On Classical Motion

C.D. McCoy[*†]

12 April 2017

Abstract

The impetus theory of motion states that to be in motion is to have a non-zero velocity. The at-at theory of motion states that to be in motion is to be at different places at different times, which in classical physics is naturally understood as the reduction of velocities to position developments. I first defend the at-at theory against the criticism raised by Arntzenius that it renders determinism impossible. I then develop a novel impetus theory of motion that reduces positions to velocity developments. As this impetus theory of motion is by construction a mirror image of the at-at theory of motion, I claim that the two theories of motion are in fact epistemically on par—despite the unfamiliar metaphysical picture of the world furnished by the impetus version.

1 Introduction

There are two intuitively appealing ways to understand the nature of motion. One is that motion is nothing over and above the occupation, by an object, of different places at different times. This is known as the *at-at theory of motion* (being *at* different places *at* different times) and is generally attributed to Russell (Russell, 1903, Ch. 54). One is inclined (because of the "nothing over and above" qualification) to say in this case that facts about motion are grounded in facts about the positions of objects; it is, in other words, usually understood as a reductionist account of motion. The other way of comprehending the nature of motion is to take motion to be the process of moving, where moving is understood as the possession, by an object, of a non-zero instantaneous quantity of motion—a non-zero instantaneous velocity. I will call this the *impetus theory of motion*, essentially in keeping with the terminology in (Arntzenius, 2000). As velocity on this account is a basic quantity possessed by objects, it is evidently a non-reductive account of motion.

[*]School of Philosophy, Psychology, and Language Sciences, University of Edinburgh, Dugald Stewart Building, 3 Charles Street, Edinburgh EH8 9AD, United Kingdom. email: `casey.mccoy@ed.ac.uk`

[†]**Acknowledgements:** Thanks to Craig Callender, John Dougherty, Gil Hersh, Nat Jacobs, Wayne Myrvold, Matt Pead, David Schroeren, and Christian Wüthrich for helpful comments and conversations on the topic of this paper. I especially appreciate Natalja Deng's thoughtful comments on the paper for the 2016 Society for the Metaphysics of Science Conference in Geneva. I also wish to thank audiences at that conference, as well as at the 90th Joint Session of the Aristotelian Society and the Mind Association in Cardiff and at the 2012 Philosophy of Logic, Mathematics, and Physics Graduate Conference at the University of Western Ontario for their comments and questions.

When stated without the reductionist qualification, neither view would seem, on the face of it, to necessarily preclude the other. The possession of an instantaneous velocity is not obviously at odds with being at different places at different times; in other words, instantaneous velocities appear to be entirely consistent with the at-at theory of motion. There is a compelling argument, however, that the at-at theory implies that there are in fact no *truly* instantaneous velocities (Russell, 1903; Arntzenius, 2000; Albert, 2000). Briefly, the argument is that the explication of instantaneous velocity using calculus makes reference to non-instantaneous position developments and hence velocity determined by position developments is not truly instantaneous. Although at each instant one can define an object's velocity at that instant, the velocity itself is fully grounded in the object's past and future behavior. Velocity is not a basic quantity possessed by objects, as the impetus theory would have it. Thus the at-at theory of motion and the impetus theory of motion furnish distinct metaphysical accounts of motion.

Significant criticisms of these two accounts of motion are raised by Arntzenius (2000). In particular, he argues that the at-at theory precludes the possibility of determinism by the way it defines velocity, whence he concludes that "surely the 'at-at' theory is wrong if it entails that" (Arntzenius, 2000, 191). The impetus theory avoids this problem, at least, by reifying instantaneous velocities, but it does so at the ontological costs of adding fundamental quantities and necessarily imposing what I will call a kinematical constraint—an additional law of nature. Since the at-at theory is sufficient for describing motion without the inclusion of instantaneous velocities, the impetus theory should, it seems, by parsimony be rejected. Given these drawbacks we might find both theories unsatisfactory, as Arntzenius himself does, leaving us bereft of an acceptable account of something so simple and familiar as motion.

In my view, however, the lie is not so bad. My aim in this paper is to relate satisfactory versions of both theories of motion, using the criticisms of Arntzenius as a foil. I should note at the outset that there are various other criticisms of the at-at theory of motion as well, based mainly on the notion that the reductive account fails to do justice to the causal role of velocity (Tooley, 1988; Lange, 2005; Easwaran, 2014). As I think there is a natural extension of what I say here to those concerns (and for the sake of some concision), I leave discussion of these concerns for another occasion and concentrate mainly on Arntzenius's arguments, for these arguments represent important challenges to any theory of motion and must be overcome if we are to have a satisfactory account thereof.

The course of the paper is as follows. In §2 I rebut Arntzenius's criticisms of the at-at theory of motion, arguing that the at-at theory only precludes one particularly strong version of determinism and does not in fact do so "by logic and definition alone," as he claims. In §3 I take up the impetus theory of motion. I do agree with the standard argument against the most familiar impetus theory of motion, one that merely super-adds instantaneous velocities to the at-at theory of motion, and rehearse this argument for convenience of exposition. There is, however, a subtle way to develop the impetus account to avoid any metaphysical profligacy. Arntzenius (2003) himself in fact moots the basic thought briefly. The idea is to treat velocity as basic (as the impetus account states) but define positions in terms of velocity developments through integration. Arntzenius quickly rejects it, albeit for what I will argue are poor reasons. I pick up the thought, developing it in detail and fully in parallel to the at-at theory, i.e. as an alternative reductive account, one that, however, instead reduces position facts to velocity facts rather than the other way around. Readers familiar with modern physical frameworks for classical motion might suspect that they suggest differing accounts of motion, so in §4 I discuss two of the best known frameworks, Hamiltonian and Lagrangian mechanics, concluding that the issues nevertheless remain the same despite initial appearances to the contrary. I briefly conclude in §5.

2 The At-At Theory

The at-at theory of motion holds that what it is to be in motion is nothing over and above being *at* different places *at* different times. To add physical content to the account, it will be convenient to set it in the simplest theoretical framework of classical mechanics. In classical mechanics motion is conventionally understood to be continuous, and velocity is defined as the time rate of change of position using calculus. Velocity is hence a derived quantity of motion depending only on objects' positions at different times. The standard presentation of classical mechanics therefore is naturally understood in the at-at way.

2.1 Instantaneous Velocity as a Neighborhood Property

Let us begin with the claim that velocity on the at-at theory is not a truly instantaneous quantity (Russell, 1903; Albert, 2000; Arntzenius, 2000), since it is important to understand the claim in order to appreciate the objection over determinism. The typical calculus-based definition of what is usually called *instantaneous velocity*, which I will denote $\dot{\mathbf{x}}$, is simply the rate of change of position $\mathbf{x}$ with respect to time t, i.e. the time derivative of position $\mathrm{d}\mathbf{x}/\mathrm{d}t$.

To compute the derivative of a function one must determine the limit of a particular sequence of functions. In the case of instantaneous velocity, the relevant sequence of functions is the sequence of average velocities over increasingly small temporal intervals approaching the instant of interest. The average velocity $\bar{\mathbf{u}}$ over an interval $2*\Delta t$ centered on time t is normally defined as follows:

DEFINITION.

$$\bar{\mathbf{u}}(t,\Delta t) := \frac{\mathbf{x}(t+\Delta t)-\mathbf{x}(t-\Delta t)}{2*\Delta t}.$$

In words, the average velocity $\bar{\mathbf{u}}$ is just the difference in positions at two times divided by the time interval between them.

Now, given a particular instant t, consider the sequence of average velocities as the temporal interval Δt decreases to zero. The limiting value of these average velocities as Δt goes to zero defines the time derivative of position $\mathrm{d}\mathbf{x}/\mathrm{d}t$, i.e. the instantaneous velocity $\dot{\mathbf{x}}$:

DEFINITION.

$$\dot{\mathbf{x}}(t) := \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} := \lim_{\Delta t\to 0}\bar{\mathbf{u}}(t,\Delta t) = \lim_{\Delta t\to 0}\frac{\mathbf{x}(t+\Delta t)-\mathbf{x}(t-\Delta t)}{2*\Delta t},$$

Let us think now about what these definitions suggest about the metaphysics of position and velocity.

It is often assumed that position functions can take any functional form in classical mechanics. In principle one may even allow discontinuous functions. Position is therefore taken to be a truly instantaneous property and hence a component of an object's instantaneous state, where we understand the state of an object as the complete description of all dynamical (changeable) properties of that object at an instant.

Is instantaneous velocity a truly instantaneous property like position? While instantaneous velocity is *defined* at particular times t, Albert and Arntzenius argue that it is not truly a quantity of an object at an instant. As Albert says, "What needs to be kept in mind is just that there is all the difference in the world between being uniquely attachable *to* some particular time and being the component of the *instantaneous physical situation of the world* at that time!" (Albert, 2000, 17). So, although an instantaneous velocity can be defined for instants that satisfy the conditions of differentiability, it would be a mistake, according to Albert and Arntzenius, to say that this velocity is a property of an object at an instant.

As this is an essential point to grasp for what follows—and since Albert and Arntzenius only offer few comments on why this is so—it is worth expanding on the underlying reasoning. The reasoning is clearest in the analogous case of average velocity. Let us ask, "of precisely what is average velocity a property?" In general, given an average velocity centered at time t and over an interval $t-\Delta t$ to $t+\Delta t$, an object in motion does not possess an instantaneous velocity at t equal to the given average velocity except perhaps at isolated instants (unless of course it is in uniform motion). One would surely not want to say that average velocity is a property of an object only at these points, since average velocity is well-defined regardless of whether it equals instantaneous velocity. One would also not want to say it is a property of the object at each instant in the given interval, since average velocity in general changes depending on the chosen interval Δt and one would end up with inconsistent properties. Indeed, this is why Δt is included explicitly as an argument of the average velocity function. What the definition suggests, then, is that average velocity ought to be understood as the property of an object over a specific interval, i.e. the interval between $t-\Delta t$ and $t+\Delta t$ (that is, if one believes that averages should be thought of as properties at all).

Similar reasoning suggests that instantaneous velocities are not truly instantaneous properties in the at-at theory. Refer to the definition of instantaneous velocity above. It necessarily makes reference to a sequence of intervals (denoted by Δt) in the neighborhood of the given instant t (the limit point). Instantaneous velocity is therefore better thought of as what Arntzenius calls a *neighborhood property*. Neighborhood properties are properties possessed by an object not at an instant but in *neighborhoods* of an instant. It is crucial to understand, however, that a neighborhood property is not to be thought of as a property of some specific interval around the instant, for it is not some particular interval to which one attributes the property but to the development of the function in *arbitrary* neighborhoods of the instant.

Arntzenius's explication of the notion of neighborhood properties is rather compressed, relying on an intimate familiarity with differential calculus. To better understand the concept of neighborhood properties and the kind of reference at work here one needs to fully grasp the concept of a limit, so let us look carefully at the usual $\varepsilon-\delta$ definition:

DEFINITION. *The function f **approaches the limit** l **near** a means: for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all x, if $0 < |x-a| < \delta$, then $|f(x)-l| < \varepsilon$.*

To put this definition to work in the case of instantaneous velocity, replace the symbols in the previous definition with the ones used above in the definition of instantaneous velocity, recalling that the instantaneous velocity is equal to the limit of average velocities in the neighborhood of t, i.e. $\dot{\mathbf{x}}(t) = \lim_{\Delta t \to 0} \bar{\mathbf{u}}(t, \Delta t)$:

DEFINITION. *The average velocity $\bar{\mathbf{u}}(t,\Delta t)$ **approaches the limit $\dot{x}$ near** t means: for every $\varepsilon > 0$ there is some $\delta > 0$ such that, for all Δt, if $0 < |\Delta t| < \delta$, then $|\bar{\mathbf{u}}(t,\Delta t) - \dot{\mathbf{x}}(t)| < \varepsilon$.*

It will help to unpack the limit definition a bit in words to get clear on the role of neighborhoods. Here is what it says. First we pick any positive real number ε we like. If the instantaneous velocity exists at t, then what the definition requires is that we can always find some neighborhood around t, e.g. a temporal interval between $t-\Delta t$ and $t+\Delta t$, such that the difference between the average velocity over the interval and the instantaneous velocity is less than ε. Since the choice of ε is up to us, we can require the difference between them to be as small as we like; if the instantaneous velocity exists, there will *still* be some choice of δ for which all temporal neighborhoods of t with intervals between $t-\delta$ and $t+\delta$ will secure the required difference (or less). Thus the instantaneous velocity does not depend on any particular choice of ε or any particular neighborhood $t \pm \Delta t$. One can always choose an ε which forces a choice of δ that "moves inside" of a particular neighborhood. The instantaneous velocity does, however, clearly depend on more than the position of the object at t to exist, namely the behavior of the position function in arbitrary temporal neighborhoods of the instant in question.

To illustrate the application of these concepts, it is useful to consider an example of discontinuous motion. Suppose that there is an object in uniform motion which at some instant t jumps discontinuously to position L before immediately returning to its previous inertial trajectory. The limits of the object's position $\mathbf{x}(t)$ and its instantaneous velocity $\dot{\mathbf{x}}(t)$ are well-defined in this example. Because it is defined by way of t's neighborhood, the limit of its position is where it would have been had it not made the jump and its velocity is the velocity it was traveling at before and after the jump. The limit of position is thus a neighborhood property here, and the example demonstrates that the limit of position does not necessarily agree with the instantaneous position at time t. Similarly, instantaneous velocity is a neighborhood property. Since this is a case of non-classical motion, it is not clear what to say about the object's "actual" state of motion at its "jump" location, except perhaps that it is moving "discontinuously" (which, observe, still requires reference to its behavior at other times) (Cf. (Jackson & Pargetter, 1988)).

Now suppose, as a second case, that the object jumped to another position $M \neq L$. In both of these cases the instantaneous positions of the two objects at time t differ but their "neighborhood positions" are identical. While it may sound strange to say that the object is at one place (in the usual sense) and, in another sense, not there, once one recognizes that neighborhood position is just a way of describing the continuity of some object's position development (in the neighborhood of an instant), it is clear that instantaneous velocity too is a way of describing an aspect of the object's position development, namely, how fast it is changing (in the neighborhood of an instant).

Before turning to Arntzenius's critique of the at-at theory, I mention that other authors have frequently made use of examples like these just mentioned, where motion is discontinuous (or not differentiable), in order to conjure intuitions about the nature of motion (Jackson & Pargetter, 1988; Tooley, 1988; Carroll, 2002). In general I will mostly restrict attention in the following to classical motion, i.e. the kind of motion described by classical mechanics. In classical mechanics, as I noted above, motion is conventionally understood to be continuous. This assumption also permits one to take motion to be smooth (infinitely differentiable), since, according to the (Stone-)Weierstrass theorem, any continuous function can be approximated as closely as one likes by a smooth one over some closed interval (here, of time). I will make essential use of this assumption of smoothness later, although for certain specific purposes I will occasionally consider cases where discontinuous motion is permitted.

2.2 Determinism and Motion

Albert and Arntzenius take it to be an objectionable consequence of a theory of motion if by definition and logic alone determinism is rendered impossible. Their objection depends on the main conclusion of the previous section, namely that instantaneous velocity is not a a truly instantaneous property. Let us see first how this objection is raised against the at-at theory of motion. I will then show that Arntzenius and Albert are mistaken: the preclusion of determinism, properly understood, is not at all a consequence of the at-at theory of motion.

The type of determinism Arntzenius and Albert have in mind is usually called *Laplacian determinism*, as it is evoked by Laplace in this oft-quoted passage:

> Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it...it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. (Laplace, 1840/1902)

Albert (2000, 10) describes the present state of the world as the instantaneous states of every object in the world at the present instant, a description that accords with Laplace's. It is also the one that Arntzenius adopts. These instantaneous states, along with the appropriate dynamical law (determined by the various classical forces at work), are the things one needs to know in classical mechanics to deduce the trajectories of all the objects in the world, if classical mechanics is deterministic in the Laplacian sense. If it is, then this is the sense in which Laplace's intelligence could know the movements of all objects for all times, past and future.

If, however, the at-at theory is correct and instantaneous velocity is not an instantaneous quantity in classical mechanics, as shown above, then classical mechanics is not deterministic in the Laplacian sense. In classical mechanics both position and velocity are required to determine future and past evolution. If velocity is not truly instantaneous, then the present state of the world only includes the positions of all the objects in it. Thus Laplace's intelligence would be left scratching its head looking at its impoverished situational report, unable to determine anything whatsoever about the future or the past on its basis—all this, according to Arntzenius, because instantaneous velocity was simply *defined* in the way recently shown, and therefore determinism is said to be rendered impossible by definition (and logic) alone.

The obvious rejoinder to Arntzenius's objection is to say that our definition of determinism (or alternately, present state of the world) is mistaken. So, if one is unperturbed by giving up Laplacian determinism, then a simple solution is obviously at hand. Just allow the Laplacian intelligence to have access to neighborhood properties as well as the truly instantaneous state of all objects. In so doing we modify the notion of "present state of the universe" to include the neighborhood of the present, i.e. determinism comes to depend on both neighborhood and properly instantaneous properties.

If we say this, though, then we may seem to find ourselves in an uncomfortable dilemma over definitions. On the one hand, we can insist that the definition of instantaneous velocity is correct and, on the other, that Laplacian determinism (or present state of the world) is defined correctly. Why prefer one definition to the other?

I say that it is a mistake to characterize the issue in this way, as a matter of definitional preference, since it is not the case that definitions alone are at issue. As Earman observes, for example, "we cannot begin to discuss the implications of physics for the truth of the doctrine of determinism until we know what determinism is; on the other hand, no precise definition can be fashioned without making substantive assumptions about the nature of physical reality" (Earman, 1986). As he rightly notes here, we have to make some substantive assumptions about the world even to begin evaluating the notion of determinism. So long as we are making substantive assumptions about the nature of physical reality, then, the issue at hand is not simply a matter of logic and definition alone.

What are the "substantive assumptions" underlying the definitions employed here then? On the face of it, they appear to depend on interpretive matters concerning laws of nature. One who believes that the laws produce new physical situations from old ones may find any relaxation of the definition of determinism from the Laplacian version objectionable. If one thinks instead that laws are merely a description of regularities that can be gleaned from the physical facts, however, then relaxing the definition of determinism is unproblematic. The state of the world does not have to be instantaneous on this view; it may be whatever the most physically salient notion of determinism requires. In the case of classical mechanics, the relevant state of the world is then naturally taken to be the neighborhood state of an instant.

That Arntzenius has the former point of view in mind follows from a related objection to the at-at theory he makes. That objection is that assuming the standard definition of instantaneous velocity imposes non-dynamical constraints on evolution (to be described below). This is objectionable, as he says, because

> ...surely our notion of a physical state is such that being in a particular physical state at some time does not by definition and logic alone put any constraints on what physical states the system can be in at other times. Physics may impose constraints on the possible developments of the physical states of systems, but surely logic and definition by itself should not do so. And that implies that neighborhood properties and neighbor hood states are not physical states, they are features of finite developments of physical states. (Arntzenius, 2000, 195)

If one has the idea that laws act on truly instantaneous states of affairs to produce new states of affairs, this concern might be quite reasonable. For this objection to be sustainable, though, (accepting this view of laws for the moment) it must be that Arntzenius is right about two things: that only physics can impose such constraints, and that physics does not impose the constraints that instantaneous velocity places on position developments. I claim that he is wrong on both counts.

What exactly are these constraints to which he objects? Arntzenius's explication is again compact, and it is worth providing a detailed exposition in order to rebut his objections. To clarify what he has in mind, consider for simplicity the case of some object constrained to move on a surface Σ. At some time let the position of the particle be $\mathbf{p}$. If position is the only instantaneous property of the object at that time, then any trajectory on the surface is kinematically possible (see Fig. 2.1), i.e. possible in advance of consideration of the constraints imposed by forces and the dynamical laws. If one allows the instantaneous velocity to be part of the state of the particle (not necessarily presupposing that this velocity is the time derivative of position $\dot{\mathbf{x}}$), then the possession of a particular velocity, say $\mathbf{u}$, restricts the possible position developments to the past and to the future (see Fig. 2.2). Only those developments which have $\mathbf{u}$ as their derivative at $\mathbf{p}$ are kinematically possible when both $\mathbf{p}$ and $\mathbf{u}$ are part of the state. Note that velocity imposes this constraint only in the neighborhood of $\mathbf{p}$—for any point $\mathbf{q}$ in the neighborhood of $\mathbf{p}$ there exists some kinematically

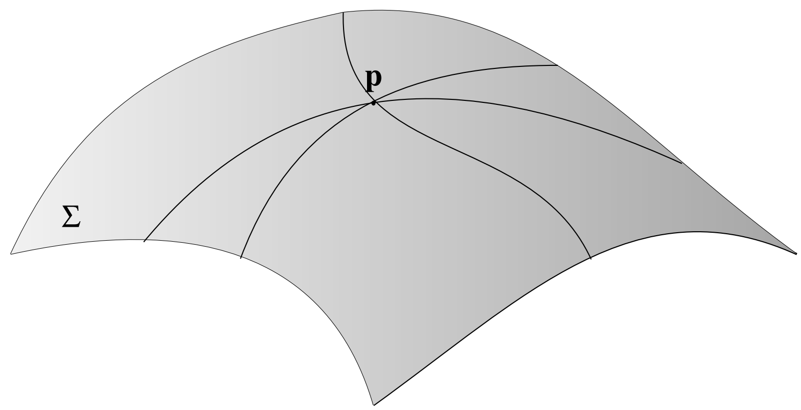

Figure 1: Some trajectories compatible with an object on surface Σ located at position p.

possible trajectory that passes through **q** from **p**—so it is in some sense a weak constraint but a constraint nonetheless (Butterfield, 2006, 724-5).

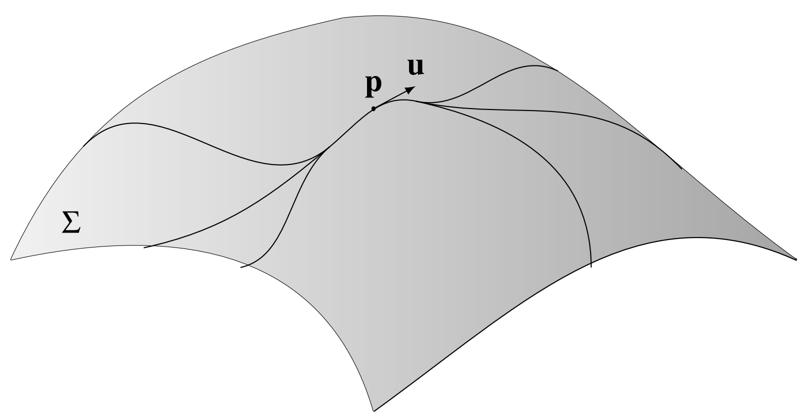

Figure 2: Trajectories compatible with an object on surface Σ located at position p with velocity $\vec{u}$.

There are, as I see it, two reasonable ways to respond to Arntzenius's charges.

First, we may suppose that the kinematical constraint imposed by instantaneous velocity is actually a kind of metaphysical constraint. This constraint need not be imposed in such a way that instantaneous velocity becomes truly instantaneous. Rather it may be imposed in much the same way that dynamical constraints are imposed, namely via laws. For example, the requirement that trajectories be smooth, which classical mechanics generally presupposes, might be considered a metaphysical law, one that is understood to be prior to any physical law. Similarly, the kinematical constraint imposed by instantaneous velocity might be considered a metaphysical law in this way.

Why think that this constraint is a metaphysical law rather than a physical law? Perhaps because it expresses "what we mean" by velocity, or maybe because it is merely a pre-condition on any physics that would describe what we call classical motion. At the very least, the possibility of this view challenges the notion that only physics can constrain position developments as Arntzenius assumes. Either one accepts this possibility—physics is, after all, obviously committed to many metaphysical presuppositions already—or one has the burden of showing why there cannot be constraints like this.

One objection to taking this course might be based on the popular view of metaphysics where only metaphysical necessity could potentially ground a metaphysical law of the kind I suggest. On this view the exhibition of a possible world where that law does not hold is sufficient to undermine its lawhood. Then one only has to consider worlds where there is discontinuous motion, such as Tooley (1988) and Carroll (2002) do, to see that the metaphysical constraint imposed by instantaneous velocity does not hold. If one allows that metaphysical laws, like special science laws, do not need to constrain all metaphysical possibilities, then such thought experiments do not militate against this suggestion however.

Nevertheless, there is a second way to respond that to some extent obviates the former, so I will not argue further for the first. That way is to recognize that the kinematical constraint imposed by instantaneous velocity already follows from the dynamical constraints imposed by the laws (in which case it ought to be actually considered a dynamical constraint). The general form of dynamical constraints in classical mechanics is given by Newton's Second Law:

$$\sum \mathbf{F} = m\mathbf{a};$$

that is to say, the instantaneous sum of forces on an object is equal to the mass of the object times its acceleration. The acceleration $\mathbf{a}$ is usually defined as the time derivative of the instantaneous velocity $\dot{\mathbf{x}}$; in other words, Newton's Second Law is equivalently expressed as

$$\sum \mathbf{F} = \ddot{\mathbf{x}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}\right),$$

where the acceleration $\ddot{\mathbf{x}}$ is the second time derivative of position. To make acceleration the second time derivative of position, one must obviously have a first time derivative of position, i.e. an instantaneous velocity, to differentiate. Given that Newton's Second Law is a dynamical constraint of the kind that Arntzenius accepts as a physical constraint, it would seem to follow as a matter of course that the kinematical constraint imposed by instantaneous velocity is a physical constraint as well—it is, from this point of view, a constraint that simply follows from Newton's Second Law. The point is even more transparent when one re-writes Newton's Second Law as two first order differential equations:

$$\sum \mathbf{F} = \dot{\mathbf{u}};$$
$$\mathbf{u} = \dot{\mathbf{x}},$$

where $\mathbf{u}$ is the instantaneous velocity. Here the "kinematical" constraint that is part of Newton's Second Law is made explicit.

Let me summarize this section. I have argued that the objections raised by Arntzenius against the at-at theory of motion are unfounded. Although I agree with Arntzenius that instantaneous velocity is not truly instantaneous according to the at-at theory and is better understood as a neighborhood property, I disagree with the claim that instantaneous velocity threatens the determinism of classical mechanics. If one does insist on holding onto the idea that only truly instantaneous properties can figure into the physical state of the world (or a system), then it is true that one must give up on the idea that the laws produce future states on this basis alone (in keeping with

Laplacian determinism). It is, in short, a basic fact of classical physics that the world cannot be deterministic in this sense. But it is not true that this is a matter of logic and definition alone, as Arntzenius insists. If one allows that the state of the world include neighborhood properties (and arguably one may do this however one chooses to interpret laws of nature), then there is a clear and substantive sense in which classical mechanics is deterministic. This idea is motivated by attending to the kinematical constraints imposed by instantaneous velocities. If they are seen as either falling under a metaphysical law or following simply from the definition of Newton's Second Law, as I claim they should, then they should be considered a part of the physically relevant state of the world. Therefore I conclude that the at-at theory is not threatened by the objections raised by Arntzenius.

3 Impetus Theory of Motion

The impetus theory of motion holds that what it is to be in motion is possessing a non-zero quantity of motion, i.e. a non-zero velocity. In the context of classical mechanics this velocity should be a truly instantaneous velocity in the same way as position is taken to be truly instantaneous. Hence it should be a component of an object's physical state. However, since it is customary in classical dynamics to call the time rate of change of position, i.e. $\dot{\mathbf{x}}$, the instantaneous velocity, I will henceforth call the truly instantaneous quantity of motion of an object its *impetus* and denote it $\mathbf{v}$. To further avoid confusion I will also refer to $\dot{\mathbf{x}}$ as the *kinematical velocity*. In this section I investigate the prospects of the impetus theory of motion in the context of classical motion. First, in agreement with Arntzenius (2000), I reject rescuing Laplacian determinism by simply supplementing classical dynamics with impetuses and quickly rehearse the basic argument to provide some useful context. I then raise a new challenge to the at-at theory of motion by re-conceiving classical mechanics in a way that takes velocity rather than position as basic and makes position a derived quantity.

3.1 Super-addition of Impetus to At-At Theory of Motion

I showed in the previous section that the Laplacian picture of determinism—a state of the universe at an instant being evolved deterministically forward by the laws of motion—is untenable on the at-at theory of motion. The issue was that velocities are non-instantaneous on the at-at theory of motion while the laws of motion require truly instantaneous velocities in order to be deterministic in the Laplacian sense. An obvious solution to this problem is to supplement the instantaneous properties of objects with a truly instantaneous velocity, i.e. an impetus. Then one may ostensibly maintain the intuitive definitions of determinism and physical state preferred, among others, by Albert and Arntzenius.

This kind of impetus theory, which superadds impetuses to the account of motion given by the at-at theory, is an unacceptable theory of motion however. Insofar as objects follow continuous trajectories in classical mechanics, the kinematical velocity $\dot{\mathbf{x}}$ exists and correctly describes the time rate of change of position; hence it is a velocity. If the impetus is to play a role in describing actual motion, then it must necessarily be equated with the kinematical velocity, i.e. it must be the case that $\mathbf{v} = \dot{\mathbf{x}}$. As the kinematical velocity and the impetus are both meant to be velocities describing motion, such a relation introduces a necessary kinematical constraint. Such constraints are to be considered, following the discussion in the previous section, as either metaphysically or physically necessary in classical mechanics. Yet, given the necessity of this kinematical constraint, impetuses appear to be incapable of doing any additional "work" in the theory over and above kinematical velocity (which, again, is given in the theory since position is taken as given). They are physically idle—one cannot say anything more with them than what can already say with position developments. It will simply not do to add physically superfluous properties and constraints solely in order to preserve a particular metaphysics of laws. To do so would make metaphysics frivolous. Therefore, one ought to preclude impetuses from classical ontology along with their associated kinematical constraints.

Although impetus is necessarily constrained to equal kinematical velocity for differentiable trajectories, perhaps one might think that the conceptual independence of the two velocities becomes manifest in a wider context, viz. one where trajectories are not continuous. This is not so. If position is not a differentiable function of time at some instants, then the impetuses must likewise be undefined at these instants, else they would be irrelevant to the description of actual motion. Suppose, for the sake of argument, that impetuses could disagree with the kinematical velocities. In the case of differentiable position functions the impetuses would not function as quantities of motion as intended, since they would give the wrong dynamical evolution of the objects' positions. In cases where the position functions results in undefined kinematical velocities at some instants, if impetuses were to possess a defined value, then they would also give the wrong dynamical evolution, as they would indicate a future motion that does not occur. Impetus is therefore entirely expendable in this wider context. Both velocities must agree when kinematical velocity is well-defined, whereas the kinematical velocity is not beholden to impetus in any way.

3.2 Position Reduced to Velocity Developments

Insofar as one makes the usual assumption that position is included in an object's instantaneous physical state, impetuses are superfluous and instantaneous velocity is not really instantaneous. If the arguments in §2 that the at-at theory of motion does not preclude determinism are correct, then one has reason to believe that the at-at theory should be our preferred theory of classical motion. Although I do believe that the at-at theory gives us a fine way of understanding the metaphysics of motion, I also believe that there exists an unappreciated impetus alternative to the at-at theory. Moreover, I claim that this alternative seriously underdetermines our interpretation of classical motion, for only epistemically inaccessible facts could decide between them.

The incipient idea is to take velocity as a quantity of an object's instantaneous state and position as a quantity derived from velocity by integration: $\mathbf{x}(t) = \int \mathbf{v}(t)\,\mathrm{d}t$. Thus, whereas the at-at theory considered above reduces velocity developments to position developments, this theory reduces position developments to velocity developments. It gives rise naturally to a kind of impetus theory, since velocity is now taken as a truly instantaneous quantity, although in this case $\mathbf{v} = \dot{\mathbf{x}}$ not because they are equal by a kinematical constraint, but because the fundamental theorem of calculus applied to the definition of position just given informs us that the terms on both sides simply refer to precisely the same property.

Arntzenius in fact briefly raises the inchoate thought behind this theory of motion in his response to (Smith, 2003) but quickly overrules it because there are "velocity developments that are incompatible with calculus"—in particular the "calculus definition of velocity" (Arntzenius, 2003, 282). In essence, he claims that one cannot allow arbitrary velocity functions (as one allows arbitrary position functions in the at-at view) because time derivatives of position functions cannot recover a (very) large class of these arbitrary velocity functions, viz. those functions which are not the derivatives of any function. Since the definition of velocity (as time derivatives of position) would limit which functions could be used to describe velocity,"logic and definition alone would still imply constraints between instantaneous states at different times "(Arntzenius, 2003, 282).[1]

Arntzenius again does not spell out the details of his objection, so to see how his objection is supposed to work let us consider his example, the pathological function known as the Dirichlet function. It is defined as follows: let $\mathbf{v}(t)$ be defined such that $\mathbf{v}(t) = 1$ for rational t and $\mathbf{v}(t) = 0$ for irrational t. The particular details of this function are not visualizable at any scale—if one tries to plot it, it just looks like two lines at $\mathbf{v} = 1$ and $\mathbf{v} = 0$. (The Thomae function is a similar alternative function that is better able to be visualized.)

Now suppose that there is a position development $\mathbf{x}(t)$, the derivative $\dot{\mathbf{x}}(t)$ of which is this just defined velocity development $\mathbf{v}(t)$. Then, recalling the definition of instantaneous velocity in §2, it must be the case that $\bar{\mathbf{v}}(t,\Delta t)$, i.e. $(\mathbf{x}(t+\Delta t) - \mathbf{x}(t-\Delta t))/2 * \Delta t$, approaches the limit $\mathbf{v}$ near t. This cannot be the case, however, since for any ε such that $1 > \varepsilon > 0$ the inequality $|\bar{\mathbf{v}}(t,\Delta t) - \dot{\mathbf{x}}(t)| < \varepsilon$ will not hold. This is because as one moves closer to t the Dirichlet function repeatedly jumps between 0 and 1 as t goes from rational to irrational numbers, etc. Thus no limit point is ever approached. Since velocity functions with characteristics similar to the Dirichlet function cannot be recovered by differentiating position functions in this way, Arntzenius intimates that this definition of velocity imposes constraints on possible velocity developments merely by "logic and definition".

I say this is a bad argument. Before explaining in full why it is, though, some preliminaries are needed. Suppose that we do take pathological functions like the Dirichlet function seriously (for the moment) as physically possible velocity developments. In the impetus view presently under consideration, position is a derived property, as it is defined via integration. Here, though, it becomes quite important precisely which notion of integration we use. The notion of integration familiar from basic calculus, Riemann integration, cannot be applied to the Dirichlet function. The function has no integral in this sense of integration. Thus it would seem that an object with the Dirichlet velocity development has no position development when position is defined via Riemann integration.

Surely, one might think, this is the problem with the proposed account of motion: the potential absence of well-defined positions. Yet this is not so. Suppose that we allow everywhere continuous but nowhere differentiable position developments in the at-at theory of motion. Then an object whose position development is described by the Weierstrass function, the most well-known function of this kind, has no velocity development. This is because the position development is nowhere differentiable; hence the object with this position development nowhere has a velocity. Such a case, I submit, is no more strange than the previous. In short, if pathological functions are a problem for the impetus theory, then equally they are for the at-at theory as well.

One might object to the cases being treated equally, saying that it is possible to imagine an object moving along a trajectory with the functional form of the Weierstrass function, but it is not possible to imagine an object failing to have a position. This objection begs the question against

[1]Understanding and meeting this objection leads into somewhat technical terrain; the reader who finds the alternative impetus theory plausible and does not care for mathematical niceties should skip ahead to below the long quotation.

the impetus view under consideration here. We are used to thinking of objects moving in space. We should not allow a mere prejudice to preclude an alternative theory of motion. Only if the theory suffers from legitimate defects, for example an inability to recover empirical content or to provide a coherent account of classical motion, should it be disfavored.

In any case, Arntzenius's own objection is not that the mere lack of a position development is problematic. Rather it is that there exist velocity developments which are not the derivatives of any position development according to the previously given definition of derivative. But why should we assume *this* definition of velocity in the present impetus theory of motion? Velocity is not and should not be defined in the impetus theory as it is taken as basic. Rather it is position which is derived and hence must be defined. Arntzenius's objection depends on duplicitously treating velocity as both basic and derived.

If one were to do the same in the context of the at-at theory, i.e. treat position as both fundamental and derived, then one would have precisely the same problem: there exist position developments which are not the integrals of any velocity development. A position development following the Weierstrass function would be precisely such a case. Thus there would be position developments that are incompatible with calculus, namely the calculus definition of position (via integration). Arntzenius's objection cuts both ways.

Nevertheless, there are alternative definitions of integration according to which functions like the Dirichlet function are in fact integrable. The well-known Lebesgue integral, for example, generalizes the notion of Riemann integration by utilizing a measure with respect to which integration is performed. Given a set X and a measure μ on the measurable subsets of X, the Lebesgue integral of a function f over the set $A \subset X$ is written $\int_A f\,\mathrm{d}\mu$. If we take the f to be the Dirichlet function, X as the set of real numbers $\mathbf{R}$, $A = [0,1]$, and μ the standard Lebesgue measure associated with the real numbers, then

$$\int_A f\,\mathrm{d}\mu = 1.$$

Thus, when one makes use of the Lebesgue integral, one finds that the integral of the Dirichlet function over this finite interval is just a constant.

If we adopt this alternative definition of integration, then it appears we have a new problem though: the usual calculus derivative of a constant is zero, which is obviously not the same as the Dirichlet function. When previously we used Riemannian integration we had the fundamental theorem of calculus to guarantee a certain duality between differentiation and integration, so that integrating a derivative or differentiating an integral returns the original function. To restore this kind of duality we need to do is modify our definition of derivative to regain an analog of the fundamental theorem. The simplest way to do this is just to define differentiation in terms of Lebesgue integration, namely as

$$\lim_{A \to 0} \frac{1}{\mu(A)} \int_A f\,\mathrm{d}\mu.$$

The Lebesgue differentiation theorem guarantees that this derivative exists and, more importantly, equals f at almost every point in X. Thus if we take f to be the Dirichlet function, one of the possible derivatives of its integral is the Dirichlet function. While it is true that other derivatives are possible, viz. functions that differ in their values on a set of measure zero, this is no problem for an impetus theory of motion based on Lebesgue integration, since the velocity developments are basic and only positions are derived. Thus it is only necessary that we can recover the Dirichlet function by differentiation, not that the derivative is unique.

Still, recalling some concerns raised earlier, it should remain puzzling why an object moving along a constant position development (derived via integration of the Dirichlet function) would have such a complicated velocity development (the Dirichlet function), as the non-zero velocity deviations in that velocity development appear to have no physical effect on the object's position. Thus it would seem that this lengthy excursion into complicated mathematical analysis is rather strained and has not further clarified motion at all. (I would insist that it was not pointless, however, since it does address several spurious objections that might be raised against the impetus theory under consideration.) Indeed, I think the discussion of pathological functions *is* rather strained in the context of classical motion. For the purposes of mechanics and motion, we would do well to heed the words of Poincaré from 1899:

> Logic sometimes makes monsters. For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. Indeed, from the point of view of logic, these strange functions are the most general; on the other hand those which one meets without searching for them, and which follow simple laws appear as a particular case which does not amount to more than a small corner.
>
> In former times when one invented a new function it was for a practical purpose; today one invents them purposely to show up defects in the reasoning of our fathers and one will deduce from them only that.
>
> If logic were the sole guide of the teacher, it would be necessary to begin with the most general functions, that is to say with the most bizarre. It is the beginner that would have to be set grappling with this teratologic museum. (as quoted in (Kline, 1972, 973))

The lesson of the tortuous course I have taken is, and what I think this quotation suggests, is that logical and mathematical possibility by no means correspond to metaphysical and physical possibility. When one reasonably restricts attention to suitably well-behaved functions to describe the motion of objects, one is able to keep the relevant physical notions before oneself in plain view. Restricting attention in this way allows that the usual classical physics fully permits the reductive impetus theory of motion proposed here just as much as it permits the reductive at-at theory of motion discussed in the previous section. The wider context of functions in mathematical analysis is from the point of view of (meta)physics a distracting teratology—for the at-at theory as much as for the impetus theory. Perhaps one might find the foregoing discussion belaboring, but it seems necessary given that many arguments in this literature, including Arntzenius's dismissal of this approach, have been based on the dishonest possibility of objects following bizarre paths.

It is now time to turn then to honest questions of ontology as a way of distinguishing this new impetus theory from the at-at theory, since logic and definition do not trouble the former any more than the latter (that is, not at all). The at-at theory of motion takes position as basic and therefore is naturally bound up with the existence of space and time. Objects have a position in virtue of their embedding in space (substantivalism) or in virtue of their spatial relations with other objects (relationalism). It is important to recognize, however, that objects only have an *absolute* position in virtue of the existence of absolute space, as is supposed in the doctrine of substantivalism. I stress that this supposition is, strictly speaking, an additional metaphysical posit above and beyond the metaphysics required for the at-at theory of motion.

What then does the reductive impetus theory under discussion suggest? The impetus theory of motion takes velocity as basic, so, in analogy to the at-at theory, it is naturally understood as depending on the existence of some sort of "velocity space" (whether understood in a way analogous to relationist or substantivalist space). Objects have a velocity, one would say, in virtue of their embedding in velocity space or in virtue of their velocity differences with other objects. Although one could also posit absolute "locational" space in addition to the assumed velocity space, it would seem to go against the idea of taking velocities as basic. Thus "locational space" is naturally a derivative concept (and perhaps in some sense emergent) according to this theory of motion.

Granted, as spatial reasoning is so familiar, this velocital view is perhaps quite strange and unintuitive. It is not thereby implausible, however. A simple imaginative illustration may be helpful. Suppose that we are down at the links watching a golf ball in flight and furthermore that we were able to perceive its instantaneous physical situation (from our point of view). Look at the ball. Is it moving? On the at-at theory we would certainly be able to say *where* it is (in relation to ourselves) but not whether it is *moving* (in relation to ourselves), since it has no instantaneous property of velocity. If we only knew the positional facts about the temporal neighborhood of that instant, though, then we could say whether it is moving or not.

On the impetus theory the situation is just the opposite. In this case we would certainly be able to say whether the ball is *moving* (with respect to ourselves) or not but we could not say *where* it is (with respect to ourselves). Similarly, if we knew the velocital facts about the temporal neighborhood of that instant, then we could say well enough where the ball is (with respect to us). Indeed, we are, I submit, usually in possession of such facts (whereas we of course never perceive instantaneous physical situations!). Nevertheless, we should accept that the two views are distinct, since they are not metaphysically intertranslatable given their different verdicts in this example—they say different things about the world at instants of time, even if they do not in ordinary circumstances.

Despite commonplace intuitions to the contrary, there is really nothing metaphysically awry with the reductive impetus theory, since it only differs from the usual way of thinking at instants. I suspect it will have already occurred to some readers, however, that there is at least one salient difference which distinguishes it from the at-at theory. That difference is the one secondary students learn between derivatives and integrals: the derivatives of well-behaved functions are determined, whereas (indefinite) integrals are determined only up to an additive constant, the constant of integration. Thus, on the at-at theory, given a temporal neighborhood, the velocities would be fully determined, whereas on the impetus theory, by contrast, given a temporal neighborhood, the positions would be determined only up to a constant.

The difference between derivatives and integrals is sufficiently well known that something must be said about this. Does this circumstance militate against the impetus theory? The reader will not be surprised that my answer to the question is "no", but it is quite instructive to see why any objection along these lines is misguided. The first thing to note is that the fundamental dynamical law in Newtonian mechanics, Newton's Second Law, does not by itself force the problem upon us. Re-expressing $\mathbf{F} = m\mathbf{a}$ in the impetus theory's basic terms yields $\mathbf{F} = m\dot{\mathbf{v}}$, which makes no necessary reference to position, so no reference to derived quantities, so no reference to integrals which could give rise to undetermined integration constants. It is therefore only when the forces themselves depend on positions, i.e. $\mathbf{F} = \mathbf{F}(\mathbf{x})$, that the Second Law necessarily becomes a second order ordinary differential equation. Then, since $\mathbf{x} = \int \mathbf{v}\,\mathrm{d}t$, there will be derived quantities determined by integrals and, hence, constants of integration which potentially become physically significant.

Let us consider a few examples to see what comes of these constants of integration. Suppose first that there is but a single inertially moving particle in the universe. In this case $\mathbf{v}$ is constant and $m\dot{\mathbf{v}} = 0 = \mathbf{F}$. What is the position of the particle? Since the particle experiences no forces, one is not forced to say anything at all about space and position. If we like, however, we can integrate the velocity to yield a position function, i.e. $\int \mathbf{v}\,\mathrm{d}t = \mathbf{x}(t) + \mathbf{X}$, where X is a constant. What is X? If there were absolute space, then X would be the difference between the relative position $\mathbf{x}$ and the absolute position of the object. But it would be contrary to reason to suppose that there are absolute positions from the point of view of the reductive impetus theory of motion. So in this case the constant of integration should be neglected, just as it would be in the relationist version of the at-at theory of motion.

Suppose next that there are two free particles in the universe. In this case the velocities of both particles are constants and both particles experience no forces. Again, we are not forced to say anything about space and position, but we can integrate the two independent velocities to yield two position functions with two constants of integration. Now you might think that there is now a problem, since even if we rid ourselves of one spurious constant of integration due to the absence of absolute positions there still remains one, which ought to represent the relative positions of the two particles. But our assumptions do not indicate any spatial relation at all between the two particles, as they are free particles which experience no forces. The only facts in such a universe are the velocity developments, from which we can infer the "distances" they travel, understanding that these distances should be measured out in independent positional spaces. This circumstance is somewhat analogous to the tangent (velocity) spaces of curved manifolds (spaces), which cannot be automatically identified as they are in Euclidean space without the addition of some way to parallel transport tangent vectors, i.e. a connection. It is of course natural to think that there is only one unified "locational" space in which motion takes place, but that intuition is unsupported in this case since the particles are free. Fortunately for our intuitions, the weirdness of independent locational spaces of motion depends on the unrealistic idealization of force free motion.

So consider the more realistic example of motion within the scope of Newton's Law of Universal Gravitation, which depends on relative positions to determine motion. For simplicity, consider the gravitational system of two objects and Newton's Law applied to one:

$$\mathbf{F}_{12} = Gm_1m_2\frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_2 - \mathbf{x}_1|^3},$$

where $\mathbf{F}_{ij}$ is the force on object i due to object j, G is Newton's gravitational constant, m_i is the mass of the object, and $\mathbf{x}_i$ is the position of the object i. In the impetus theory of motion the positions are derived quantities, so we would want to re-express Newton's Law of Universal Gravitation as follows:

$$\mathbf{F}_{12} = Gm_1m_2\frac{\int(\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t + \mathbf{X}}{|\int(\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t + \mathbf{X}|^3},$$

where $\mathbf{X}$ is a constant equal to the sum of the two constants of integration introduced by re-expressing the relative positions as indefinite integrals. What the gravitational force at a particular instant is clearly depends on what $\mathbf{X}$ is. Presumably this is the circumstance that the impetus theory's detractor envisions as fatal for the view.

Plainly, if one were to have the view that laws act on instantaneous states, then the integration constants would apparently be essential for securing determinism, since different choices of $\mathbf{X}$—essentially different choices of relative distance between the two particles—would entail different forces, accelerations, and future motions. The reductive impetus theory of motion, however, like the at-at theory of motion, does not rescue Laplacian determinism. Instantaneous velocities in concert with Newton's Second Law cannot fully determine future and past states alone, just as instantaneous positions in concert with Newton's Second Law cannot, as I argued at length in the previous section. So, while the desire to save this kind of determinism motivates some to consider super-added impetuses, it is certainly not a motivation to consider the reductionist impetus theory.

If one does not expect the impetus theory to rescue Laplacian determinism, then there is in fact no underdetermination of the physically relevant integration constants. An extended example would perhaps usefully illustrate why, but the following general argument is much more straightforward.

Consider that classical mechanics in the abstract is a theory of possible motions, where a motion is a specification of each object's position and velocity at each instant. One begins with the set of kinematically possible motions; a motion in this set is a collection of smooth trajectories of the objects in the system. What dynamical laws do is select from this set the dynamically possible motions—dynamical laws are constraints on the set of kinematically possible motions. In physically reasonable cases dynamical evolution is deterministic in the specific sense that these dynamically possible motions never "cross" (do not ever share all the same positions and velocities for all trajectories).

It is not enough to pick out a particular motion to specify just the velocities of each object at a particular instant, since many motions have the same velocities at an instant but different position specifications. This is why there is the constant in the gravitational force function above: specifying the velocities at a time alone is insufficient for picking out the forces at that time. Now it is well known that it *is* enough to specify the positions of each object at two instants to pick out a particular motion. This is essentially because Newton's Second Law is a second order differential equation. But there is nothing special about positions here: it is just as well to specify the velocities of each object at two instants to pick out a particular motion. It is also enough to specify the positions and velocities of all the objects at a single instant. But to do this one needs neighborhood properties on both the at-at and impetus theory, since velocities and positions, respectively, are reduced to position and velocity developments (respectively).

Thus, returning to the example of Newtonian gravitation, if we express Newton's Law of Universal Gravitation with definite integrals (which represent position displacements by integrating velocity developments), i.e.

$$\mathbf{F}_{12} = Gm_1m_2 \frac{\int_{t-\Delta t}^{t+\Delta t} (\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t}{|\int_{t-\Delta t}^{t+\Delta t} (\mathbf{v}_2 - \mathbf{v}_1)\,\mathrm{d}t|^3},$$

then the constants of integration disappear and the gravitational forces are fully determined by the given velocity developments.

Since the physically significant integration constants are required to fully specify the position developments, it might appear that some additional information is somehow yet required in the impetus theory in comparison with the at-at theory. But again, this is so only at a single instant. Since picking out a particular motion (by, for example, specifying velocities at two instants or even some small initial interval of the velocity development) determines an entire position development, whatever physically relevant integration constants are included automatically in those determined position developments. Thus, despite initial appearances, there is no conceptual problem created by the presence of integration constants at all. Indeed, only persisting in believing the myth that classical motion is Laplacian deterministic will lead one to find a problem with the at-at theory or the present impetus theory of motion.

Therefore there are sufficient grounds to claim that the reductive impetus view is coherent and sufficiently dual to the at-at theory of motion to metaphysically underdetermine theory choice. The at-at theorist denies that kinematical velocity is truly instantaneous, giving a reductive account of kinematical velocity in terms of position developments. One adds impetuses at the cost of ontological redundancy. In the reductive impetus theory the roles of position and velocity are exactly reversed. By the same arguments one finds that velocities are truly instantaneous and that positions are not; the latter are instead reduced to velocity developments. One could add truly instantaneous positions back to try to regain Laplacian determinism, but the argument against them would mirror the argument against truly instantaneous velocities being added to the at-at theory of motion. Thus we are left with two reasonable alternatives: the at-at theory of motion and the reductive impetus theory of motion (where, note, the reduction mentioned here is of position developments to velocity developments, not motion).

Furthermore, I claim that there is no evident reason to prefer one over the other as an account of classical motion. Indeed, Carroll has argued that the at-at theory is irreproachable because of our epistemological limitations. As he says, knowing that the at-at theory is false would "require powers of discrimination well beyond us" (Carroll, 2002, 64), since any counter-example concerns goings-on at infinitesimal temporal intervals. If he is right, and it certainly seems to be hard to dispute it, then the impetus theory of motion that I have presented is similarly irreproachable. There is at bottom nothing askew with the impetus version of reality; it describes a classical motion of the same kind as the at-at theory would, differing only on what is happening in epistemically inaccessible infinitesimal temporal intervals. So far as I can see, due to the symmetry of the two views there are no obvious reasons to favor one theory over the other, just long custom and convenience on which to base a preference for the at-at theory. Recognizing that there are actually two dual, viable interpretations of classical motion is, it seems, of considerable interest for what it suggests about the possibilities for the metaphysical grounds of motion (particularly vis-à-vis the spaces in which motion occurs).

4 Analytical Mechanics

Although classical mechanics continues to be understood most familiarly through Newton's Laws, most physicists understand the subject in terms of the more sophisticated Hamiltonian or Lagrangian approaches. As philosophers have become acquainted with these approaches through some recent work that makes use of them, e.g. (North, 2009; Curiel, 2014), I offer some further remarks in this section on the preceding arguments by discussing motion in these frameworks.[2]

[2]It would also be worth including some remarks on the Hamilton-Jacobi approach, as it provides a distinctive picture of mechanics (Butterfield, 2005). Indeed, it even strongly suggests the equivalent defensibility of the two

Although the frameworks are suggestive of different interpretations, I claim that the issues raised in the previous sections nevertheless remain entirely the same.

First, Hamiltonian mechanics. The basic quantities in Hamiltonian mechanics are (generalized) positions $\mathbf{q}$ and (generalized) momenta $\mathbf{p}$. These quantities (for each object) are taken to compose the state of the system. The equations of motion are given by solving the following two first order ordinary differential equations, Hamilton's equations:

$$\dot{\mathbf{p}} = -\frac{\mathrm{d}H}{\mathrm{d}\mathbf{q}} \qquad\qquad \dot{\mathbf{q}} = \frac{\mathrm{d}H}{\mathrm{d}\mathbf{p}}, \tag{1}$$

where H is the Hamiltonian, a function defined on the phase space (the space of states) of a system. The phase space of Hamiltonian mechanics is $6n$-dimensional, where n is the number of objects in the system (3 for the spatial degrees of freedom and 3 for the momentum of each object). On the face of it, the quantities $\mathbf{q}$ and $\mathbf{p}$ are on a par: they are both basic quantities in the framework, Hamilton's equations are nearly identical for each (they differ only by a minus sign), and there are no apparent kinematical constraints that break the equal footing on which they stand.

How then, one might ask, can Hamiltonian mechanics give the same classical motions as the simpler Newtonian approach, since it seemingly allows non-classical possibilities, i.e. motions that do not satisfy the kinematical constraint $\dot{\mathbf{q}} = \mathbf{p}/m$, which links momentum and change in position? The only place where constraints like this can be imposed in the framework is in the Hamiltonian. Indeed, only Hamiltonians that satisfy a couple requirements will result in classical motions: H must be a function that is second order in the momenta and otherwise solely a function of the positions. In systems treated by classical mechanics the Hamiltonian can then be understood as the total energy of the system.

Arntzenius (2000) mentions Hamiltonian mechanics as a modern example of an impetus theory. Given appearances alone this would be the natural classification, since both position and momentum are taken as basic quantities. Yet one should not be too easily taken in by formalism—supposing that the Hamiltonian respects the requirements mentioned in the previous paragraph, the second of Hamilton's equations above is actually just $\dot{\mathbf{q}} = \mathbf{p}/m$, i.e. merely equates the product of mass and velocity with momentum. It is in essence just a physically-motivated definition of momentum, not a dynamical law per se.

Now one should be able to see that the interpretive situation is not so simple. One may insist that the Hamiltonian framework affords the possibility of a complete instantaneous state, one including both position and momentum (velocity). The second of Hamilton's equation, however, suggests two interpretations. One is that it imposes a kinematical constraint, in which case we have once more the problems of the super-additive impetus theory of motion of §3.1. The other is that it merely reveals momentum as velocity in disguise, i.e. merely as a neighborhood property, in which case we have once more the at-at theory of motion. Integrating Hamilton's equations leads one to the dual possibility of the impetus theory of motion of §3.2. Thus Hamiltonian mechanics in fact affords precisely the same two possibilities discussed in the previous sections: the at-at theory of motion and the reductive impetus theory of motion.

Second, Lagrangian mechanics. Here the relevant quantities are called the generalized coordinates (positions) and generalized velocities, commonly labeled $\mathbf{q}$ and $\dot{\mathbf{q}}$. The Lagrangian, L, is

---

theories, as it furnishes a neat demonstration that only one half of the kinematical quantities (position, velocity) are actually required to completely describe motion in classical mechanics.

a function defined as the difference between kinetic energy and potential energy of the system. Now, it may seem as if Lagrangian mechanics builds in the usual kinematical connection between position and velocity at the start, since notationally at least $\dot{\mathbf{q}}$ is the time derivative of $\mathbf{q}$. There is a slight subtlety however. The two different classes of variables (the velocities and positions) are actually on a par in how they are initially treated in the Lagrangian framework. When one derives the equations of motion from the Lagrangian analog of Newton's Second Law, i.e. the Euler-Lagrange equation—

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}\dot{\mathbf{q}}} - \frac{\mathrm{d}L}{\mathrm{d}\mathbf{q}} = 0 \tag{2}$$

—one takes an independent derivative of the Lagrangian with respect to each of $\mathbf{q}$ and $\dot{\mathbf{q}}$. In the derivation of the equations of motion of a particular system they are treated as if they had no dependence at all on one another (whereas normally the chain rule of calculus would have to be applied). It is only after the equations of motion are derived that $\dot{\mathbf{q}}$ becomes understood as the time derivative of $\mathbf{q}$. Thus, as with Hamiltonian mechanics, the Lagrangian framework ostensibly allows non-classical motions, i.e. ones which do not satisfy the kinematical constraint between time derivatives of position and velocity. Only those motions that do satisfy the Euler-Lagrange equation end up satisfying this constraint, which is why the constraint is understood to be enforced after the derivation of the equations of motion. So one has, once more, an at-at theory of motion. Naturally, one could re-interpret that constraint in terms of integration rather than differentiation, in which case one has, once more, the reductive impetus theory of motion.

Although there is therefore no novelty in these frameworks vis-á-vis a theory of motion, there is perhaps a novel issue to be found here with respect to the status of the non-classical motions which is worth mentioning. These motions, to repeat, are those that do not enforce the relation $\mathbf{v} = \dot{\mathbf{x}}$, i.e. the relation that velocity is the time rate of change of position (or the impetus theory analog). Are these mathematical fictions, introduced merely for the purpose of formulating the powerful analytic techniques of Hamiltonian and Lagrangian mechanics? Or should we countenance these as genuine metaphysical possibilities? My inclination is towards the former, since it is difficult to understand what $\mathbf{v}$ is meant to represent, if it is not representing the velocity determined by the object's trajectory. Could it perhaps be, though, some sort of unactualized "causal power"? It just so happens, of course, that in our world—insofar as it is classical—all such powers are fully actualized. One would like to know why *that* would be so. That they should be is enshrined in the principle of least action (or Hamilton's principle), but little philosophical work has been done to explain why the principle should be true.[3]

5 Conclusion

I have defended two theories of motion, the well-known at-at theory and a novel impetus theory of motion. I claim that the choice between these two theories is epistemically underdetermined. It may indeed simply be a matter of convention whether one understands motion in one way or the other given the epistemic irreproachability of each theory. Assuming that there is some fundamental "space" that grounds the basic properties of the two theories, it would also perhaps then be

[3] Although there has been some debate about the metaphysical role of such a principle, in particular its implications for dispositional essentialism: (Katzav, 2004; Ellis, 2005; Smart & Thébault, 2015).

a matter of convention whether one sees the world as fundamentally spatial or as fundamentally velocital, as they differ only in their instantaneous ontologies.

To review the argument, in the first part of the paper I defended the at-at theory of motion against the criticism that it makes determinism impossible by fiat. The impossibility of the Laplacian version of determinism is indeed a feature of classical mechanics, but it is not so merely by how velocity is defined. While I agree with the critics that velocities should not be thought of as really instantaneous in the at-at theory of motion, that they are still reasonably considered part of the physical state of a system follows from the nature of the laws, whether these laws be metaphysical laws or physical laws. The kind of determinism relevant to mechanics is thus not Laplacian determinism, but determinism based on states including instantaneous and neighborhood properties.

In the second part of the paper I considered two impetus theories of motion. I rehearsed the argument against an impetus theory of motion that makes velocity truly instantaneous by fiat. Kinematical velocity is a derived quantity in the at-at theory of motion, but one that fully accounts for motion without reference to some super-added impetus. Thus we are led to conclude that this impetus theory should be discarded. The second, novel impetus theory of motion turns the at-at theory on its head by assuming that velocity is basic and truly instantaneous and position is derived (reduced to velocity developments). I defended this view from two major criticisms, namely that it is at odds with the definition of velocity, and that integration introduces undetermined constants of integration. I concluded that the this impetus theory of motion and the at-at theory are dual theories of motion, sufficiently symmetric so that there is no evident way to decide between them as accounts of motion, at least on any epistemic grounds.

Finally, I wish to note that, although we are quite used to thinking of the world in spatial terms, it is not inconceivable that people could have thought of motion in the velocital way which I suggest. Indeed, Borges provides an intriguing report of the people of Tlön which may readily be interpreted in such terms (although his favored analysis of their culture is in terms of a kind of idealism):

> It is no exaggeration to state that the classic culture of Tlön comprises only one discipline: psychology. All others are subordinated to it. I have said that the men of this planet conceive the universe as a series of mental processes which do not develop in space but successively in time. Spinoza ascribes to his inexhaustible divinity the attributes of extension and thought; no one in Tlön would understand the juxtaposition of the first (which is typical only of certain states) and the second—which is a perfect synonym of the cosmos. In other words, they do not conceive that the spatial persists in time. The perception of a cloud of smoke on the horizon and then of the burning field and then of the half-extinguished cigarette that produced the blaze is considered an example of association of ideas. (Borges, 1961, 116)

Indeed, the perception of motion is direct and foremost from the point of view of psychology. It might seem only natural that one has recourse to the concept of space only by specifying forces of interaction that give rise to a unified spatial description, and thus only in certain, specified states. It is, I submit, a conceptually coherent way of proceeding in theorizing about the world, albeit one that has so far not found favor, neither in physics nor in everyday life.

References

Albert, D. (2000). *Time and Chance*. Cambridge, MA: Harvard University Press.

Arntzenius, F. (2000). Are there really instantaneous velocities? *The Monist*, *83*(2), 187–208.

Arntzenius, F. (2003). An arbitrarily short reply to sheldon smith on instantaneous velocities. *Studies in History and Philosophy of Modern Physics*, *34*, 281–282.

Borges, J. L. (1961). Tlön, uqbar, orbis tertius. In S. Richardson, & C. Smith (Eds.) *New World Writing*, vol. 18, (pp. 110–125). Philadelphia: J. B. Lippincott & Co.

Butterfield, J. (2005). On hamilton-jacobi theory as a classical root of quantum theory. In *Quo Vadis Quantum Mechanics?*, The Frontiers Collection, (pp. 239–273). Berlin: Springer.

Butterfield, J. (2006). Against *Pointillisme* about mechanics. *British Journal for the Philosophy of Science*, *57*, 709–753.

Carroll, J. W. (2002). Instantaneous motion. *Philosophical Studies*, *110*, 49–67.

Curiel, E. (2014). Classical mechanics is lagrangian; it is not hamiltonian. *British Journal for the Philosophy of Science*, *65*, 269–321.

Earman, J. (1986). *A Primer on Determinism*. Dordrecht, Netherlands: D. Reidel Publishing Company.

Easwaran, K. (2014). Why physics uses second derivatives. *British Journal for the Philosophy of Science*, *65*, 845–862.

Ellis, B. (2005). Katzav on the limitations of dispositionalism. *Analysis*, *65*, 90–92.

Jackson, F., & Pargetter, R. (1988). A question about rest and motion. *Philosophical Studies*, *53*, 141–146.

Katzav, J. (2004). Dispositions and the principle of least action. *Analysis*, *64*, 206–214.

Kline, M. (1972). *Mathematical Thought from Ancient to Modern Times*, vol. 3. Oxford: Oxford University Press.

Lange, M. (2005). How can instantaneous velocity fulfill its causal role? *Philosophical Review*, *114*(4), 433–468.

Laplace, P.-S. (1840/1902). *A Philosophical Essay on Probabilities*. New York: John Wiley & Sons.

North, J. (2009). The ”structure” of physics: A case study. *The Journal of Philosophy*, *106*, 57–88.

Russell, B. (1903). *The Principles of Mathematics*. Cambridge: Cambridge University Press.

Smart, B., & Thébault, K. (2015). Dispositions and the principle of least action revisited. *Analysis*, *75*, 386–395.

Smith, S. (2003). Are instantaneous velocities real and really instantaneous. *Studies in History and Philosophy of Modern Physics*, *34*, 261–280.

Tooley, M. (1988). In defense of the existence of states of motion. *Philosophical Topics*, *16*, 225–254.