strictly increasing* if $x \succsim y \Rightarrow f(x) \succsim' f(y)$ and $x \succ y \Rightarrow f(x) \succ' f(y)$; a *represen-*

[3]In the context of values, see e.g. Zhou (1997), attributing the idea to Harsanyi (1967–68); for beliefs, see e.g. Herzberg (2015).

[4]The literature on this topic is largely shaped by the approaches of Ramsey (1928) (opposing impatience), and of Koopmans (1960) and Diamond (1965) (requiring impatience). Our results are compatible with both approaches.

tation of $\succsim$ if $x \succsim y \Leftrightarrow f(x) \succsim' f(y)$; an *order embedding* if it is an injective representation; and an *order isomorphism* if it is a bijective representation.

A *preordered vector space* is a real vector space V with a (possibly incomplete) preorder $\succsim_V$ that is *linear* in the sense that $v \succsim_V v' \Leftrightarrow \lambda v + w \succsim_V \lambda v' + w$, for all $v, v', w \in \mathbb{V}$ and $\lambda > 0$.[5] Note that we allow vector spaces to have infinite dimension. A *partially ordered* vector space is a preordered vector space in which the linear preorder is a partial order; that is, it is anti-symmetric. An *ordered* vector space is a partially ordered vector space in which the partial order is an order; that is, it is complete. When $L\colon V \to V'$ is a linear map between partially ordered vector spaces, L is increasing if and only if it is *positive*, in the sense that $v \succsim_V 0 \Rightarrow Lv \succsim_{V'} 0$; and L is strictly increasing if and only if it is *strictly positive*, meaning that $v \succsim_V 0 \Rightarrow Lv \succsim_{V'} 0$ and $v \succ_V 0 \Rightarrow Lv \succ_{V'} 0$.

In this paper we will be exclusively concerned with representations with values in partially ordered vector spaces.[6] We provide an axiomatic basis for using such representations for preference relations in Lemma 16, and for comparative likelihood relations in Lemma 23 below, but for now we focus on examples. The set $\mathbb{R}$ of real numbers, with the usual ordering, is a simple example of a partially ordered vector space, so our representations include familiar real-valued ones. Allowing for arbitrary partially ordered vector spaces allows for natural representations of incomplete and discontinuous preorders, as the following examples illustrate.

Example 1. Consider bundles of three goods (say fame, love, and money) represented by points in $X = \mathbb{R}^3_+$. Let $V = \mathbb{R}^3$, with the following linear partial order:

$$(x_1, x_2, x_3) \succsim_V (y_1, y_2, y_3) \iff (x_1 \geq y_1,\ x_2 \geq y_2, \text{ and, if } x_1 = y_1 \text{ and } x_2 = y_2, \text{ then } x_3 \geq y_3).$$

Let $f\colon \mathbb{R}^3_+ \to V$ be the function $f(x_1, x_2, x_3) = (\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{1}{3}x_1 + \frac{2}{3}x_2, x_3)$. Suppose person A's preference relation $\succsim_A$ over simple lotteries over $\mathbb{R}^3_+$ is represented by expectations of f, thereby satisfying the expected utility axiom of strong independence.[7] It violates the completeness axiom, because, for example, $(1, 0, 0) \curlywedge_A (0, 1, 0)$, reflecting the fact that A finds fame and love only roughly comparable. (Note that the incomparability is limited in the sense that two units of fame are preferred to one unit of love, and vice versa.) The preferences violate standard decision theoretic notions of continuity, such as the Archimedean axiom of Blackwell and Girshick (1954) and the mixture continuity axiom of Herstein and Milnor (1953), reflecting the fact that while A sees money as valuable, she finds fame (likewise love) infinitely more valuable.

Example 2. Consider a sphere S divided into open northern and southern hemispheres S_+ and S_-, and equator S_0. Let μ_+, μ_-, and μ_0 be the uniform probability measures on S_+, S_-, and S_0, respectively. For every measurable set $A \subset S$ define $f(A) = (\mu_+(A \cap S_+), \mu_-(A \cap S_-), \mu_0(A \cap S_0)) \in V$, for $V = \mathbb{R}^3$ as in the previous example. This f represents a likelihood preorder on the algebra X of measurable subsets of S. The preorder is incomplete, since, for example, the hemispheres S_+ and S_- are incomparable. (Here we have allowed the incomparability to be unlimited, in the sense that any positive-measure subsets of S_+ and S_- are incomparable.) Moreover, the equator S_0, though more likely than the null set, is less likely than the interior of any spherical triangle, no matter how small. Correspondingly, the likelihood preorder violates standard continuity axioms for comparative likelihood, such as the Archimedean axiom C6 of Fine (1973), or the monotone continuity axiom of Villegas (1964) and its weakening C8 in Fine (1973). For both of these reasons, the likelihood preorder cannot be represented by a standard ($[0, 1]$-valued) probability measure.

[5] A linear preorder, in our sense, is sometimes called a vector preorder. See section 4 for generalisation to vector spaces over ordered fields other than $\mathbb{R}$.

[6] See however Remark 5 for comments relevant to merely preordered vector spaces.

[7] A preorder $\succsim$ on a convex set X satisfies strong independence if for each $\alpha \in (0, 1)$, $x, y, z \in X$, $x \succsim y$ if and only if $\alpha x + (1 - \alpha)z \succsim \alpha y + (1 - \alpha)z$.

In Example 1, the domain $X = \mathbb{R}^3_+$ is a convex set and the representation f is *mixture preserving*, i.e. $f(\alpha x + (1-\alpha)x') = \alpha f(x) + (1-\alpha)f(x')$ for all $x, x' \in X, \alpha \in [0,1]$. When $V = \mathbb{R}$, von Neumann-Morgenstern expected utility representations of preferences are the paradigmatic representations of this type. In Example 2, the domain X is a Boolean algebra of sets, and the representation f is a *vector measure*, meaning that it is finitely additive: $f(x \cup x') = f(x) + f(x')$ for disjoint $x, x' \in X$. When $V = \mathbb{R}$, probability measures are the paradigmatic representations of this type.[8]

The applications of our aggregation theorems we provide in sections 3.1 and 3.2 involve mixture-preserving and vector-measure representations with values in arbitrary partially ordered vector spaces. The next examples illustrate how such representations generalize some standard ways of representing incomplete or discontinuous preorders.

Example 3 (Multi-representation)**.** Suppose a preorder $\succsim$ on a set X is represented in the following way. There is a family $\{ f_i \mid i \in \mathcal{I} \}$ of functions from X to $\mathbb{R}$ such that $x \succsim x'$ if and only if $f_i(x) \geq f_i(x')$ for all $i \in \mathcal{I}$. This can be rewritten as a representation by a single function $F\colon X \to \prod_{i\in\mathcal{I}} \mathbb{R}$ when $\prod_{i\in\mathcal{I}} \mathbb{R}$ is equipped with the product partial order,[9] making it a partially ordered vector space, and F is defined by $(F(x))_i = f_i(x)$. Note that if the f_i are mixture preserving on convex X, then F is also mixture preserving; and if the f_i are probability measures on an algebra X, then F is a vector measure.

Multi-representations like this are used in expected utility theory to represent the preferences of agents with incomplete values;[10] in decision theory to represent the preferences of agents with incomplete beliefs;[11] and in probability theory to represent agents with incomplete beliefs.[12]

Example 4 (Lexicographic representation)**.** Suppose a preorder $\succsim$ on a set X is represented in the following way. There is a finite vector $(f_1, \ldots, f_n)$ of functions from X to $\mathbb{R}$ such that $x \succsim x'$ if and only if $f_j(x) = f_j(x')$ for all j or $f_j(x) > f_j(x')$ for the least j such that $f_j(x) \neq f_j(x')$. The f_j can again be combined into a single function $F\colon X \to \mathbb{R}^n$, with $(F(x))_j = f_j(x)$. As in Example 3, if the f_j are mixture-preserving functions or probability measures, then F is a mixture-preserving function or a vector measure. Moreover, this F represents $\succsim$ if we endow $\mathbb{R}^n$ with the 'lexicographic' order. To give the general picture, let $(\mathcal{J}, \succsim_{\mathcal{J}})$ be an ordered set, and let $\mathbb{R}^{\mathcal{J}}_{\mathrm{wo}}$ be the subspace of $\mathbb{R}^{\mathcal{J}}$ whose members have well-ordered support.[13] The lexicographic order $\succsim_{\mathrm{lex}}$ on $\mathbb{R}^{\mathcal{J}}_{\mathrm{wo}}$ is defined by the property that $f \succsim_{\mathrm{lex}} f'$ if and only if $f = f'$ or $f(j) > f'(j)$ for the $\succsim_{\mathcal{J}}$-least j such that $f(j) \neq f'(j)$. This makes $\mathbb{R}^{\mathcal{J}}_{\mathrm{wo}}$ what Hausner and Wendel (1952) call a 'lexicographic function space'. It is an ordered vector space. In the original example we can identity $\mathbb{R}^n$ with $\mathbb{R}^{\mathcal{J}}_{\mathrm{wo}}$, for $\mathcal{J} \coloneqq \{1, 2, \ldots, n\}$ with the usual ordering.

Representations with values in lexicographic function spaces are used in expected utility theory to represent the preferences of agents with discontinuous values;[14] in decision and game

[8]Standard treatments of probability theory assume countable additivity, but the motivation for this further requirement is reduced given that we will not be assuming continuity axioms like Monotone Continuity.

[9]In general, for a family $\{(X_i, \succsim_i) : i \in \mathcal{I}\}$ of preordered sets, the product preorder $\succsim_{\mathrm{P}}$ on $\prod_{i\in\mathcal{I}} X_i$ is defined by the condition that $(x_i)_{i\in\mathcal{I}} \succsim_{\mathrm{P}} (y_i)_{i\in\mathcal{I}}$ if and only if $x_i \succsim_i y_i$ for each $i \in \mathcal{I}$.

[10]See e.g. Aumann (1962); Fishburn (1982); Seidenfeld *et al* (1995); Shapley and Baucells (1998); Dubra *et al* (2004); Baucells and Shapley (2008); Evren (2008); Manzini and Mariotti (2008); Evren (2014); Galaabaatar and Karni (2012, 2013); McCarthy *et al* (2017a).

[11]See e.g. Gilboa and Schmeidler (1989); Seidenfeld *et al* (1995); Bewley (2002); Ghirardoto *et al* (2003); Nau (2006); Gilboa *et al* (2010); Ok, Ortoleva and Riella (2012); Galaabaatar and Karni (2013).

[12]For entries to a vast literature, see Halpern (2003) and Stinchcombe (2016); on multi-representation of incomplete comparative likelihood preorders, discussed further in section 3.2, see Insua (1992) and Alon and Lehrer (2014).

[13]That is, $\mathbb{R}^{\mathcal{J}}_{\mathrm{wo}} = \{ f \in \mathbb{R}^{\mathcal{J}} \mid \{ j \in \mathcal{J} \mid f(j) \neq 0 \}$ is well-ordered by $\succsim_{\mathcal{J}} \}$.

[14]See Hausner (1954); Fishburn (1971); Blume, Brandenburger and Dekel (1989); Borie (2016); Hara, Ok and Riella (2016); McCarthy *et al* (2017b).

theory to represent the preferences of agents with discontinuous beliefs;[15] and in probability theory to represent agents with discontinuous beliefs.[16]

The constructions in these examples can be combined, yielding representations of possibly incomplete and discontinuous relations that take values in $\prod_{i\in\mathcal{I}}\mathbb{R}^{\mathcal{J}_i}_{\mathrm{wo}}$, a partially ordered vector space where each component has the lexicographic order and their product has the product partial order.[17] In fact, this construction is fully general: every partially ordered vector space can be order-embedded in such a space.[18]

Finally, we note that our approach is also compatible with the use of representations with values in non-Archimedean ordered fields, such as non-standard models of the real-numbers.[19] These provide natural ways of representing discontinuous relations; see Example 10 below. We discuss this topic further in section 4.

Remark 5 (Preordered Vector Spaces)**.** We focus on representations in partially ordered vector spaces, rather than merely preordered ones, because it seems desirable for objects that are ranked equally to be assigned *the same* value. But for some purposes it is useful to consider preordered vector spaces instead. Our results have implications for this more general case, insofar as any preordered vector space has a partially ordered vector space as a quotient.[20] Lemma 18 and Theorem 19 illustrate some of the extensions this makes available.

2.2 Framework and axioms

We assume throughout that X and $\mathbb{I}$ are nonempty, possibly infinite, sets. For each $i \in \mathbb{I} \cup \{0\}$, let $\succsim_i$ be a preorder on X. In applications, $\mathbb{I}$ is typically a population, and for each i in $\mathbb{I}$, $\succsim_i$ expresses the values or beliefs of individual i, while $\succsim_0$ expresses those of the social observer.

Say that a family $\{f_i : X \to V_i\}_{i\in\mathcal{I}}$ of functions with values in vector spaces is *co-convex* if their joint range $(f_i)_{i\in\mathcal{I}}(X) \subset \prod_{i\in\mathcal{I}} V_i$ is convex. Equivalently, the f_i are co-convex if there is a surjective map q from X onto a convex set $\bar{X}$, and mixture-preserving functions $\bar{f}_i : \bar{X} \to V_i$, such that $f_i = \bar{f}_i \circ q$. Thus co-convexity is a modest generalisation of the assumption that X is a convex set and the f_i are mixture preserving.

The central assumption of our main theorems will be that for $i \in \mathbb{I} \cup \{0\}$, the $\succsim_i$ have co-convex representations f_i with values in partially ordered vector spaces V_i. In the special case in which $\mathbb{I}$ is finite and each V_i is the real numbers with the usual ordering, the requirement of co-convex representations was introduced by De Meyer and Mongin (1995).

Any preordered set can be shown to have a representation with values in some partially ordered vector space (McCarthy *et al*, 2018, Lemma A.6), so the question is the significance of co-convexity. We postpone this question, though, until sections 3.1 and 3.2, where we present contexts in which co-convex representations naturally arise.

We consider the following Pareto-style axioms.[21]

[15] See Blume, Brandenburger and Dekel (1991a,b); Brandenburger, Friedenberg and Keisler (2008).

[16] See Halpern (2010); Brickhill and Horsten (2018).

[17] If desired, the $\mathcal{J}_i$ can be enlarged so that the representation may be taken into $\prod_{i\in\mathcal{I}}\mathbb{R}^{\mathcal{J}}_{\mathrm{wo}}$. Elements in this space can be seen as $\mathcal{I}\times\mathcal{J}$ matrices, with the row space lexicographically ordered, and one matrix ranking higher than another if it ranks higher in each row.

[18] That every ordered vector space can be order-embedded in a lexicographic function space was shown by Hausner and Wendel (1952); the extension to partially ordered vector spaces is given in Hara, Ok and Riella (2016) and McCarthy *et al* (2017b).

[19] See e.g. Hammond (1994a,b, 1999); Herzberg (2009); Halpern (2010); Pivato (2014); Benci, Horsten and Wenmackers (2018a); Benci, Horsten, and Wenmackers (2018b); Brickhill and Horsten (2018).

[20] That is, if $\succsim_V$ is a linear preorder on V, then it determines a linear partial order on $V/{\sim_V}$; the quotient map $V \to V/{\sim_V}$ is a representation of $\succsim_V$.

[21] See e.g. Weymark (1993, 1995) for discussion in the context of Harsanyi's theorem.

P1 If $x \sim_i y$ for all $i \in \mathbb{I}$, then $x \sim_0 y$.
P2 If $x \succsim_i y$ for all $i \in \mathbb{I}$, then $x \succsim_0 y$.
P3 If $x \succsim_i y$ for all $i \in \mathbb{I}$, and $x \succ_j y$ for some j, then $x \succ_0 y$.
P4 If $x \succsim_i y$ for all $i \in \mathbb{I} \setminus \{j\}$ and $x \curlywedge_j y$, then $x \not\precsim_0 y$.

When $\mathbb{I}$ is finite and the $\succsim_i$ are preference relations, P1 is Pareto indifference; P1–P2 together are sometimes called semi-strong Pareto; and P1–P3 are strong Pareto. Since we are allowing for incompleteness, P4 is a natural supplement. Given P2 and a sufficiently rich domain, P4 follows from the simpler condition that, if $x \sim_i y$ for all $i \in \mathbb{I} \setminus \{j\}$ and $x \curlywedge_j y$, then $x \curlywedge_0 y$. But to avoid domain conditions, we will appeal to P4 as formulated. P1–P4 are also natural conditions under other interpretations of the $\succsim_i$, such as when they are comparative likelihood relations.

2.3 Results

Suppose we are given co-convex representations $f_i \colon X \to V_i$ of the $\succsim_i$, $i \in \mathbb{I}$. Set $V_\mathbb{I} := \prod_{i \in \mathbb{I}} V_i$ and $f_\mathbb{I} := (f_i)_{i \in \mathbb{I}} \colon X \to V_\mathbb{I}$. We are interested in the question of whether $\succsim_0$ has a representation of the form $Lf_\mathbb{I}$, with L a linear map from $V_\mathbb{I}$ into some partially ordered vector space V. When $\mathbb{I}$ is finite, we can write

$$Lf_\mathbb{I} = \textstyle\sum_{i \in \mathbb{I}} L_i f_i$$

where $L_i \colon V_i \to V$ is the ith component of L;[22] thus $Lf_\mathbb{I}$ is an additive representation. But when $\mathbb{I}$ is infinite, L is not determined by its components; relatedly, the sum over i does not make sense.[23] In any case, by endowing $V_\mathbb{I}$ with the product partial order $\succsim_\mathrm{P}$, we can consider the positivity of L. Note that if L is positive, or strictly positive, then so is every L_i, but if $\mathbb{I}$ is infinite, the converse need not hold.

Since $f_\mathbb{I}$ and $Lf_\mathbb{I}$ are automatically co-convex, a necessary condition for the existence of a representation of $\succsim_0$ of the form $Lf_\mathbb{I}$ is that there exists *some* representation $f_0 \colon X \to V_0$ of $\succsim_0$ such that all the f_i, including f_0, are co-convex. Our first theorem says that this necessary condition is also sufficient, and lays out the connection between the Pareto-style axioms and the positivity properties of L.

Theorem 6. *Assume that for $i \in \mathbb{I} \cup \{0\}$ the $\succsim_i$ have co-convex representations f_i.*
(a) P1 *holds if and only if $\succsim_0$ has a representation of the form $Lf_\mathbb{I}$, with L linear.*
(b) P1–P2 *hold if and only if $\succsim_0$ has a representation of the form $Lf_\mathbb{I}$, with L linear and positive.*
(c) P1–P3 *hold if and only if $\succsim_0$ has a representation of the form $Lf_\mathbb{I}$, with L linear and strictly positive.*
(d) P1–P4 *hold if and only $\succsim_0$ has a representation of the form $Lf_\mathbb{I}$, with L linear and strictly positive, and every L_i an order embedding.*

An obvious question is whether *every* representation $f_0 \colon X \to V_0$ of $\succsim_0$, co-convex with $f_\mathbb{I}$, has the form $Lf_\mathbb{I}$. Note that, if f_0 is a representation, then so is $f_0 + b$, for any constant $b \in V_0$, but at most one of these will be of the form $Lf_\mathbb{I}$. So a more reasonable question is whether we can write every f_0 in the form $Lf_\mathbb{I} + b$. The following result gives an affirmative answer, subject to the following domain richness condition.

DR $V_\mathbb{I} = \mathrm{Span}(f_\mathbb{I}(X) - f_\mathbb{I}(X))$.

[22] In other words, regardless of whether $\mathbb{I}$ is finite, $L_i := L1_i$, where 1_i is the natural embedding of V_i into $V_\mathbb{I}$.
[23] Nonetheless, the linearity of L does entail a kind of finite additivity; see section 3.1 for further discussion in the context of Harsanyi's theorem.

In other words, $f_{\mathbb{I}}(X)$ is not contained in any proper affine subspace of $V_{\mathbb{I}}$; equivalently, $V_{\mathbb{I}}$ is the affine hull of $f_{\mathbb{I}}(X)$.[24]

Theorem 7. *Assume that for $i \in \mathbb{I} \cup \{0\}$ the $\succsim_i$ have co-convex representations f_i. For the left-to-right directions of parts (b)–(d), assume* DR.

(a) P1 *holds if and only if $f_0 = Lf_{\mathbb{I}} + b$ for some $b \in V_0$ and L linear.*
(b) P1–P2 *hold if and only if $f_0 = Lf_{\mathbb{I}} + b$ for some $b \in V_0$ and L linear and positive.*
(c) P1–P3 *hold if and only if $f_0 = Lf_{\mathbb{I}} + b$ for some $b \in V_0$ and L linear and strictly positive.*
(d) P1–P4 *hold if and only if $f_0 = Lf_{\mathbb{I}} + b$ for some $b \in V_0$ and L linear and strictly positive, with every L_i an order embedding.*

In the special case in which $\mathbb{I}$ is finite and $V_i = \mathbb{R}$ for each $i \in \mathbb{I} \cup \{0\}$ (so that P4 is vacuous), this is proved in De Meyer and Mongin (1995, Prop. 1) without assuming DR.[25]

So far we have allowed the representations f_i to have values in different partially ordered vector spaces. One might wish to construct a single partially ordered vector space V in which all of the different preorders are represented, and in which the representation of $\succsim_0$ is essentially the sum of the representations of the $\succsim_i$. 'Essentially' is required here since we cannot literally sum over $\mathbb{I}$ when it is infinite. But say that a map $S\colon \prod_{i\in\mathbb{I}} V \to V$ *extends summation* if it restricts to the summation map $\bigoplus_{i\in\mathbb{I}} V \to V$; equivalently, each component S_i of S is the identity map on V.

Theorem 8. *Assume that for $i \in \mathbb{I} \cup \{0\}$ the $\succsim_i$ have co-convex representations f_i. Then* P1–P4 *hold if and only if there exists a partially ordered vector space V, representations $g_i\colon X \to V$ of the $\succsim_i$ for $i \in \mathbb{I} \cup \{0\}$, and a linear map $S\colon \prod_{i\in\mathbb{I}} V \to V$ such that*

(a) $g_0 = Sg_{\mathbb{I}}$
(b) *S extends summation and is strictly positive on* $\prod_{i\in\mathbb{I}} \mathrm{Span}(g_i(X) - g_i(X))$[26]
(c) *$f_{\mathbb{I}}$, f_0, $g_{\mathbb{I}}$, and g_0 are together co-convex.*

If DR *holds as well as* P1–P4*, we can further require $V = V_0$ and $g_0 = f_0$.*

2.3.1 Domain assumptions

DR is a stronger domain assumption than is needed for Theorem 7(b)–(d) and the last claim in Theorem 8,[27] and we adopt it for its simplicity. It cannot simply be dropped from these theorems, even when $\mathbb{I}$ is finite, as the following example shows.

Example 9. Let $X = \mathbb{R}$ and $\mathbb{I} = \{1, 2\}$. Let $\succsim_1$ and $\succsim_2$ be represented by $f_1(x) = x$ and $f_2(x) = -x$. Thus $V_{\mathbb{I}} = \mathbb{R}\times\mathbb{R}$, $f_{\mathbb{I}}(x) = (x, -x)$, and $\mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X)) = \mathbb{R}(1, -1)$. Let $V_0 = \mathbb{R}$ with the 'trivial' linear partial order such that any two distinct elements are incomparable (thus

[24] To interpret DR, it is worth noting that, by Lemma 25 below, and taking into account the fact that $f_{\mathbb{I}}(X)$ is convex, $\mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X)) = \{\lambda(f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y)) : \lambda > 0, x, y \in X\}$. Thus when the $\succsim_i$ are preference relations, so that the V_i may be seen as utility spaces, DR says that any logically possible profile of utility differences (i.e. any element of $V_{\mathbb{I}}$) is realized as the utility difference between some $x, y \in X$, at least up to scale. Indeed, DR is equivalent to the conjunction of the claims (i) that $V_i = \mathrm{Span}(f_i(X) - f_i(X))$, so that that none of the utility spaces V_i is gratuitously large (in terminology introduced in section 2.4, this means that f_i is pervasive); and (ii) that $\mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X)) = \prod_{i\in\mathbb{I}} \mathrm{Span}(f_i(X) - f_i(X))$.

[25] In this special case, our methods give an alternative and arguably simpler DR-free proof of part (a), but they do not appear to help at all with DR-free proofs of the other parts. See however the discussion at the end of section 2.3.1.

[26] Here positivity refers to the product partial order on $\prod_{i\in\mathbb{I}} V$, restricted to $\prod_{i\in\mathbb{I}} \mathrm{Span}(g_i(X) - g_i(X))$.

[27] For example, for Theorem 7(b)(c), DR could be replaced by the weaker assumption that $\mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X))$ contains the positive cone $\{ v \in V_{\mathbb{I}} \mid v \succsim_{\mathrm{P}} 0 \}$; for part (d) and for the last part of Theorem 8 one could assume that $\mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X))$ contains both the positive cone and $\bigoplus_{i\in\mathbb{I}} V_i$. However, even these assumptions are stronger than necessary.

$x \succsim_{V_0} 0$ if and only if $x = 0$). Let $\succsim_0$ be represented by $f_0(x) = x$. Thus $(f_{\mathbb{I}}, f_0)(X)$ is convex, $\succsim_0$ satisfies P1–P4, but DR fails.

Suppose $f_0 = Lf_{\mathbb{I}} + b$, with $L\colon V_{\mathbb{I}} \to V_0$ positive. We have $(1, 0) \succsim_{\mathrm{P}} 0$ and since L is positive, $L((1, 0)) = 0$; and similarly, $L((0, 1)) = 0$. Linearity of L implies $L = 0$, contradicting $f_0 = Lf_{\mathbb{I}} + b$. Thus parts (b)–(d) of Theorem 7 do not hold. Similarly for the last claim of Theorem 8: there is no representation g_1 of $\succsim_1$ with values in V_0.

When $\mathbb{I}$ is infinite, DR is perhaps surprisingly strong.

Example 10. Suppose that $\mathbb{I} = \mathbb{N}$, $X = \prod_{i\in\mathbb{I}}[0, 1]$, $V_i = \mathbb{R}$, $f_i(x) = x_i$ for $i \in \mathbb{I}$. Then $V_{\mathbb{I}} = \prod_{i\in\mathbb{I}} \mathbb{R}$ is the space of sequences of real numbers, but $\mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X))$ is the subspace of bounded sequences, so DR fails.

Theorem 7(b)–(d) also fail. As in Benci, Horsten and Wenmackers (2018a), choose a uniform probability measure μ on $\mathbb{N}$ with values in a non-Archimedean ordered field F that extends the reals; as such, it is an ordered real vector space. Let V_0 be the ordered real vector space of finite (including infinitesimal) elements of F; thus for every x in V_0 there is some natural number n with $n \succsim_{V_0} x$. Note that μ has values in V_0. Let $f_0\colon X \to V_0$ map each element of X to its μ-expectation; that is $f_0(x) = \sum_{i\in\mathbb{I}} \mu(\{i\})x_i$. The $\succsim_0$ so represented satisfies P1–P4 (note that the $\succsim_i$ are complete, so P4 is vacuous). All the $\succsim_i$ satisfy strong independence, but $\succsim_0$ violates both the Archimedean and mixture continuity axioms of expected utility.

Suppose $f_0 = Lf_{\mathbb{I}} + b$ for some linear mapping $V_{\mathbb{I}} \to V_0$, implying $b = 0$. Consider the sequence $v = (1, 2, 3, 4, \dots)$ in $V_{\mathbb{I}}$, and for each natural number n, the bounded sequence $v_n = (1, 2, \dots, n-1, n, n, n, n, \dots)$. By linearity of L we must have $L(v_n) \succsim_{V_0} n - 1$ (since $f_0(\frac{1}{n}v_n)$ is infinitesimally close to 1). But $v \succsim_{\mathrm{P}} v_n$ for every natural number n. So if L is positive, we must have $L(v) \succsim_{V_0} n$ for every n. But that is impossible, by construction of V_0. A similar argument shows that the last claim of Theorem 8, involving DR, also fails.

An interesting question therefore is how far DR can be weakened in Theorems 7 and 8. We plan to take up that question in other work, but for now we note that one can sometimes bypass DR by allowing L in Theorem 7 to be defined only on a subspace of $V_{\mathbb{I}}$ that contains $f_{\mathbb{I}}(X)$. For example, by adding constants to the f_i, we can assume that $f_{\mathbb{I}}(X)$ is contained in $Y := \mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X))$. Without any need for DR, Theorem 7(a)–(c) hold if L is only required to be defined on Y. Part (d) also holds under the further assumption that Y contains $\bigoplus_{i\in\mathbb{I}} V_i$, which is needed for the components L_i to be defined.[28] This last assumption is weaker than DR when $\mathbb{I}$ is infinite, and is satisfied in Example 10. To illustrate, f_0 in that example extends uniquely to a strictly positive linear $L\colon Y \to V_0$ that maps each bounded sequence to its μ-expectation, with every L_i an order embedding, and we have $f_0 = Lf_{\mathbb{I}}$. But the example shows that L cannot be extended to a positive linear map $V_{\mathbb{I}} \to V_0$.

We will not pursue domain questions any further. For simplicity, we use DR to discuss the uniqueness of the representations discussed section 2.3, to which we now turn. But we avoid it in the applications we present in section 3.

2.4 Uniqueness

We first address the general question of to what extent representations with values in partially ordered vector spaces are unique. Say that a function $f\colon X \to V$ is *pervasive* if $V = \mathrm{Span}(f(X)-$

[28]The proofs of these claims are trivial variations on the proof of Theorem 7, so we omit them. The situation for the last statement in Theorem 8 is slightly less straightforward, but, in short, we can again replace DR by the assumption that Y contains $\bigoplus_{i\in\mathbb{I}} V_i$, if we only require in part (b) that S is strictly positive on $Y' := \mathrm{Span}(g_{\mathbb{I}}(X) - g_{\mathbb{I}}(X))$. That is to say, assuming that Y contains $\bigoplus_{i\in\mathbb{I}} V_i$, P1–P4 hold if and only if there exist representations $g_i\colon X \to V_0$ of $\succsim_i$ for $i \in \mathbb{I}$, and a linear map $S\colon \prod_{i\in\mathbb{I}} V_0 \to V_0$, such that $f_0 = Sg_{\mathbb{I}}$, S extends summation, S is strictly positive on Y', and $f_{\mathbb{I}}$, f_0, and $g_{\mathbb{I}}$ are together co-convex.

$f(X)$); equivalently, V is the affine hull of $f(X)$. The restriction to pervasive representations in the next result is mild, since by adding a constant to f we can always obtain a pervasive representation of the preorder on X that is represented by f: for any $x_0 \in X$, the representation $f^*: X \to \mathrm{Span}(f(X) - f(X))$ defined by $f^*(x) = f(x) - f(x_0)$ is pervasive.

Lemma 11. *Suppose given co-convex representations $f: X \to V$ and $g: X \to V'$ of a preorder $\succsim$ on X. Suppose that f and g are pervasive. Then there exists a unique linear order isomorphism $L: V \to V'$ and unique $b \in V'$ such that $g = Lf + b$.*

The following result explains the sense in which the type of representation of $\succsim_0$ discussed in Theorem 6 is unique.

Proposition 12. *Assume that for $i \in \mathbb{I} \cup \{0\}$ the $\succsim_i$ have co-convex representations f_i such that DR holds. Suppose that $L: V_{\mathbb{I}} \to V$ and $L': V_{\mathbb{I}} \to V'$ are linear maps to partially ordered vector spaces such that $Lf_{\mathbb{I}}$ and $L'f_{\mathbb{I}}$ represent $\succsim_0$. Then there is a unique linear order isomorphism $M: L(V_{\mathbb{I}}) \to L'(V_{\mathbb{I}})$ such that $L' = ML$.*

The proof establishes in part that $Lf_{\mathbb{I}}: X \to L(V_{\mathbb{I}})$ and $L'f_{\mathbb{I}}: X \to L'(V_{\mathbb{I}})$ are co-convex, pervasive representations of $\succsim_0$, so that Lemma 11 applies. Note that Proposition 12, unlike Theorem 6, assumes DR. Here is an example to illustrate why.

Example 13. Let $X = \mathbb{R}$, let $\mathbb{I} = \{1, 2\}$, and let $\succsim_0, \succsim_1, \succsim_2$ all equal the standard ordering on $\mathbb{R}$. Let $V_1, V_2 = \mathbb{R}$ with the standard ordering, and let $f_1, f_2: X \to \mathbb{R}$ both be the identity map. Thus $V_{\mathbb{I}} = \mathbb{R} \times \mathbb{R}$ and DR fails. Let $L: V_{\mathbb{I}} \to V := \mathbb{R}$ map $(x, y) \mapsto x + y$, and let $L': V_{\mathbb{I}} \to V' := V_{\mathbb{I}}$ be the identity map. Then $Lf_{\mathbb{I}}$ and $L'f_{\mathbb{I}}$ both represent $\succsim_0$, but there is no linear map $M: V \to V'$ such that $L' = ML$.

Next we establish that the L and b in Theorem 7 are unique.

Proposition 14. *Assume that for $i \in \mathbb{I} \cup \{0\}$ the $\succsim_i$ have co-convex representations f_i such that DR holds. Then there at most one linear map $L: V_{\mathbb{I}} \to V_0$ and one $b \in V_0$ such that $f_0 = Lf_{\mathbb{I}} + b$.*

Here, DR cannot be dropped, even in the special case in which $\mathbb{I}$ is finite and $V_i = \mathbb{R}$ for each $i \in \mathbb{I} \cup \{0\}$, by Fishburn (1984, Cor. 1).

In Theorem 8, given DR, we can choose g_0 to be a pervasive representation of $\succsim_0$. (Choose f_0 to be pervasive, using the construction before Lemma 11, and then apply the last statement of Theorem 8.) Thus we assume that g_0 and g'_0 are pervasive in the following result.

Proposition 15. *Assume that for $i \in \mathbb{I} \cup \{0\}$ the $\succsim_i$ have co-convex representations f_i such that DR holds. Suppose $g_i: X \to V$ and $S: \prod_{i \in \mathbb{I}} V \to V$ satisfy the conditions (a)–(c) of Theorem 8, as do $g'_i: X \to V'$ and $S': \prod_{i \in \mathbb{I}} V' \to V'$. Assume also that g_0 and g'_0 are pervasive. Then there exists a unique linear order isomorphism $L: V \to V'$ and unique constants $b_i \in V'$ such that, for all $i \in \mathbb{I} \cup \{0\}$, $g'_i = Lg_i + b_i$. Moreover, $S'(v) = LS((L^{-1}v_i)_{i \in \mathbb{I}})$ for all $v \in \bigoplus_{i \in \mathbb{I}} V' + \mathrm{Span}(g'_{\mathbb{I}}(X) - g'_{\mathbb{I}}(X))$.*

3 Applications

We now present two applications in which co-convex representations naturally arise.

3.1 Preference aggregation

Let us assume that the $\succsim_i$ are preference relations, with $\succsim_0$ that of the social observer. In this context, co-convex representations arise naturally from the main expected utility axiom of strong independence (see note 7).

Lemma 16. *Suppose X is a convex set.*

(i) A preorder on X satisfies strong independence if and only if it has a mixture-preserving representation. The representation may be chosen to be pervasive.

(ii) Any family $\{f_i \colon X \to V_i\}_{i \in \mathcal{I}}$ of mixture-preserving functions is co-convex.

Given that the $\succsim_i$ satisfy strong independence, pervasive (and, by Lemma 11, essentially unique) mixture-preserving representation f_i may be chosen for each of them, and these representations are automatically co-convex. The results of section 2 therefore apply.[29] Here, for example, is a corollary of Theorem 8.

Theorem 17. *Suppose X is convex, and for all $i \in \mathbb{I} \cup \{0\}$, $\succsim_i$ satisfies strong independence. Assume* P1–P4. *Then there exists a partially ordered vector space V, mixture-preserving representations $f_i \colon X \to V$ of the $\succsim_i$, and a linear map $S \colon \prod_{i \in \mathbb{I}} V \to V$ that extends summation, such that $f_0 = S f_{\mathbb{I}}$.*

In this context, our results generalize the celebrated social aggregation theorem of Harsanyi (1955). In his framework, the population $\mathbb{I}$ is finite, and X is the set of simple probability measures on a given set of social outcomes. In contrast, we allow $\mathbb{I}$ to be infinite, and we allow X to be an arbitrary convex set, which may in particular be infinite dimensional.[30] This allows for a wide range of models of uncertainty. For example, in the setting of objective risk, X may be any convex set of probability measures on a measurable space; in the setting of objective risk and subjective uncertainty, it may be the set of Anscombe-Aumann acts; in the setting of subjective uncertainty, it may be the set of Savage acts when those are equipped with convex structure, as in for example Ghirardoto *et al* (2003);[31] it may be a set of simple lotteries with nonstandard probabilities; or it may be an arbitrary mixture space, and hence isomorphic to a convex subset of a vector space as noted by Hausner (1954).[32]

One version of Harsanyi's result is that, if P1–P3 hold, as well as strong independence, continuity and completeness for each $\succsim_i$, then the social preorder can be represented by the sum of real-valued mixture-preserving individual utility functions.[33] Theorem 17 shows that

[29] For applications of Theorem 7, it may be useful to note that, in this context, DR is equivalent to the following domain richness condition, stated directly in terms of X and the $\succsim_i$:

DR′ Suppose given $x_i, y_i \in X$ and $\lambda_i > 0$, for each $i \in \mathbb{I}$. Then there exist $z, w \in X$ and $\lambda > 0$ such that, for all $i \in \mathbb{I}$, $\frac{\lambda}{\lambda+\lambda_i} w + \frac{\lambda_i}{\lambda+\lambda_i} x_i \sim_i \frac{\lambda}{\lambda+\lambda_i} z + \frac{\lambda_i}{\lambda+\lambda_i} y_i$.

(Heuristically: the difference in value for i between z and w is λ_i/λ times the difference between x_i and y_i.)

[30] The dimension of X is, by definition, the dimension of the smallest affine space that contains it; equivalently, the dimension of the vector space $\mathrm{Span}(X - X)$.

[31] As is well known, however, allowing for subjective uncertainty is likely to lead to impossibility results; see the end of section 4.

[32] For a discussion of the relationship between mixture spaces in the sense of Hausner and mixture *sets* in the sense of Herstein and Milnor (1953), see Mongin (2001); Mongin constructs a natural map from each mixture set onto a convex set, and shows that it is an isomorphism if and only if the mixture set is a mixture space (or in his terminology 'non-degenerate').

[33] For discussion of different variations of Harsanyi's result, see e.g. Weymark (1993). Note that Harsanyi's result is usually stated in terms of a *weighted* sum of individual utility functions; but one can absorb the weights into the utility functions to get an unweighted sum. Of course, in both the weighted and unweighted versions, the representation of the social preorder is a linear transformation of the profile of individual utilities. Roughly speaking, Theorems 6 and 7 generalize the formulation with weighted sums, while Theorem 8 and its corollaries Theorems 17 and 19 generalize the unweighted version.

essentially the same conclusion holds without assuming continuity or completeness, but assuming P4 (which is vacuous given completeness), and allowing the utility functions to be vector-valued. In light of the discussion after Example 4, and especially note 17, the conclusion of Theorem 17 may be made visibly closer to Harsanyi's by taking utility values to be matrices of real numbers; Harsanyi's conclusion is then the case where the matrices are one-by-one.

The only caveat is that, when $\mathbb{I}$ is infinite, the linear map S used to combine the individual utility functions f_i does not simply sum them up. However, we still get separability, or 'finite additivity'. If $\mathbb{I} = \mathbb{J} \sqcup \mathbb{K}$, then $V_{\mathbb{I}} = V_{\mathbb{J}} \times V_{\mathbb{K}}$; denoting restrictions in the obvious way, we have $Sf_{\mathbb{I}} = S_{\mathbb{J}} f_{\mathbb{J}} + S_{\mathbb{K}} f_{\mathbb{K}}$. So much follows from the linearity of S; since S also extends summation, $S_{\mathbb{J}}$ is simply the summation map when $\mathbb{J}$ is finite.

Our results also yield fully additive representations in some important cases when $\mathbb{I}$ is infinite. Suppose, for example, that in the context of Theorem 17, each element of X is a gamble in which only finitely many people from $\mathbb{I}$ have any chance to exist. For each $i \in \mathbb{I}$ and $x, x' \in X$, it is natural to suppose that $x \sim_i x'$ if i is certain not to exist in either one; thus $f_i(x) = f_i(x')$. By subtracting a constant from f_i, we may assume that $f_i(x) = f_i(x') = 0$. (Note that renormalizing f_i in this way changes neither the fact that f_i represents $\succsim_i$ nor the fact that $Sf_{\mathbb{I}}$ represents $\succsim_0$.) The upshot of this construction is that, for each $x \in X$, $f_i(x) = 0$ for all but finitely many $i \in \mathbb{I}$. We can therefore write $Sf_{\mathbb{I}} = \sum_{i \in \mathbb{I}} f_i$, a fully additive representation of the social preorder.

Suppose now that X is a convex set of probability measures on a measurable space Y. It is natural to ask whether the representations constructed in Theorem 17, for example, can be written as integrals over Y, in the style of expected utility theory.

To make the question precise, suppose we are given a vector space V and a separating vector space V' of linear functionals $V \to \mathbb{R}$.[34] A function $U \colon Y \to V$ is *weakly X-integrable with respect to V'* if there exists $f \colon X \to V$ such that $\int_Y \Lambda \circ U \, \mathrm{d}\mu = \Lambda \circ f(\mu)$ for all $\Lambda \in V', \mu \in X$. In particular, every $\Lambda \circ U$ must be Lesbesgue integrable against every $\mu \in X$. The *Pettis* or *weak* integral is defined by setting $\int_Y U \, \mathrm{d}\mu := f(\mu)$. When $f \colon X \to V$ can be written in this form, we say that f is *expectational*. The question, then, is whether the representations f_i, including f_0, can be chosen to be expectational functions.

In the most common case, where X is a convex set of *finitely supported* probability measures on a measurable space Y with measurable singletons, there is a straightforward positive answer: *any* mixture-preserving, vector-valued function on X is expectational (independently of how V' is chosen). However, in the general case, it turns out that we need to consider representations with values in *preordered* vector spaces (cf. Remark 5). Indeed, the following result (McCarthy *et al*, 2018, Lemma 4.3) contrasts with Lemma 16(i).

Lemma 18. *Let X be an arbitrary convex set of probability measures. A preorder on X satisfies strong independence if and only if it has an expectational (and not merely mixture-preserving) representation with values in a preordered (but not necessarily partially ordered) vector space.*

The next result uses this to elaborate on Theorem 8.

Theorem 19. *Suppose X is a convex set of probability measures on a measurable space Y, and, for all $i \in \mathbb{I} \cup \{0\}$, $\succsim_i$ satisfies strong independence. Assume* P1–P4. *Then there exists a preordered vector space V equipped with a separating vector space V' of linear functionals; for $i \in \mathbb{I} \cup \{0\}$, functions $U_i \colon Y \to V$ that are weakly X-integrable with respect to V', such that $\mu \mapsto \int_Y U_i \, \mathrm{d}\mu$ represents $\succsim_i$; and a linear map $S \colon \prod_{i \in \mathbb{I}} V \to V$ that extends summation such that $U_0 = SU_{\mathbb{I}}$.*

[34]Endowing V with the weak topology with respect to V' makes it a locally convex topological vector space whose dual is V' (Rudin, 1991, 3.10).

Thus when $\mathbb{I}$ is finite, for example, without assuming continuity or completeness or interpersonal comparisons, we find that the social preorder is represented by the mapping $\mu \mapsto \int_Y \sum_{i\in\mathbb{I}} U_i \, \mathrm{d}\mu$; in other words, by expected total utility.

3.2 Opinion pooling

Here we assume that X is a Boolean algebra of events, understood as sets of states of nature, and the $\succsim_i$ are comparative likelihood relations on X, expressing the beliefs of each individual, with $\succsim_0$ expressing those of the social observer. So for events $A, B \in X$, $A \succsim_i B$ has the interpretation that, according to i, A is at least as likely as B.

Generalising ordinary probability measures, we will be considering representations by vector measures,[35] with values in partially ordered vector spaces. In Lemma 23(i) below, we give a necessary and sufficient condition for a preorder $\succsim$ on X to be representable by a vector measure f. But in terms of standard axioms of comparative probability (see e.g. Alon and Lehrer, 2014), the existence of such a representation entails that $\succsim$ satisfies Reflexivity, Transitivity, and Generalized Finite Cancellation. Positivity and Non-Triviality are equivalent respectively to the further conditions that $f(A) \succsim_V 0$ for all $A \in X$ and that $f(A) \succ_V 0$ for some $A \in X$. We treat Positivity and Non-Triviality as natural but optional assumptions about $\succsim$, rather than imposing these conditions on f. The main remaining axioms—Completeness and Monotone Continuity—are the ones which our use of vector measures is intended to avoid.

Our Pareto-style axioms P1–P4 may seem as plausible here as in the context of preference aggregation. However, consider

Example 20. Let $\mathbb{I} = \{1, 2\}$. A ball is going to be drawn randomly from an urn containing three balls, red, yellow and blue. Individual 1 privately observes that the ball is not red, and concludes $\{B\} \sim_1 \{R, Y\}$. Individual 2 privately observes that the ball is not yellow, and concludes $\{B\} \sim_2 \{R, Y\}$. The social observer, privy to each individual's private information, will conclude $\{B\} \succ_0 \{R, Y\}$, contrary to P1.

The natural reply, though, is that P1–P4 are not designed for the problem in which the observer's task is to aggregate the opinions represented by $(\succsim_i, \mathcal{I}_i)$, where $\mathcal{I}_i$ is i's private information. They are meant for circumstances in which the only data the observer has, or considers relevant, is the $\succsim_i$, say when all private information is either hidden from the observer, or has been made common knowledge. Thus we assume that P1–P4 are applicable in at least an important range of cases; see Dietrich and List (2016) for related discussion.

Linear opinion pooling is the idea that society's beliefs should be represented by a linear combination of individual beliefs with nonnegative coefficients, and goes back at least to Stone (1961). When each $\succsim_i$ can be represented by a probability measure on X, linear pooling was axiomatized by McConway (1981).[36] An alternative axiomatization using Pareto-style conditions was given in Mongin (1995) and De Meyer and Mongin (1995), applying the Lyapunov convexity theorem. Further references are given in section 4. Here we give linear pooling results that replace ordinary probability measures with vector measures. We will first extend the approach based on Lyapunov's theorem. However, this requires a finite population, as well as some technical restrictions, so we will go on to explain an alternative approach, still using Pareto-style conditions, that mixes objective and subjective probability, in the spirit of Anscombe-Aumann decision theory.

[35] These were defined following Example 2.

[36] In this special case, the linear combination is typically normalized to a convex combination.

3.2.1 Lyapunov

Our aggregation theorems apply if we assume that the $\succsim_i$ are represented by *co-convex* vector measures. This assumption follows under certain conditions from the Lyapunov convexity theorem. The following version is proved in Armstrong and Prikry (1981) (their Theorem 2.2, which allows X more generally to be an F-algebra).

Theorem 21 (Lyapunov)**.** *Suppose X is a σ-algebra, and $\{f^i \colon X \to \mathbb{R}\}_{i \in \mathcal{I}}$ is a finite family of finitely additive, bounded, non-atomic signed measures on X. Then the f^i are co-convex.*

With this in mind, we say that a vector measure $f\colon X \to V$ is *admissible* if V is finite-dimensional, and, with respect to some (hence any) basis of V, each component of f is bounded and non-atomic.[37] To illustrate, these conditions are fulfilled by f in Example 2. Then Lyapunov's theorem ensures that $f(X)$ is convex; it also ensures that any finite number of admissible vector measures are co-convex.

We thus obtain (for example) this application of Theorem 6.

Theorem 22. *Suppose that $\mathbb{I}$ is finite, and that, for $i \in \mathbb{I} \cup \{0\}$, $\succsim_i$ is a preorder on a σ-algebra X that can be represented by an admissible vector measure $f_i \colon X \to V_i$. Suppose also that* P1–P4 *hold. Then there exists a finite dimensional partially ordered vector space V and, for each $i \in \mathbb{I}$, a linear order embedding $L_i \colon V_i \to V$, such that the admissible vector measure $\sum_{i \in \mathbb{I}} L_i f_i$ represents $\succsim_0$.*

So, under the stated assumptions, the social observer's beliefs are given by a linear pooling of individual beliefs.

3.2.2 Convexification

Here we give an approach that allows the population to be infinite and that provides an axiomatic basis for the use of vector measures. It mixes objective and subjective probabilities in the style of Anscombe and Aumann (1963).

We proceed by embedding X in a convex set $\overline{X}$. Here we only assume that X is a Boolean algebra on a set S of states of nature. Say that an *extended event* is a function $F\colon S \to [0,1]$ that is constant on each cell of a finite partition $\mathcal{A} \subset X$ of S. We identify each $A \in X$ with the extended event χ_A given by the characteristic function of A. Let $\overline{X}$ be the set of extended events; it is a convex subset of the vector space of all functions $S \to \mathbb{R}$.

A preorder $\overline{\succsim}$ on $\overline{X}$ can be understood as a comparative likelihood relation in the following way. Suppose that, for each $p \in [0,1]$, a coin with bias p is going to be tossed, independently of the events encoded in X; let H_p be the event that it lands heads.[38] Given $F \in \overline{X}$, constant on each cell of some partition $\{E_1, \ldots, E_n\} \subset X$ of S, we associate the event $H_F = E_1 H_{p_1} \vee \cdots \vee E_n H_{p_n}$, where p_j is the value that F takes on E_j. Thus H_F occurs whenever nature selects some event E_j from the partition, and, independently, the coin with bias p_j lands heads. $F \mathrel{\overline{\succsim}} F'$ holds just in case H_F is judged at least as likely as $H_{F'}$.[39] So interpreted, $\overline{\succsim}$ should appropriately take into

[37] Each component of f is automatically a finitely additive signed measure on X. Following Armstrong and Prikry (1981), a finitely additive signed measure f^j is said to be non-atomic if for every $\epsilon > 0$ there is a finite partition $\{A_1, \ldots, A_n\}$ of X such that for all k the total variation $|f^j|(A_k)$ of A_k under f^j is less than ϵ. Note that non-atomicity in this sense has nothing to do with the partial order on V.

[38] At a cost in abstraction, we could avoid the need for infinitely many coins by considering a single sample from $[0,1]$ with the uniform measure.

[39] While emphasizing that our conceptual approach is not decision theoretic, $\overline{\succsim}$ should match the preferences of an agent who gets a prize on heads.

account the objective probabilities of the coin tosses; we suggest that the appropriate constraint is strong independence.

The following proposition allows us to generate representations of likelihood relations on X by vector measures.

Lemma 23. *Let X be a Boolean algebra and $\overline{X}$ the set of extended events.*

(i) If a preorder $\overline{\succsim}$ on $\overline{X}$ satisfies strong independence, its restriction $\succsim$ to X can be represented by a vector measure. Conversely, a preorder $\succsim$ on X can be represented by a vector measure only if it arises in this way.

(ii) If a mixture-preserving function $\overline{f}\colon \overline{X} \to V$, with V a vector space, satisfies $\overline{f}(\chi_\varnothing) = 0$, its restriction f to X is a vector measure. Conversely, a function $f\colon X \to V$ is a vector measure only if it arises in this way.

Lemma 23, in combination with Lemma 16, yields the following application of Theorem 6.

Theorem 24. *Let X be a Boolean algebra, and $\overline{X}$ the space of extended events. For $i \in \mathbb{I} \cup \{0\}$, let $\overline{\succsim}_i$ be a preorder on $\overline{X}$ whose restriction to X is $\succsim_i$. Assume that each $\overline{\succsim}_i$ satisfies strong independence, and that the $\overline{\succsim}_i$ together satisfy* P1–P4.

Then there is for each $i \in \mathbb{I} \cup \{0\}$ a vector measure $f_i\colon X \to V_i$ that represents $\succsim_i$, and a strictly positive linear map $L\colon V_\mathbb{I} \to V_0$, with each L_i an order embedding, such that $f_0 = Lf_\mathbb{I}$.

Let us contrast the two routes to linear pooling. Convexification allows the population to be infinite, and avoids the restriction of the Lyapunov approach to $\succsim_i$ with admissible representations. In addition, Lemma 23 provides necessary and sufficient conditions for a preorder $\succsim$ on X to be represented by a vector measure, whereas no such result has been given for $\succsim$ to be represented by an admissible vector measure. On the other hand, convexification requires extending X to $\overline{X}$, along with Pareto and strong independence for the extended preorders. It is hard to see an objection to this extension of Pareto,[40] but strong independence has its critics (even while remaining very popular) in the standard decision theoretic version of the Anscombe-Aumann framework.[41] Those with similar doubts in our comparative likelihood framework (cf. note 39) might prefer the Lyapunov approach. Finally, for those who prefer to avoid objective probabilities, we note the possibility of imposing convex structure directly on X; compare Ghirardoto *et al* (2003).

4 Related literature

We conclude by relating our results to extant work.

In section 2.1 we noted that our use of partially ordered vector spaces generalises some other forms of representation. These include representations with values in a non-Archimedean ordered field F, often used to model failures of continuity (see note 19 for references). Here, F is generally assumed to be an extension of the real numbers, and as such is an ordered real vector space (cf. Example 10). However, advocates of this approach may prefer to rework our results using F, rather than $\mathbb{R}$, as the basic field. This would mean interpreting such conditions as linearity, convexity, and strong independence using coefficients drawn from F.[42] We do not pursue this project, but the only results that we do not expect to extend almost verbatim are Theorems 19 and 22.

[40] We note, however, that if the $\succsim_i$ satisfy P1–P4, and are extended to strongly independent preorders $\overline{\succsim}_i$ on $\overline{X}$, it does not follow that the $\overline{\succsim}_i$ satisfy P1–P4.

[41] See e.g. Gilboa (2009) for discussion.

[42] Thus, for example, a subset X of an F-vector space is convex over F if and only if, for all $x, y \in X$ and all $\alpha \in F$ with $0 < \alpha < 1$, we have $\alpha x + (1 - \alpha)y \in X$.

As already noted, in the special case in which $\mathbb{I}$ is finite and $V_i = \mathbb{R}$ for each $i \in \mathbb{I} \cup \{0\}$, Theorem 7 is proved in De Meyer and Mongin (1995, Prop. 1), without requiring DR; note that P4 is vacuous under these assumptions. This is the first result we know of that emphasizes the usefulness of co-convexity in the context of social aggregation. They apply their result to preference aggregation to obtain Harsanyi's theorem, and to opinion pooling using Lyapunov, in the special case in which each $\succsim_i$ is represented by a standard probability measure.

Zhou (1997) allows $\mathbb{I}$ to be infinite, and considers the special case in which X is a mixture set and the f_i are mixture-preserving representations with values in $\mathbb{R}$ (i.e. von Neumann-Morgenstern expected utility functions),[43] so that the $\succsim_i$ satisfy strong independence, continuity and completeness. Under these assumptions, his Theorems 1 and 2 are essentially our Theorem 7(a) and (b) respectively.[44] He also raises the question of whether, in our terminology, DR can be dropped from Theorem 7(b), and gives an affirmative answer under yet further assumptions. Our examples in section 2.3.1 show that the answer is negative in general. But we emphasise that our Theorem 6 and the main part of Theorem 8 do not assume DR.

Despite the importance of the problem, the results of Danan *et al* (2015) are the only ones we know of that generalize Harsanyi's theorem by dropping completeness. They assume a finite population, and take X to be the set of probability measures on a finite set of outcomes. They assume that the $\succsim_i$ satisfy strong independence and a slight strengthening of mixture continuity. It follows that the $\succsim_i$ have expected multi-utility representations (Shapley and Baucells, 1998; Dubra *et al*, 2004), and they present their results in such terms. Recall from Example 3 than any multi-representation can be re-interpreted as a representation in a partially ordered vector space. Our Theorem 7(a) and (b) say that, given P1 or P1 and P2, the vector-valued representation of $\succsim_0$ is linearly related to those of the other $\succsim_i$. Danan *et al* prove essentially this same result in their setting, and go slightly further in describing the components of the linear relation in terms of the original multi-representations. They also explain how these results extend to the case of an infinite population, generalizing Zhou's results to allow for incompleteness (although, unlike Zhou, assuming a finite set of outcomes).

Our results differ from theirs in three main ways. First, we do not require continuity; one motivation for this is that in the absence of completeness, continuity is a more problematic assumption than it appears.[45] Second, as emphasized in section 3.1, we allow for a much wider range of interpretations of the domain X. Third, while they only consider the axioms P1–P2, we consider the effects of adding the standard strong Pareto-style axiom P3, and in addition, the apparently novel P4. This whole package of Pareto-style axioms is plausible, in our view, and essential to those of our results that are arguably the fullest generalizations of Harsanyi. Of course, our results do not wholly replace those of Danan *et al* (2015), as they study details that arise specifically in the context of continuity and multi-utility representations. But we believe that it is important to consider the full range of Pareto-style conditions, even in that context; our results provide a framework for doing so, although we do not pursue it here.

As in Harsanyi's theorem, the results of Zhou (1997), Danan *et al* (2015) and this paper do not assume interpersonal comparisons. Immediately after the statement of his main theorem, though, Harsanyi (1955) introduces interpersonal comparisons and a form of anonymity, leading to the further conclusion that the social preorder can be represented by the sum of real-valued mixture-preserving individual utility functions that have been normalized to reflect interpersonal comparisons. Generalizations of this version of Harsanyi's result, that assume interpersonal

[43]As Mongin (2001) proves, each mixture set X maps naturally onto to a convex set $\overline{X}$ such that any mixture-preserving function on X comes from one on $\overline{X}$; thus the use of mixture-preserving functions on mixture sets rather than on convex sets does not give any greater generality.

[44]The exact statement of Zhou's Theorem 2 is closer to the variant of Theorem 7(b) outlined in section 2.3.1.

[45]See Dubra (2011) and McCarthy and Mikkola (2018), extending a classic observation of Schmeidler (1971).

comparisons and anonymity at the outset but drop one or more of continuity, completeness, strong independence, and the requirement of a finite population, are given in Fleurbaey (2009), Pivato (2013, 2014), and McCarthy *et al* (2018).[46]

Generalizations of Harsanyi to the case where preferences are over Anscombe-Aumann or Savage acts, allowing individuals to have different beliefs and values, are well known to lead to impossibility results.[47] Our treatment of opinion pooling separately from preference aggregation may therefore be seen as following the advice of, for example, Mongin (1998, pp. 352–3) to aggregate opinion prior to aggregating preferences.

The classic axiomatization of linear opinion pooling, when the population is finite and the p_i are standard ($[0,1]$-valued) probability measures on a σ-algebra, was given in McConway (1981). An axiomatization using analogues of the Pareto-style conditions P1–P3 was given in Mongin (1995); see also De Meyer and Mongin (1995). Chambers (2007) extends that approach to axiomatize linear pooling for ordinal probabilities. Extending McConway's approach, linear pooling results for finitely additive probability measures and an infinite population are given in Herzberg (2015) and Nielsen (2019); the latter giving an equivalent result using analogues of P1 and P2.

5 Proofs

We begin with some simple observations about convex sets, cones and vector preorders.

Lemma 25. *Let Z be a nonempty convex set. Then* $\mathrm{Span}(Z-Z) = \{\, \lambda(z-z') \mid \lambda \in \mathbb{R}, \lambda > 0, z, z' \in Z \,\}$.

Proof. The right-hand side is clearly contained in and spans the left-hand side. It suffices to show that it equals its own span. It is clearly closed under scalar multiplication. To show that it is closed under addition, suppose given $\lambda(z-z')$ and $\mu(w-w')$ with $\lambda, \mu > 0$ and $z, z', w, w' \in Z$. Then it is easy to check $\lambda(z-z') + \mu(w-w') = \nu(v-v')$, for $\nu := \mu + \nu$, $v := \frac{\lambda}{\nu} z + \frac{\mu}{\nu} w$, and $v' := \frac{\lambda}{\nu} z' + \frac{\mu}{\nu} w'$. Note that $\nu > 0$ and that v, v' are elements of Z, since it is convex. □

Given a vector space V, recall that $C \subset V$ is a *convex cone* if $0 \in C$, $C + C \subset C$ and $\lambda C \subset C$ for all $\lambda > 0$. Clearly if $C, C' \subset V$ are convex cones, then so is $C + C'$. The following is well-known; see e.g. Ok (2007, G.1.3).

Lemma 26. *Let $C \subset V$, where V is a vector space. The binary relation $\succsim_V$ on V defined by $v \succsim_V w \Leftrightarrow v - w \in C$ is a linear preorder if and only if C is a convex cone. Conversely, any linear preorder on V is of this form.*

Lemma 27. *Suppose Z is a nonempty subset of a vector space V, and $\succsim$ is a preorder on Z that satisfies strong independence. Let $C := \{\, \lambda(x-y) \mid \lambda \in \mathbb{R}, \lambda > 0, x, y \in Z, x \succsim y \,\}$. Then C is a convex cone in V. Equip V with the linear preorder $\succsim_V$ defined by $v \succsim_V w \iff v - w \in C$. Then the inclusion $\iota\colon Z \to V$ represents $\succsim$.*

Proof. It is clear that $0 \in C$, and that for $\lambda > 0$, $\lambda C \subset C$. To show that $C + C \subset C$, let $v, w \in C$. Then $v = \lambda(x-y)$ and $w = \mu(x'-y')$ for some $\lambda, \mu > 0$, $x, x', y, y' \in Z$ with $x \succsim y$ and $x' \succsim y'$.

[46] While it would be straightforward to add interpersonal comparisons and anonymity to the results of this paper when $\mathbb{I}$ is finite, there is a well known incompatibility between full anonymity and Pareto when $\mathbb{I}$ is infinite. This is avoided in Pivato (2014) by restricting to finite anonymity, and in McCarthy *et al* (2018) by restricting to social lotteries in which only finitely many individuals have a chance of existing.

[47] From among a very large body of literature, see e.g. Broome (1990); Mongin (1995, 1998); Mongin and Pivato (2015); and Zuber (2016).

We have
$$v + w = (\lambda+\mu)\Big[\big(\tfrac{\lambda}{\lambda+\mu}x + \tfrac{\mu}{\lambda+\mu}x'\big) - \big(\tfrac{\lambda}{\lambda+\mu}y + \tfrac{\mu}{\lambda+\mu}y'\big)\Big].$$
Let $\alpha = \frac{\lambda}{\lambda+\mu}$. Since $\succsim$ satisfies strong independence, we have $\alpha x + (1-\alpha)x' \succsim \alpha y + (1-\alpha)x'$ and $\alpha y + (1-\alpha)x' \succsim \alpha y + (1-\alpha)y'$, and thus $\alpha x + (1-\alpha)x' \succsim \alpha y + (1-\alpha)y'$. The displayed equation then shows that $v + w \in C$, establishing that C is a convex cone.

We now show that for $x, y \in Z$, $x \succsim y \Leftrightarrow x \succsim_V y$. Clearly $x \succsim y \Rightarrow x - y \in C \Rightarrow x \succsim_V y$. Conversely, suppose $x \succsim_V y$. Then $x-y \in C$, hence there exist $\lambda > 0$, $x', y' \in Z$ with $x' \succsim y'$ and $x - y = \lambda(x'-y')$. Letting $\alpha = \frac{1}{1+\lambda}$, this rearranges to $\alpha x + (1-\alpha)y' = \alpha y + (1-\alpha)x'$. Since $x' \succsim y'$ and $\succsim$ satisfies strong independence, we have $\alpha x + (1-\alpha)x' \succsim \alpha x + (1-\alpha)y' = \alpha y + (1-\alpha)x'$. By strong independence again, we must have $x \succsim y$. □

Recall that we write $1_i \colon V_i \to V_{\mathbb{I}}$ for the natural embedding of V_i into $V_{\mathbb{I}}$.

Proof of Theorem 6. First let us verify that, in each case, the right-hand side entails the left-hand side. So suppose we have a partially ordered vector space $(V, \succsim_V)$ and a linear $L \colon V_{\mathbb{I}} \to V$ such that $Lf_{\mathbb{I}}$ represents $\succsim_0$. Recall that we equip $V_{\mathbb{I}}$ with the product partial order $\succsim_{\mathrm{P}}$.

Suppose first that $x \sim_i y$ for every $i \in \mathbb{I}$. Then $f_{\mathbb{I}}(x) = f_{\mathbb{I}}(y)$, hence $Lf_{\mathbb{I}}(x) = Lf_{\mathbb{I}}(y)$. Since $Lf_{\mathbb{I}}$ represents $\succsim_0$, we find that $x \sim_0 y$. Therefore P1 holds. Next, suppose that L is positive, and suppose $x \succsim_i y$ for every $i \in \mathbb{I}$. Then $f_{\mathbb{I}}(x) \succsim_{\mathrm{P}} f_{\mathbb{I}}(y)$, hence $Lf_{\mathbb{I}}(x) \succsim_V Lf_{\mathbb{I}}(y)$, so $x \succsim_0 y$. Thus P2 holds as well as P1. Similarly, suppose that L is strictly positive. If $x \succsim_i y$ for all $i \in \mathbb{I}$ and $x \succ_j y$ for some $j \in \mathbb{I}$, then $f_{\mathbb{I}}(x) \succ_{\mathrm{P}} f_{\mathbb{I}}(y)$, whence $Lf_{\mathbb{I}}(x) \succ_V Lf_{\mathbb{I}}(y)$ and $x \succ_0 y$; thus P3 holds. This covers the right-to-left directions in (a)–(c).

As for (d), suppose that L is strictly positive and each L_i is an order embedding. Suppose that $x \succsim_i y$ for all $i \in \mathbb{I} \setminus \{j\}$ and $x \curlywedge_j y$. This implies that $f_i(x) \succsim_{V_i} f_i(y)$ for all $i \in \mathbb{I} \setminus \{j\}$ and $f_j(x) \curlywedge_{V_j} f_j(y)$. We can therefore write $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) = v_1 + v_2$ with $v_1 \succsim_{\mathrm{P}} 0$ and $v_2 = 1_j(f_j(x) - f_j(y)) \curlywedge_{\mathrm{P}} 0$. Since L is strictly positive, we have $L(v_1) \succsim_V 0$; since L_j represents $\succsim_{V_j}$, we have $L(v_2) \curlywedge_V 0$. Therefore $L(f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y)) = L(v_1) + L(v_2) \not\precsim_V 0$. This implies $x \not\precsim_0 y$. This shows that P4 must hold as well as P1–P3.

Conversely, now, we show that the left-hand side entails the right in each case.

Assume P1. Define subsets of $V_{\mathbb{I}}$:

$$\begin{aligned} C_0 &= \{\, \lambda(f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y)) \mid \lambda > 0, x, y \in X, x \succsim_0 y \,\} \\ C_P &= \{\, v \mid v \in V_{\mathbb{I}}, v \succsim_{\mathrm{P}} 0 \,\} \\ C &= C_0 + C_P. \end{aligned}$$

We first prove that these are convex cones. Let $X' = (f_{\mathbb{I}}, f_0)(X) \subset V_{\mathbb{I}} \times V_0$. By the co-convexity assumption, X' is convex. Let $\pi_{\mathbb{I}}$ and π_0 be the projections of $V_{\mathbb{I}} \times V_0$ onto $V_{\mathbb{I}}$ and V_0 respectively. Let $\succsim'$ be the preorder on X' represented by the restriction of π_0 to X': $x \succsim' y \iff \pi_0(x') \succsim_{V_0} \pi_0(y')$. This $\succsim'$ satisfies strong independence, as in Lemma 16(i). Define a subset of $V_{\mathbb{I}} \times V_0$:

$$C_0' = \{\, \lambda(x' - y') \mid \lambda > 0, x', y' \in X', x' \succsim' y' \,\}.$$

By Lemma 27, C_0' is a convex cone. Note that, for $x' = (f_{\mathbb{I}}, f_0)(x)$ and $y' = (f_{\mathbb{I}}, f_0)(y) \in X'$, we have
$$x' \succsim' y' \iff \pi_0(x') \succsim_{V_0} \pi_0(y') \iff f_0(x) \succsim_{V_0} f_0(y) \iff x \succsim_0 y. \tag{1}$$
This shows that $C_0 = \pi_{\mathbb{I}}(C_0')$, and therefore that C_0 is a convex cone. Meanwhile, C_P is a convex cone by Lemma 26. Therefore the sum C is also a convex cone.

Next we prove
$$x \succsim_0 y \iff f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) \in C_0. \tag{2}$$

The left-to-right direction is obvious from the definition of C_0. For the converse, we first show that $\pi_{\mathbb{I}}$ is injective on $Y \coloneqq \mathrm{Span}(X' - X')$. Since $\pi_{\mathbb{I}}$ is linear and Y is a vector space, it suffices to show that if $\pi_{\mathbb{I}}(v) = 0$ for some $v \in Y$, then $v = 0$. So suppose $\pi_{\mathbb{I}}(v) = 0$. By Lemma 25, we can write $v = \lambda(a' - b')$, with $a', b' \in X'$, $\lambda > 0$; by definition of X', we can write $a' = (f_{\mathbb{I}}(a), f_0(a))$ and $b' = (f_{\mathbb{I}}(b), f_0(b))$ with $a, b \in X$. Then $\pi_{\mathbb{I}}(v) = 0$ implies $f_{\mathbb{I}}(a) = f_{\mathbb{I}}(b)$; so, by P1, $a \sim_0 b$, and hence $f_0(a) = f_0(b)$. Therefore $a' = b'$ and $v = 0$, establishing injectivity of $\pi_{\mathbb{I}}$ on Y.

Now suppose $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) \in C_0$. Let $x' = (f_{\mathbb{I}}, f_0)(x)$ and $y' = (f_{\mathbb{I}}, f_0)(y)$. Note that $\pi_{\mathbb{I}}(x' - y') = f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) \in C_0$. Since $x' - y' \in Y$, $C_0 = \pi_{\mathbb{I}}(C_0')$, $C_0' \subset Y$, and $\pi_{\mathbb{I}}$ is injective on Y, we can conclude that $x' - y' \in C_0'$. By Lemma 27, however, $x' - y' \in C_0'$ implies $x' \succsim' y'$, and by (1) we obtain $x \succsim_0 y$. This establishes the right-to-left direction in (2).

With these preliminaries, we now prove the left-to-right direction in part (a). Assuming P1, we define a partially ordered vector space $(V, \succsim_V)$ as follows. We first let V be the quotient of $V_{\mathbb{I}}$ by the subspace $C_0 \cap -C_0$, and let $L\colon V_{\mathbb{I}} \to V$ be the quotient map. $L(C_0)$ is a convex cone in V, so it defines a linear preorder $\succsim_V$ on V, by Lemma 26. Namely, we have

$$L(v) \succsim_V L(w) \iff L(v - w) \in L(C_0) \iff v - w \in C_0.$$

It follows that $\succsim_V$ is a partial order: if $L(v) \sim_V L(w)$ then $v - w \in C_0 \cap -C_0$, so $L(v) = L(w)$. We claim that $L\colon V_{\mathbb{I}} \to V$ is the required map for part (a): that is, we claim that $Lf_{\mathbb{I}}$ represents $\succsim_0$. Suppose first that $x \succsim_0 y$. Then $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) \in C_0$, so $Lf_{\mathbb{I}}(x) \succsim_V Lf_{\mathbb{I}}(y)$. Conversely, if $Lf_{\mathbb{I}}(x) \succsim_V Lf_{\mathbb{I}}(y)$, then $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) \in C_0$. Therefore, using (2), we find $x \succsim_0 y$, as desired.

For part (b), further assuming P2, we instead define V to be the quotient of $V_{\mathbb{I}}$ by $C \cap -C$. We let $L\colon V_{\mathbb{I}} \to V$ be the quotient map, and now we equip V with the linear partial order $\succsim_V$ defined by

$$L(v) \succsim_V L(w) \iff L(v - w) \in L(C) \iff v - w \in C. \tag{3}$$

It is clear from the fact that C contains C_P that L is positive. It remains to prove that $Lf_{\mathbb{I}}$ represents $\succsim_0$. Suppose first that $x \succsim_0 y$. Then $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) \in C_0 \subset C$, so $Lf_{\mathbb{I}}(x) \succsim_V Lf_{\mathbb{I}}(y)$. Conversely, if $Lf_{\mathbb{I}}(x) \succsim_V Lf_{\mathbb{I}}(y)$, then $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) \in C$. We may therefore write $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) = v_0 + v_P$ for some $v_0 \in C_0, v_P \in C_P$. Solving this equation for v_P, we find $v_P \in \mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X))$. Since $f_{\mathbb{I}}(X)$ is convex, Lemma 25 implies $v_P = \lambda(f_{\mathbb{I}}(x') - f_{\mathbb{I}}(y'))$ for some $\lambda > 0$, $x', y' \in X$. Since $v_P \in C_P$, P2 implies $x' \succsim_0 y'$, hence, by (2), $v_P \in C_0$. This implies $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) \in C_0$, and hence by (2) again, $x \succsim_0 y$.

For parts (c) and (d), we use the same V, L, and $\succsim_V$ as in part (b). For part (c), it only remains to show that if P1–P3 hold, then L is strictly positive. As before, L is positive, and $\succsim_V$ is a partial order; because of this, it suffices to show that, for $v \in V_{\mathbb{I}}$, $v \succ_P 0$ rules out $Lv = 0$. Suppose on the contrary that $Lv = 0$. Then, by the way L was defined, we must have $v \in C \cap -C$. We can therefore write $-v = v_0 + v_P$ with $v_0 \in C_0$ and $v_P \in C_P$. Since $v_0 \in C_0$, we can further write $v_0 = \lambda(f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y))$ with $\lambda > 0$, $x \succsim_0 y$. Rearranging, we find $f_{\mathbb{I}}(y) - f_{\mathbb{I}}(x) = \frac{1}{\lambda}(v + v_P)$. So if $v \succ_P 0$, we have $f_{\mathbb{I}}(y) - f_{\mathbb{I}}(x) \succ_{\mathrm{P}} 0$. P3 yields $y \succ_0 x$, a contradiction.

Finally, for part (d), it suffices to show that, if P1–P4 hold, then every L_i is an order embedding: $v \succsim_{V_i} 0 \Leftrightarrow L_i(v) \succsim_V 0$. Since, as just established, L is strictly positive, so is each L_i, and it remains to show that if $L_i(v) \succsim_V 0$, then $v \succsim_{V_i} 0$. Suppose therefore that $L_i(v) \succsim_V 0$. That is, $L1_i v \succsim_V 0$, so by (3), $1_i v \in C$ and we can write $1_i v = v_0 + v_P$ for some $v_0 \in C_0$, $v_P \in C_P$. We may further write $v_0 = \lambda(f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y))$ for some $\lambda > 0$, $x \succsim_0 y$. Rearranging, we find $f_{\mathbb{I}}(y) - f_{\mathbb{I}}(x) = \frac{1}{\lambda}(-1_i v + v_P)$. Therefore, $f_j(y) \succsim_{V_j} f_j(x)$ for all $j \in \mathbb{I} \setminus \{i\}$. By P3 and P4 respectively, we will have $y \not\precsim_0 x$ (a contradiction) if $f_i(y) \succ_{V_i} f_i(x)$ or $f_i(y) \curlywedge_{V_i} f_i(x)$; therefore we must have $f_i(x) \succsim_{V_i} f_i(y)$. Now, by choice of v_0 and v_P, $v = \lambda(f_i(x) - f_i(y)) + (v_P)_i$; both terms on the right are $\succsim_{V_i} 0$, so we find $v \succsim_{V_i} 0$, as desired. ☐

The proof of Theorem 7 rests on the following 'abstract Harsanyi theorem'.

Theorem 28. *Let X be a nonempty set. Let Y and Z be vector spaces and let $f\colon X \to Y$, $g\colon X \to Z$ be co-convex. Then*

$$g(x) = g(x') \implies f(x) = f(x') \text{ for all } x, x' \in X \tag{4}$$

if and only if $f = Lg + y_0$ for some linear $L\colon Z \to Y$ and some $y_0 \in Y$. Moreover, the restriction of L to $\mathrm{Span}(g(X) - g(X))$ is uniquely determined.

Proof. To take the last statement first, suppose that $f = Lg + y_0$ and also $f = L'g + y_0'$. Subtracting, we see that L and L' differ by the constant $y_0' - y_0$ on $g(X)$, and therefore they are equal on $\mathrm{Span}(g(X) - g(X))$.

For the first statement, it is clear that (4) holds if f is of the form $f = Lg + y_0$. For the converse, let $X' \coloneqq (f, g)(X)$; it is a convex set by assumption. Let $A \coloneqq \mathrm{Span}(X' - X')$; it is a linear subspace of $Y \times Z$. In light of Lemma 25 applied to X', the condition (4) is equivalent to the condition that A contains no elements of the form $(y, 0)$ with $y \neq 0$. Since A is a linear subspace, we find that

$$(y, z), (y', z) \in A \implies y = y'.$$

A is therefore the graph of a partial function L from Z to Y. By definition, the domain of L is the projection of A to Z, namely $\mathrm{Span}(g(X) - g(X))$, and L is characterized by the equation

$$A = \{(L(z), z) \colon z \in \mathrm{Span}(g(X) - g(X))\}.$$

Also, L is a linear function since A is a linear subspace. Extend L arbitrarily to a linear function from Z to Y. Fix $(y, z) \in X'$ and set $y_0 = y - L(z)$. Then for any $x \in X$, we have $f(x) = L(g(x)) + y_0$. ☐

Proof of Theorem 7. Note that, if $Lf_{\mathbb{I}} + b$ represents $\succsim_0$, then so does $Lf_{\mathbb{I}}$. Thus the right-to-left direction in each part is a special case of the corresponding claim in Theorem 6.

For the left-to-right directions, assume for the remainder that P1 holds. This implies $f_{\mathbb{I}}(x) = f_{\mathbb{I}}(x') \implies f_0(x) = f_0(x')$ for $x, x' \in X$. Co-convexity of the f_i is equivalent to co-convexity of $f_{\mathbb{I}}$ and f_0. Theorem 28 therefore yields a linear map $L\colon V_{\mathbb{I}} \to V_0$ and some $b \in V_0$ such that $f_0 = Lf_{\mathbb{I}} + b$, establishing part (a).

Let $v \in V_{\mathbb{I}}$. By the co-convexity assumption, $f_{\mathbb{I}}(X)$ is convex, so DR and Lemma 25 imply that there exist $x, y \in X$, $\lambda > 0$ such that $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) = \lambda v$. Since $Lf_{\mathbb{I}}$ represents $\succsim_0$,

$$x \succsim_0 y \iff Lv \succsim_{V_0} 0.$$

We have $v \succsim_{\mathrm{P}} 0 \Leftrightarrow x \succsim_i y$ for every $i \in \mathbb{I}$. If P2 holds, $v \succsim_{\mathrm{P}} 0 \Rightarrow x \succsim_0 y$; if P3 holds $v \succ_{\mathrm{P}} 0 \Rightarrow x \succ_0 y$. Using the displayed equivalence, if P2 holds, L is positive, and if P2–P3 hold, L is strictly positive. This establishes (b)–(c).

Now assume P1–P4 hold. By Theorem 6(d), there exists a partially ordered vector space $(V', \succsim_{V'})$ and a linear map $L'\colon V_{\mathbb{I}} \to V'$ such that $L'f_{\mathbb{I}}$ represents $\succsim_0$, with every L_i' an order embedding. Let $v \in V_i$. By convexity of $f_{\mathbb{I}}(X)$, DR and Lemma 25 again, there exist $x, y \in X$, $\lambda > 0$ such that $f_{\mathbb{I}}(x) - f_{\mathbb{I}}(y) = \lambda 1_i v$. Then $v \succsim_{V_i} 0 \iff L_i'v \succsim_{V'} 0 \iff L'f_{\mathbb{I}}(x) \succsim_{V'} L'f_{\mathbb{I}}(y) \iff x \succsim_0 y \iff f_0(x) \succsim_{V_0} f_0(y) \iff Lf_{\mathbb{I}}(x) \succsim_{V_0} Lf_{\mathbb{I}}(y) \iff L_i v \succsim_{V_0} 0$. This shows that L_i represents $\succsim_{V_i}$. Since V_0 and V_i are partially ordered, it also shows that $\ker L_i = \{0\}$: $L_i v = 0 \iff L_i v \sim_{V_0} 0 \iff v \sim_{V_i} 0 \iff v = 0$. Therefore L_i is injective, and L_i is an order embedding, establishing (d). ☐

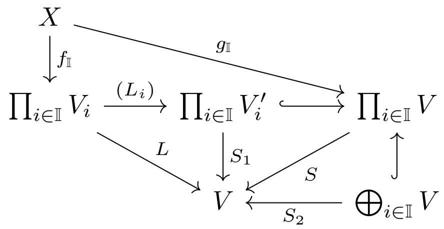
Proof of Theorem 8. For the right-to-left direction, suppose we are given V, g_i, and S satisfying the three conditions (a)–(c), or even just (a) and (b). Suppose $x, y \in X$ with $x \succsim_i y$ for all $i \in \mathbb{I}$.

Since g_i represents $\succsim_i$, this implies $g_{\mathbb{I}}(x) - g_{\mathbb{I}}(y) \succsim_{\mathrm{P}} 0$, where now $\succsim_{\mathrm{P}}$ is the product partial order on $\prod_{i\in\mathbb{I}} V$. Since $g_{\mathbb{I}}(x) - g_{\mathbb{I}}(y) \in Y \coloneqq \prod_{i\in\mathbb{I}} \mathrm{Span}(g_i(X) - g_i(X))$, and S is assumed strictly positive on Y, we find $Sg_{\mathbb{I}}(x) \succsim_V Sg_{\mathbb{I}}(y)$, and since $Sg_{\mathbb{I}}$ represents $\succsim_0$, this implies $x \succsim_0 y$. This establishes P1 and P2. If $x \succ_i y$ for some $i \in \mathbb{I}$, then a similar argument yields $x \succ_0 y$, and hence P3.

For P4, suppose $x \succsim_i y$ for all $i \in \mathbb{I} \setminus \{j\}$ and $x \curlywedge_j y$. Let 1_j be the inclusion of V into $\prod_{i\in\mathbb{I}} V$ as the jth factor. Set $v \coloneqq g_j(x) - g_j(y)$, so that $v \curlywedge_V 0$. We may write $g_{\mathbb{I}}(x) - g_{\mathbb{I}}(y) = v_P + 1_j v$ with $v_P \succsim_{\mathrm{P}} 0$, and $v_P, 1_j v \in Y$. Since S is strictly positive on Y, we have $Sv_P \succsim_V 0$, and since S extends summation we have $S1_j v = v \curlywedge_V 0$. Therefore (since $\succsim_V$ is a linear preorder) we have $Sv_P + S1_j v \not\precsim_V 0$, and since $Sg_{\mathbb{I}}$ represents $\succsim_0$ we find $x \not\precsim_0 y$, as desired.

Conversely, suppose that P1–P4 hold. From Theorem 6(d), there is a strictly positive linear map $L\colon V_{\mathbb{I}} \to V$ with values in some partially ordered vector space V, such that $Lf_{\mathbb{I}}$ represents $\succsim_0$ and every L_i is an order embedding.[48] This implies that $g_i \coloneqq L_i f_i$ is a representation of of $\succsim_i$ with values in V for each $i \in \mathbb{I}$. Together the g_i define $g_{\mathbb{I}}\colon X \to \prod_{i\in\mathbb{I}} V$.

We now define the map $S\colon \prod_{i\in\mathbb{I}} V \to V$. The construction is illustrated by the following commutative diagram.



Here, let $V_i' \coloneqq L_i(V_i)$; it is a subspace of V order-isomorphic to V_i. (Note that L_i is injective, since it is an order embedding.) The L_i together define an isomorphism $(L_i)\colon \prod_{i\in\mathbb{I}} V_i \to \prod_{i\in\mathbb{I}} V_i'$. On the other hand, L is a map from $V_{\mathbb{I}} = \prod_{i\in\mathbb{I}} V_i$ to V. We therefore obtain a map $S_1\colon \prod_{i\in\mathbb{I}} V_i' \to V$ by $S_1 = L \circ (L_i)^{-1}$.

Next, we have the summation $S_2\colon \bigoplus_{i\in\mathbb{I}} V \to V$. The domains of S_1 and S_2 are both subspaces of $\prod_{i\in\mathbb{I}} V$. The intersection of their domains is $\bigoplus_{i\in\mathbb{I}} V_i'$. Given $v \in V_i'$, $L(1_i L_i^{-1}(v)) = v$ (where here 1_i is the inclusion of V_i into $\prod_{i\in\mathbb{I}} V_i$). This shows that S_1 and S_2 coincide on each V_i', and therefore on $\bigoplus_{i\in\mathbb{I}} V_i'$. Therefore there exists a linear map $S\colon \prod_{i\in\mathbb{I}} V \to V$ that extends both S_1 and S_2.

By construction, $g_{\mathbb{I}} = (L_i) f_{\mathbb{I}}$ and therefore $Sg_{\mathbb{I}} = Lf_{\mathbb{I}}$. Therefore $g_0 \coloneqq Sg_{\mathbb{I}}$ represents $\succsim_0$, as required for part (a).

For part (b) it remains to show that S is strictly positive on Y. Note that in the commutative diagram defining S, the horizontal map $(L_i)\colon \prod_{i\in\mathbb{I}} V_i \to \prod_{i\in\mathbb{I}} V$ is an order embedding, under which Y is the image of $Y' \coloneqq \prod_{i\in\mathbb{I}} \mathrm{Span}(f_i(X) - f_i(X))$. Therefore S is strictly positive on Y if and only if L is strictly positive on Y'. But L is strictly positive on $V_{\mathbb{I}}$, which contains Y'.

Part (c) is straightforward from the fact that $f_{\mathbb{I}}$ and f_0 are co-convex and $g_{\mathbb{I}}$ and g_0 are linear transforms of $f_{\mathbb{I}}$.

For the last claim of the theorem, given DR as well as P1–P4, Theorem 7(d) provides a linear $L\colon V_{\mathbb{I}} \to V_0$ such that $f_0 = Lf_{\mathbb{I}} + b$ for some $b \in V_0$. Repeating the construction above, now with $V = V_0$, we obtain g_i and S satisfying conditions (b) and (c) but instead of (a) we have $f_0 = Lf_{\mathbb{I}} + b = Sg_{\mathbb{I}} + b$. However, picking any $i \in \mathbb{I}$, we can replace g_i by $g_i - b$ to obtain $f_0 = Sg_{\mathbb{I}}$. □

[48] Anticipating the proof of Theorem 19, we note that the following construction of the g_i and S does not depend on the V_i or V being partially ordered, rather than merely preordered.

Proof of Lemma 11. We have $f(x) = f(x') \iff x \sim x' \iff g(x) = g(x')$. Since f and g are pervasive, Theorem 28 tells us that there is a unique linear $L\colon V \to V'$ and $b \in V'$ such that $g = Lf + b$, and also a unique linear $L'\colon V' \to V$ and $b' \in V$ such that $f = L'g + b'$. (Note that here b and b' are uniquely determined as well as L and L'.) Together we find $f = L'Lf + (L'b + b')$. We also have $f = \mathrm{id}_V f$, so the uniqueness statement in Theorem 28 implies that $L'L = \mathrm{id}_V$. A similar argument gives $LL' = \mathrm{id}_{V'}$, showing that L is bijective. It remains to show that L is an order isomorphism: for all $v_1, v_2 \in V$, we want $v_1 \succsim_V v_2 \iff Lv_1 \succsim_{V'} Lv_2$. By Lemma 25, and the fact that f is pervasive, there exist $x_1, x_2 \in X$ and $\lambda > 0$ such that $v_1 - v_2 = \lambda(f(x_1) - f(x_2))$. We have $v_1 \succsim_V v_2 \iff f(x_1) \succsim_V f(x_2) \iff x_1 \succsim x_2 \iff g(x_1) \succsim_{V'} g(x_2) \iff Lf(x_1) \succsim_{V'} Lf(x_2) \iff L(v_1) \succsim_{V'} L(v_2)$, as desired. □

Proof of Proposition 12. $\mathrm{Span}(Lf_{\mathbb{I}}(X) - Lf_{\mathbb{I}}(X)) = L\,\mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X))$ by the linearity of L, and $\mathrm{Span}(f_{\mathbb{I}}(X) - f_{\mathbb{I}}(X)) = V_{\mathbb{I}}$, by DR. Thus $\mathrm{Span}(Lf_{\mathbb{I}}(X) - Lf_{\mathbb{I}}(X)) = L(V_{\mathbb{I}})$, showing that $Lf_{\mathbb{I}}\colon V_{\mathbb{I}} \to L(V_{\mathbb{I}})$ is pervasive. So too is $L'f_{\mathbb{I}}\colon V_{\mathbb{I}} \to L'(V_{\mathbb{I}})$.

We next show that $Lf_{\mathbb{I}}$ and $L'f_{\mathbb{I}}$ are co-convex. Fix any $x_1, x_2 \in X$ and $\alpha \in [0,1]$. Since $f_{\mathbb{I}}(X)$ is convex, there exists $x_3 \in X$ such that $f_{\mathbb{I}}(x_3) = \alpha f_{\mathbb{I}}(x_1) + (1-\alpha) f_{\mathbb{I}}(x_2)$. Therefore $Lf_{\mathbb{I}}(x_3) = \alpha Lf_{\mathbb{I}}(x_1) + (1-\alpha)Lf_{\mathbb{I}}(x_2)$, and similarly $L'f_{\mathbb{I}}(x_3) = \alpha L'f_{\mathbb{I}}(x_1) + (1-\alpha)L'f_{\mathbb{I}}(x_2)$. Therefore $(Lf_{\mathbb{I}}, L'f_{\mathbb{I}})(x_3) = \alpha(Lf_{\mathbb{I}}, L'f_{\mathbb{I}})(x_1) + (1-\alpha)(Lf_{\mathbb{I}}, L'f_{\mathbb{I}})(x_2)$. Thus $Lf_{\mathbb{I}}$ and $L'f_{\mathbb{I}}$ are co-convex.

By Lemma 11 there is therefore a unique linear order isomorphism $M\colon L(V_{\mathbb{I}}) \to L'(V_{\mathbb{I}})$ and $b \in L'(V_{\mathbb{I}})$ such that $L'f_{\mathbb{I}} = MLf_{\mathbb{I}} + b$. Suppose given $v \in V_{\mathbb{I}}$; by DR and Lemma 25 there are $x_1, x_2 \in X$, and $\lambda > 0$, such that $v = \lambda(f_{\mathbb{I}}(x_1) - f_{\mathbb{I}}(x_2))$. From the fact that $L'f_{\mathbb{I}} = MLf_{\mathbb{I}} + b$ it follows that $L'v = MLv$, as required. □

Proof of Proposition 14. Suppose that $f_0 = Lf_{\mathbb{I}} + b$ and also $f_0 = L'f_{\mathbb{I}} + b'$, so we have to show $L = L'$ and $b = b'$. By DR and Lemma 25, we can write any $v \in V_{\mathbb{I}}$ in the form $v = \lambda(f_{\mathbb{I}}(x_1) - f_{\mathbb{I}}(x_2))$, with $x_1, x_2 \in X$ and $\lambda > 0$. Applying L or L' we find $Lv = \lambda(f_0(x_1) - f_0(x_2)) = L'v$; therefore $L = L'$. Moreover, we have $b = f_0(x_1) - Lf_{\mathbb{I}}(x_1) = f_0(x_1) - L'f_{\mathbb{I}}(x_1) = b'$. □

Proof of Proposition 15. First we claim that g_0 and g_0' are co-convex. Fix $x_1, x_2 \in X$ and $\alpha \in [0,1]$. Since $f_{\mathbb{I}}$ and g_0 are co-convex, there exists $x_3 \in X$ such that $f_{\mathbb{I}}(x_3) = \alpha f_{\mathbb{I}}(x_1) + (1-\alpha)f_{\mathbb{I}}(x_2)$ and $g_0(x_3) = \alpha g_0(x_1) + (1-\alpha)g_0(x_2)$. Similarly there exists $x_3' \in X$ such that $f_{\mathbb{I}}(x_3') = \alpha f_{\mathbb{I}}(x_1) + (1-\alpha)f_{\mathbb{I}}(x_2)$ and $g_0'(x_3') = \alpha g_0'(x_1) + (1-\alpha)g_0'(x_2)$. By Theorem 8, P1 holds. Since $f_{\mathbb{I}}(x_3') = f_{\mathbb{I}}(x_3)$, we must, by P1, have $g_0(x_3') = g_0(x_3)$. Therefore $g_0(x_3') = \alpha g_0(x_1) + (1-\alpha)g_0(x_2)$, establishing that g_0 and g_0' are co-convex.

Thus, by Lemma 11, there is a unique linear order isomorphism $L\colon V \to V'$ and unique $b_0 \in V'$ such that $g_0' = Lg_0 + b_0$.

By assumption, for each $i \in \mathbb{I}$, f_i and g_i are co-convex, and (by DR) f_i is pervasive. As described before Lemma 11, for any $x_0 \in X$, $g_i - g_i(x_0)$ is a pervasive map $X \to \mathrm{Span}(g_i(X) - g_i(X))$. Since f_i is also pervasive, Lemma 11 gives a unique linear order isomorphism $M_i\colon V_i \to \mathrm{Span}(g_i(X) - g_i(X))$ and unique $c_i \in V$ such that $g_i - g_i(x_0) = M_i f_i + c_i$. Thus $g_i = M_i f_i + c_i + g_i(x_0)$. It is easy to verify from the uniqueness claim in Lemma 11 that both M_i and $c_i + g_i(x_0)$ are uniquely determined by this equation, independently of x_0. As i varies these together define $a \in \prod_{i\in\mathbb{I}} V$ and a linear order isomorphism $M\colon V_{\mathbb{I}} \to \prod_{i\in\mathbb{I}} \mathrm{Span}(g_i(X) - g_i(X))$ such that $g_{\mathbb{I}} = Mf_{\mathbb{I}} + a$. Now fix $i \in \mathbb{I}$ and $x \in X$. By DR and Lemma 25, there exist $y, z \in X$ and $\lambda > 0$ such that $\lambda(f_{\mathbb{I}}(y) - f_{\mathbb{I}}(z)) = 1_i f_i(x)$, where 1_i is the inclusion of V_i into $V_{\mathbb{I}}$. We therefore have $\lambda(g_{\mathbb{I}}(y) - g_{\mathbb{I}}(z)) = \lambda(Mf_{\mathbb{I}}(y) - Mf_{\mathbb{I}}(z)) = M1_i f_i(x) = 1_i M_i f_i(x) = 1_i(g_i(x) - a_i)$, where in the last two terms 1_i is the inclusion of V into $\prod_{i\in\mathbb{I}} V$ as the ith factor. Applying S and using the fact that it extends summation we find $g_0(y) - g_0(z) = Sg_{\mathbb{I}}(y) - Sg_{\mathbb{I}}(z) = (g_i(x) - a_i)/\lambda$.

A parallel argument gives $g'_0(y) - g'_0(z) = (g'_i(x) - a'_i)/\lambda$ for some $a'_i \in V'$. Since $g'_0 = Lg_0 + b_0$, we have $g'_0(y) - g'_0(z) = Lg_0(y) - Lg_0(z)$; so in combination we find

$$Lg_i(x) - La_i = g'_i(x) - a'_i. \tag{5}$$

Rearranging, $g'_i = Lg_i + (a'_i - La_i)$. We set $b_i = (a'_i - La_i) \in V'$. Note that by (5), $b_i = g'_i(x) - Lg_i(x)$ for every $x \in X$, and is uniquely determined by this equation.

For the last statement, suppose first that $v \in \bigoplus_{i\in\mathbb{I}} V'$. We have

$$LS((L^{-1}v_i)_{i\in\mathbb{I}}) = L(\sum_{i\in\mathbb{I}} L^{-1}(v_i)) = \sum_{i\in\mathbb{I}} LL^{-1}(v_i) = \sum_{i\in\mathbb{I}} v_i = S'(v),$$

where every sum has finitely many non-zero terms. On the other hand, suppose $v \in \operatorname{Span}(g'_{\mathbb{I}}(X) - g'_{\mathbb{I}}(X))$. Using Lemma 25 we can write v in the form $v = \lambda(g'_{\mathbb{I}}(x) - g'_{\mathbb{I}}(y))$ with $x, y \in X$ and $\lambda > 0$. By equation (5), we have $L^{-1}v_i = \lambda(g_i(x_i) - g_i(y_i))$, so $S((L^{-1}v_i)_{i\in\mathbb{I}}) = \lambda(g_0(x) - g_0(y))$ and then $LS((L^{-1}v_i)_{i\in\mathbb{I}}) = \lambda(g'_0(x) - g'_0(y)) = S'v$. □

Proof of Lemma 16. Suppose that $\succsim$ is a preorder on X. It is straightforward to check that $\succsim$ satisfies strong independence if it has a mixture-preserving representation. Conversely, if $\succsim$ satisfies strong independence, let V be the vector space $\operatorname{Span} X$ and $\iota\colon X \to V$ the inclusion. Taking $Z = X$ in Lemma 27, $\succsim$ and ι define a convex cone C in V and a linear preorder $\succsim_V$ such that ι then represents $\succsim$. Let $\overline{V}$ be the quotient of V by the subspace $C \cap -C$, and let $L\colon V \to \overline{V}$ be the quotient map. Define a linear partial order $\succsim_{\overline{V}}$ on $\overline{V}$ by $L(v) \succsim_{\overline{V}} L(w) \Leftrightarrow v - w \in C$. It is clear that L is a representation of $\succsim_V$, hence $L\iota$ is a representation of $\succsim$, with values in the partially ordered vector space $\overline{V}$. To complete the proof of part (i), it suffices to apply the construction of pervasive representations described before Lemma 11.

For part (ii), fix any $x, y \in X$ and $\alpha \in [0,1]$. Let $z = \alpha x + (1-\alpha)y$. From the fact that the f_i are mixture preserving, it follows that, for all i, $f_i(z) = \alpha f_i(x) + (1-\alpha)f_i(y)$, and therefore $\alpha(f_i)_{i\in\mathcal{I}}(x) + (1-\alpha)(f_i)_{i\in\mathcal{I}}(y) = (f_i)_{i\in\mathcal{I}}(z)$, establishing that $(f_i)_{i\in\mathcal{I}}(X)$ is convex. □

Proof of Theorem 17. By Lemma 16, we can choose co-convex, mixture-preserving representations $F_i\colon X \to V_i$, for $i \in \mathbb{I} \cup \{0\}$. We use these F_i as the 'f_i' in Theorem 8, which then yields V, $g_i\colon X \to V$ and $S\colon \prod_{i\in\mathbb{I}} V \to V$. For our current purposes we define $f_i := g_i$.

The only thing left to show is that the g_i are mixture-preserving. This follows from the construction of the g_i in Theorem 8, but here is a direct argument. Fix $x, y \in X$ and $\alpha \in [0,1]$. By Theorem 8(c), g_i and F_i are co-convex. So there exists $z \in X$ such that $g_i(z) = \alpha g_i(x) + (1-\alpha)g_i(y)$ and $F_i(z) = \alpha F_i(x) + (1-\alpha)F_i(y)$. Since F_i is mixture preserving, the second of these equations holds if and only if $F_i(z) = F_i(\alpha x + (1-\alpha)y)$, or equivalently if and only if $z \sim_i \alpha x + (1-\alpha)y$. But then, since g_i represents $\succsim_i$, $g_i(z) = g_i(\alpha x + (1-\alpha)y)$, and therefore $g_i(\alpha x + (1-\alpha)y) = \alpha g_i(x) + (1-\alpha)g_i(y)$, as desired. □

Proof of Theorem 19. Lemma 18 yields an expectational representation $f_i\colon X \to V_i$ of each $\succsim_i$; suppose $f_i(\mu) = \int_Y u_i \, \mathrm{d}\mu$. Let $\overline{V}_i = V_i/\sim_{V_i}$ be the partially ordered quotient of V_i, and $\overline{f}_i\colon X \to \overline{V}_i$ be the composition of f_i with the quotient map; $\overline{f}_i$ also represents $\succsim_i$. Since the f_i are expectational, they are mixture preserving, and it follows from Lemma 16 that they are co-convex; then the $\overline{f}_i$ are co-convex as well. By Theorem 6(d) there is a partially ordered vector space $\overline{V}$ and a strictly positive linear map $\overline{L}\colon \overline{V}_{\mathbb{I}} \to \overline{V}$ such that $\overline{L f}_{\mathbb{I}}$ represents $\succsim_0$, and such that the components $\overline{L}_i\colon \overline{V}_i \to \overline{V}$ are order embeddings. Now define a preorder $\succsim_V$ on $V := V_{\mathbb{I}}$ by $x \succsim_V y \iff \overline{L}\overline{x} \succsim_{\overline{V}} \overline{L}\overline{y}$, where $\overline{x}, \overline{y} \in \overline{V}_{\mathbb{I}}$ are the images of $x, y \in V_{\mathbb{I}}$. Let L denote the set-theoretic identity map $V_{\mathbb{I}} \to V$. Since $\overline{L}$ is strictly positive, so is L. Since each $\overline{L}_i$ is an order-embedding, so is each L_i (note L is injective). Moreover, $Lf_{\mathbb{I}}$ represents $\succsim_0$. If V'_i is the given separating vector space of linear functionals on V_i, then $V' \coloneqq \bigoplus_{i\in\mathbb{I}} V'_i$ is a separating vector space of linear functionals on $V_{\mathbb{I}}$, so on V; with respect to V', we have $f_{\mathbb{I}}(\mu) = \int_Y u_{\mathbb{I}}\,\mathrm{d}\mu$. Because L is the set-theoretic identity map we have $Lf_{\mathbb{I}}(\mu) = L\int_Y u_{\mathbb{I}}\,\mathrm{d}\mu = \int_Y Lu_{\mathbb{I}}\,\mathrm{d}\mu$.

We now appeal to the proof of the left-to-right direction of Theorem 8: the construction (which works even though V_i and V are now merely preordered vector spaces) takes as input $f_{\mathbb{I}}$ and L and yields $g_i\colon X \to V$ representing $\succsim_i$ (for $i \in \mathbb{I}\cup\{0\}$) and $S\colon \prod_{i\in\mathbb{I}} V \to V$ extending summation such that $g_0 = Sg$. Moreover, by construction $g_i = L_i f_i$ for $i \in \mathbb{I}$, and $g_0 = Lf_{\mathbb{I}}$. We find $g_i(\mu) = L_i\int_Y u_i\,\mathrm{d}\mu = \int_Y L_i u_i\,\mathrm{d}\mu$, and, as already stated, $g_0(\mu) = \int_Y Lu_{\mathbb{I}}\,\mathrm{d}\mu$. Defining $U_0 = Lu_{\mathbb{I}}$ and $U_i = L_i u_i$ for $i \in \mathbb{I}$, it remains to show that $U_0 = SU_{\mathbb{I}}$, i.e. that $Lu_{\mathbb{I}} = S(L_i u_i)$. And indeed S is constructed so that $S \circ (L_i) = L$. □

Proof of Theorem 22. Since Lyapunov's Theorem (Theorem 21) implies that the f_i are co-convex, Theorem 6 yields the vector space V and linear map L with components L_i. The only additional points to be checked are that V can be taken to be finite-dimensional and that $\sum_{i\in\mathbb{I}} L_i f_i$ is an admissible vector measure.

On the first point, one can harmlessly replace V by its subspace $L(V_{\mathbb{I}})$, since the latter contains the image of every L_i; since, under the hypotheses of the theorem, $V_{\mathbb{I}}$ is finite-dimensional, so is $L(V_{\mathbb{I}})$. (Alternatively, V as constructed in the proof of Theorem 6 will already be finite dimensional.)

On the second point, given that the f_i are admissible, it is straightforward to verify that their product $f_{\mathbb{I}}\colon X \to V_{\mathbb{I}}$ is admissible, and so then is the linear transform $Lf_{\mathbb{I}} = \sum_{i\in\mathbb{I}} L_i f_i$ of $f_{\mathbb{I}}$. □

Proof of Lemma 23. We begin with part (ii). Given $\overline{f}$, we have to check that its restriction f to X is additive. For A, B disjoint in X we have

$$\begin{aligned}\tfrac12 f(A\cup B) = \tfrac12\overline{f}(\chi_A+\chi_B) = \tfrac12\overline{f}(\chi_A+\chi_B) + \tfrac12\overline{f}(\chi_\emptyset) &= \overline{f}(\tfrac12\chi_A + \tfrac12\chi_B)\\ &= \tfrac12\overline{f}(\chi_A) + \tfrac12\overline{f}(\chi_B) = \tfrac12(f(A)+f(B)).\end{aligned}$$

Hence $f(A\cup B) = f(A) + f(B)$. Here we have used the fact that X is embedded into $\overline{X}$ by $A \mapsto \chi_A$, the assumption that $\overline{f}(\chi_\emptyset) = 0$, the mixture preservation property (twice), and again the embedding of X.

Conversely, suppose given a vector measure $f\colon X \to V$. We essentially define $\overline{f}\colon \overline{X} \to V$ by setting $\overline{f}(F) = \int_S F\,\mathrm{d}f$ for each $F \in \overline{X}$. Explicitly, suppose F is constant on each cell of the partition $\{E_1,\ldots,E_n\} \subset X$ of S, taking value p_j on E_j. Then set $\overline{f}(F) = \sum_{j=1}^n p_j f(E_j)$. This $\overline{f}$ is clearly mixture preserving, and the restriction of $\overline{f}$ to X is f.

For part (i), suppose given $\overline{\succsim}$ satisfying strong independence; by Lemma 16, it admits a mixture-preserving representation $\overline{f}\colon \overline{X} \to V$. Subtracting a constant, we can assume $\overline{f}(\chi_\emptyset) = 0$. Thus, by part (ii), $\overline{f}$ restricts to a vector measure f on X, which automatically represents $\succsim$.

Conversely, if $\succsim$ can be represented by a vector measure $f\colon X \to V$, then, by part (ii), f extends to a mixture-preserving function $\overline{f}\colon \overline{X} \to V$; the preorder $\overline{\succsim}$ on $\overline{X}$ represented by $\overline{f}$ satisfies strong independence, by Lemma 16, and its restriction to X is $\succsim$. □

Proof of Theorem 24. From Lemma 16 we get co-convex mixture-preserving representations $\overline{f}_i\colon \overline{X} \to V_i$ of each $\overline{\succsim}_i$. Subtracting a constant in each case, we can assume $\overline{f}_i(\chi_\emptyset) = 0$.

Theorem 6(d) yields a strictly positive linear map $L\colon V_{\mathbb{I}} \to V$, for some partially ordered vector space V, with every L_i an order embedding, and such that $L\overline{f}_{\mathbb{I}}$ represents $\overline{\succsim}_0$. By Lemma 23(ii), the $\overline{f}_i$ restrict to representations f_i of the $\succsim_i$ by vector measures. We recover the statement of the theorem by redefining $V_0 \coloneqq V$ and $\overline{f}_0 \coloneqq L\overline{f}_{\mathbb{I}}$. □

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