WITTGENSTEIN ON THE FOUNDATIONS OF MATHEMATICS

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ABSTRACT

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Wittgenstein on the Foundations of Mathematics

This thesis comprises two main parts, which are on different aspects of Wittgenstein's philosophy of mathematics.

In Part I, an attempt is made to survey the original source material on which any detailed assessment of Wittgenstein's remarks on the foundations of mathematics from his middle and later periods ought to be based. This survey is presented within the context of a sketch of Wittgenstein's biography, which also mentions some of the major developments in his thinking. In addition, certain main themes are emphasized; these have to do primarily with the Kantian aspects of Wittgenstein's thought and with his mysticism or the 'religious point of view'.

In Part II, Kreisel's critique of Wittgenstein's remarks on the foundations of mathematics, which has been developed since 1958 in a series of published articles, receives close examination, and, in connection with this, different approaches to the philosophical investigation of mathematics are considered which represent genuine alternatives to Wittgenstein's approach. There are separate sections on Lakatos' *Proofs and Refutations* and Bourbaki's 'L'Architecture des Mathématiques'.

Finally, besides a bibliography which surveys the reception of Wittgenstein's views on the foundations of mathematics, there are two substantial appendices, which are supplemental to Part I. The first of these gives the manuscript sources for typescripts **221** and **222-4**, and the correspondences in both directions between these typescripts. The second appendix is part of a chronological version of von Wright's catalogue of Wittgenstein's papers, beginning in 1929.

Ich glaube meine Stellung zur Philosophie dadurch zusammengefasst zu haben, indem ich sagte: Philosophie dürfte man eigentlich nur *dichten*. Daraus muss sich, scheint mir, ergeben, wie weit mein Denken der Gegenwart, Zukunft, oder der Vergangenheit angehört. Denn ich habe mich damit auch als einen bekannt, der nicht ganz kann, was er zu können wünscht.

Wittgenstein

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Preface

According to its original plan my thesis had three main parts: (I) a survey of the primary sources for the study of Wittgenstein's thought on the foundations of mathematics; (II) an examination of Wittgenstein's thought on this subject in his three major works: *Tractatus Logico-Philosophicus*, *Philosophical Grammar* and *Philosophical Investigations*; and (III) a discussion of ways in which Wittgenstein's thought in his later writings might be developed.

The present thesis consists only of versions of Parts I and III according to the original plan. There are two main reasons for this. Firstly, the work on Part I, which I regard as an essential preliminary to Part II, became more involved than I had anticipated. Secondly, the Philosophy Panel of this University decided that I should not be allowed to complete a D.Phil. thesis, which had been my hope, and as a result I had neither the space nor the time to complete all three parts. Consequently, the two parts of my thesis cohere less than I should have liked; they represent a stage in the development of a larger project.

Acknowledgement is due to a number of people for their help. I am grateful, above all, to Dr Gordon Baker, whose patient guidance and assistance over the last four years have been invaluable. My initial supervisor, Dr Daniel Isaacson, who first indicated to me the extent of Kreisel's writings on Wittgenstein, also deserves special thanks. While studying in Finland, I have benefited from discussions with Professor Georg Henrik von Wright, mainly on the origin and composition of Wittgenstein's writings, and he has also been most generous in allowing me access to the materials available in Helsinki for research on Wittgenstein. I should also like to mention R.R. Rockingham Gill, who first introduced me to Wittgenstein's philosophy of mathematics and to the subject of mathematical logic, while I was an undergraduate at St David's University College, Lampeter.

Acknowledgement is also due to my college for its support; and I am grateful, in particular, for the award of a Meyricke Graduate Scholarship, which helped sustain me during my first years at Oxford.

Introduction

G.E. Moore once described Wittgenstein as 'the greatest philosopher since Kant'[1]. Like many other philosophers, I should agree with this assessment; but, I should also add that the depth and interest of their philosophical work has its origin in the same set of fundamental ideas. The most important of these are the idea of a limit to thought or language, the idea that philosophy is essentially negative in character, and the idea that there is an absolute boundary to be drawn between science and religion.

Wittgenstein definitely approved of Kant's philosophy. For example, commenting on Kant's critical method 'without the peculiar applications Kant made of it'[2] he once said:

> 'This is the right sort of approach. Hume, Descartes and others had tried to start with one proposition such as "Cogito ergo sum" and work from it to others. Kant disagreed and started with what we know to be so and so, and went on to examine the validity of what we suppose we know'.[3]

Of course, Wittgenstein could not have agreed with the content of the 'Transcendental Deduction' in which Kant examined 'the validity of what we suppose we know'; but it can be argued that what Kant's transcendental deductions

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[1] Professor W.B. Gallie communicated this fact to his audience at St David's University College, Lampeter on 2nd May, 1985 during his lecture "Philosophy and Philosophers".

[2] *Wittgenstein's Lectures, Cambridge 1930-1932* (Oxford, Blackwell, 1980) edited by D. Lee, section CV, A.

[3] Ibid.

are intended to perform is performed in an analogous fashion by Wittgenstein's logical analysis or logical clarification of language.

Kant and Wittgenstein disagreed, however, about the necessity of a critical investigation of mathematics. Kant writes:

> 'There is no need of a critique of reason in its empirical employment, because in this field its principles are always subject to the test of experience. Nor is it needed in mathematics, where the concepts of reason must be forthwith exhibited *in concreto* in pure intuition, so that everything unfounded and arbitrary in them is at once exposed.'[4]

Wittgenstein, in contrast, thought that mathematics was replete with philosophical confusions[5], and at least one third of his writings are devoted to their investigation.

Genuine students of Wittgenstein tend, nevertheless, to neglect a proper study of his philosophy of mathematics. One reason for this is that very few remarks directly on this subject appear in either of his major works, the *Tractatus* or the *Investigations*. Another reason is that his researches into the philosophy of mathematics are not apparently integral to his other concerns. McGuinness writes in his biography of Wittgenstein:

> '...there is a certain puzzle to be resolved - and one that ran through Wittgenstein's philosophical life,

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[4] *Critique of Pure Reason*, A 711/B 739.

[5] Volume II (MS **106**), p. 58: 'There is no religious denomination in which the misuse of metaphysical expressions has been responsible for so much sin as it has in mathematics'.

> namely the puzzle of the connexion between his passion for the philosophy of mathematics and his other interests, or indeed passions. He was never a mathematician. For him as an engineer, mathematics was a tool. His mathematical education and sophistication barely qualified him to discuss the foundations of mathematics in the way he did. So it was not by difficulties in his everyday work that he was led to these problems. Yet perhaps half of all that he wrote was concerned with mathematics. And on two occasions, this one of his coming to Cambridge [in 1911], and the later one of his resuming passionate philosophical discussion after long silence in 1928, it was a problem in the foundations of mathematics that excited him (here Frege and Russell's difficulties with the paradoxes, there Brouwer's exposition of intuitionism). These problems were not only unconnected with his technical concerns as an engineer; at first sight they also seem to be quite different from his other preoccupations. He was a musician. He read passionately works of literature that "said something to him" - something, that is, about human life. He brooded over his own defects and difficulties. He was a fierce critic of failings, especially those of honesty, in others. He came to write a book whose main point (he said himself) was an ethical one. What had the foundations of mathematics in common with propensities like these?'[6]

We do not know exactly how or why Wittgenstein first became interested in the foundations of mathematics. This interest, which was never purely technical, might simply have been aroused by his reading the works of Frege and Russell. It will have been sustained by a number of external factors. Besides Frege, whom Wittgenstein admired greatly throughout his life, and Russell, his early mentor, the influence of other philosopher-mathematicians will have served to maintain his interest in the philosophy of mathematics, not least Ramsey and certain members of the Vienna Circle.

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[6] *Wittgenstein: A Life. Young Ludwig, 1889-1921* (Berkeley, University of California Press, 1988), pp. 76-7.

Wittgenstein was also a moralist and he saw no intrinsic value in scientific work, whether in mathematics or natural science. For this reason, he was particularly concerned to expose those philosophical confusions which led scientists and philosophers to overestimate the value of science. His sustained interest in the philosophy of mathematics is thus partly due to his belief that in writings about mathematics philosophical confusions are peculiarly rife and that these lead to false conceptions of its value. Wittgenstein thought that the same was true of psychology, which he also discussed at great length. The mathematical part of his work is advertised in the following remark from *Philosophical Investigations* (1953):

> 'The confusion and barrenness of psychology is not to be explained by calling it a "young science"; its state is not comparable with that of physics, for instance, in its beginnings. (Rather with that of certain branches of mathematics. Set Theory.) For in psychology there are experimental methods and *conceptual confusion*. (As in the other case conceptual confusion and methods of proof.)
>
> [...]
>
> An investigation is possible in connexion with mathematics which is entirely analogous to our investigation of psychology. It is just as little a *mathematical* investigation as the other is a psychological one. It will *not* contain calculations, so it is not for example logistic. It might deserve the name of an investigation of the "foundations of mathematics".'[7]

Wittgenstein had intended to include as part of his projected book *Philosophical Investigations* just such an investigation of the 'foundations of mathematics'; but, unfortunately, no major work was produced, nothing

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[7] The final remark of Part II.

certainly that stands comparison with the first part of the final, printed version of *Philosophical Investigations* (TS 227).

What we have are a number of manuscripts and typescripts, as well as dictations, correspondence, and notes taken during Wittgenstein's lectures and conversations. The complexity of this source material creates serious problems for the student of Wittgenstein's remarks on the foundations of mathematics.

Wittgenstein published only two works of philosophy during his lifetime, his early masterpiece *Tractatus Logico-Philosophicus* and 'Some Remarks on Logical Form', which is an attempt, made soon after returning to academic philosophy in 1929, to modify some of his earlier doctrines. None of the fruits of Wittgenstein's research between this time and his death in 1951 were published and no finished work was left to his literary executors. For Wittgenstein's views after 1929, including those on the foundations of mathematics, it is necessary, therefore, to study his *Nachlass*. Fortunately, Wittgenstein's thought in the middle period can, for many purposes, be represented by a single text of outstanding importance, *Philosophical Grammar* (TS 213). The later period, following the commencement of Wittgenstein's work on *Philosophical Investigations*, presents greater problems. An early version of the *Investigations* (TSS 220-221) had included a second half on the foundations of mathematics, and this work is of considerable importance. However, later versions of Wittgenstein's book contain very little on mathematics, although there do exist at this time a number of substantial unrevised manuscripts on the subject. In addition, the second half of the early version of the *Investigations* later underwent considerable revision, producing typescripts 222-224. An understanding of the development of Wittgenstein's writings on the foundations of mathematics during the period of the composition of the *Investigations* seems to be required simply to determine the principal sources for Wittgenstein's later view.

Besides this essential work, the interpretation of Wittgenstein's thought on the foundations of mathematics from *all* periods is, in my judgement, facilitated greatly by work which clarifies the details of the origin and composition of his writings in their historical and biographical context. In Part I, an attempt is made to survey in a comprehensive fashion the development of Wittgenstein's writings on the foundations of mathematics in the context of his changing philosophical project, from 1926 until his death. Wittgenstein's biography is described in outline, with an emphasis on those aspects which are of significance to his development as a philosopher of mathematics. Certain large themes which help clarify Wittgenstein's thought at the broadest level of interpretation have also been emphasized; and important statements by Wittgenstein on the main features of his philosophy and on the most significant developments in his thought have been included.

The value of notes such as these ought to be obvious to anyone who has ever attempted a close study of Wittgenstein's *Nachlass*. Researchers using these original materials need some sort of guide, if they are not to be overwhelmed by their complexity. Also, any assessment of the significance of an individual text is clearly hindered, if its place in the development of Wittgenstein's thought is not understood; and the value of the extensive lecture notes and notes of conversations which survive, is clearly enhanced, if they can be tested against Wittgenstein's own contemporary writings.

The development of Wittgenstein's thought should, I believe, be traced primarily in his major works, *Tractatus Logico-Philosophicus*, *Philosophical Grammar* and *Philosophical Investigations*[8], each of which represents a well defined philosophical position. The interpretation of these works can then draw on Wittgenstein's other writings, which are also essential sources for understanding his thought in transition. Having said this, Waismann's notes of Wittgenstein's conversations in Vienna between 1929 and 1932, and the notes, taken by various students, of the 1939 lecture series in Cambridge, both have a measure of

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[8] Including the associated texts on mathematics.

independent significance for the assessment of Wittgenstein's thought on the foundations of mathematics.

The project which I have outlined above, and which is represented here by Part I, is so far incomplete, even as regards the basic chronological questions. Also, Wittgenstein's early period is not mentioned; and this ought to be recognised as a severe limitation, given that nearly all of Wittgenstein's later reflections on mathematics have their origin in his early work on the *Tractatus*.

One individual who features significantly in Part I is the mathematician Georg Kreisel. Part II is an examination of Kreisel's critique of Wittgenstein's remarks on the foundations of mathematics, which Kreisel has developed over several decades in a series of essays, reviews and memoirs. Consideration is also given here to a number of different approaches to the philosophical investigation of mathematics which represent genuine alternatives to Wittgenstein's approach. Lakatos's *Proofs and Refutations* and Bourbaki's 'L'Architecture des Mathématiques' receive individual scrutiny.

The basic means of reference to Wittgenstein's works that is employed here is the numbering system used by von Wright in his catalogue 'The Wittgenstein Papers'. These numbers ought to be as familiar to students of

Wittgenstein's philosophy as the numbers of important ancient bones might be to an archaeologist. The standard abbreviations for Wittgenstein's published works, *TLP*, *PG*, *PI*, *RFM*, etc. are used when this is considered convenient. Reference is always made to the third, revised edition of *Remarks on the Foundations of Mathematics*, unless indicated otherwise.

I Wittgenstein's Writings on the Foundations of Mathematics[1]

1.1 The Middle Period: 1926-1936[2]

Wittgenstein was gradually drawn back into philosophy in the late 1920s. Having ended his teaching career in difficult circumstances in April 1926, Wittgenstein at first considered becoming a monk, but was dissuaded and instead worked for a brief period as a gardener with the monks at Hütteldorf, near Vienna. During this time he was asked by his sister Margaret Stonborough and by his friend, the architect Paul Engelmann to work jointly with Engelmann on the construction of a mansion for Margaret in Vienna. He agreed, and from the Autumn of 1926 he devoted all of his energy to the mansion, which became, as Engelmann

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[1] In Part I, I rely heavily on these two biographies of Wittgenstein: *Wittgenstein: A Life* (Berkeley, University of California Press, 1988) by B.F. McGuinness; and *Ludwig Wittgenstein: The Duty of Genius* (London, Jonathan Cape, 1990) by R. Monk. The following biographical sketches have also been of considerable use: 'Ludwig Wittgenstein: A Biographical Sketch' in *Wittgenstein* (Oxford, Blackwell, 1982) by G.H. von Wright; and 'Ludwig Wittgenstein' by M. Nedo in *Wittgenstein: Biographie, Philosophie, Praxis* (Wiener Secession, 1989).

[2] Besides the biographical works mentioned in the previous footnote, I have also made use in this section of the Preface to *Ludwig Wittgenstein and the Vienna Circle* (Oxford, Blackwell, 1979) by B.F. McGuinness, 'Verehrung und Verkehrung: Waismann and Wittgenstein' by G.P. Baker, in *Wittgenstein: Sources and Perspectives* (Hassocks, Harvester Press, 1979), and the 'Nachwort' to Waismann's *Logik, Sprache, Philosophie* (Stuttgart, Reclam, 1976) by G.P. Baker and B.F. McGuinness.

states[3], entirely his own achievement.[4] 'Its beauty', in von Wright's words[5], 'is of the same simple and static kind that belongs to the sentences of the *Tractatus*'. It was through Margaret Stonborough, who was well known in Viennese society, that Moritz Schlick finally managed to meet Wittgenstein. Schlick had sent Wittgenstein some of his own work and proposed that they should meet, with one or two others, to discuss logical problems. In February 1927 Margaret replied on Wittgenstein's behalf that 'he still feels quite unable to concentrate on logical problems' but that if he were able to meet with Schlick alone 'he might be able to discuss such matters'[6]. A first meeting with Schlick was apparently followed by others in which Wittgenstein met with Schlick alone. By the summer of 1927, however, Wittgenstein had been persuaded to meet and have discussions with a select group of the members of Schlick's Circle[7], which besides Schlick, included Friedrich Waismann (1896-1959), Rudolf Carnap[8] (1891-1970)

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[3] P. Engelmann - F.A. von Hayek, 16.2.1953. *Wittgenstein: Sein Leben in Bildern und Texten* (Frankfurt, Suhrkamp, 1983) edited by M. Nedo and M. Ranchetti, 288-294.

[4] See *The Architecture of Ludwig Wittgenstein* (The Press of the Nova Scotia College of Art and Design, Chatham, 1973) by B. Leitner.

[5] *Wittgenstein*, p. 24.

[6] Margaret Stonborough - Schlick, 19.2.1927. *Wittgenstein: Sein Leben in Bildern und Texten*, 301.

[7] Schlick - Wittgenstein, 15.8.1927.

[8] Carnap gives an account of these conversations in his 'Autobiography' in *The Philosophy of Rudolf Carnap* (Illinois, Open Court, 1963) edited by P.A. Schilpp.

and Herbert Feigl (1902-). No record of these discussions in 1927 and 1928 seems to have been kept, but we do know that Wittgenstein sometimes read poetry to his audience, including passages from Rabindranath Tagore; thus leaving them in no doubt about the importance for him of the mysticism expressed in the *Tractatus*[9]. We also know that a certain number of these early discussions concerned the foundations of mathematics.

One topic for discussion was provided by Frank Ramsey's paper 'The Foundations of Mathematics', which had been read to the London Mathematical Society in November 1925.[10] Ramsey's object, as he states in the Preface, is 'to give a satisfactory account of the Foundations of Mathematics in accordance with the general method of Frege, Whitehead and Russell':

> 'Following these authorities, I hold that mathematics is part of logic, and so belongs to what may be called the logical school as opposed to the formalist and intuitionist schools. I have therefore taken *Principia Mathematica* as a basis for discussion and amendment; and believe myself to have discovered how, by using the work of Mr Ludwig Wittgenstein, it can be rendered free from the serious objections which have caused its rejection by the majority of German

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[9] About Wittgenstein Carnap remarks, ibid. pp. 24-30, that he 'had not paid sufficient attention to the statements in his book about the mystical' and he was surprised to find that Wittgenstein's 'attitude towards people and problems, even theoretical problems, were much more similar to those of a creative artist than to those of a scientist'.

[10] The paper was published in the Society's *Proceedings*, Series 2, 129 and later reprinted in *The Foundations of Mathematics and Other Logical Essays* (Routledge and Kegan Paul, London, 1931), edited by R.B. Braithwaite with a preface by G.E. Moore.

authorities, who have deserted altogether its line of approach.'

In the summer of 1927 Ramsey sent a copy of his paper to Schlick, who then showed it to Wittgenstein. Wittgenstein prepared a reply, which was typed, supplied with handwritten corrections, and sent via Schlick as the main, philosophical part of a letter to Ramsey[11]. A carbon copy of the uncorrected typescript, headed 'Wittgenstein an Ramsey, Juni 1927' (TS **206**[12]), was retained by Waismann. Wittgenstein's typescript is a lengthy criticism of Ramsey's definition of identity, which is central to the argument of his paper[13]. Ramsey, who felt he had been misunderstood, replied both via Schlick[14] and then directly to Wittgenstein[15].

In March 1928 Wittgenstein was persuaded by Waismann and Feigl to attend Brouwer's lecture 'Mathematik, Wissenschaft und Sprache', which was to be delivered in Vienna at the Academy of Sciences. The three men spent several hours in a café after the lecture discussing what

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[11] Wittgenstein - Ramsey, 2.7.1927. *Briefe* (Frankfurt, Suhrkamp, 1980), 189.

[12] Reprinted in *'On Ramsey's Definition of Identity'*, *Ludwig Wittgenstein and the Vienna Circle*, VI.

[13] Wittgenstein also criticizes Ramsey's definition of identity in *Philosophical Grammar* (TS **213**), section 113 (*PG*, II, 16).

[14] Schlick - Wittgenstein, 15.8.1927. In 'On Ramsey's Definition of Identity', *Ludwig Wittgenstein and the Vienna Circle*, VI.

[15] Ramsey - Wittgenstein [7/8.1927]. *Briefe*, 190.

they had heard. According to Feigl, the lecture was a great stimulus to Wittgenstein's thinking; he says 'that evening marked the return of Wittgenstein to strong philosophical interest and activities'[16].

Wittgenstein's architectural project was completed by the Autumn of 1928, and believing now that he could again do creative work in philosophy, he went early in the next year to Cambridge, planning to stay for a couple of terms to 'work on visual space and other things'[17]. His official position at Cambridge, where Moore was now Professor, was that of an Advanced Student reading for the Ph.D. Ramsey was appointed as his supervisor and the two soon met frequently for lengthy discussions. Besides renewing old acquaintances at Cambridge, with Moore, Johnson and others, Wittgenstein also made new friends, especially among the students. One of these new friends was Maurice Drury, whose notes of his conversations with Wittgenstein constitute a valuable record.[18] At the beginning of February 1929 Wittgenstein began writing down philosophical

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[16] *The Philosophy of Wittgenstein* (New Jersey, Prentice-Hall, 1964) by G. Pitcher, p. 8. Feigl's report together with the fact that Wittgenstein returned to philosophical research soon after this event can and has lead people to exaggerate the importance of Brouwer's influence on Wittgenstein, which in reality was quite minimal.

[17] Wittgenstein - Schlick, 18.2.1929.

[18] 'Some Notes on Conversations with Wittgenstein', *Acta Philosophica Fennica* 28 (1976), by M.O'C. Drury is reprinted in *Recollections of Wittgenstein* (Oxford, Blackwell, 1981) edited by R.Rhees, alongside his 'Conversations with Wittgenstein'.

remarks in a large manuscript volume, entitled 'Band I. Philosophische Bemerkungen.' (MS **105**). Volumes I and II (MSS **105-106**), which only together form a complete text[19], were to be the first in a long series of numbered manuscript volumes (MSS **105-122**) written over a period of approximately eleven years.[20] These volumes and similar ones written later are the main source for the remarks which Wittgenstein used in the creation of his more finished texts, and they are usually based themselves on remarks in less substantial pocket notebooks[21].

Scattered among the philosophical remarks in these volumes are various diary entries, some of which are in code. In Volume I (MS **105**), Wittgenstein writes about his 'delightful discussions' with Ramsey. He says:

> 'There is nothing more pleasant to me than when someone takes my thoughts out of my mouth, and then, so to speak, spreads them out in the open.'

He adds, 'I don't like taking walks through the fields of science alone'. Wittgenstein will certainly have benefited

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[19] The writing in the first volume, MS **105**, begins on the right-hand pages, on 2 February, and after page 131 is continued on the right-hand pages of the second volume, MS **106**, up to page 296. MS **106** contains no dates. The writing now continues on the left-hand pages of MS **106** up to page 298 and then in the left-hand pages of MS **105** up to page 132 and from there to the end, page 135.

[20] This series can be traced horizontally in Appendix II from the first manuscript, MS **105**, begun on 2 February 1929, to the last, MS **122**, which is continued in MS **117**(5) until 18 April 1940.

[21] The pocket notebooks can also be traced horizontally in Appendix II, beginning with MS **153a**, which is the first to have survived.

from Ramsey's mathematical expertise; and it might have been through discussions with Ramsey that Wittgenstein first became fully acquainted with the intuitionism of Hermann Weyl (1885-1955).[22] Wittgenstein is also known to have had conversations at Cambridge with the mathematician G.H. Hardy (1877-1947), who also attended some of his lectures.

Wittgenstein spent the Easter vacation in Vienna, and, having left Volume I (MS **105**) in Cambridge, he continued his writing in Volume II (MS **106**).[23] Wittgenstein probably also attended meetings of the Round Table.[24]

After Otto Weininger (1880-1936)[25], the next person on Wittgenstein's list of seminal influences on his thinking[26] is Oswald Spengler (1880-1936), whose book *The Decline of the West* (1918) he greatly admired.[27] This influence is nowhere more apparent than in the following coded remark

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[22] Ramsey's particular interest in Weyl's views on mathematics is apparent from a short series of paragraphs headed 'Principles of Finitist Mathematics', which he wrote in 1929.

[23] This would explain the discontinuities in the writing in Volumes I and II, and we know, from the dated entries, that vacations account for similar discontinuities in later volumes.

[24] Schlick - Wittgenstein, 18.2.1929.

[25] The author of *Sex and Character*.

[26] MS **154**, p. 43 (*CV*, pp. 18-19).

[27] See Drury's 'Conversations with Wittgenstein' at the end of 1929 and the beginning of 1930.

from Volume II (MS 106), pp. 253-255:

> 'I believe that in the last century mathematics has had a period of a quite particular loss of instinct from which it will suffer for a long time. I believe this loss of instinct is connected with the decline of the arts, they correspond to the same cause.'

Spengler's thought tended to reinforce Wittgenstein's own pessimism about developments in Western culture and society. More importantly, under Spengler's influence Wittgenstein came to believe that alterations in our way of life, and thus in the fundamentals of our language, might even eliminate some of the philosophical problems which concerned him.[28]

In order to obtain financial support for his work, Wittgenstein was persuaded to apply for a research grant at Trinity College. His application for a grant was given the full support of Frank Ramsey who, on Moore's request[29], wrote a report on Wittgenstein's progress. Ramsey says:

> 'From his work more than that of any other man I hope for a solution of the difficulties that perplex me both in philosophy generally and in the foundation of Mathematics in particular. It seems to be, therefore, peculiarly fortunate that he should have returned to research.'

He continues:

> 'During the last two terms I have been in close touch with his work and he seems to me to have made remarkable progress. He began with certain questions in the analysis of propositions which have now led him

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[28] For a useful discussion see von Wright's 'Wittgenstein in Relation to his Times', in his *Wittgenstein*.

[29] 'Wittgenstein's Lectures in 1930-33', *Mind*, 63, p. 3.

> to problems about infinity which lie at the root of current controversies on the foundations of Mathematics. At first I was afraid that lack of mathematical knowledge and facility would prove a serious handicap to his working in this field. But the progress he had made has already convinced me that this is not so, and that here too he will probably do work of first importance.'[30]

On 18 June Wittgenstein was awarded the Ph.D., based on the *Tractatus*, and the next day he was awarded a one year research grant.

By the summer of 1929 Wittgenstein had prepared a paper, 'Some Remarks on Logical Form', which was to be read by him, on 13 July, at the Annual Joint Session of the Aristotelian Society and the Mind Association, held that year in Nottingham. Wittgenstein attempts in this, his only philosophical article to overcome a problem in the *Tractatus* conception of elementary proposition, which was probably suggested by Ramsey's criticisms. The paper was disowned by Wittgenstein almost as soon as it was printed and, hoping that Russell would attend his lecture, Wittgenstein decided to speak instead on generality and infinity in mathematics[31]. It was at this conference that Wittgenstein met and became friends with the Oxford philosopher Gilbert Ryle[32] (1900-1976).

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[30] *Wittgenstein: Sein Leben in Bildern und Texten*, 321.

[31] Wittgenstein - Russell, [7.1929]. *Letters to Russell, Keynes and Moore*, R. 54.

[32] Following the publication of Wittgenstein's *Remarks on the Foundations of Mathematics*, Ryle published 'The Work of an Influential but Little-Known Philosopher of Science: Ludwig Wittgenstein', *Scientific American*, 197.

Wittgenstein was in Austria for the rest of the summer vacation, but neither Waismann nor Schlick were available for discussions. Schlick, who had been in Stanford for the whole summer, returned to Vienna only after Wittgenstein had himself returned to Cambridge. He writes in October[33]:

> 'Mr Waismann has given me your Remarks on Logical Form and told me that you are preparing two further publications on the foundations of mathematics. I look forward to these works with keen expectation - but even more to your next visit to Vienna.'

As the subject of the Nottingham lecture indicates, Wittgenstein had indeed now moved on to work on the foundations of mathematics. Publications in the form of further articles were not, however, to be the result of this work.

Wittgenstein began writing in Volume III (MS **107**) in October 1929. On 17 November he gave a lecture to the Heretics Club, a Cambridge society, in which he attempted to give a popular account of his ethical standpoint in the *Tractatus*. This was the so-called 'Lecture on Ethics', which survives both as a manuscript, MS **139a**[34], and, in a slightly modified version, as a typescript, TS **207**[35]. Besides the discussions with Ramsey, Wittgenstein also had discussions at Cambridge with the Italian economist Piero

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[33] Schlick - Wittgenstein, 24.10.[1929].

[34] A second version, MS **139b**, is now lost.

[35] This typescript, which has the title 'Lecture on Ethics', is based on a manuscript loaned by Wittgenstein to one of his students, R. Townsend. It is from this typescript that the lecture was published in *Philosophical Review*, 74 (1965).

Sraffa (1899-1983), who was then at King's College. Ramsey had already helped Wittgenstein to see certain mistakes in the *Tractatus*, but it was Sraffa, the next person on Wittgenstein's list of primary influences, who provided seed for the development of his mature philosophy. The new 'anthropological' viewpoint, a fundamental change in Wittgenstein's thinking, which he attributed to Sraffa's influence[36], was gradually to emerge in the early 1930s.

While Wittgenstein was in Cambridge, Schlick's Circle had begun to organize itself into a distinct philosophical school.[37] At the Prague conference on the Theory of Knowledge in the Exact Sciences, in September 1929, the Vienna Circle, as it was now called, sold a brochure containing an account of their central doctrines and philosophical antecedents. This publication, *Die Wissenschaftliche Weltauffassung* (Der Wiener Kreis, 1929)[38], also contains a sketch by Waismann of the contents of the *Tractatus* and the announcement of a forthcoming book by him, *Logik, Sprache, Philosophie*[39], which is described as an introduction to the ideas of the *Tractatus*. The

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[36] According to Rhees. See *Wittgenstein: The Duty of Genius*, p. 261.

[37] See Wittgenstein - Waismann, [6/7.1929], a response to Waismann - Wittgenstein, 5.7.29, for Wittgenstein's opinion on this development.

[38] It is published in an English translation as 'The Scientific Conception of the World' in *Empiricism and Sociology* (Dordrecht, 1973) by O. Neurath.

[39] Parts of an early draft survive among Waismann's papers.

authors of *Die Wissenschaftliche Weltauffassung*, already had great expectations for Wittgenstein's thought in the area of the foundations of arithmetic and set theory. They write:

> 'Today in this area three different schools stand opposed to one another; beside the "logicism" of Russell and Whitehead stands the "formalism" of Hilbert, who conceives arithmetic as a formula-game with definite rules, and the "intuitionism" of Brouwer, according to whom arithmetical knowledge is based on a fundamental intuition of two-oneness. The dispute between these three schools is followed in the Vienna Circle with the greatest interest. What the final outcome will be is not yet foreseeable; in any case it will include a decision on the structure of logic; hence the importance of this problem for the scientific conception of the world. Many are of the opinion that the three schools are not quite so distant as it appears. They suppose that essential trends in the three schools will approach each other during further development and, probably by making use of the far-reaching thought of Wittgenstein, become united in the final solution.'

Wittgenstein spent the Christmas vacation in Vienna, and, having left Volume III in Cambridge, he continued his writing in the new Volume IV (MS 108). A large proportion of the remarks in this volume are on mathematics. Suspicious of the Circle's transformation into a philosophical school, but nevertheless keen to communicate the results of his new research, Wittgenstein now met only with Schlick and Waismann. These meetings, which took place at Schlick's house, included, besides discussion, some straightforward exposition by Wittgenstein. The proceedings were recorded by Waismann, who was allowed to make use of his notes to communicate Wittgenstein's views, which he did in various lectures and publications.[40] We know of six meetings which took place during the Christmas vacation, the first on 18 December. Besides Wittgenstein's research on visual space, the proposition and his 'Lecture on Ethics', etc., a number of topics in the philosophy of mathematics were also discussed.[41] Each of the major themes in Wittgenstein's later thought on the foundations of mathematics, which were to be developed over the coming years, are already present in his exposition at this time, and each of these themes clearly has its origin in the *Tractatus*. One major theme is the confusion of mathematical and physical propositions: mathematical generality is confused with other types of generality[42], mathematical discovery with physical discovery, and so on. References occur to the Dedekindian definition of infinite set, which was used by Russell and Whitehead in *Principia*, to the views of Brouwer in 'Mathematik, Wissenschaft und Sprache'[43] and to the views of Weyl in 'Philosophie der Mathematik und Naturwissenschaft'[44] and 'Die heutige

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[40] Waismann's notes, which are preserved in seven school exercise-books, were published in *Ludwig Wittgenstein and the Vienna Circle* (Oxford, Blackwell, 1979) edited by B.F. McGuinness.

[41] *Ludwig Wittgenstein and the Vienna Circle*, I.

[42] Cf. *Tractatus*, 6.031.

[43] *Monatshefte für Mathematik und Physik*, 36 (1929).

[44] *Handbuch der Philosophie* (1927) published by R. Oldenbourg. A revised version is published as part of *Philosophy of Mathematics and Natural Science* (Princeton, 1949).

Erkenntnislage in der Mathematik'[45]. Wittgenstein had begun to emphasize that the meaning of a proposition is its method of verification and, against both the logicists and the intuitionists here mentioned, that mathematical so-called propositions have sense only within a mathematical system.

Back in Cambridge on 10 January 1930 Wittgenstein continued his writing again in Volume III (MS **107**) returning to Volume IV (MS **108**) on 16 February. On 18 January Frank Ramsey died, aged only 26 years. On the next day Wittgenstein began his first official lecture as part of a twice weekly series of lectures and discussion classes[46]. These lectures[47], like most of those given by Wittgenstein in subsequent years, were advertised in the *Cambridge University Reporter* simply as 'Philosophy'. The first lecture begins:

> 'Philosophy is the attempt to be rid of a particular kind of puzzlement. This "philosophic" puzzlement is one of the intellect and not of instinct. Philosophic puzzles are irrelevant to every-day life. ...'[48]

Wittgenstein's lecturing style remained basically the same

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[45] Symposion I (1927).

[46] Notes taken by two of Wittgenstein's students J.E. King and H.D.P. Lee are published in *Wittgenstein's Lectures, Cambridge 1930-1932* (Oxford, Blackwell, 1980). Moore also took notes at Wittgenstein's lectures from 1930-1933 and they appear in a thematic arrangement in his 'Wittgenstein's Lectures in 1930-1933', *Mind*, 249-51.

[47] *Wittgenstein's lectures, Cambridge 1930-1932*, AI-AVIII; 'Wittgenstein's Lectures in 1930-33', *passim* (I).

[48] *Wittgenstein's Lectures, Cambridge 1930-1932*, AI.

throughout his career; he used no notes when giving his lectures, although he did sometimes make preparatory notes, and the topics are those of his current or recent writings. The remarks on mathematics made during his first lectures repeat some of those recorded in Waismann's notes. Both sources help to clarify how Wittgenstein's views emerged from those which are expressed in the *Tractatus*. Wittgenstein also went frequently to the Moral Sciences Club, where Moore was Chairman, and on 31st January gave a short lecture on 'Evidence for the Existence of Other Minds'.

In order to persuade the Council of Trinity College to renew Wittgenstein's research grant, a report was required by an expert on Wittgenstein's progress. Bertrand Russell, who was then busy teaching at Beacon Hill School in West Sussex, was asked by Moore to provide the report[49]. The mathematicians Hardy and Littlewood were appointed as examiners. In the middle of March Wittgenstein went to meet Russell and he tried to explain his new ideas. It was decided that Wittgenstein should produce a synopsis of his recent work on which Russell could base a report. This 'loathsome work'[50] occupied Wittgenstein during the Easter vacation; and as a result there was only one meeting at

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[49] Moore - Russell, 9.3.1930. See also Russell - Moore, 11.3.1930 and Moore - Russell, 13.3.1930. *Wittgenstein Sein Leben in Bildern und Texten*, 330.

[50] Wittgenstein - Moore, [3/4.1930]. *Letters to Russell, Keynes and Moore*, M.13.

Schlick's house.[51] Wittgenstein talked on this occasion mainly about the notion of a hypothesis. TS **208** a selection of remarks dictated from Wittgenstein's MSS **105-108(1)** and MS **108(2)** up to the last week of April 1930, was rearranged to produce *Philosophical Remarks*, TS **209**[52]. About one half of the work, sections X-XIX out of I-XXII in Rhees's division of the text, is directly on the foundations of mathematics. The volume was left with Bertrand Russell[53] who sent it on to the Council of Trinity College with a report in favour of the renewal of Wittgenstein's research grant. Summarizing Wittgenstein's work, Russell says:

> 'He uses the words "space" and "grammar" in peculiar senses, which are more or less connected with each other. He holds that if it is significant to say "This is red", it cannot be significant to say "This is loud". There is one "space" of colours and another "space" of sounds. These "spaces" are apparently given a priori in the Kantian sense, or at least not perhaps exactly that, but something not so very different. Mistakes of grammar result from confusing "spaces". Then he has a lot of stuff about infinity, which is always in danger of becoming what Brouwer has said, and has to be pulled up short whenever this danger becomes apparent. His theories are certainly important and certainly very original. Whether they are true, I do not know; I devoutly hope not, as they

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[51] *Ludwig Wittgenstein and the Vienna Circle*, II.

[52] TS **209** was created by pasting cuttings from a carbon copy of TS **208** into a black ledger book. It was published as *Philosophische Bemerkungen* (Oxford, Blackwell, 1964), and in an English translation as *Philosophical Remarks* (Oxford, Blackwell, 1975), both edited by Rhees. See the review by N. Malcolm in *Philosophical Review*, 76 (1967).

[53] Russell - Moore, 5.5.1930. *Wittgenstein: Sein Leben in Bildern und Texten*, 334.

> make mathematics and logic almost incredibly difficult.'[54]

It is not surprising that a hasty reading of Wittgenstein's remarks on infinity should lead Russell at this time to believe that Wittgenstein is close to saying what Brouwer has said, but this interpretation is not justified by the text.

The Preface to *Philosophical Remarks*, in which Spengler's influence is again evident, contains an extremely important statement explaining how Wittgenstein saw his work in relation to his times:

> 'This book is written for such men as are in sympathy with its spirit. This spirit is different from the one which informs the vast stream of European and American civilization in which all of us stand. That spirit expresses itself in an onwards movement, in building ever larger and more complicated structures; the other in striving after clarity and perspicuity in no matter what structure. The first tries to grasp the world by way of its periphery - in its variety; the second at its centre - in its essence. And so the first adds one construction to another, moving on and up, as it were, from one stage to the next, while the other remains where it is and what it tries to grasp is always the same.
>
> I would like to say "This book is written to the glory of God", but nowadays that would be chicanery, that is, it would not be rightly understood. It means the book is written in good will, and in so far as it is not so written, but out of vanity, etc., the author would wish to see it condemned. He cannot free it of these impurities further than he himself is free of them.'

Here, I believe, part of the sense in which Wittgenstein's philosophy has a 'religious point of view' is clarified. Wittgenstein's critique of mathematics is not a

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[54] Russell - Moore, 5.5.30. *Wittgenstein: The Duty of Genius*, p. 293.

contribution to the progressive science of mathematics, and certainly not to the solution of an important problem for 'the scientific conception of the world'. Wittgenstein aimed rather at achieving philosophical clarity in mathematics. Here, he believed, we are up against the limits of language, which in the *Tractatus* meant the boundary between the mundane and the mystical. In 1930 and afterwards, Wittgenstein's philosophical method, when applied in the right spirit, still had for him a deep, perhaps religious significance.[55] Despite the fact that the volume was provided with this preface, *Philosophical Remarks* is clearly a premature synthesis of Wittgenstein's thought; it is by no means a finished work.

Wittgenstein's Easter term lectures[56] contain a good deal on mathematics. They concern mathematical generality, mathematical discovery, generality in geometry and mathematical proof (rules of substitution in equations and induction).[57]

In Austria Waismann continued working on his exposition of Wittgenstein's ideas in the *Tractatus*. An early stage is represented by his 'Einführung zu Wittgenstein', which is a list of paraphrases of remarks in

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[55] See p. 35 below.

[56] *Wittgenstein's Lectures, Cambridge 1930-1932*, AIX-AXI and 'Wittgenstein's Lectures in 1930-33', *passim* (I).

[57] Cf. Waismann's notes from the Christmas vacation of 1929 and *Philosophical Remarks* (1964), XIII & XIV.

the *Tractatus* grouped under various headings. This is followed by a revised version entitled 'Theses', which seems to have been composed during the course of 1930. Several versions of 'Theses' are known to have circulated among Waismann's friends.[58] In March 1930 Waismann gave a lecture on Wittgenstein entitled 'Das Wesen der Logik', which is also preserved, and his forthcoming book *Logik, Sprache, Philosophie* was advertised again in the first issue of *Erkenntnis* (1930-31), the newly acquired journal of the Vienna Circle.

In the summer of 1930 Waismann was invited to give a lecture on Wittgenstein's philosophy of mathematics at the second conference on Theory of Knowledge in the Exact Sciences, which was to be held in Königsberg in September. Wittgenstein seems to have been enthusiastic about this idea, and at a meeting of 19 June he set out what Waismann ought to say at Königsberg.[59] Waismann's lecture, 'Über das Wesen der Mathematik: der Standpunkt Wittgensteins', was one of a series of four lectures which included Carnap on logicism, Heyting on intuitionism and von Neumann on formalism. This was also the conference at which Gödel announced the discovery of his first incompleteness theorem. Waismann's lecture was not published along with

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[58] The latest version preserved is printed as Appendix B to *Ludwig Wittgenstein and the Vienna Circle*.

[59] *Ludwig Wittgenstein and the Vienna Circle*, III.

the others in *Erkenntnis* 2[60], but its content can be reconstructed to some extent from the discussion, a record of which is published in that volume[61], and from a typescript of the first part of the lecture, which survives among Waismann's papers[62]. Waismann introduces the lecture as a sketch of ideas which are still developing. He outlines a method whose goal is a 'clarification of our understanding of mathematics', and whose foundations go back to the general logical considerations of the *Tractatus*. The method has two elements: first, 'in order to ascertain the meaning of a mathematical concept, one must pay attention to the *use* that is made of it; that is to say, one must pay attention to what the mathematician really does in his work'; second, 'in order to visualise the significance of a mathematical proposition one must make clear how it is *verified*'. From this latter proposition it follows that 'mathematical propositions and their proofs cannot be separated from one another'. The

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[60] English translations of the lectures published in this volume were published in *Philosophy of Mathematics: Selected Readings* (Oxford, Blackwell, 1964) edited by P. Benacerraf & H. Putnam.

[61] Hans Hahn, a prominent member of the Vienna Circle, speaks of a polemic by Wittgenstein and the intuitionists 'against the view that the world consists of individuals, properties of individuals, properties of these properties, and so on and the axioms of logic are statements about this world.'

[62] This typescript was published in Waismann's *Lectures on the Philosophy of Mathematics* (Amsterdam, Rodopi, 1982) edited by W. Grassl. An English translation, 'The Nature of Mathematics: Wittgenstein's Standpoint' appears in *Ludwig Wittgenstein: Critical Assessments*, Vol. III (London, Croom-Helm, 1986) edited by S.G. Shanker.

lecture originally had four parts: (1) The nature of numbers; (2) The idea of infinity; (3) The concept of set; (4) The principle of complete induction. None of the last three parts have survived.

At about this time and before the end of 1930, Waismann is known to have circulated some notes on mathematics as a transcript of Wittgenstein's views.[63] They consist, like 'Theses', largely of explanations of the 'highly syncopated'[64] remarks in the *Tractatus*, only this time of the remarks on logic and mathematics. Engelmann's extracts from these notes are inscribed 'Orally from L.W., taken down before 1930'. It is possible that they originate in conversations which took place in the Easter vacation of 1929, or earlier, in Vienna.

Having filled Volume IV, MS **108**, Wittgenstein continued his writing on 11 August in Volume V, MS **109**. The latter part of Volume IV, which was written after Wittgenstein had selected the remarks for TS **208**, is the source for TS **210**. Roughly the first half of **210** is almost entirely on mathematics.

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[63] Waismann's copy supplemented with extracts taken by Stein and Engelmann is reprinted in *Ludwig Wittgenstein and the Vienna Circle*, Appendix A.

[64] See Drury's 'Conversations with Wittgenstein' for 1949.

The Lectures for Michaelmas term[65] contain little directly on mathematics. Wittgenstein does, however, compare understanding a proposition with being able to follow a rule, which from now on is an important theme and one which he was to integrate closely with his criticism of various philosophical conceptions of mathematical truth. Wittgenstein was awarded a five year research fellowship at Trinity College on 5 December; and he was thus afforded a long and stable period in which to 'carry on his researches on the foundations of Mathematics'[66]. He returned to the rooms in Whewell's Court which he had occupied before the war.

Wittgenstein spent the Christmas vacation in Vienna with his family, and on 10 December he began Volume VI, 'Philosophische Bemerkungen' (MS **110**). In this volume on 10 February he writes:

> 'The limit of language is shown by its being impossible to describe the fact which corresponds to (is the translation of) a sentence, without simply repeating the sentence.
>
> (This has to do with the Kantian solution of the problem of philosophy.)'

To recognize a limit to what can be said, or described, and to regard philosophy as a tendency to transgress this limit is, I think, what Wittgenstein means here by 'the Kantian solution of the problem of philosophy'. Wittgenstein here,

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65 *Wittgenstein's Lectures, Cambridge 1930-1932*, BVIII-BXV.

66 Moore - Russell, 9.3.1930. *Wittgenstein: Sein Leben in Bildern und Texten*, 330.

in effect, acknowledges Kant as a main philosophical antecedent.

Several of the meetings with Schlick and Waismann that took place over the Christmas vacation of 1930-31 are taken up with a discussion of Hilbert's 'Neubegründung der Mathematik'.[67] Waismann's notes of these meetings are an important source for Wittgenstein's current views on Hilbert's formalism, especially for his views on the consistency problem. The discussion moves on from this subject to the subject of rule-following. An addendum to Waismann's notes contains an explanation by Wittgenstein of his criticism of Russell's definition of number in terms of equinumerousity, which he says he had already given to his Cambridge audience.

Back in Cambridge Wittgenstein continued his writing in Volume V (MS **109**) which was completed at the beginning of February. Wittgenstein's Lent Term lectures are mainly on the concept of a proposition, logic and grammar; he 'returned again and again'[68] to the explanation of the sense in which the rules of grammar are arbitrary.

Wittgenstein had no philosophical conversations over the Easter vacation, probably due to mental exhaustion. For the Easter Term, and from then on, his lectures were

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[67] *Ludwig Wittgenstein and the Vienna Circle*, IV.

[68] 'Wittgenstein's Lectures in 1930-33', II, p. 299.

held in his rooms in Whewell's Court, and because of the cramped conditions the notes taken by some of his students are less complete[69]. His writing is continued again in Volume VI, MS **110**, and from this time onwards some of his first draft notebooks are also retained. The first of these are the pocket notebooks **153a** ('Anmerkungen'), **153b**, **154** and **155**. MSS **154** and **155** are sources for two manuscript volumes written in the second half of 1931 (MSS **111** and **112**).

In March Waismann gave a lecture in Vienna entitled 'Logik, Sprache, Philosophie'. An account of the lecture appears in *Erkenntnis* 2 (1931) along with his article 'Logik und Sprache'.

During the summer holiday in Austria, which was spent mainly on the Hochreit, Wittgenstein continued the process of selecting remarks for the typescript stage from his latest manuscript volumes. He was writing manuscript material at the same time, and on 7 July, having completed Volume VI (MS **110**), he continued his writing in Volume VII (MS **111**), which was itself completed by 13 September[70]. He had only one meeting in Vienna, on 21 September, and this time it was with Waismann alone in the house in the

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69 *Wittgenstein's Lectures, 1930-1932*, p. xii.

70 Volume VII contains the first of Wittgenstein's remarks on Frazer's *Golden Bough*. See Drury's 'Conversations with Wittgenstein' for 1931.

Argentinierstrasse.[71] Wittgenstein showed Waismann the typescript material he had just produced, which was currently 90 pages in length, and he gave a verbal summary of his current position. He says that he is concerned centrally with what it is to understand the meaning of a proposition and is opposed to the ordinary view that meaning is a psychological process. A proposition is understood when it can be applied. Waismann raises a number of questions arising out of earlier conversations on the philosophy of mathematics. These concern: existence proofs, consistency, contradiction, equation and substition rule, and indirect proof. Wittgenstein spent some time in Norway in the Autumn, contemplating marriage with a young Swiss lady whom he had known for some years.

For the coming academic year, 1931-32, Wittgenstein was granted leave from his official lecturing responsibilities, which he had requested in order to concentrate on his own work. He did, however, continue to give unpaid 'discussion classes' for interested students once a week.[72] His writing in large manuscript volumes was continued on 5 October in Volume VIII, 'Bemerkungen zur philosophischen Grammatik' (MS **112**), which was completed by

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[71] *Ludwig Wittgenstein and the Vienna Circle*, V.

[72] Moore says ('Wittgenstein's lectures in 1930-33', I, p. 4) that in the Michaelmas term of 1931 and the Lent term of 1932 he ceased attending Wittgenstein's lectures, for a reason he cannot remember, but still went to the discussion classes. Moore had probably forgotten that there weren't any 'lectures' during those terms.

28 November. He then continued immediately in Volume IX, 'Philosophische Grammatik' (MS **113**). Near the beginning of Volume VIII, he writes:

> 'Perhaps what is inexpressible (what I find mysterious and am not able to express) is the background against which whatever I could express has its meaning.'

The expression of Wittgenstein's mysticism in the *Tractatus* is bound up with doctrines about what can and cannot be said. In the later period, doctrines about language are eschewed, and Wittgenstein's mysticism is no longer explicit in his philosophical writings. To understand the mysticism of the later writings, is to understand the background against which those writings have the depth of significance which their author clearly intended. This, I believe, is to understand their 'religious point of view'.

For part of Michaelmas term Wittgenstein explained his views to his audience by reference to lectures on 'The Elements of Philosophy' by his Cambridge contemporary C.D. Broad (1887-1971)[73]. Broad had distinguished a type of critical philosophy which, according to Lee[74], may be characterised as 'Kant's critical method without the peculiar applications Kant made of it'; Wittgenstein remarked:

> 'This is the right sort of approach. Hume, Descartes and others had tried to start with one proposition such as "Cogito ergo sum" and work from it

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[73] *Wittgenstein's Lectures, Cambridge 1930-1932*, CV-CVII.

[74] Ibid., CV, A.

> to others. Kant disagreed and started with what we know to be so and so, and went on to examine the validity of what we suppose we know.'

Wittgenstein's position in relation to Kant at this time is clarified further by other remarks made during the lectures, including these ones on idealism and realism:

> 'Idealists were right in that we never transcend experience. Mind and matter is a division *in* experience. Realists were right in protesting that chairs do exist. They get into trouble because they think that sense-data and physical objects are causally related.
>
> Idealists saw that a hypothesis was not something outside experience. Realists saw that a hypothesis was not merely a proposition about experience.'[75]

A similar pattern is to be found in many of Wittgenstein's philosophical positions; Wittgenstein's philosophy, like Kant's, is a middle way. Interpretations of Wittgenstein as a sceptical, idealist or realist thinker show a total lack of appreciation of the nature of his philosophy.

For the winter semester of 1931-32 Schlick was in California. In November Wittgenstein wrote to him expressing reservations about the form of the book being planned by Waismann: 'a lot of things', he says, 'will be presented quite differently from the way I think right'. He adds that there are 'many, many formulations' in the *Tractatus* with which he is no longer in agreement. Wittgenstein spent the Christmas holiday, as usual, in Vienna, and during this time he had only one discussion

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[75] Ibid., CV, D.

with Waismann.[76] Waismann's notes for 9 December under the heading 'On Dogmatism' contain an important expression of what is, perhaps, the single most important development in Wittgenstein's thought which separates the earlier and the later work. Wittgenstein says that at the time he wrote the *Tractatus* he had not yet understood clearly enough that in philosophy you cannot discover anything, that there can be no surprizes in philosophy. He is clear now that we needn't wait for any discovery in philosophy, we have already got everything.[77] With reference to Waismann's *Theses* he says:

> 'If there were theses in philosophy, they would have to be such that they do not give rise to disputes. ... As long as there is a possibility of having different opinions and disputing about a question, this indicates that things have not yet been expressed clearly enough. ... I once wrote, The only correct method of doing philosophy consists in not saying anything and leaving it to another person to make a claim.[78] That is the method I now adhere to.'

The method of philosophy is to 'tabulate grammatical rules'; one should 'not talk of sense and what sense is at all', but 'remain entirely within grammar'. 'The point is to draw essential, fundamental distinctions.'

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[76] *Ludwig Wittgenstein and the Vienna Circle*, VI.

[77] In a lecture not long afterwards (*Wittgenstein's Lectures, Cambridge 1930-1932*, B XII, 3) he put it by saying: 'In logic nothing is hidden'. It is regrettable that Norman Malcolm's study of Wittgenstein's criticism of his early thought, *Nothing is Hidden* (Oxford, Blackwell, 1986), like so many other studies by those friends and former pupils of Wittgenstein who were best placed to influence the reception of his views, leaves out discussion of his remarks on the foundations of mathematics.

[78] Cf. *Tractatus*, 6.53.

> 'In my book I still proceeded dogmatically. Such a procedure is legitimate only if it is a matter of capturing the features of the physiognomy, as it were, of what is only just discernible - and that is my excuse. I saw something from far away and in a very indefinite manner, and I wanted to elicit from it as much as possible. But a rehash of such theses is no longer justified.'

It was probably at this time that Waismann altered his original conception of *Logik, Sprache, Philosophie*; it was no longer to be merely an exposition of the *Tractatus*. During this meeting, Wittgenstein also clarified his views on certain mathematical subjects: infinity, Ramsey's definition of identity, and consistency. In Waismann's notes, in Notebook 5, there now appears the heading 'Insertion from dictation' which is followed by further sections on mathematics. They are on consistency, mathematical discovery, generality in geometry and indirect proof. They might have come from something which is known to have been dictated by Wittgenstein at Christmas and which was to be sent to Schlick. One dictation for Schlick from around this time which has survived is the so-called 'Diktat für Schlick' (TS **302**). After these 'Insertions from dictation' there follow in the remainder of Notebook 5 and the beginning of Notebook 6 extracts from a manuscript or typescript of Wittgenstein's coinciding partly with Volume IV (MS **108**) and partly with *Philosophical Remarks* (TS **209**).[79]

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[79] Waismann's Notebooks 6 and 7 have the sub-title '(Math.)'. Besides the record of one more meeting, the rest of Notebook 6 and the whole of Notebook 7 consists of extracts from the sections on mathematics in *Philosophical Grammar* (TS **213**).

Wittgenstein's Lent Term lectures[80] were on grammar and grammatical rules. There were no meetings in Vienna during the Easter vacation.

Wittgenstein's lectures in the Easter term[81] were on the philosophy of mathematics, except for one[82] in which Wittgenstein responded to a brief paper read out in the class by Moore, criticizing his use of the word 'grammar'. In these lectures Wittgenstein talks about the variety of mathematical calculi, the nature of unsolved mathematical problems, following a rule, looking for something in mathematics, and infinity. Wittgenstein's writing in Volume IX (MS **113**) was completed by 23 May and he then continued his writing in Volume X, 'Philosophical Grammar'(MS **114**) on 27 May and until 5 June 1932.

During the summer of 1932 Wittgenstein was involved in a disagreement over priority with Rudolf Carnap. Carnap's article 'Die Physikalische Sprache als Universalsprache der Wissenschaft', which had been published in *Erkenntnis* 2 (1931), contains no acknowledgment of ideas which Wittgenstein believed Carnap had received from him through

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[80] *Wittgenstein's Lectures, 1930-1932*, CVIII-CXIV. 'Wittgenstein's Lectures in 1930-33', *passim* (III).

[81] *Wittgenstein's Lectures, Cambridge 1930-1932*, CXVI-CXXII. 'Wittgenstein's Lectures in 1930-33', *passim* (III).

[82] *Wittgenstein's Lectures, Cambridge 1930-1932*, CXVI.

the meetings in Vienna.[83] As a result of this dispute, Wittgenstein no longer allowed his views to be communicated to the Vienna Circle in the previous manner. The last recorded conversation on 1 July 1932, in Notebook 6, is an attempt by Wittgenstein to refute Carnap's claim that the notion of hypothesis described in his article comes from Poincaré.[84]

Wittgenstein continued to meet frequently with Waismann to discuss the work on *Logik, Sprache, Philosophie*. His meetings with Schlick over the coming years, which included a holiday in Italy in the summer of 1933, resulted in a number of dictations in Schlick's hand (**303-307**)[85]. **305**, a single page, is on mathematical conjecture.

Wittgenstein's own preparation for the publication of his views now reached a new stage. He first completed the typescript which he had begun during the previous summer by adding to it a selection of remarks from his latest manuscript volumes. The resulting typescript, **211**, is a selection of remarks from Volumes V-X(1)(MSS **109-114(1)**).[86]

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[83] Wittgenstein - Carnap, 8.8.1932. *Wittgenstein: Sein Leben in Bildern und Texten*, 359.

[84] *Ludwig Wittgenstein and the Vienna Circle*, VII.

[85] A typescript version of 'Grosses Format' (MS **140**) was not included in the catalogue.

[86] The remarks on Frazer's Golden Bough selected for this typescript from Volume VII (MS **111**) were published as Part I of *Remarks on Frazer's* Golden Bough (Brynmill Press,

Typescripts **208**, **210** and **211** together thus form a complete selection from the manuscript volumes which Wittgenstein had been writing since returning to Cambridge in 1929. Wittgenstein then proceeded to make an arrangement of this material by cutting the typescripts into slips and clipping the slips together into bundles to form chapters. There are also numerous additions and deletions in handwriting. These bundles of cuttings, which were grouped together in various folders, make up typescript **212**.

In this year Wittgenstein also began writing in the pocket notebooks **156a** and **156b**, which contain revisions of the beginning of **212**. On page 118 of **156a** there are some mathematical formulae, and the writing from here to the end in **156a** and in the first few pages of **156b** is on mathematics. Also, on page 43 of **156b** we find the remark:

> '"What is mathematics?" - Well, that which is written in books on mathematics'. But what then is its relation to all these calculations? Now that is very difficult to describe and not very interesting at all.'

This is just one example of the many interesting remarks on the foundations of mathematics hidden within the huge mass of Wittgenstein's writings.

For the academic year 1932-1933 Wittgenstein gave two sets of weekly lectures, 'Philosophy' and 'Philosophy for Mathematicians'. Notes from both sets of lectures have

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1979) edited by R. Rhees. Part II consists of notes made by Wittgenstein after he was given a copy of Frazer's book by one of his pupils in 1936 (MS **146**).

been preserved[87]. The former series, which includes some lectures on religion, aesthetics and Freudian psychoanalysis, also includes lectures on mathematics.[88] The latter series[89] is devoted mainly to a critical examination of Hardy's *Pure Mathematics*, which was the standard university text at that time[90]. Reference is also made to Hardy's views as they are expressed in his article 'Mathematical Proof', *Mind*, 1929. The lectures begin:

> 'Is there a substratum on which mathematics rests? Is logic the foundation of mathematics? In my view mathematical logic is simply part of mathematics. Russell's calculus is not fundamental; it is just another calculus. ...'

Wittgenstein goes on to criticize Russell's definition of number, and after that to discuss real numbers and periodicity, and Hardy's definition of real number in *Pure Mathematics*. He also talks about mathematical proof, criticizing Hardy's statement that he believes Goldbach's conjecture to be true, and ends with a discussion of Russell's Theory of Types, to which Hardy had stated an objection.

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[87] 'Wittgenstein's Lectures in 1930-33', *passim* (III). *Wittgenstein's Lectures, Cambridge 1932-1935*, edited by Alice Ambrose, is based on the notes of Ambrose and Margaret MacDonald.

[88] *Wittgenstein's Lectures, Cambridge 1932-1935*, Part I, 7-10. 'Wittgenstein's Lectures in 1930-1933', C (III).

[89] *Wittgenstein's Lectures, 1932-1935*, Part IV. Moore did not attend these lectures.

[90] Opposite the title page of the original English edition of the *Tractatus* there is a list of 'Volumes arranged' in the same series, which include's a book by Hardy: *Mathematics for Philosophers*.

Among those attending the lectures was a promising undergraduate mathematician named Francis Skinner, who was then in his third year at Trinity. Skinner became devoted to Wittgenstein and spent the following three years, during which time he held a graduate scholarship at Trinity, working with Wittgenstein on the preparation of his work for publication.

In March 1933 Wittgenstein responded to an article by Richard Braithwaite[91], which he believed had misrepresented his views, by writing an open letter to *Mind*[92]. Wittgenstein says here that the publication of his work is being retarded by 'the difficulty of presenting it in a clear and coherent form'.

During the summer vacation, which he spent on the Hochreit, Wittgenstein laboured up to seven hours a day dictating to a typist on the basis of typescript **212**. The result of this work is the most important statement of Wittgenstein's philosophy in his middle period, *Philosophical Grammar* (TS **213**)[93], and the typescripts **214-218**. Sections 108-140 (pp. 529ff) of typescript **213** are on the philosophy of mathematics. The main headings are:

The Foundations of Mathematics (108-114)
On Cardinal Numbers (115-118)

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[91] 'Philosophy', in *University Studies* (London, Ivor Nicholson and Watson, 1933) edited by H. von Wright.

[92] *Mind*, 42 (1933), pp. 415-16.

[93] The so-called 'Big Typescript'.

Mathematical Proof (119-125)
Inductive Proofs and Periodicity (126-135)
Infinity in Mathematics (136-140)

Typescript **219**, which contains a fair amount on mathematics, was also produced at about this time.

Philosophical Grammar, although it has the appearance of a finished work, is really no more than a stage, albeit a significant one, in the ordering of Wittgenstein's philosophical remarks. In Volume VI (MS **110**), in June, Wittgenstein wrote: '(My book might be called: Philosophical Grammar. This title would no doubt have the smell of a textbook title but that doesn't matter, for behind it there is the book.)'. *Philosophical Grammar* has none of the literary qualities which Wittgenstein achieved in the *Tractatus* and which he wanted now still to achieve; it is far from Wittgenstein's ideal of philosophy written as poetry. He started immediately on the job of revising the first half in the pocket notebooks **156a** and **156b** and with additions and alterations to the typescript itself.

Wittgenstein began Michaelmas term, as he had in the previous year, by giving two sets of lectures, 'Philosophy'[94] and 'Philosophy for Mathematicians'[95]. After three or four weeks, however, he decided that, rather than lecturing to a large audience, he would prefer to

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[94] *Wittgenstein's Lectures, Cambridge 1932-1935*, Part II, 'Lectures preceding dictation of *The Blue Book*'.

[95] *Wittgenstein's Lectures, Cambridge 1932-1935*, Part IV.

dictate his thoughts to a small group of selected students, allowing these dictations then to be copied and handed out to others. The select group of students included: Francis Skinner, Louis Goodstein, H.S.M. Coxeter, Alice Ambrose and Margaret Masterman.[96] The dictation resulted in what became known as the *Blue Book*, because of the colour of its paper covers.[97] Notes of Wittgenstein's 'lectures and informal discussions in the intervals between dictation of the *Blue Book*' are also preserved, in the so-called *Yellow Book* (MS **311**).[98] These, like the *Blue Book* itself, contain some material on mathematics.

Wittgenstein's writing, meanwhile, was continued in a new series of manuscript volumes, large notebooks, which have become known as 'C1', 'C2', etc.[99] In C1, C2 and the first part of C3 (MSS **145**, **146** and **147(1)**), Wittgenstein continues the process of revising the text of the first part of *Philosophical Grammar* (TS **213**). The second part of C3 (**147(2)**) contains notes for Wittgenstein's lectures.

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[96] R.L. Goodstein was later Professor of Mathematical Logic at Leicester University and in some of his work he claims an indebtedness to Wittgenstein. H.S.M. Coxeter also became a professional mathematician.

[97] Wittgenstein later gave copies to Schlick and Russell (Wittgenstein - Russell, [1935-1936], *Letters to Russell, Keynes and Moore*, R. 56).

It was published as the first part of *The Blue and Brown Books* (Oxford, Blackwell, 1958).

[98] Selections by Ambrose from these notes are included in *Wittgenstein's Lectures, Cambridge 1932-1935*, Part II.

[99] The notation is due to Rhees.

Wittgenstein's first, although uncompleted, attempt at the presentation of his work for publication was carried out in Volume X(2) (MS **114(2)**) and Volume XI(1) (MS **115(1)**) under the heading 'Revision'. Wittgenstein here draws on the first draft revisions of *Philosophical Grammar* in the pocket notebooks **156a** and **156b**, and in the large notebooks C1-C3 (**145-147**). In 'Grosses Format' (MS **140**) under the heading 'Second Revision' there is a revision of an early part of the 'Revision'.[100]

In 1934, if not before, Wittgenstein began thinking seriously about emigrating to the Soviet Union. Once his fellowship and Skinner's scholarship had ended in 1936, Wittgenstein planned that they should live together in the Soviet Union and take on work as manual labourers. The two men took Russian lessons together in preparation.[101]

By the Easter vacation of 1934 the form of Wittgenstein's co-operation with Waismann on *Logik, Spache, Philosophie* had altered; they had agreed to be the book's

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[100] A typescript version, dictated by Wittgenstein, was found among Waismann's papers under the title 'Wi MS'.

Rhees's interpretation of the revisions in MSS **114(2)**, **115(1)** and **140** of the first part of *Philosophical Grammar* (TS **213**) was published as Part I of *Philosophical Grammar* (Blackwell, Oxford, 1969). Part II of Rhees's *Philosophical Grammar* places consecutively the relatively unrevised sections from TS **213** on logical inference, generality and the foundations of mathematics. See the review by G.P. Baker and P.M.S. Hacker in *Mind*, 85 (1976).

[101] Their Russian teacher Fania Pascal wrote an interesting memoir of Wittgenstein, which is reprinted in *Recollections of Wittgenstein*.

joint authors. Wittgenstein would sketch its form and continue to provide manuscripts and typescripts for Waismann, whose task would be to write everything out in a coherent form. Waismann was, however, to find collaborating with Wittgenstein in this way extremely difficult.

Wittgenstein began pocket notebook **157a** on 4 June. In September he visited Drury in Ireland with Skinner.

For the academic year 1934-35 Wittgenstein held only one lecture course, entitled 'Philosophy'[102], but he also spent four days a week dictating to Skinner and Ambrose. The dictations were not this time meant as a substitute for lectures, but were part of a second major attempt by Wittgenstein to give his work a 'clear and coherent form'. There existed originally only three typewritten copies of these dictations, but others began to circulate, against Wittgenstein's will, and they became known as the *Brown Book* (TS **310**).[103] Volume C6 (MS **150**), which is continued immediately in Volume C7 (MS **151**), consists mainly of notes for Part II of the *Brown Book*. (MS **141**, eight loose sheets, is a version of the beginning of the *Brown Book* sketched in German.)

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[102] *Wittgenstein's Lectures, Cambridge 1932-1935*, Part III.

[103] It was published as the second part of *The Blue and Brown Books* (Oxford, Blackwell, 1958).

Wittgenstein's lectures for the academic year 1934-35 have been preserved in detailed notes taken by Ambrose and Masterman[104]. The Easter term lectures are almost entirely on logic and mathematics. Wittgenstein's manuscript volume C4 (MS **148**), which is continued immediately in C5 (MS **149**), consists mainly of notes made in preparation for the lectures in this academic year, and they include a special section on mathematics written in preparation for the Easter term lectures (MS **148(2)**[105]). Both sources are of value for understanding the transition from the remarks on mathematics in *Philosophical Grammar* to the later remarks on mathematics. In July Wittgenstein sent Schlick a long letter about Gödel's theorem.[106] He recommends that rather than allowing oneself to be astonished at the prose formulation of this, or any other mathematical result, one should instead go through the proof carefully to see what it actually proves.

In September Wittgenstein travelled to the Soviet Union to see for himself whether it would be possible for him to live there. He began in Leningrad, where he met the

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[104] *Wittgenstein's Lectures, Cambridge 1932-1935*, Part III.

[105] After page 10 of MS **148** there are various geometrical diagrams, followed by pages numbered independently from 1 to 47. These pages, which I have called MS **148(2)**, contain the notes on mathematics for the Easter Term lectures. They were used by Ambrose in *Wittgenstein's Lectures, Cambridge 1932-1935* to supplement the notes taken at the lectures.

[106] Wittgenstein - Schlick, 31.7.1935. *Wittgenstein: Sein Leben in Bildern und Texten*, 369.

professor of philosophy at the University, Tatiana Gornstein, and in Moscow he had discussions with the professor of mathematical logic, Sophia Janovskaya. With Janovskaja he later carried on a correspondence. As a result of this visit, Wittgenstein was offered a chair in philosophy at Kazan University, where Tolstoy had once studied, and later a teaching post at the University of Moscow. He seems, however, to have been convinced that he would find it very difficult to live in the Soviet Union.

In Michaelmas Term began the final year of Wittgenstein's fellowship. During this year, he held a weekly seminar on the 'Philosophy of Psychology', which concentrated on the topics of private experience and sense data, but also on the foundations of mathematics. Volumes C5 (MS **149**), C7 (MS **151**) and the six loose sheets headed 'Privacy of Sense Data' (MS **181**) consist mainly of notes for these lectures.[107] Volume C7 includes draft material

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[107] 'Wittgenstein's Notes for Lectures on "Private Experience" and "Sense Data"' (*Philosophical Review*, 77 (1968)) is a selection by Rhees from Wittgenstein's notes for these lectures, which draws from MSS **149**, **151**, and the earlier MS **148**. The selection excludes Wittgenstein's remarks on mathematics. 'The Language of Sense Data and Private Experience' (*Philosophical Investigations*, 7 (1984)) contains Rhees's notes of Wittgenstein's lectures 'from the middle of February, 1936, to the end of the session in June'. In the earlier article, p. 272, Rhees says that at the end of January and the beginning of February 1936, Wittgenstein gave four or five lectures on the foundations of mathematics; so Rhees's published notes also exclude the majority of Wittgenstein's remarks on mathematics for this year.

for a lecture on mathematics.[108] 'Notes for the "Philosophical Lecture"' (MS 166), which is on the same topics in the philosophy of psychology, probably also dates from this academic year.

Wittgenstein was determined not to continue as an academic at Cambridge after his fellowship had expired, and he seriously considered training to become a psychiatrist. He was particularly interested in Freud, whose book *The Interpretation of Dreams* he had read, and he believed that he might have an aptitude for this work. As a consequence, he went with Skinner in June to visit Drury, who was in Dublin studying medicine.

It was here that Wittgenstein heard of the assasination of Moritz Schlick. Schlick's death was for Wittgenstein a great personal loss, and partly because Schlick had been the main motivating force behind the work on *Logik, Sprache, Philosophie*, collaboration on this book soon came to an end. Waismann subsequently decided to finish the book himself; although, due to various circumstances, it was not published in his lifetime.[109]

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[108] At the bottom of page 15 there is a note indicating that the writing is continued on page 24. The intervening pages contain notes for a lecture on the philosophy of mathematics.

[109] It was first published in an English translation in 1965 under the title *Principles of Linguistic Philosophy*. An earlier German version was published in 1976.

Waismann's *Einführung in das Mathematische Denken*[110], which incorporates some of Wittgenstein's ideas[111], was published in Vienna in 1936.

Still, it seems, uncertain about what to do, Wittgenstein decided in 1936, as he had done in 1913, to retire to his hut in Norway and work on completing his book.

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110 Translated into English as 'Introduction to Mathematical Thinking' (London, Hafner, 1951).

111 Waismann gives a detailed description of his indebtedness to Wittgenstein in the Epilogue.

1.2 The Later Period: 1936-1951[1]

In August 1936, after a brief holiday in France, Wittgenstein withdrew to his hut in Skjolden, Norway. From here, a short while later, he wrote to Moore:

> 'I do believe that it was the right thing for me to come here thank God. I can't imagine that I could have worked anywhere as I do here. It's the quiet and, perhaps, the *wonderful* scenery; I mean, its quiet seriousness'[2].

By the end of August Wittgenstein had begun in Volume XI(2) (MS **115(2)**) a revision, in German, of the *Brown Book*, which he entitled 'Philosophische Untersuchungen, Versuch einer Umarbeitung'[3]. He soon became dissatisfied with this work[4], however, and after about two months he concluded his writing with the remark: 'Dieser ganze "Versuch einer Umarbeitung" vom Seite 118 bis hierher ist *nichts wert*'. Wittgenstein then began a new revision on 5 November[5], drawing on the work in Volume XI(2) (MS **115(2)**) and on new

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[1] Here I make considerable use of von Wright's essay 'The Origin and Composition of the *Philosophical Investigations*', in his *Wittgenstein* (Oxford, Blackwell, 1982); although my account differs from his one on certain points.

[2] Wittgenstein - Moore, [10.1936]. *Letters to Russell, Keynes and Moore*, M.29.

[3] Wittgenstein's revisions together with an independent translation of the rest of the *Brown Book* were published as *Eine philosophische Betrachtung* in L. Wittgenstein *Schriften 5* (Suhrkamp, Frankfurt, 1970).

[4] Wittgenstein - Moore, 20.11.[1936]. *Letters to Russell, Keynes and Moore*, M.31.

[5] Wittgenstein wrote 'Neue Umarbeitung begonnen' in his pocket calendar on this date.

work in the large notebook C8 (MS **152**). He called this new revision, manuscript **142**, *'Philosophische Untersuchungen'*.

On 8 December Wittgenstein left Skjolden for Vienna, where he spent Christmas, and at the beginning of the new year he also visited Cambridge. While alone in Norway Wittgenstein had been thinking seriously about both logic and his sins, and, as a test of his own personal courage and integrity, which he regarded as essential to his work, he now 'confessed' these sins to his friends in Austria and England.

Wittgenstein returned to Skjolden at the end of January. A pocket notebook begun in 1934, **157a**, was continued on 9 February with work relating to manuscript **142**. This work was then continued immediately on 27 February in pocket notebook **157b**. His work having gone badly during this period[6], Wittgenstein left at least as early as May to spend the summer in Vienna and then Cambridge.

In Cambridge Wittgenstein dictated typescript **220** on the basis of manuscript **142** to Francis Skinner. This work, with successive minor alterations, was to form the beginning of all subsequent versions of *Philosophical Investigations*.

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[6] Wittgenstein - Moore, 4.3.[1937]. *Letters to Russell, Keynes and Moore*, M.34.

An important new feature of typescript **220**, compared to the dictations in the *Brown Book*, is the inclusion of remarks, mainly derived from *Philosophical Grammar* (TS **213**), which explain Wittgenstein's views on the nature of philosophy and which describe his philosophical method. For example, Wittgenstein now writes:

> 'A main source of our failure to understand is that we do not *command a clear view* of the use of our words. - Our grammar is lacking in this kind of perspicuity. A perspicuous representation produces just this sort of perspicuity. A perspicuous representation produces just that understanding which consists in 'seeing connexions'. Hence the importance of finding *intermediate cases*.
>
> The concept of a perspicuous representation is of fundamental significance for us. It earmarks the form of account we give, the way we look at things. (A kind of "Weltanschauung" that seems to be typical of our time. Spengler.)
>
> Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.
>
> For it cannot give it any foundation either.
>
> It leaves everything as it is.
>
> It also leaves mathematics as it is, and no mathematical discovery can advance it.
>
> A "leading problem in mathematical logic" (Ramsey) is a problem of mathematics like any other.'[7]

Wittgenstein also expressed this conception, as he had done earlier to Waismann[8], by saying that there are no theses in philosophy.[9] This central doctrine is, perhaps, the most obvious indication that Wittgenstein's philosophy of

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[7] The quotation here is of remarks 100-101 (102-103) from typescript **220**, which is based directly on the lost manuscript **142**.

[8] See 'On Dogmatism' in *Ludwig Wittgenstein and the Vienna Circle, VI*.

[9] Typescript **220**, 104d.

mathematics is not a form of intuitionism, in either Brouwer's or Weyl's sense.

Wittgenstein now wanted to continue typescript **220** with remarks on the foundations of mathematics. The first new remarks on this subject are contained in pocket notebook **157b(2)**, and it is here that Wittgenstein first writes:

> 'The mathematician is not a discoverer, but an inventor.'

This remark encapsulates much of what Wittgenstein wants to say in response to the mathematician who is inclined to stress 'the objectivity and reality of mathematical facts'[10]; though it cannot, of course, be understood in isolation from Wittgenstein's detailed investigations of this tendency.

Wittgenstein returned to Norway in the summer, arriving in Skjolden on 16 August. His main work on the continuation of typescript **220** was begun in the new Volume XIV (MS **118**) on 18 August[11]. There are also many coded diary entries in this volume and in Wittgenstein's other writing from this period, and these show him to be suffering from bouts of extreme anxiety and depression, which affected his work. On 11 September Wittgenstein began transferring the remarks on mathematics which he had

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[10] *Philosophical Investigations* (1953), Part I, § 254.

[11] The first few remarks in Volume XIV, which begins in code on 13 August, were written during the journey.

written in Volume XIV (MS **118**) into the new Volume XIII (MS **117**).[12] His new writing was continued in Volume XIV (MS **118**), but he was now struggling to make progress; his writing was, he said, 'too uneasy, much too constrained':

> 'If I must write in this way, is it then better not to write a book, but to restrict myself here *tant bien que mal* to writing remarks, which might be published after my death?
>
> The remarks which I write enable me to teach philosophy well, but not to write a book.' (12.9.37)

A few days later, he explains:

> 'If I am thinking about a topic just for myself and not with a view to writing a book, I jump about all round it; that is the only way of thinking that comes naturally to me. Forcing my thoughts into an ordered sequence is a torment for me. Is it even worth attempting now?' (15.9.37)

Despite these doubts, Wittgenstein did continue with the attempt to write a book, though not one with a *single* ordered sequence of remarks. The writing on mathematics in Volume XIV (MS **118**), which ends with a series of remarks on Gödel, was continued in the new Volume XV (MS **119**) on 24 September.

Wittgenstein put his work on mathematics aside on 12 October, when he continued his writing in Volume XV (MS **119**) with a new set of remarks under the title 'Zu Ursache und Wirkung, intuitivem Erfassen'[13]. Wittgenstein concluded this work on 22 October and on the next day he

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[12] See Appendix I, A.

[13] Parts of this work anticipate Wittgenstein's later writings on epistemology. It was published as "Ursache und Wirkung: Intuitives Erfassen", and in translation as "Cause and Effect: Intuitive Awareness", in *Philosophia*, 3-4 (1976).

began work in Volume XII(1) (MS **116(1)**)[14] selecting remarks from *Philosophical Grammar* (TS **213**), or 'separating the wheat from the chaff'[15], as he put it. Remarks from the writing in Volume XV (MS **119**), which was continued simultaneously, are included among these selections from his 'old typescript'. Similarly, Volume XII(2) (MS **116(2)**) draws on remarks in Volume XVI (MS **120**), where Wittgenstein's writing was continued on 19 November.

Wittgenstein spent the Christmas of 1937 in Vienna, where he stayed until the middle of January before returning to Cambridge. In February 1938 he deposited a number of manuscripts and typescripts with Trinity College Library, and gave to the college the publication rights in the event of his death. He then travelled to Dublin to visit Drury, whose medical training was being completed with a period of residence at the City of Dublin Hospital. Wittgenstein seems still to have been tempted by the thought of becoming a psychiatrist and with Drury's help he managed to visit some psychiatric patients at St Patrick's Hospital. He was now also writing in pocket notebook **158**[16]. On p. 68 he remarks:

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[14] The dating of Volume XII(1) (MS **116(1)**) was a problem raised by von Wright in his essay on 'The Origin and Composition of *Philosophical Investigations*'. The solution was later provided by Stephen Hilmy in his book *The Later Wittgenstein* (Oxford, Blackwell, 1987).

[15] Volume XV (MS **119**), 23.10.1937.

[16] See Volume XVI (MS **120**), 11.3.1938.

> 'I am not teaching you anything; I'm trying to persuade you to *do* something.'
>
> 'What we do is much more akin to psychoanalysis than you might be aware of.'

This work was later continued in pocket notebook **159(1)**, which contains a fair amount on logic and mathematics, particularly on Cantor. In March 1938 came the news of the German annexation of Austria. Wittgenstein returned to Cambridge later in that month, and in order to avoid becoming a German Jew under Nazi law, he decided to apply for British citizenship. To help ensure the success of his application, Wittgenstein now sought a lecturing post at the University.

In Cambridge Wittgenstein now lived with Francis Skinner. His writing in Volume XVI (MS **120**) was continued there on 26 April in Volume XVII(1) (MS **121(1)**), which after p. 53 is on mathematics. Work on the foundations of mathematics was also resumed in Volume XIII (MS **117**) under the heading 'Ansätze' (MS **117(2)**). Wittgenstein's literary executors remark that: 'It must have been Wittgenstein's intention...to attach appendices on Cantor's theory of infinity and Russell's logic to the contributions on problems of the foundations of mathematics that he planned to include in the "Philosophical Investigations". Under the heading "Additions" he wrote a certain amount on the problems connected with set theory: about the diagonal procedure and the different kinds of number concept.'[17]

In the summer of 1938 Wittgenstein began lecturing to a select group of students, which included Rush Rhees, Yorick Smythies and James Taylor, who all became close friends of Wittgenstein, and also Casimir Lewy and Theodore Redpath. Two sets of classes, held in Taylor's rooms, were concerned respectively with aesthetics and religious belief[18]. Remarks made by Wittgenstein during these lectures which establish the connection between his thought on aesthetics and theology and his thought on the foundations of mathematics are of particular interest.[19] According to Wittgenstein, both aesthetic judgement and religious belief are misunderstood when compared respectively to scientific modes of description and belief; and he saw the tendency to misunderstand aesthetics and theology in this way as part of a prevalent 'style of thinking'[20]. In describing Jeans' popular scientific book *The Mysterious Universe* he talks about 'a kind of idol

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[17] From the editors preface to *Remarks on the Foundations of Mathematics*. *Remarks on the Foundations of Mathematics*, II, reproduces MS **117(2)** and a selection of the remarks from MS **121(1)**.

[18] A compilation of notes taken by Smythies, Rhees and Taylor at both of these sets of classes are published in the first and third parts of *Lectures and Conversations on Aesthetics, Psychology and Religious Belief*, edited by C. Barrett (Oxford, Blackwell, 1966).

[19] Ibid., 'Lectures on Aesthetics', III, sections 35-41.

[20] Ibid., sections 37 & 41.

worship, the idol being Science and the Scientist'[21]. Wittgenstein identifies this phenomenon also in the work of Cantor and he mentions here the 'charm' exercised by Cantor's famous proof:

> 'If I describe the surroundings of the proof, then you may see that the thing could have been expressed in an entirely different way; and then you see that the similarity of $\aleph_0$ and a cardinal number is very small. The matter can be put in a way which loses the charm it has for many people.'[22]

This aspect of the motivation for Wittgenstein's critique of the foundations of mathematics cannot be ignored, if his work is to be properly understood. It is not simply that Wittgenstein saw contemporary thought in the philosophy of mathematics as a particularly good subject area for the application of his philosophical method. The kind of interest that was being taken in the subject represented for him one aspect of a general state of moral and religious decline in Western civilization.

In August Rush Rhees met every day for three weeks with Wittgenstein in order to discuss the topic of continuity[23]; and at around this time, Wittgenstein and Smythies together produced a translation into modern, idiomatic English of part of Tagore's English version of his religious play *The King of the Dark Chamber*.

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[21] Ibid., section 36.

[22] Ibid., section 39, footnote.

[23] 'On Continuity: Wittgenstein's Ideas, 1938', in *Discussions of Wittgenstein* (London, Routledge & Kegan Paul, 1970), was written from Rhees's notes of these conversations.

Most importantly, it was in the summer of 1938 that Wittgenstein dictated typescript **221** on the basis of the remarks on the foundations of mathematics in Volumes XIII-XV (MSS **117-119**), which he had written during the previous Autumn in Norway.[24] This typescript is a continuation of typescript **220** and together they form a book with 396 consecutively numbered remarks. The preface, typescript **225**, is dated 'Cambridge, August 1938'. (The first draft of this preface is contained in pocket notebook **159(2)** and later drafts are contained in Volume XIII(3) (MS **117(3)**).) Wittgenstein offered the book to Cambridge University Press on 30 September, and they agreed to publish the German original along with a parallel English translation. However, in October the press was informed that Wittgenstein was 'uncertain' about the publication of his book but was 'making arrangements with a translator'. The translator was Rush Rhees, who had been recommended for the job by Moore. Rhees worked throughout Michaelmas term on the translation, meeting with Wittgenstein regularly to discuss any difficulties that arose. Wittgenstein, meanwhile, continued the process of selecting remarks from *Philosophical Grammar*, which he had begun in Volume XII(1) and continued in Volume XII(2) (MSS **116(1 & 2)**, in Volume XIII(4) (MS **117(4)**).

In January 1939 Wittgenstein extended typescript **221** with writing on mathematics from two sources: firstly, from

[24] See Appendix I, A.

part of the earlier manuscript **115**; and, secondly, from writing which he had begun in Volume XVII(2) (MS **121(2)**) on 25 December and had then continued in pocket notebook **162a**. The resulting, final version of typescript **221**, which I shall sometimes refer to as 'the mathematical typescript', has the following structure[25]:

MS Source	Remarks	Pages
117	162-297	1-69
118, 119	298-395[26]	70-119
115	397-421	120-130
121, 162a	422-442	131-135

This early version of the *Philosophical Investigations* was not published in 1939 or later, and this would seem to be because the book in this form was not yet genuinely ready for publication. Wittgenstein had decided to apply for the post of Professor of Philosophy at Cambridge, which had become vacant on Moore's resignation, and he wanted to submit his book as part of the application. For this reason, it seems, the book was completed in haste, and includes none of the additional material contained in Volume XII(1 & 2) (MS **116(1 & 2)**) or Volume XIII(2 & 4) (MS **117(2 & 4)**). Wittgenstein was also not satisfied with the

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[25] See also Appendix I, A.

[26] Remark 396 has no manuscript source.

translation of the first half of the 'first volume'[27] of his book, which Rhees had left with him in January.[28]

Wittgenstein was elected as Professor of Philosophy at Cambridge on 11 February, and he became a British citizen on 14 April. His writing on mathematics was continued in MS **162a** and then immediately in MS **162b** until 17 April.

In the Lent and Easter terms of 1939 Wittgenstein gave a series of twice weekly lectures on the foundations of mathematics[29]. He attracted a large audience, which included: Rush Rhees, Yorick Smythies and Casimir Lewy; G.H. von Wright[30] and Norman Malcolm[31], who were to become his close friends; the English mathematician Alan Turing (1912-1954); Alistair Watson, a student of mathematics; and

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27 Wittgenstein - Keynes, 1.2.39. *Letters to Russell, Keynes and Moore*, K.28.

28 Rhees's translation, with corrections by Wittgenstein, survives as typescript **226**. See Wittgenstein - Moore, 2.2.39. *Letters to Russell, Keynes and Moore*, M.40.

29 These lectures are described in *Ludwig Wittgenstein: A Memoir*, by N. Malcolm and also briefly in 'Ludwig Wittgenstein', *Australasian Journal of Philosophy*, 29 (1951), by D.A.T. Gasking and A.C. Jackson.

30 Georg Henrik von Wright was a graduate of the University of Helsinki, where he had been a pupil of Eino Kaila (1890-1958).

31 Malcolm, like Ambrose, had been a pupil of Oets Bouwsma in Nebraska. He had come to Cambridge in Michaelmas 1938 to study with Moore.

also R.G. Bosanquet, J.N. Findlay, D.A.T. Gasking[32], Marya Lutman-Kokoszynska and John Wisdom. Notebook **161(1)** consists of remarks in English made in connection with these lectures. Lecture notes have also been preserved[33]. Wittgenstein begins the first, introductory lecture with remarks locating his subject matter:

> 'I am proposing to talk about the foundations of mathematics. An important problem arises from the subject itself: How can I - or anyone who is not a mathematician - talk about this? What right has a philosopher to talk about mathematics?
>
> [...]
>
> I can as a philosopher talk about mathematics because I will only deal with puzzles which arise from the words of our ordinary everyday language, such as "proof", "number", "series", "order", etc.
>
> Knowing our everyday language - this is one reason why I can talk about them. Another reason is that all the puzzles I will discuss can be exemplified by the most elementary mathematics - in calculations which we learn from ages six to fifteen, or in what we easily might have learned, for example, Cantor's proof.'

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32 Gasking later published 'Mathematics and the World', an article influenced by Wittgenstein's lectures, which was reprinted in *Philosophy of Mathematics: Selected Readings* (Oxford, Blackwell, 1964), edited by P. Benacerraf and H. Putnam.

33 *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939*, from the notes of R.G. Bosanquet, Norman Malcolm, Rush Rhees, and Yorick Smythies, which was edited by Cora Diamond with Rhees's assistance, forms an excellent companion to the mathematical typescript (TS 221). Smythies' notes were the only ones used by Diamond which had not been written up to some extent after Wittgenstein's lectures. Rush Rhees's original notes were found among his papers after his death and they are now at the University College of Swansea.

A pirated version of Malcolm's notes was published as *Math Notes* in San Francisco in 1954. Bosanquet's version also came into circulation and reached Oxford in early 1950 along with the *Blue Book* and the *Brown Book* (See the beginning of M.A.E. Dummett's article 'Reckonings: Wittgenstein on Mathematics', *Encounter*, L (1978)).

Having introduced the subject, Wittgenstein devotes the remaining lectures to a presentation of his thoughts in the mathematical typescript (TS **221**). The content of the lectures is also determined by the development of the discussion, which consists largely of a dialogue between Wittgenstein and Turing. Turing was at this time also giving classes in Cambridge under the title 'Foundations of Mathematics'.

The fate of Wittgenstein's family in Austria had been a cause of anxiety to him for some time.[34] During the summer, he travelled to Vienna, Berlin and New York on a mission to help secure the safety of his sisters.

Britain and France declared war on Germany on 3 September. Wittgenstein wanted now to do some kind of war work. For the first two years of the war, however, he was unable to find suitable employment; so he had to remain as a lecturer at Cambridge. He moved back again into his old rooms in Whewell's Court. Pocket notebook **160** is dated first on 14 September.

In Michaelmas term Wittgenstein held a seminar with the title 'Philosophical Investigations'. From this time on he also became more active in the Moral Science Club. He lectured there on 2 February, and on 19 February he

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[34] Wittgenstein - Moore, 19.10.1938. *Letters to Russell, Keynes and Moore*, M.37.

lectured to the Mathematical Society. His research on the foundations of mathematics was continued in the new Volume XVIII (MS **122**) on 16 October and then in the final part of Volume XIII (MS **117(5)**) between 3 February and 18 April 1940[35]. In these extensive writings Wittgenstein 'keeps on renewing the attempt to elucidate his thoughts on the nature of mathematical proof: what it means, for example, to say that a proof must be surveyable; that it presents us with a new picture; that it creates a new concept; and the like. His effort is to declare "the motley of mathematics" and to make clear the connexion between the different techniques of calculation. In so striving he simultaneously sets his face against the idea of a "foundation" of mathematics, whether in the form of a Russellian calculus or in that of the Hilbertian conception of a meta-mathematics. The idea of contradiction and of a consistency proof is extensively discussed.'[36] Between 10 April and 21 August Wittgenstein made some entries in pocket notebook **162b(2)**. On 2 July he writes:

> 'If we look at things from an ethnological point of view, does that mean we are saying that philosophy is ethnology? No, it only means that we are taking up a position right outside so as to be able to see things *more objectively*.'

Volume XVIII (**122**) is the last in the long series of numbered manuscript volumes which Wittgenstein had begun after his return to philosophical research in 1929.

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[35] A selection of remarks from this work was published as *Remarks on the Foundations of Mathematics*, III.

[36] From the editors preface to *Remarks on the Foundations of Mathematics*.

In this writing and in the other writing on the philosophy of mathematics which occupied Wittgenstein during the war years, his thought in the mathematical typescript (TS **221**) undergoes a certain amount of revision. For example, in section 216, he had said:

> 'When I say "This proposition follows from that one", that is to accept a rule. The acceptance is *based* on the proof. ...'

However, reflecting on 'proofs of constructability', or 'geometrical proofs', he writes in Volume XVIII (MS **122**):

> 'I should now like to say: the sequence of signs in the proof does not necessarily carry with it any kind of acceptance. If however it's to be a matter of accepting, this does not have to be "geometrical" acceptance.'

Numerous examples of such minor alterations could be given. Most of Wittgenstein's new work, however, seems to *expand* on the mathematical typescript, particularly in the areas of philosophical grammar which he considered relevant to the three main schools: logicism, formalism and intuitionism.

In the academic year 1940-41 Wittgenstein held seminars on 'Philosophical Investigations' and, in addition, some private discussions on aesthetics. Between 25 September and 23 November 1940 Wittgenstein wrote in notebook **123(1)**[37], which is mainly on the topics of following a rule and mathematical proof. This work was resumed in notebook **123(2)** on 16 May 1941 and then

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[37] This writing is not reproduced in *Remarks on the Foundations of Mathematics*.

continued immediately in volume **124(1)** between 6 June and 4 July. Pocket notebook **161(2)** is draft material for **124(1)**. In these writings from the late spring and early summer of 1941 Wittgenstein discusses 'the relationship between mathematical and empirical propositions, between calculation and experiment, treats the concept of contradiction and consistency anew and ends in the neighbourhood of the Gödelian problem'[38]. Later in the year, between 22 June and 29 September, Wittgenstein was writing on a similar range of topics in pocket notebook **163**.[39]

Two pocket notebooks, **164**, 'perhaps the most satisfactory presentation of Wittgenstein's thoughts on following a rule'[40] and **165** were written *circa* 1941-1944, but have not been dated more precisely.

Wittgenstein's friend and companion Francis Skinner died on 11 October 1941.[41]

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[38] From the Preface to *Remarks on the Foundations of Mathematics*.

[39] A selection from **123(2)** and **124(1)** and the early part of **163** forms the first part, sections 1-23, of Part VII of *Remarks on the Foundations of Mathematics*.

[40] Preface to the revised edition of *Remarks on the Foundations of Mathematics*. Pocket notebook **164** was published as Part VI of this work.

[41] R.L. Goodstein's Preface to his book *Constructive Formalism: Essays on the Foundations of Mathematics* (Leicester 1951), ends:

'My last word is for my dear friend Francis Skinner, who died at Cambridge in 1941, and left no other

Through his friendship with Gilbert Ryle, Wittgenstein finally managed to obtain war work. From October 1941 he worked as a dispensary porter at Guy's Hospital in London, where Ryle's brother was already employed as a consultant. Despite the new job, which he found physically exhausting, Wittgenstein continued to write on the foundations of mathematics. Pocket notebook **125**[42] was completed between 28 December 1941 and 16 October 1942. Wittgenstein also travelled to Cambridge at the weekends to hold discussion classes.

In the summer of 1942, following minor surgery, Wittgenstein stayed with Rhees in Swansea, and the two men had a number of discussions on Freud. Wittgenstein discussed the same topic with Rhees during 1943 and again in 1946[43]. He clearly believed that there were significant similarities between his own philosophical method and Freud's psychoanalytical technique. This similarity is described well by Monk, who writes: 'Freud's explanations have more in common with a mythology than with science; for example, Freud produces no evidence for his view that

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record of his work and of his great gifts of heart and mind than lies in the recollections of those who had the good fortune to know him.'

[42] *Remarks on the Foundations of Mathematics*, IV derives mainly from MS **125** but with some additions from MSS **126** and **127**.

[43] These discussions were recorded by Rhees and later published as the second part of *Lectures and Conversations on Aesthetics, Psychology and Religious Belief*, edited by C. Barrett (Oxford, Blackwell, 1966).

anxiety is always a repetition of the anxiety we felt at birth, and yet "it is an idea which has a marked attraction":

> "It has the attraction which mythological explanations have, explanations which say that this is all a repetition of something that has happened before. And when people do accept or adopt this, then certain things seem clearer and easier for them."[44]

Freud's explanations, then, are akin to the elucidations offered by Wittgenstein's own work. They provide, not a causal, mechanical theory, but:

> "...something which people are inclined to accept and which makes it easier for them to go certain ways: it makes certain ways of behaving and thinking natural for them. They have given up one way of thinking and adopted another."

It was in this respect that Wittgenstein described himself to Rhees at this time as a follower of Freud.'[45]

At the beginning of this year, 1942, a young student of mathematics named Georg Kreisel, who was later to become

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[44] *Lectures and Conversations*, '*Conversations on Freud*', II, Summer 1942.

[45] *Ludwig Wittgenstein: the Duty of Genius*, p. 438. Already in *Philosophical Grammar* (TS **213**), section 122, Wittgenstein had written:

> 'A mathematician is bound to be horrified by my mathematical comments, since he has always been trained to avoid indulging in thoughts and doubts of the kind I develop. He has learned to regard them as something contemptible and, to use an analogy from psycho-analysis (this paragraph is reminiscent of Freud), he has acquired a revulsion from them as infantile. That is to say, I trot out all the problems that a child learning arithmetic, etc., finds difficult, the problems that education represses without solving. I say to those repressed doubts: you are quite correct, go on asking, demand clarification!'

a leading figure in the development of mathematical logic and an influential critic of Wittgenstein's work, had come as an undergraduate to Trinity College. Kreisel, who was originally from Graz, had by this time already developed an interest in the foundations of mathematics, and sometime during his first year at Cambridge made inquires on this subject to his coach, the mathematician A.S. Besicovitch. Besicovitch contacted John Wisdom, a lecturer in philosophy at Trinity College, and this led to Wittgenstein giving a series of lectures on the foundations of mathematics in the Autumn of 1942. After the fourth lecture, he explained in a letter to Rhees that they 'will probably go on moderately well as long as I am able - as I am now - to do a little work on the subject. I'm quite certain this won't be for very long'.[46] Not long after the lectures had begun, Wittgenstein invited Kreisel to walk with him and have private conversations.[47] Kreisel remarks: 'So far this was not strange as I had in his (and my) eyes, at least among the seminar participants, a virtue: I did not study philosophy.'[48] These conversations - generally a monologue

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[46] Wittgenstein - Rhees, 4.11.1942. Wittgenstein says that about ten people are coming to his lectures, but no notes seem to have been taken.

[47] My account of Kreisel's contact with Wittgenstein is based on his 'Zu Einigen Gesprächen mit Wittgenstein' in *Ludwig Wittgenstein: Biographie, Philosophie, Praxis* (Wiener Secession, 1989). See also his earlier 'Zu Wittgenstein's Gesprächen und Vorlesungen über die Grundlagen der Mathematik', *Proceedings of the 2nd International Wittgenstein Symposium* (Vienna, Hölder-Pichler-Tempsky, 1978).

[48] 'Zu Einigen Gesprächen', p. 131.

by Wittgenstein - gave Kreisel greater pleasure than the lectures, which he found 'tense and often incoherent'[49] and even 'rather comical'[50]. Wittgenstein proposed that they should read Hardy's *Pure Mathematics* together; and Kreisel remembers that Wittgenstein commented on everything in the book 'forcibly' and 'impressively': 'It was lively and relaxed; never more than two proofs per conversation, never more than half an hour.'[51] Kreisel also remembers that Wittgenstein reformulated some of Hardy's proofs. The consecutive writing which was begun in pocket notebook **126** on 20 October and continued in pocket notebook **127(1)** ('Mathematik und Logik')[52] on 6 January is partly based on the notes which Wittgenstein was making at this time in the margin of his copy of Hardy's book.[53] Wittgenstein also lent Kreisel a copy of the *Blue Book*, which Kreisel returned after the summer vacation of 1942, expressing at

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49 'Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939', *Bulletin of the American mathematical Society* (Vol. 84, No. 1, 1978), footnote 2.

50 'Zu Einigen Gesprächen', p. 131.

51 Ibid., p. 136.

52 The 'F' marked on the cover is not, I think, a label for the pocket notebook but the 'F' of the remarks on seeing an aspect, which already appear in Wittgenstein's writings by this time. There is, therefore, no cause to conjecture about a whole series of alphabetically marked volumes ending in 'Band S' (MS **138**).

53 *Remarks on the Foundations of Mathematics*, V is based on MSS **126** and **127(1)**.

the same time certain misgivings about some of its main ideas[54].

During 1942 Wittgenstein had become interested in the work of a clinical research unit based at Guy's Hospital, which was investigating wound shock. The research unit moved to the Royal Victoria Infirmary in Newcastle in November 1942 and soon afterwards Wittgenstein accepted an offer to join it there as a laboratory technician. Wittgenstein left Guy's on 17 March 1943, but before taking up his new post, he visited Rhees in Swansea, and here the two men resumed their conversation on Freud.[55] Wittgenstein had by this time decided to deposit his copy of *Philosophical Investigations* (TSS **220** & **221**) in a bank, where he thought it would be in less danger of destruction from bombing. Rhees remembers that a Swansea lawyer was to be empowered to withdraw it and send it on to someone, if Wittgenstein should die. Before depositing it, however, Wittgenstein read the text again and found that certain passages were 'foul'. These he cut down and revised, and he then got Rhees to type out the corrected version. The revisions survive in the form of 12 pages of the revised

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[54] See 'Zu Einigen Gesprächen', section 2, and also Kreisel's review of *The Blue and Brown Books* in 'Wittgenstein's Theory and Practice of Philosophy', *British Journal for the Philosophy of Science*, 34 (1960).

[55] See the two sets of 'notes following conversations in 1943' in the second part of *Lectures and Conversations*.

text (TS 237), corresponding to 16 consecutive pages from typescript 220 (TS 238).[56]

Wittgenstein took up his new post at Newcastle on 29 April 1943. He held no further seminars at Cambridge and it seems that he did little or no philosophical research at this time. Besides working assiduously as a laboratory technician, Wittgenstein also developed his interest in the research itself, which he discussed with the two doctors in charge of the project, Grant and Reeve. Wittgenstein seems to have been particularly interested in Dr Grant's approach to the problem of the nature and treatment of shock. Grant believed that 'shock' was in fact an unhelpful diagnosis, which ought to be replaced with a detailed record of the patients developing condition and the treatment administered. The word 'shock' was indeed 'a hindrance to unbiased observation and a cause of misunderstanding'[57]. There is a parallel here to Hertz's approach to the problems of 'force' in physics, and it was this philosophical aspect of the research which mainly interested Wittgenstein. After the war, The Medical Research Council reported that Grant's work:

> '...threw grave doubt upon the value of attacking the "shock" problem as if wound "shock" were a single clinical and pathological entity. In consequence, several lines of investigation started for the Committee at the beginning of the war were abandoned.'

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[56] The source for the information in this paragraph is a letter from Rhees to von Wright, dated 13.11.1973.

[57] *Observations on the General Effects of Injury in Man.*

Here we have an instance where Wittgenstein applied his philosophical method to aid the development of a particular area of scientific knowledge. Wittgenstein's writings on the foundations of mathematics, by contrast, were not written with the purpose of aiding the development of particular areas of mathematics by eliminating specific conceptual problems.[58]

Sometime in 1943 Wittgenstein re-read the *Tractatus Logico-Philosophicus* with his long-standing friend, the Russian philologist Nicholas Bachtin. He wrote later:

> 'It suddenly seemed to me that I should publish those old thoughts and the new ones together: that the latter could be seen in the right light only by contrast with and against the background of my old way of thinking.'[59]

Wittgenstein contacted Cambridge University Press in September, suggesting that they publish his *Philosophical Investigations* together with a reprint of the *Tractatus*. Cambridge University Press confirmed their acceptance of this offer on 14 January 1944.

In December Wittgenstein wrote to Malcolm: 'I am feeling rather lonely here & may try to get to some place where I have someone to talk to. E.g. to Swansea where

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[58] Wittgenstein's underlying attitude is made explicit in this remark from 1949: 'I may find scientific questions interesting, but they never really grip me. Only *conceptual* and *aesthetic* questions do that. At bottom I am indifferent to the solution of scientific problems; but not the other sort' (MS **138**, p. 5).

[59] Preface to the final, printed version of *Philosophical Investigations*.

Rhees is a lecturer in philosophy'. Rhees, he believed, had a 'real talent for philosophy'.[60] Wittgenstein left his job at Newcastle on 16 February 1944, and, having been granted leave of absence in Cambridge, he then moved to Swansea to work on his book. Here on 27 February he resumed work in the pocket notebook entitled '*Mathematik und Logik*', which he had begun at the beginning of the previous year (MS **127(2)**). This writing was continued in volume **124(2)** between 5 March and 19 April. MS **127(3)** was probably begun next. In March he wrote to his friend Roland Hutt: 'I am now in Swansea working at my book. Whether with success or not, God knows.'[61]

Wittgenstein had probably by now withdrawn his copy of *Philosophical Investigations* from the bank where it had been deposited in late 1942 or early 1943, and had begun the task of revising it for publication. Typescript **239** is a revision of the whole of the first half of the early version of the *Investigations* (TS **220**), which incorporates the revisions already made in typescript **238**. In conjunction with this work, and most probably at the same time or soon afterwards, Wittgenstein also began a revision of the second half of the early version of the

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[60] Wittgenstein - Malcolm, 7.12.43. *Ludwig Wittgenstein: A Memoir* (Oxford, Oxford University Press, 1984), 8.

[61] Wittgenstein - Hutt, 17.3.44

Investigations (TS **221**).[62] This revision survives as a collection of *Zettel*, containing numerous handwritten additions and deletions, which Wittgenstein had clipped together into organized bundles. I shall sometimes refer to it as 'the revised mathematical typescript'. It is catalogued as three different items, according to the separation into bundles[63]: TS **222**, corresponds to remarks 162-375 and 397-442 of the original, excluding those remarks in TS **224**; TS **223**, corresponds to remarks 376-396; and finally TS **224**, corresponds to remarks 298-317.[64] (TS **240** consists of a few pages of **221** with some handwritten corrections. These pages were not included in TS **222**.)

Towards the end of volume **128**, which was written at about this time, there is a revised version of the Preface to Wittgenstein's book and on the last page a new title:

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[62] The numberings and renumberings of remarks at the beginning of TS **222** start at around the same number, 200, as the numbering and renumbering of remarks at the end of **239** finish. Also, during my research in Helsinki, I have discovered bracketed numbers next to a few of the remarks in TS **222** which clearly make reference to remark numbers in TS **239**.

[63] Pages 120-135, which are on negation etc., were not cut-up. They do not have a separate catalogue number, because they were not separated from the main bundle of clippings in the same way as TSS **223** and **224**.

[64] See Appendix I, B.

The revised mathematical typescript was published as Part I, including appendices, of *Remarks on the Foundations of Mathematics*. The remarks on negation, originating in MS **115**, were published as the first appendix to Part I, while typescripts **224** and **223** were published as the second and third appendices respectively.

> 'Philosophische Untersuchungen der Logisch-Philosophischen Abhandlung entgegengestellt'.

The motto by Hertz is also replaced by one from Johann Nestroy:

> 'It is in the nature of all progress that it looks much greater than it really is.'

This motto seems to echo the end of Wittgenstein's Preface in the *Tractatus*, where he writes:

> '...the value of this work...is that it shows how little is achieved when these problems are solved.'

In addition, the context of the quoted sentence in Nestroy makes it clear that Wittgenstein is also echoing here his thoughts about 'progress' in the Preface to *Philosophical Remarks*.[65]

Volume XII(3) (MS **116(3)**) was probably also written in 1944.

In June Wittgenstein wrote to his friend Hutt: 'My work hasn't been going too well, which makes my future rather doubtful'.[66] It was at about this time that Wittgenstein began writing which was to become the source for a whole new continuation of the first half of the *Investigations* (TS **239**). Volume **124(3)**, which was begun on 3 July, begins with the first new remarks for this

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[65] See 'The Motto' in Baker and Hacker's *Analytical Commentary on Wittgenstein's* Philosophical Investigations (Oxford, Blackwell, 1980).

[66] Wittgenstein - Hutt, 8.6.44.

continuation[67], and further work is contained in the pocket notebooks **179** and **180a**. Wittgenstein's final draft of the new continuation is contained in volume **129(1)**, which was begun on 17 August. In special pages at the beginning of that volume, **129(2)**, there are several draft prefaces to the *Investigations*.

In October Wittgenstein returned to Cambridge, where Russell had now also returned after a period of six years working in the United States. Wittgenstein had no significant contact with his former teacher, who had a low opinion of his later thought, but he did have discussions every few weeks with Moore. In November, Wittgenstein took over Moore's chairmanship of the Moral Science Club. Wittgenstein had thought of taking William James's *Principles of Psychology*[68] as a text for his Michaelmas term lectures, but he chose instead to discuss topics on the philosophy of psychology taken from his current work on the *Philosophical Investigations*.[69] Describing his audience to Rhees in November, he mentions 'a woman, a Mrs so & so[70] who calls herself Miss Anscombe, who certainly is

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[67] Volume **124(3)** also contains revisions of two remarks in typescript **221/222**. These were incorporated into the text of Part I of the revised edition of *Remarks on the Foundations of Mathematics* (§§ 115-116).

[68] James's *Principles of Psychology* played a similar role in Wittgenstein's thinking about the philosophy of psychology as did Hardy's *Pure Mathematics* in his thinking about the philosophy of mathematics.

[69] Wittgenstein - Rhees, 17.10.1944.

[70] Geach.

intelligent, though not of Kreisel's caliber'.[71] Elizabeth Anscombe had studied at St Hugh's, Oxford and had come to Cambridge as a postgraduate student in 1942. She was to become a close friend of Wittgenstein's and, along with Rush Rhees and G.H. von Wright, one of his three literary executors.

Kreisel, for whom Wittgenstein clearly had by now considerable respect[72], had been working on hydrodynamics since the beginning of 1944, work which had application to the design of artificial harbours and the measurement of sea swells in coastal regions. Experts had hoped to determine movements on the surface of the sea by means of pressure gauges on the sea bed. Kreisel believed, however, that this approach overlooked the possibility that small differences in the deep pressure distribution might leave undetermined significant features of the surface phenomena. This story was related to Wittgenstein, who apparently found it satisfying, and it became for Kreisel an important metaphor, which he used as a defence against, what he calls, 'visionaries of the depths' in mathematics. One can easily imagine that Wittgenstein would have been pleased by the tendency of these thoughts and also impressed by

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[71] Wittgenstein - Rhees, 28.11.44.

[72] Monk reports that: 'In 1944...Wittgenstein shocked Rhees by declaring Kreisel to be the most able philosopher he had ever met who was also a mathematician. "More able than Ramsey?" Rhees asked? "Ramsey?!" replied Wittgenstein. "Ramsey was a mathematician!"' (*Wittgenstein: The Duty of Genius*, p. 498.)

Kreisel's use of a vivid new metaphor to oppose older ones which emphasize the significance of foundations.

In a letter to Rhees of 17 October 1944, Wittgenstein had remarked: 'I have hope to get a typist soon, but no hope whatever to finish my book in the near future'. To the typist mentioned here Wittgenstein probably dictated typescript **241**[73], which is based on volume **129(1)**, and typescript **243**, which is based on the last revised preface contained in **129(2)**. Despite his remark to Rhees, having returned from Swansea after the Christmas vacation, Wittgenstein does seem, momentarily at least, to have had the intention to publish. The new preface (TS **243**) is dated, in handwriting: 'Cambridge, Januar 1945'.

There are two alternative works which Wittgenstein might have considered publishing at around this time; both would have been based on recent work, and both would have excluded a large amount of work already existing in manuscript form. Firstly, Wittgenstein might have been thinking about publishing a revised early version of the *Investigations* in the form: TSS **243-239-221/222**. This would explain how he could have changed his mind so quickly about the prospect of publishing, and it would also explain the reference to work on the foundations of mathematics which persists in the typescript preface. Secondly, and

[73] TSS **241** and **242** share the same underlying typescript but contain different corrections in Wittgenstein's hand.

perhaps this is more likely, he might have been thinking about publishing a work in the form: TSS **243-239-241**.

Whichever of these alternatives is correct, we do know that, if he had not already done so, Wittgenstein soon made a new typescript from typescripts **243**, **239** and **241**, and that this was to be the basis for his future work.[74] This new typescript is the one which von Wright later reconstructed, and which he called the 'Intermediate Version of the *Investigations*'[75]. Its basic structure, ignoring minor manuscript and typescript sources, is as follows:

TS Source	Remarks	Pages
243		1-4
239	1-188	5-134
221	189-198	135-143
241	198-300	144-195

It is notable that a fragment of the mathematical typescript (TS **221**) is used to join the new continuation in typescript **241** to the end of the revised first half of the early version; so that neither typescript **221** nor

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[74] Typescripts **243** and **239** were later presented together as a gift to Yorick Smythies.

[75] It does not exist intact but only as part of typescript **227** and in typescript **242**, pages which Wittgenstein gave to von Wright in 1949 (or 1950). See 'The Intermediate Version of Part I of the *Investigations*' in von Wright's 'The Origin and Composition of the *Philosophical Investigations*', in his *Wittgenstein*.

typescript **222** could now be used without some modification as a further continuation of the text.

The first part of the early version of the *Investigations* (TS **220**) and the revised version (TS **239**) both end with remarks continuing the theme, already introduced in both typescripts, of following a rule. They both end finally with this remark:

> "But *are* the steps then *not* determined by the algebraic formula?" - The question contains a mistake.[76]

The first remarks of typescript **221**, the second half of the early version of the Investigations, continues with remarks on the sense in which an algebraic formula might be said to determine the steps of a calculation; and from here we are led into remarks on the inexorability of mathematics, the nature logical inference, and further topics in the foundations of mathematics. In the new typescript, the first few remarks of **221** follow, with very slight modifications, and these are then continued by further remarks from **221** on the rule following theme[77], beginning:

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[76] TS **220**, 161c; TS **239**, 206; *Philosophical Investigations* (1953), 189a. See Appendix I, B.

[77]

TS **221**	TS **222**	PI(1953)
342	121	---
343	122	193
344	123	191
345	124	192
346	125	194
347	126	195
348	127	196
349	128	---
350	129	---
351	130	197

> '"It is as if we could grasp the whole use of the word in a flash". ...'

This continuation is itself followed by remarks on the idea of a private language, beginning with the linking question of the first remark:

> 'How do words *refer* to sensations?'

The revised first part of the early version of the *Investigations* was in this way increased to about twice its previous length. The two themes of rule following and private language are now combined: Wittgenstein examines the idea of following a rule privately in the case of the private ostensive definition of words referring to particular private sensations.

These changes in the continuation of the first part of the *Investigations* were accompanied by a shift in Wittgenstein's main area of research from the foundations of mathematics to the philosophy of psychology. This change, which had already begun in the summer of 1944, lasted until Wittgenstein's last efforts to improve his book in 1949. Various isolated remarks and sets of remarks on mathematics were written after 1944, but none with the purpose of producing a revised mathematical typescript embodying the writing produced during the war years. As the likelihood of completing a second volume on the foundations of mathematics gradually diminished over the coming years, Wittgenstein probably saw the task of writing it as belonging more and more to some future writer working under the inspiration of his own remarks.[78]

Following the decision not to publish early in 1945, Wittgenstein began work on expanding and extending his book with a selection of remarks taken mainly from earlier revisions of both *Philosophical Grammar* and *Philosophical Investigations*[79] and from more recent work on the philosophy of psychology[80]. Volume **130(1)** and the fourth and final part of Volume XII (MS **116**), dated 'May 1945', are the most recent sources used by Wittgenstein. The work of dictating the selection for a typescript is mentioned in a letter to Rhees:

> 'The Term's over and my thoughts travel in the direction of Swansea. I've been working fairly well since Easter. I am now dictating some stuff, remarks, some of which I want to embody in my first volume (if there'll ever be one). This business of dictating will take another month or six weeks. After that I could leave Cambridge.'[81]

The result of this lengthy period of dictation, which in fact kept Wittgenstein in Cambridge until mid-August, is

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[78] In the summer of 1944, while Wittgenstein was writing on the concept of a private language, Rhees is said to have asked him: 'What about your work on mathematics?' And Wittgenstein is said to have replied with a wave of his hand: 'Oh, someone else can do that' (*Wittgenstein: The Duty of Genius*, p. 466).

[79] Specifically, from the revision of *Philosophical Grammar* in 1933-34 (MSS **114(2)** & **115(1)**) from the revision of the so-called *Brown Book* in 1936 (**115(2)**) and from the revisions of *Philosophical Grammar* in 1937 (MS **116(1)**).

[80] From MSS **116(3)**, **129**; and MSS **130(1)**, **116(4)**.

[81] Wittgenstein - Rhees, 13.6.1945. See also Wittgenstein - Moore, 22.7.1945.

'Bemerkungen I' (TS **228**).[82] Wittgenstein used this typescript together with a list (MS **182**) of the remarks which he had decided to include in his 'first volume' to complete the work of revision. The final result is typescript **227**[83], which has 693 numbered sections and is more than twice the length of the intermediate version. Some of the additions, although these are only a small proportion of the total, are on the foundations of mathematics. Some of the remarks on negation from typescript **221/222** are included here in a revised form (§§ 552-557); but there are also scattered remarks from different sources on a whole range of topics: the concept of number (§§ 67-68)[84], contradiction and consistency (§ 125)[85], mathematical reality (§§ 254)[86], mathematical conjecture (§ 334)[87], set-theory (§ 426)[88], looking for something in mathematics (§§ 462-3)[89], intuitionism (§§

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[82] The sources for this typescript in order of entry are: MS **116(1-3)**; MS **129**; TS **213**, and MSS **114**, **117**, **119**; MSS **114(1)**, **115**; and MSS **130(1)** and **116(4)**.

[83] It was published as Part I of *Philosophical Investigations* (Oxford, Blackwell, 1953). The manuscript sources are listed in an unpublished work by André Maury, which is available in Helsinki.

[84] § 67 is from *Philosophical Grammar* (TS **213**).

[85] From volume **130(1)**.

[86] From volume **129(1)**.

[87] From **115(1)**.

[88] From volume **116(2)**.

[89] From volume **114(2)**.

514-517)[90], and so on. These remarks, like those on mathematics which appear in his lectures on the philosophy of psychology, for example, indicate the extent to which Wittgenstein saw parallels between philosophical problems in different subject areas. Wittgenstein's continued reference to a 'first volume' in the letter to Rhees suggests that he was still contemplating a second volume on the philosophy of mathematics.[91]

In August Wittgenstein thought that he might publish by Christmas, and in September he believed that his book was 'nearing its final form'.[92] He is probably referring here to the work on typescript **227**. However, many of the remarks in 'Bemerkungen I' were not included in this typescript and at some point Wittgenstein also made an independent arrangement of 'Bemerkungen I', which he called 'Bemerkungen II' (TS **230**). (There also exist two lists of correspondences between 'Bemerkungen I' and 'Bemerkungen II' (TS **231**).)

In the academic year 1945-46 Wittgenstein gave twice weekly lectures on the philosophy of psychology. He also gave a talk at the Moral Science Club on 14 November:

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[90] §§ 515-517 are from pocket notebook **163**.

[91] Earlier references to a 'first volume' in letters to Keynes (1.2.39), Moore (2.2.39) and von Wright (19.3.39) support this view.

[92] Wittgenstein - Malcolm, 20.9.45. *Ludwig Wittgenstein: A Memoir*, 15.

'roughly, on what I believe philosophy is, or what the method of philosophy is'[93]. The Christmas and Easter vacations were spent in Swansea, where Wittgenstein had discussions with Rhees[94]. The effort involved in lecturing seems to have been one reason for the abandonment of any immediate plans for publication; and for this, and other reasons, Wittgenstein soon began thinking about resigning his professorship at Cambridge and completing his book elsewhere. His writing was continued, after a long hiatus[95], in volume **130(2)** between 26 May and 9 August; and then consecutively in volume **131** on 10 August and volume **132** on 9 September.

In the academic year 1946-47 Wittgenstein lectured once a week on the philosophy of psychology, using Wolfgang Köhler's *Gestalt Psychology* (1929) as a text. Stimulated by conversations which he had been having again with Kreisel, Wittgenstein added to these lectures further classes on the philosophy of mathematics. Notes by students attending the lectures on the philosophy of

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[93] Wittgenstein - Moore, 14.11.46. *Letters to Russell, Keynes and Moore*, M.49.

[94] Rhees's notes following conversations in 1946 completes the chapter on Freud in *Lectures and Conversations*.

[95] Wittgenstein - Malcolm, 25.4.46. *Ludwig Wittgenstein: A Memoir*, 21.

psychology have been preserved.[96] Interestingly, the first lecture begins in the same manner as did the first of his lectures on the foundations of mathematics in 1939:

> 'These lectures are on the philosophy of psychology. And it may seem odd that we should be going to discuss matters arising out of, and occurring in a science, seeing that we are not going to do the science of Psychology and we have no particular information about the sort of things that are found when the science is done. But there are questions, puzzles that naturally suggest themselves when we look at what psychologists may say, and what non-psychologists (and we) may say.'

There is, however, an important difference between psychology and mathematics which should be noted in this connection. Psychology can be studied at one level by reflecting on phenomena familiar from everyday life; and these phenomena provide a rich source from which the intelligent layman can draw, if he chooses to criticize the writings of certain psychologists, for example, Freud. In contrast, for those who do not have a thorough education in mathematics, there is no similarly rich source on which to draw when investigating what mathematicians have to say about their subject. I believe that Wittgenstein, whether he realized it or not, came up against this problem when trying to improve his remarks on the foundations of mathematics. He once remarked: 'With my full philosophical rucksack I can only climb slowly up the mountain of mathematics.'[97] Perhaps he never reached sufficient

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[96] *Wittgenstein's Lectures on the Philosophy of Psychology: 1946-7* (Hassocks, Harvester Press, 1988) is a compilation of the notes taken by P.T. Geach, A.C. Jackson and K.J. Shah.

[97] Volume III (MS **107**), 11.9.1929.

altitude to conduct a satisfactory survey of its topography.

Wittgenstein's writing on the philosophy of psychology was continued consecutively in volumes **133**, on 22 October, **134**, on 28 February 1947, and **135**, on 12 July. At the end of February, he wrote:

> 'Oh, why is it demanded of me that I write philosophy as if I were writing a poem? It is in this case, as if there were here a little thing which had a wonderful meaning. Like a leaf, or a flower.'

Wittgenstein felt a deep need to express himself in a religious manner, both in his lecturing and in his writing, and for him this meant an artistic manner of expression. The connection between religion and art which Wittgenstein had made in the *Tractatus*[98] persists in his later thought:

> 'One might say: art *shows* us the miracles of nature. It is based on the *concept* of the miracles of nature. (The blossom, just opening out. What is *marvellous* about it?) We say: "Just look at it opening out!".' (10.3.1947)

Besides clarifying features of Wittgenstein's style, understanding this connection also clarifies his attitude towards mathematics. Recalling earlier remarks about the mathematician Pascal, Wittgenstein continues here:

> 'The mathematician too can wonder at the miracles (the crystal) of nature of course; but can he do so once a problem has arisen about *what* it actually is he is contemplating? Is it really possible as long as the object that he finds astonishing and gazes at with awe is *shrouded* in a philosophical fog?
>
> I could imagine somebody might admire not only real trees, but also the shadows or reflections that they cast, taking them for trees. But once he has told himself that these are not really trees after all

[98] 6.421.

> and has come to be puzzled at what they are, or at how they are related to trees, his admiration will have suffered a rupture that will need healing.'

Later, in 1949, Wittgenstein again mentions the 'beauty of mathematical demonstrations, as experienced by Pascal'.[99]

> 'Within *that* way of looking at the world these demonstrations did have *beauty* - not what superficial people call beauty. Again, a crystal is not beautiful in just any "setting"...' (MS **138**, 18.1.1949)

We are meant to understand, I believe, that Pascal's way of experiencing mathematics has been lost.[100] And here, perhaps, we have at least part of the explanation of why Wittgenstein felt that mathematics had declined along with the arts.[101] The work of both the artist and of the mathematician had become shrouded in philosophical fog.

Besides the conversations with Kreisel, Wittgenstein also had weekly meetings with Anscombe, who was now a

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[99] Earlier Wittgenstein had written:
'The mathematician (Pascal) who admires the beauty of a theorem in number theory; it's as though he were admiring a beautiful natural phenomenon. It's marvellous, he says, what wonderful properties numbers have. It's as though he were admiring the regularities in a kind of *crystal*.'
'One might say: what wonderful laws the Creator built into numbers!' (MS **125**, 4.1.1942)

[100] Cf. Russell in 'The Study of Mathematics':
'Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.' (*Mysticism and Logic* (London, Allen & Unwin, 1917), p. 60.)

[101] See p. 17 above.

research fellow at Somerville, Oxford, and with Malcolm[102], who had been back in Cambridge since August on a Guggenheim Fellowship. Anscombe - whom he regarded as '*more* than just *intelligent*'[103] - came to Cambridge for tutorials on the philosophy of religion; but with Malcolm Wittgenstein read through the *Philosophical Investigations* (TS **227**), paragraph by paragraph, 'so there will be at least one person who will understand my book when it is published'[104]. In May, Wittgenstein spoke to the Jowett Society in Oxford.

At the end of term Wittgenstein went to Swansea, and in August he went to Dublin to visit Drury, who was now a psychiatrist at St Patrick's Hospital.[105] By the end of August he was back in Cambridge, and, following a trip to Vienna in September, he resigned his professorship. He wanted now to retire to solitude, either in Norway or Ireland, 'in order', as he wrote to von Wright, 'to think and, *if possible*, to finish a part of my book'.[106] Michaelmas term was taken as sabbatical leave and he spent

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[102] Wittgenstein - Rhees, 15.10.46.

[103] Wittgenstein - Malcolm, 30.10.45. *Ludwig Wittgenstein: A Memoir*, 17.

[104] *Ludwig Wittgenstein: A Memoir*, p. 44.

[105] In 1973, Drury published *The Danger of Words*, a collection of essays on philosophical problems in psychiatry.

[106] Wittgenstein - von Wright, 27.8.47. 'Some Hitherto Unpublished Letters from Ludwig Wittgenstein to Georg Henrik von Wright', 9, in *The Cambridge Review* (28 February 1983).

his time extending 'Bemerkungen I' (TS **228**) by dictating typescript **229**[107] on the basis of volumes **130-135(1)**[108]. He now had all of his recent work on the philosophy of psychology in a 'handy form'. He continued his writing in volume **135(2)** from 9 November until 18 December.

In early December Wittgenstein moved to Ireland, where he was to live for the next eighteen months. He stayed initially at Ross's Hotel in Dublin, before moving, on 9 December, to lodgings at a farmhouse in Red Cross, County Wicklow. His writing on the philosophy of psychology was continued in 'Band Q'[109] (MS **136**) between 18 December and 25 January 1948. He later told Drury: 'I now see clearly that it was the right thing for me to give up the professorship. I could never have got this work done while I was in Cambridge.'[110] He continued his writing in the new 'Band R' (MS **137**) on 2 February.

On 28 April after a period of about five weeks in which he was unable to work, Wittgenstein moved to Rosro, an isolated cottage on the west coast of Ireland, in

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[107] This typescript was published as *Remarks on the Philosophy of Psychology, Volume I* (Oxford, Blackwell, 1980).

[108] Wittgenstein - von Wright, 6.11.47. 'Some Hitherto Unpublished Letters', 10.

[109] The three manuscript volumes which Wittgenstein wrote in Ireland with a view to completing a part of his book are each marked with letters of the alphabet, beginning with 'Q'.

[110] 'Conversations with Wittgenstein', 1948.

Connemarra. Within a month of staying there, he had managed to resume his writing on the philosophy of psychology in 'Band R'. He left Rosro in August and stayed with Drury and then Ben Richards before leaving, in September, for Vienna, where his sister Hermine was seriously ill with cancer. Back in Cambridge in late September, Wittgenstein spent a couple of weeks dictating typescript **232**[111] on the basis of volumes **135,136** and **137** up to the entry of 23 August. His writing in volume **137** was continued until 9 January.

Wittgenstein returned to Ireland in the middle of October. He decided that for the winter months he would stay in Dublin at Ross's Hotel. During December he was visited by Anscombe and Rhees, and with each of them he read through his recent work and described how he wanted to revise part of his book. He began 'Band S' (MS **138**) on 15 January 1949 and his writing was continued in this volume until 22 March. (Pocket notebook **168** is a fair manuscript copy of some remarks from volumes **136-138** on general subjects.)

Wittgenstein saw a great deal of Drury during this period in Dublin, and they had frequent discussions. On one occasion they talked about the history of philosophy, and Wittgenstein's remarks on this occasion, like those

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[111] This typescript was published as *Remarks on the Philosophy of psychology, Volume II* (Oxford, Blackwell, 1980).

made during the lectures on Broad in Michaelmas term 1931, help to characterise his general philosophical position:

> 'WITTGENSTEIN: Kant and Berkeley seem to me to be very deep thinkers.
>
> DRURY: What about Hegel?
>
> WITTGENSTEIN: No, I don't think I would get on with Hegel. Hegel seems to me to be always wanting to say that things which look different are really the same. Whereas my interest is in showing that things which look the same are really different. I was thinking of using as a motto for my book a quotation from *King Lear*: "I'll teach you differences".
>
> [...]
>
> DRURY: It is remarkable that Kant's fundamental ideas didn't come to him until he was middle-aged.
>
> WITTGENSTEIN: My fundamental ideas came to me very early in life.
>
> DRURY: Schopenhauer?
>
> WITTGENSTEIN: No; I think I see quite clearly what Schopenhauer got out of his philosophy - but when I read Schopenhauer I seem to see to the bottom very easily. He is not deep in the sense that Kant and Berkeley are deep.

Wittgenstein's respect for Kant, and for Kant's Irish predecessor, Berkeley, was as great as that which he had for any other philosopher, including Frege. Wittgenstein's discussions with Drury were, however, more usually on a religious topic. Wittgenstein once remarked:

> 'I am not a religious man but I cannot help seeing every problem from a religious point of view'[112]

And on another occasion he said:

> 'Bach wrote on the title page of his *Orgelbücheln*, "To the glory of the most high God, and that my neighbour may be benefited thereby". That is what I would have liked to say about my work.'[113]

The religious, mystical perspective of Wittgenstein's philosophy and its basic Kantian formation are two fundamental features, which survive the transformations separating the *Tractatus* from the *Philosophical Investigations*. Neither feature can be ignored, I believe, if Wittgenstein's thought is going to be properly understood.

In April, Wittgenstein travelled to Vienna, and while there he continued his writing on the philosophy of psychology in the new pocket notebook **169**. One of the entries reads:

> 'I want to call the enquiries into mathematics that belong to my Philosophical Investigations "Beginnings of Mathematics"[114].'

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[112] 'Some Notes on Conversations', p. 79. In his last, unfinished manuscript *Wittgenstein: Philosophy from a Religious View*, Norman Malcolm tried, unsuccessfully, I believe, to understand the sense of this important remark. In my chronological account of Wittgenstein's life and works, I have tried to indicate how, I believe, this remark should be understood, not least because it is a theme which can easily be overlooked, just as the logical positivists overlooked the mysticism of the *Tractatus*.

[113] 'Conversations with Wittgenstein', 1949. Cf. the Preface to *Philosophical Remarks* (TS **209**).

[114] Russell and Whitehead use this phrase, 'beginnings of mathematics', in the first paragraph of the Preface to *Principia Mathematica*.

He returned to Dublin on 16 May. There is one additional entry in 'Band S' (MS **138**) on 20 May.

Wittgenstein concluded his work in Ireland by producing manuscript **144**, which is a handwritten selection of remarks from all of the volumes on the philosophy of psychology which he had written since 1946 (MSS **130(2)-138**). In addition to this ordered series of remarks, there also exists a collection of cuttings or *Zettel* (TS **233**)[115], nearly all of which were taken from typescripts **228**, **229** and **232**, and which are, therefore, nearly all based on either the same or closely related manuscript material as that on which manuscript **144**[116] is based. It seems that Wittgenstein wanted to use this material and manuscript **144** together to improve and complete the latter part of typescript **227**.[117] (Pocket notebook **170** was probably also written in 1949.)

In late June Wittgenstein was in Cambridge as a guest of von Wright, who now held the chair in philosophy at the University. Here Wittgenstein dictated typescript **234**[118]

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[115] An arrangement of this material, by Peter Geach, was published as *Zettel* (Oxford, Blackwell, 1967).

[116] TS **229** is based on MS **130(2)-135(1)** and TS **232** is based on MS **135(2)-138**.

[117] See the remarks by Anscombe and Rhees in the 'Editor's Note' to *Philosophical Investigations* (1953).

[118] It was from this typescript that Part II of *Philosophical Investigations* (Oxford, Blackwell, 1953) was printed.

on the basis of manuscript **144**. With the completion of this work, in July, Wittgenstein's efforts to improve his book came to an end. Even now, however, Wittgenstein saw an investigation of the foundations of mathematics as part of the ideal completion of his book. A loose slip of paper which had been placed between the pages of typescript **234** reads:

> 'An investigation is possible in connexion with mathematics which is entirely analogous to our investigation of psychology. It is just as little a *mathematical investigation* as the other is a psychological one. It will *not* contain calculations, so it is not for example logistic. It might deserve the name of an investigation of the "foundations of mathematics".'

During his last two years, Wittgenstein lived mainly with his friends, in Cambridge and Oxford. While he was still alive he wanted to continue writing philosophy. Due to illness, however, there were long periods when he was unable to work.

In the spring of 1949 Wittgenstein accepted an invitation from Norman Malcolm, who was now at Cornell, to visit him in the United States. Wittgenstein left England on the *Queen Mary* on 21 July. At Cornell, he had discussions with various members of the philosophy department, individually and in seminars, including Max Black[119], Stuart Brown, Willis Doney and John Nelson. He

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119 Author of *The Nature of Mathematics* (London, Routledge & Kegan Paul, 1933) and *A Companion to Wittgenstein's* Tractatus (New York, Cornell University Press, 1964).

also had discussions with Oets Bouwsma, who had come from Nebraska especially to meet Wittgenstein.[120] Bouwsma had been Malcolm's tutor, and it was he who had originally encouraged both Malcolm and Ambrose to study with Moore in Cambridge. By the time of Wittgenstein's visit, Bouwsma had already made a close study of the *Blue Book*. One topic of the group discussions was Frege's philosophy, which Wittgenstein also discussed with Bouwsma alone. Bouwsma recalls:

> '...he had begun to talk about the difficulty of discussing Frege, and explained how Frege had come from the problems of math, and now talked and wrote about all sorts of problems without making the proper distinctions. So, pointing to a house along Cayuga Heights Road: The couple who live in that house - well, there may be no such couple. But we know how to find out. We'll stop and see. But in mathematics, there are expressions of the same sort - the least converging series and we may show that there is no such convergent series. But this is not an empirical matter. Frege did not make this sort of distinction.'[121]

Wittgenstein's concern to expose the frequent confusion of mathematical and empirical generality, especially in the writings of Frege and Russell, remained constant throughout the entire course of the development of his thought on the foundations of mathematics.

With Malcolm Wittgenstein started by reading through the *Philosophical Investigations*, as he had done recently with Rhees and Anscombe in Ireland. Soon, however they

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[120] Bouwsma's notes of his conversations with Wittgenstein were published in *Wittgenstein Conversations 1949-1951* (Indianapolis, Hackett, 1986).

[121] *Wittgenstein Conversations 1949-1951*, 17 August.

moved on to a discussion of Moore's articles 'Proof of an External World' and 'Defence of Common Sense', which Malcolm had recently criticized in a published article. Wittgenstein had discussed the topic of 'certainty' with Moore in the past[122], and there are important series of remarks on this topic in TS **234**.[123] However, these conversations with Malcolm were to stimulate Wittgenstein to undertake a far more intensive study of the questions raised by Moore's articles, and these researches form a large portion of the writings done by him in the last eighteen months of his life.[124]

At the beginning of the Autumn Wittgenstein fell ill and was admitted to hospital for an examination. After returning to England in October he was diagnosed as having cancer of the prostate.

Wittgenstein stayed at von Wright's house in Cambridge until Christmas, when he visited his family in Vienna.

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122 'A Discussion Between Moore and Wittgenstein on *Certainty*', which survives among Malcolm's papers, contains Malcolm's verbatim record of part of one such discussion.

123 Two series containing remarks on certainty in mathematics are found in *Philosophical Investigations*, Part II (TS **234**), pp. 221, 224-227. See also *Zettel* (TS **233**), sections 401-417.

124 Wittgenstein's writings on epistemology were published as *On Certainty* (Oxford, Blackwell, 1969). They include numerous remarks on mathematics, including, according to the editors numbering: 10, 25-28, 30, 34, 38-39, 43-50 and 55 from MS **172(2)**; 77, 111 and 113 from MS **174**; 212 and 217 from MS **175(1)**; and 303-304, 337b, 340, 350, 375, 392, 446-448, 563b, 650-658 and 664 from MSS **175(2)**, **176(2)** and **177**.

Here in February 1950 Wittgenstein's sister Hermine died of cancer. Wittgenstein was now reading Goethe's *Farbenlehre*, a work which soon stimulated him to write, in MS **172(1)**, the first of his remarks on colour, .[125] The logic of colour words was to be a second major topic of his final years. In MS **172(2)** these remarks are followed by the first of his new writings on epistemology.[126] Wittgenstein left Vienna on 23 March and stayed briefly in London before returning, on 4 April, to von Wright's house in Cambridge. Wittgenstein here chose to decline an invitation which he had received to give the John Locke Lectures in Oxford. The remarks on colour begun in Vienna were continued in notebook **173** on 24 March[127].

Wittgenstein moved to Elizabeth Anscombe's house in Oxford on 25 April. The writings on epistemology begun in Vienna were continued in notebook **174**[128] in the summer and then in notebook **175(1)** until 23 September.[129] Besides Anscombe and Smythies, who was working at the University's Forestry Library, Wittgenstein also had discussions at this

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125 Wittgenstein - Malcolm, 16.1.50. *Ludwig Wittgenstein: A Memoir*, 47. These writings were published as Part II of *Remarks on Colour* (Oxford, Blackwell, 1977).

126 These writings were published as §§ 1-65 of *On Certainty*.

127 These writings from the spring of 1950 were published as Part III of *Remarks on Colour*.

128 The writing on epistemology in **174** was published as §§ 66-192 of *On Certainty*.

129 The writing on epistemology in **175(1)** was published as §§ 193-299 of *On Certainty*.

time with Bouwsma, who was in Oxford to give the John Locke Lectures.

At the beginning of October Wittgenstein visited Skjolden with his friend Ben Richards. Richards took with him J.L. Austin's recently published translation of Frege's *Grundlagen der Arithmetic*, and they read and discussed Frege's book for much of their time together. Wittgenstein returned to Anscombe's house in Oxford in November[130].

In January 1951 Wittgenstein made out his will, giving to 'Mr R. Rhees, Miss G.E.M. Anscombe, and Professor von Wright of Trinity College all the copyright in all my unpublished writings and also all the manuscripts and typescripts thereof to dispose of as they think best...'.[131]

In February, his condition having worsened, Wittgenstein accepted an offer to stay at the house of his physician, Dr Bevan, in Cambridge. At the end of February he was given only a few months to live. He told Mrs Bevan: 'I am going to work now as I have never worked before'; and during the last two months of his life he managed to write

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[130] Bouwsma's notes of his conversations with Wittgenstein at this time are in the section entitled 'Oxford' in *Wittgenstein Conversations 1949-51*.

[131] For an account of how the literary executors discharged this responsibility, see the section on the 'The Posthumous Publications' in von Wright's 'The Wittgenstein Papers', in his *Wittgenstein*.

over half of his new remarks on epistemology. These writings were begun in pocket notebook **175(2)** on 10 March and continued consecutively in notebooks **176(2)** and **177**[132]. Notebook **176(1)** is a selection and revision of the earlier material on colour.[133] He wrote on 16 March:

> 'I believe it might interest a philosopher, one who can think himself, to read my notes. For even if I have hit the mark only rarely, he would recognize what targets I had been ceaselessly aiming at.'

He might have said the same about the other manuscript material which was left to us in a similarly unfinished state, and especially about the writings on the foundations of mathematics, which are so poorly represented in his two major works.

The last entry in notebook **177** is dated 27 April. Wittgenstein lost consciousness on 28 April and died the following day. His last words, to his friends, were: 'I've had a wonderful life'.

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[132] They were published as §§ 300-676 of *On Certainty*.

[133] It was published as Part I of *Remarks on Colour*.

II The Philosophical Investigation of Mathematics

2.1 Kreisel and Wittgenstein

Kreisel got to know Wittgenstein during 1942, while he was still an undergraduate mathematician at Trinity College, Cambridge, and their friendship lasted until Wittgenstein's death in 1951.[1] After this time, as Wittgenstein's later work was gradually being made widely available through the publications of his literary executors, Kreisel wrote a number of articles in which he criticized Wittgenstein's views on the philosophy of mathematics.

The first of these articles was 'Wittgenstein's Remarks on the Foundations of Mathematics', a review of *Remarks on the Foundations of Mathematics*, published in *The British Journal for the Philosophy of Science* (Vol. IX, No. 34, 1958). A review of *The Blue and Brown Books*, 'Wittgenstein's Theory and Practice of Philosophy', was published two years later in the same journal (Vol. XI, No. 43, 1960). These two early reviews were followed in the late 1970s by four further articles: the essay 'Der

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[1] The story of Kreisel's friendship with Wittgenstein is sketched in section 1.2, pp. 70-3, 80-1 and 88.

unheilvolle Einbruch der Logik in die Mathematik'[2] in *Essays on Wittgenstein in Honour of G. H. von Wright* (Vol. 28, Nos. 1-3, 1976); a lecture delivered on the 25th anniversary of Wittgenstein's death, later published as 'The Motto of "Philosophical Investigations" and the Philosophy of Proofs and Rules' in *Grazer Philosophische Studien* (Vol. 6, 1978); a further review article 'Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939', published in the *Bulletin of the American Mathematical Society* (Vol. 84, No. 1, 1978); and finally, a brief memoir 'Zu Wittgensteins Gesprächen und Vorlesungen über die Grundlagen der Mathematik', delivered as a lecture at the 2nd International Wittgenstein Symposium and printed in the *Proceedings* (Vienna, 1978). A decade later, Kreisel published a more detailed memoir 'Zu Einigen Gesprächen mit Wittgenstein', which was published in *Wittgenstein: Biographie, Philosophie, Praxis*, (*Wiener Secession*, 1988).[3]

With his having had the benefit of numerous conversations with Wittgenstein, Kreisel's critique in these articles might have been expected to be especially accurate and well informed, certainly when compared to the contributions by other mathematicians. Regrettably, Kreisel's interpretation of Wittgenstein in his first two reviews shares all the major faults of the contemporary

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[2] It is in English.

[3] Kreisel's two memoirs of Wittgenstein were used in section 1.2 and they are not examined below.

interpretations by other prominent mathematical logicians and philosophers of mathematics[4]. In particular, Wittgenstein is criticized for addressing technical questions beyond his mathematical competence, he is seen to be especially concerned with elementary mathematics and he is understood to be recommending a form of strict finitism.

These rather crude misunderstandings are partly explained in Kreisel's case by the strong feeling of reaction which he had against Wittgenstein some years after Wittgenstein's death, and which seems to have been connected with his getting to know Kurt Gödel: Kreisel felt that he had been deceived by Wittgenstein's 'sparkling mind' into overestimating the value of philosophy. After many years, he came to believe that this earlier reaction was rather exaggerated. Kreisel's change in attitude together with what he describes as his '"conversion" to the view of the silent majority'[5] seems to account for the improved quality of his later interpretation and criticism of Wittgenstein.

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[4] Paul Bernays, who published 'Betrachtungen zu Ludwig Wittgenstein's *Bemerkungen über die Grundlagen der Mathematik*', *Ratio* (Vol. II, No. 1, 1959), and Michael Dummett, who published 'Wittgenstein's Philosophy of Mathematics' (*Philosophical Review*, 68, 1959), were the most significant and influential of these authors. Their articles were both reprinted, the former in an English translation, in *Philosophy of Mathematics: Selected Readings* (Oxford, Blackwell, 1964), edited by P. Benacerraf and H. Putnam.

[5] In footnote 5 to the Appendix 'Proofs and Rules' in 'The Motto'. See also Kreisel's 'Autobiographical Remarks' in 'Der unheilvolle Einbruch', pp. 173-4.

These developments in Kreisel's understanding of Wittgenstein explain the dual nature of my interest in his work. I wish, firstly, to help demonstrate the baselessness of those early interpretations of Wittgenstein, due to Bernays, Kreisel and Dummett, in particular, which helped ensure the widespread misunderstanding of the remarks on the foundations of mathematics. Secondly, I wish to use some of the criticisms which appear in Kreisel's later articles to suggest various ways in which post-Wittgensteinian philosophy of mathematics might fruitfully develop.

Wittgenstein's success in the philosophy of mathematics ought to be measured, I believe, by his influence on the practice of mathematicians. For this reason, the attitude of mathematicians towards his work ought to be of particular interest to anyone wishing to promote his views. Kreisel's criticism of Wittgenstein has a special interest, therefore, because the opinions he expresses seem often to represent those of the working mathematician. Kreisel's position as both a respected former pupil of Wittgenstein and a mathematically informed critic of his work, also lends any study of the two men a certain amount of historical interest. Only R.L. Goodstein[6], another pupil of Wittgenstein who went on to specialize in the foundations of mathematics, is in a comparable position.

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[6] See section 1.2, p. 45.

The main difficulties which Kreisel's articles present to the philosophical reader are their rather offhand style and the richness of detail, mathematical and otherwise, which they contain. Neither of these features ought, however, to be used as an excuse for their neglect; although, in my case they are used as an excuse for a rather selective survey.

Kreisel's first and longest article on Wittgenstein is the review of *Remarks on the Foundations of Mathematics*, which was published shortly after the appearance of the first edition of that work. This article is examined particularly closely in what follows, because it introduces all of the themes which are developed in Kreisel's later articles, and because it treats in some mathematical detail the supposed technical deficiencies of Wittgenstein's work, thus allowing the charge of technical incompetence to be thoroughly examined.

Kreisel begins by making a distinction between 'philosophy of mathematics', concerning 'philosophically interesting differences between various parts or aspects of mathematics' and 'general philosophy', concerning 'philosophically interesting differences between mathematics and other intellectual activities'. According to Kreisel, Wittgenstein made contributions in both of these subject areas, and it is with this conception of the proper aims of 'general philosophy' and 'philosophy of mathematics' in mind that Kreisel aims in his review to draw attention to those 'observations' of Wittgenstein's 'which seem novel and stimulating, and to bring out their limitations'.

In fact, if Wittgenstein did make any contributions to the 'philosophy of mathematics', where that discipline concerns such topics as '"what is a constructive proof" or "what is a predicative concept"', they were clearly incidental to his main concerns. Wittgenstein's later remarks on the foundations of mathematics are, perhaps, best described as an application of his philosophical method in a particular subject area. The same method, according to Wittgenstein, is applied in his philosophical investigation of psychology. Kreisel's interpretation of Wittgenstein's remarks as contributions to the 'philosophy of mathematics' is a source of misunderstanding which, unfortunately, persists throughout all of his articles on Wittgenstein.

Kreisel's review divides into two unequal parts. The first part, headed 'General Philosophy', contains one section, 2, which has the aim of 'suggesting a framework for reading Wittgenstein's remarks on general philosophy'. The second part, headed 'Philosophy of Mathematics', although it contains more on 'general philosophy', is subdivided as follows: sections 3-4 contain criticisms of Wittgenstein's 'general views'; sections 5-6 contain background material, which, Kreisel believes, puts Wittgenstein's contribution in 'proper perspective'; sections 7-8, on strict-finitism and proof theory respectively, examine what are supposed to be Wittgenstein's 'significant contributions to the philosophy of mathematics', and in sections 9-12 Kreisel criticizes Wittgenstein's 'uninformed' comments on higher mathematics: Cantor's diagonal argument, section 9; Gödel's first incompleteness theorem, section 10; the consistency problem, section 11; and the paradoxes, section 12.[7] Finally in section 13, Kreisel adds a personal note.

(2)[8]. In the sections on 'general philosophy', Kreisel admits that he goes by 'impressions and quotations out of context' (1.2d). On the basis of more substantial research, i.e. a careful reading of Wittgenstein, it is readily apparent that Kreisel's account is unsatisfactory even as a 'rough review' (2.4). The theme he picks out, 'the limits of empiricism', is an important theme in Wittgenstein's writings, but his inadequate treatment of it makes detailed criticism inappropriate. I shall, however, remark on a number of the points which are made.

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[7] In a postscript to this review written in 1970, Kreisel mentions that 'systematic expositions of the points adumbrated in §§ 5-6, 9-12' are contained in 'Mathematical logic: What has it Done for the Philosophy of Mathematics' in *Bertrand Russell, Philosopher of the Century*, edited by R. Schoenman (London, Allen & Unwin, 1967).

[8] Bracketed numbers heading a paragraph in this way refer to Kreisel's section headings, which, unless otherwise stated, are the source for any quotations used in the commentary on that section.

Kreisel states that Wittgenstein 'is not prepared to use the notions of mathematical object and mathematical truth as tools in philosophy'[9], and he responds to Wittgenstein's objections in this way:

> 'To me the real objection to these notions is that, at any rate as far as I know, there does not exist a single significant development in philosophy based on them; in fact, some uses of these notions seem quite contentless such as the familiar "explanation" of the consistency of our mathematical results by saying that our results agree because we deal with the same objects. As I see it, the position is similar to that of the notions of the atom or absolute simultaneity at the time of the Greeks: neither of them could be used for understanding the world at the stage of technical and conceptual development of that time. ... In other words, the notion of a mathematical object is defective because one has no clue for using it to provide satisfactory answers to the (philosophically significant) questions which it should answer, but no case has been made that it cannot do so-rather like philosophy itself.'

These remarks make clear, straightaway, an important difference between the philosophical viewpoints of Kreisel and Wittgenstein. Wittgenstein believed that among contemporary mathematical logicians the notions of 'mathematical object' and 'mathematical truth' were often used in an aberrant, confused way which, according to his own conception, was typically philosophical. By removing such confusions, he believed that the philosopher makes a contribution towards the achievement of philosophical clarity in mathematics. Kreisel, by contrast, does not recognize the peculiar philosophical usage described by Wittgenstein; and so he does not recognize a need for

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[9] No doubt, Kreisel means to refer to Wittgenstein's treatment of 'what a mathematician is inclined to say about the objectivity and reality of mathematical facts' (*Philosophical Investigations* (1953), § 254).

critical methods to rid mathematics of philosophical confusion: any idea which is currently employed in an attempt to understand mathematics might, with the advancement of scientific knowledge, come to have genuine significance.

Wittgenstein often 'reminds' us in his writings of the relation between the meaning of a mathematical proposition and its proof. Kreisel comments (2.2):

> 'As Gödel has pointed out to me...the doctrine is supported at the level of computations where one considers symbolic operations with numerals in contrast to assertions about numbers considered as characteristics of sets: '5 + 7 = 12', at this level, means that this equation is the last of some sequence of equations obtained by the application of certain rules, and the proof goes just the one step further of exhibiting this sequence. But as soon as one regards numbers as characteristics of sets one can meaningfully ask whether certain computational rules are correct, and to this extent statements about numbers have meaning independent of the rules of proof considered. Quite generally, it is simply not true that proof is primary and theorem derived, that only the proof determines the content of a theorem. In fact, Wittgenstein is wrong in saying that generally we change our way of looking at a theorem during the proof (*RFM*, IV, 30)[10], but equally often we change our way of looking at the proof as a result of restating the theorem; e.g. if we are accustomed to the principle of proof that the totality of all subsets of a set is itself a set, we may reject it when it is pointed out to us that it is only valid for the notion of a combinatorial set and not, e.g. for the notion of a set as a rule of construction.'

The mathematical content of Gödel's suggestion seems clear enough. It is a mathematical result, or one might say a

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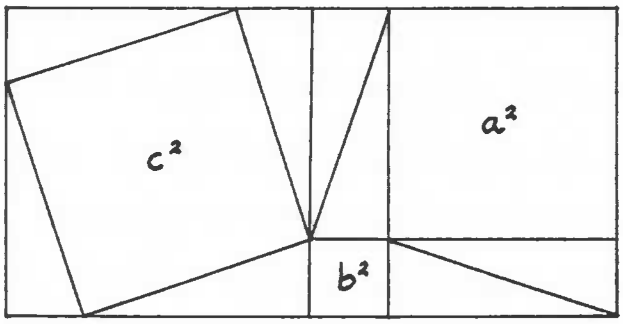
[10] Kreisel's page and section references to the first edition of *Remarks on the Foundations of Mathematics* have been replaced by chapter, section and, where appropriate, paragraph references to the third, revised edition of that work.

statement of several mathematical results, that certain axiomatizations of arithmetic are such that their theorems have corresponding theorems in an axiomatization of set theory. And if a definition of 'number' is given within an axiomatization of set theory, the properties of numbers so defined can certainly be compared with those defined within an axiomatization of arithmetic. This would be like comparing the properties of numbers in ordinary arithmetic with those in modulo arithmetic, for example. The possibility of using a correspondence between two axiomatic systems as a standard of correctness in the construction of one of the axiomatic systems is not, however, an objection to Wittgenstein's remarks about the conceptual relation between theorem and proof. Wittgenstein was attempting to combat a picture of mathematics in which proof was represented as altogether inessential to our understanding of mathematical propositions.

Kreisel's second point does not constitute a valid objection either. Firstly, although Kreisel is surely correct when he states that it is not *only* the proof which determines the content of a theorem, this point is explicitly recognised by Wittgenstein (*RFM*, II, 7; VII, 10e). Secondly, the example he takes from set theory does not invalidate Wittgenstein's claim that the meaning of a theorem is to be understood primarily by examination of its proof. The converse of this claim is that the meaning of a proof is to be understood primarily by examination of the theorem which it proves. In Kreisel's case, a theorem is restated, and so a certain principle employed in its proof becomes irrelevant. This is not a case where closer examination of a theorem enables us to understand its proof.

Genuine examples of symmetry between proof and theorem can be found, however, in diagrammatic proofs in arithmetic and geometry. Consider the following diagram, for example:



Couldn't this diagram be used both as the statement and as the proof that in a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides? This example is interesting, but the original philosophical problem seems now to have been forgotten. The mistake was already made when Kreisel formulated Wittgenstein's 'reminder' as a thesis about the general relation between theorem and proof in mathematics.

(3 & 4). Kreisel next examines Wittgenstein's 'general conclusions': that the traditional aims of philosophy are unattainable, that there can be no

mathematical foundation of mathematics and that philosophy should aim at a clarification of the grammar of mathematics (1.3b). Kreisel says: 'I do not accept his conclusions since I do not think that they are fruitful for further research', and he aims to show that: 'the value of the book does not lie in a new point of view, but in penetrating observations on a limited subject matter'.

In opposition to Wittgenstein, Kreisel believes that the aims of the traditional schools of philosophy are not to be rejected entirely but modified and then pursued. Any response to this suggestion depends, of course, on how the aims of the traditional schools of philosophy are formulated. The aim of refuting universal scepticism about mathematical truth, which concerned some philosophers following the discovery of the paradoxes, most notably Frege and Hilbert, is a meaningless one, whose incoherence is properly brought out, I believe, by the kinds of method which Wittgenstein developed. In contrast, the aim of improving the standards of mathematical rigour, an aim which Frege, Russell and Hilbert all shared, is obviously an intelligible one for mathematicians to pursue.

Kreisel states (3.2):

> 'Wittgenstein's views on mathematical logic are not worth much because he knew very little and what he knew was confined to the Frege-Russell line of goods. But it is true that the methods of mathematical logic have not been applied successfully to the subject of elementary computations, and this is precisely the subject which interested him most.'

While it is true to say that Wittgenstein's knowledge of mathematical logic was limited largely to the work of Frege and Russell, it is also true to say that his work consists largely in criticism of the work of these authors. However, the most satisfactory response to Kreisel's claim, is to indicate the precise nature of Wittgenstein's interest in mathematical foundations. Anticipating objectors who might doubt his competence to discuss the foundations of mathematics, Wittgenstein explained himself to his Cambridge audience of 1939 in this way:

> 'I can as a philosopher talk about mathematics because I will only deal with puzzles which arise from our ordinary, everyday language, such as "proof", "number", "series", "order", etc.
>
> Knowing our everyday language - this is one reason why I can talk about them. Another reason is that all the puzzles I will discuss can be exemplified by the most elementary mathematics - in calculations which we learn from ages six to fifteen, or in what we easily might have learned, for example, Cantor's proof.'[11]

Wittgenstein's frequent choice of elementary examples was governed by his belief that the type of phenomenon which concerned him can be illustrated at this level. To think that Wittgenstein was particularly concerned with elementary computations is therefore a definite mistake.

Kreisel also reveals in this section a prejudice which I am sure disinclines many mathematicians from approaching Wittgenstein's work. Remarking on 'the whole programme of clarifying the grammar', Kreisel says that it reminds him of the 'soft options' at school:

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[11] *Wittgenstein's Lectures*, Lecture I, p. 14.

> '...even if this conception turns out to be useful, there is no clear reason for rejecting the others, no more than that human geography should exclude scientific geology. There are such traditional problems as the genesis of our mathematical concepts, the justification of proofs, ie. what makes them correct rather than what makes them interesting, which are the fundamental concepts and which derived: we need a conceptual apparatus to formulate these questions in a satisfactory way, and we do not have such an apparatus. But it seems unlikely that the concepts favoured by Wittgenstein will provide it. It is by no means clear that these questions are ripe for precise formulation, any more than the general (and natural) questions of present day mathematics were ripe for a formulation at the time of the Greeks.'

Kreisel here underestimates the difficulty of the philosophical investigation of mathematics undertaken by Wittgenstein. It is by no means an *easy* option. One will not recognize this, of course, unless one appreciates the rigour of Wittgenstein's own thinking and along with that the highly wrought nature of his more finished texts, such as the revised mathematical typescript (TS **222**). Where Wittgenstein's ideas have been effectively employed by other philosophers, by Malcolm, for example, in the philosophy of psychology or by Winch in his investigation of social science, it has been due to their skill and sensitivity in the handling of philosophical questions and not their ability to repeat various of Wittgenstein's 'slogans' in the appropriate contexts. It must also be said that the concepts favoured by Wittgenstein are not designed to provide a framework for the formulation of the 'traditional problems' listed by Kreisel; they are designed to provide a framework for the criticism of incoherent responses to these problems.

(4). Kreisel continues the previously quoted passage as follows:

> 'There is another, less austere conception of the philosophy of mathematics, which Wittgenstein ignores too. Since this conception, it seems to me, underlies most of current work in mathematical logic, he implicitly rejects it by rejecting mathematical logic. As mathematics has grown, a variety of different methods of proof, definitions, theorems have accumulated. By the light of nature we see differences, groupings within one branch, and similarities between different branches of mathematics. One may see one aim of a philosophy of mathematics in getting a clear understanding of these connections, and there is no reason in advance why this should be done only by reference to "applications", and not, e.g. by mathematical properties, by mathematical characterisations. From this point of view it is a contribution to the philosophy of mathematics if a new aspect of the methods of mathematics has been noticed...; here there is no one fundamental problem. I regard the "rival" philosophies of mathematics in this light: not as contradictory in substance, but as emphasising different aspects of mathematics...'

Despite what Kreisel says, it is not at all obvious that this 'less austere' conception of the philosophy of mathematics is inconsistent with any stated doctrine of Wittgenstein's, although it does seem to fall outside of the conception of philosophy which he developed. Wittgenstein did not reject mathematical logic; it would be more accurate to say that he believed that certain aspects of the influence of contemporary mathematical logic on mathematics were harmful. His critique must be understood in the precise form in which it was stated, and that is largely as a criticism of the works of Frege and Russell.

(5 & 6). In the following two sections, Kreisel attempts to give 'some rough indications of research into

the foundations of mathematics' in order to 'put Wittgenstein's observations in perspective'. Kreisel believes that 'the aspects emphasized by him are few among many' (1.1).

Kreisel's approach here accords with his determination to regard the 'rival' philosophies of mathematics 'not as contradictory in substance, but as emphasising different aspects of mathematics'. This is possible in the case of logicism, formalism and intuitionism, I would maintain, because technical developments are standardly referred to in the statement of those philosophies. In contrast, Wittgenstein's philosophy of mathematics, being entirely critical, has no associated technical developments. This explains why Kreisel is forced to regard Wittgenstein's remarks merely as 'penetrating observations and questions on a limited subject matter', i.e. as suggestions towards technical developments in strict finitist mathematics.

(5). Kreisel says:

> 'Abstract set theory provides the most famous of all foundations of mathematics. The remarkable fact is this: each known branch of mathematics has a model in abstract set theory, and frequently, a most natural model. Thus, e.g. the question "what is a number" to which it is hard to give a natural meaning, gets the answer: an element of the set which is the intersection of all inductive sets.'

He goes on:

> '...the development of arithmetic within set theory has not only helped us to understand arithmetic better, but has had important repercussions on the study of set theory and logic generally, e.g. it permits the application of Gödel's incompleteness

> theorem to systems of set theory, and related results.'

The latter discoveries do not, Kreisel observes, satisfy the philosopher 'who seeks a "simpler" foundation, who looks for the fundamental concepts in mathematics and wants to build up derived ones'. Kreisel is, however, clearly suspicious of this tendency: there is 'no obvious order in which abstract sets precede numbers' and 'as e.g. Poincaré pointed out with great lucidity, the reduction of arithmetic to set theory itself requires the processes of arithmetic'. Kreisel states that, by contrast, Wittgenstein 'emphasises strongly' the mathematical significance of the selection of fundamental concepts.

Wittgenstein did not deny the mathematical interest of the development of arithmetic within set theory, although he did deny that it had enabled some of his contemporaries to understand arithmetic better. Also, Wittgenstein's objections to logicism (see, for example, *RFM*, III, 14) clearly resemble Poincaré's objection, and Kreisel can only have failed to recognise this because of his desire to ascribe to Wittgenstein a constructivist viewpoint. Finally, in his later writings Wittgenstein nowhere emphasizes the mathematical significance of the selection of fundamental concepts. On the contrary, he rejects the very notion that there are fundamental concepts in mathematics. This contrasts with his earlier position in the *Tractatus*, where 'number' was introduced as being fundamental.[12]

(6). Kreisel attempts in the next section to place Wittgenstein's work in the context of the various constructivist tendencies in the foundations of mathematics. He distinguishes between intuitionism, as developed by Brouwer and Heyting, finitism, and strict finitism, 'as described by Bernays in "Sur le platonisme dans les mathématiques"'. He remarks:

> 'Wittgenstein's views seem related and favourable to intuitionism, probably mainly because of common features such as the objection to the idea of a mathematical object, the priority attached to proofs over theorems..., and the use of Brouwer's household example of the decimal expansion of pi (*RFM*, V, 9 or 19 or 27). But a closer look shows that this similarity is superficial, and that Wittgenstein's views on mathematics are near those of strict finitism; or, perhaps one should say, he concentrates on the strictly finitist aspects of mathematics.'

The similarity between the views of Brouwer and Wittgenstein is certainly superficial, although there are some genuine connections: both reacted against Cantorian set theory and Hilbert's formalist conception of mathematics. Kreisel provides no justification, however, for saying that Wittgenstein concentrates on the 'strictly finitist aspects of mathematics', as I shall demonstrate.

Kreisel contrasts intuitionism and finitism, pointing out that intuitionism 'goes beyond finitism because it

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[12] See *Tractatus* 6 - 6.022 and *Philosophical Grammar* (TS 213), section 109, p. 540 (*PG*, II, 12, p. 296).

makes statements concerning all possible constructions'. He says that 'it is unprofitable to compare Wittgenstein's views and intuitionism' for 'in his simple computational examples we are dealing with strictly combinatorial processes, and the typically intuitionist concepts do not apply: he leaves off before intuitionism starts'. He adds that 'all the mathematics which Wittgenstein considers clear (not e.g. the completeness of the set of real numbers) fits comfortably within the narrow framework of finitist mathematics', and that 'an even narrower aspect of mathematics is considered by him': strict finitism[13].

(7 & 8). The following two sections thus examine 'Wittgenstein's significant contributions to the philosophy of mathematics', which concern 'very elementary computations' (1.1). Wittgenstein is supposed to be concerned to distinguish between 'constructions which consist of a finite number of steps and those which can actually carried out, or between configurations which consist of a finite number of discrete parts and those which can actually be kept in mind (or surveyed)'.

Kreisel notes that 'within any degree of sophistication of proofs a clarification according to degree of complexity is most natural', and in his only

[13] Bernays and Dummett both agree: Bernays says that 'Wittgenstein maintains everywhere a standpoint of strict finitism' (*Philosophy of Mathematics*, p. 519); while Dummett talks of a 'constructivism, more severe than any version yet proposed' (Ibid., p. 505).

reference to Wittgenstein's text, he says: 'Wittgenstein stresses (*RFM*, III, 2) the further point that explicit definitions and new notations may convert a piece of mathematics which is not strictly finitist into one'.

It is clear that Kreisel has misread Wittgenstein's remarks on surveyability. The passage he refers to reads:

> 'I want to say: if you have a proof-pattern that cannot be taken in, and by a change in notation you convert it into one that can, then you are producing a proof, where there was none before.'

Wittgenstein attempts in Volumes XVIII and XIII(5) (MSS **122** & **117(5)**), from which this remark is taken, to characterize a general feature of mathematical proofs, and he refers to this feature when attempting to formulate objections to the view that the system of *Principia Mathematica* is a reduction of elementary arithmetic. Wittgenstein's point is easily clarified by reference to other remarks in Volumes XVIII and XIII(5), for example (*RFM*, III, 39):

> '"Proof must be capable of being taken in" really means nothing but: a proof is not an experiment. We do not accept the result of a proof because it results once, or because it often results. But we see in the proof the reason for saying that this *must* be the result.'

There is here no suggestion that the standards of mathematical proof should be revised in favour of strict finitist standards, and this would, anyway, be clearly against the spirit of Wittgenstein's critical philosophy, against the spirit of not denying anything.

(8). Kreisel's remarks in his next section are premissed on the assumption that Wittgenstein is a strict finitist. Here Kreisel attempts to apply what Wittgenstein says to 'the general notion of "equivalence of proofs" or "content of proofs"' (7.1).

Referring to Wittgenstein's 'criticism of the reduction of arithmetic to logic and to some remarks of his on non-constructive existence proofs in analysis', Kreisel remarks (8.1):

> 'Wittgenstein's first point concerning the reduction of numerical arithmetic to logic with identity is this: we do not really have a reduction here because, by the methods of logic alone, we could not decide whether a particular formula of logic corresponds to some given formula of numerical arithmetic.'

Kreisel thus believes that what Wittgenstein says is 'acceptable' and 'familiar': 'it concerns the metamathematical methods used for investigating relations between two systems. We do not speak of a "reduction" unless the metamathematical methods are weaker in some suitable sense or, at least, more evident than the methods studied'.

This clearly is not Wittgenstein's point. In the passage mentioned (*RFM*, III, 3) Wittgenstein is asking whether a correspondence between two proofs in different calculi has been demonstrated when the proof in one calculus is no longer surveyable, i.e. no longer a proof, in the other. To the rhetorical suggestion that they correspond, because one has been derived from the other by a given method, Wittgenstein responds with a doubt which applies to any method regardless of its strength. No method can derive an unsurveyable proof from a surveyable one, for 'if I look at it again half an hour later, may it not have altered?'.

Kreisel also says (8.2): 'Wittgenstein repeatedly raises the question of characterising the equivalence of proofs in contrast to equivalence of results (*RFM*, III, 8, 9 and 3c)'. Wittgenstein did not, however, wish to characterize the equivalence of proofs in *contrast* to the equivalence of results; he wanted to insist, with certain qualifications, that equivalence of results just *is* equivalence of proofs. He was not, therefore, interested in developing technical criteria for determining the equivalence of proofs.

There are, however, unresolved difficulties in Wittgenstein's account, which would probably be illuminated by a wider survey of realistic examples. In particular, there is the problem of how to understand the situation where a theorem has two different proofs (*RFM*, III, 58ff). And here there are two different cases, where the proofs are in different systems[14] and where the proofs are in the same system (*RFM*, III, 60).

[14] Shoenfield's case, which Kreisel mentions, is a variation on this case, where in one system the theorem is an axiom, and so has a trivial proof.

Kreisel proceeds (8.3a) to comment on a section from one of the wartime notebooks (*RFM*, V, 46) where Wittgenstein mentions non-constructive existence proofs. Wittgenstein remarks here on the vagueness of the notion of mathematical understanding, and he continues:

> 'Hence the issue whether an existence-proof which is not a construction is a real proof of existence. That is, the question arises: Do I *understand* the proposition "There is ..." when I have no possibility of finding where it exists? And here there are two points of view: as an English sentence for example I understand it, so far, that is, as I can explain it (and note how far my explanation goes). But what can I do with it? Well, not what I can do with a constructive proof. And in so far as what I can do with the proposition is the criterion of understanding it, thus far it is not clear *in advance* whether and to what extent I understand it.'

And part of Wittgenstein's objection to the influence of mathematical logic within mathematics is that because we can write it in a mathematical symbolism we 'feel obliged to understand it' (cf. *RFM*, V, 13c).

In his discussion of this remark, Kreisel uses as examples the proofs of (Ex)Ax and (Ex)(y)B(x,y), where the variables range over the natural numbers. In the first case I can expect, if A is recursive, to read off from the proof instructions for calculating an n such that A(n). In the second case, -(x)(Ey)-B(x,y) is proved by *reductio ad absurdum*, and here it is a reformulation of the proof eliminating the non-constructive use of the quantifier which shows, according to Kreisel, what we can do with it[15]. Kreisel believes (8.3b), however, that this is a special case: 'Suppose I have a finitist and a non-finitist proof of a universal formula (x)A(x): what can I "do" with the former which I cannot do with the latter?' And referring to the 'fan theorem', he remarks: 'I do not see any "practical purpose" or considerations of "usefulness" which could decide between the two proofs'.

Wittgenstein believed that the essential difference between constructive and non-constructive existence proofs was not properly recognized by many contemporary mathematical logicians and that recognition of this difference was hindered by the use of modern logical notation in the statement of mathematical theorems. He thus wanted to emphasize the difference between the two types of proof, and he suggested that we direct our attention towards their 'use', or their 'employment'. Whether Wittgenstein would have felt that the finitist proof of the fan theorem was clear or not we do not know. We do know, however, that he would not want us to 'decide between' two proofs on the basis of their 'usefulness'; he would want rather that we be sensitive to the difference in their meanings. In extreme cases we might question whether two proofs ought really to be regarded as proofs of the same theorem (*RFM*, III, 62).

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[15] Kreisel thus objects to Wittgenstein's remark (*RFM*, V, 52) that a philosopher should not reformulate proofs.

Kreisel's next remark is rather more perceptive (8.4):

> 'It is my impression that the emphasis on "application" or "on what we do with it" aims at unifying our point of view: we are to look at two conceptions like the classical and intuitionist conception of mathematics, and find a place for each from our point of view, namely: according to their applications.'

Mathematicians, Kreisel believes, pretend to a similar criterion of judgement, namely: 'mathematical fruitfulness'. The truth in Kreisel's remark is that Wittgenstein would not have recognized intuitionist mathematics as separable from *mathematics*, and the same holds for Hilbert's 'metamathematics' and Cantorian set theory. In each case an application of mathematical calculi which was fully intelligible remained to be found, and in the most extreme case it was only the formal (geometrical) aspects of a mathematical calculus, the 'game with signs according to rules', which was as yet fully intelligible (*RFM*, V, 5 & 7 *passim*).

(9-12). The remainder of Kreisel's review deals with what he describes as 'isolated topics': Cantor's diagonal argument, Gödel's first incompleteness theorem, the consistency problem, and the paradoxes. It is clear from a careful reading of Wittgenstein that these topics are not in fact isolated from the rest of Wittgenstein's discussion at all, but are deliberately chosen to illustrate his major themes and the application of his critical method. Kreisel, however, makes 'no attempt to relate them to a general point of view', and so fails in each case, as I will show, to understand Wittgenstein's point.

(9). Referring to Wittgenstein's discussion of Cantor in *RFM* II, Kreisel argues:

> 'Wittgenstein says (*RFM*, II, 29-32) that it was the diagonal argument which gave sense to the assertion that the set of all sequences (of natural numbers) is not enumerable. The definition is: a set of sequences a(1),...a(m),... is enumerable if there is a double sequence s(n,1),... s(n,m)..., n = 1,2,... with the following property: for each sequence of the set, there is an n_a such that a(1),... is identical with s(n_a,1),..., i.e. for all m, a(m) = s(n_a,m) And the diagonal argument states, that the set of all sequences is not enumerable because (i) if s(n,m) is a double sequence, then s(n,n) + 1 is also a sequence and so (ii) any proposed enumeration s(n,m) fails to include one sequence, namely s(n,n) + 1. One could only wish that all one's assertions had as much sense as the assertion of the non-enumerability of the set of all sequences before its proof!'

Assuming that we are employing the definition of enumerability which Kreisel presents here and that we are ignorant of Cantor's diagonal method, what ought we to say if that method were now used to provide us with a diagonal sequence? Ought we to accept that the diagonal sequence is a 'sequence'? And that it is 'different' from all the others? (*RFM*, II, 34).

One problem with restating Cantor's diagonal argument, or at least, stating it in a form which Wittgenstein does not discuss, as Kreisel has done, is that Wittgenstein's remarks concern the precise form in which that argument was presented and understood by various of his contemporaries, and it was quite particular features of this presentation which Wittgenstein emphasized in his own discussion.

The remarks of Wittgenstein's which Kreisel refers to in this passage are about expansions understood as numbers and not merely as sets of sequences, and these remarks concern the question whether the diagonal procedure *shows* us a real number different from all those in the system or whether it *gives sense* to the statement 'real number different from all those in the system':

> 'Cantor shows that if we have a system of expansions it makes sense to speak of an expansion that is different from them all.--But that is not enough to determine the grammar of the word "expansion"'. (*RFM*, II, 30)

That is to say, a comprehensive grammar of real numbers is not established by the diagonal procedure, and so it cannot be said that this procedure has introduced us to a new real number. Wittgenstein believed his contemporaries were inclined to misunderstand this and so misunderstand the nature of mathematical concept formation.

Wittgenstein allows, of course, that there is a proper, 'sober' statement of the result of Cantor's diagonal method applied to the real numbers (*RFM*, II, 20), but he was concerned to bring out aspects of the contemporary lack of philosophical clarity. The recurrent themes of the relation between proof and proposition and, as in the following passage, of the distinction between the mathematical and the physical are again prominent:

> 'The usual expression creates the fiction of a procedure, a method of ordering which, though applicable here, nevertheless fails to reach its goal because of the number of objects involved, which is greater than the number of all cardinal numbers[16].
>
> If it were said: "Consideration of the diagonal procedure shews you that the *concept* 'real number' has much less analogy with the concept 'cardinal number' than we, being misled by certain analogies, are inclined to believe", that would have a good and honest sense. But just the *opposite* happens: one pretends to compare the "set" of real numbers in magnitude with that of the cardinal numbers. The difference in kind between the two conceptions is represented, by a skew form of expression, as difference of extension. I believe, and hope, that a future generation will laugh at this hocus pocus.' (*RFM*,II,22)

In general, Kreisel fails to recognize that Wittgenstein's critique cannot be understood independently of its object, and that the cogency of his critique must be judged, first of all, on that basis. Its value in the future might come from the fact that a similar critique is still appropriate or from its being a good example of the application of a critical method. (Of course, it might be that the application is seen to have been misguided.) A tendency to picture mathematics as if it were 'the natural history of mathematical objects', or perhaps more generally, a tendency to conflate the mathematical and the physical, seems likely to be a permanent feature of our thinking about mathematics, although even here the precise form of the critique of this tendency at any time must be sensitive to the precise form in which it receives expression at that time.

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[16] Wittgenstein's usage of the term 'cardinal number' is not standard and corresponds to 'finite cardinal number'.

(10). In these last sections on higher mathematics, Kreisel is notably most extreme in his criticism of Wittgenstein's remarks on Gödel's theorem[17]:

> 'Wittgenstein criticises Gödel's first incompleteness theorem; or, at least, the part which states that if a suitable system of arithmetic is consistent then there is a true formula of the form (n)A(n) which is not provable in the system; the formula is one with number q which states: for every n, n is not the number of a proof of the formula with number q, i.e. a proof of itself. The arguments are wild, including such points as an inconsistency wouldn't matter (*RFM*, I, App. III, 11), or how do we know that this is the correct translation of the arithmetic formula (n)A(n) (*RFM*, I, App. III, 10), or what does it mean to suppose that a formula is provable (*RFM*, VII, 22g). Even if an inconsistency didn't "matter", one cannot hope to discuss significantly on this basis a result which explicitly supposes consistency of the system. Why does he think that the elaborate details of Gödel's paper are needed? Just because Gödel has to show that on the assumption of consistency the proposed translation is correct. And finally, one of the major purposes of considering formal systems (and it is formal systems which Gödel considers) is that a clear combinatorial (geometric) meaning is given to a formula being provable.'

Once again, Kreisel has paid too little attention to what Wittgenstein actually said. It is clear that Wittgenstein does not criticize Gödel's proof of the incompleteness theorem but Gödel's philosophical interpretation of its result, specifically the claim that the proof involves the construction of a true but unprovable proposition. He thus criticizes what is essentially the prose formulation of the result which is contained in the first part of Gödel's famous paper 'Über formal unentscheidbare Sätze der

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[17] Similarly, Dummett says that Wittgenstein's passages on this topic (and on the topic of consistency) are among those which 'are of poor quality or contain definite errors', *Philosophy of Mathematics*, p. 491.

Principia Mathematica und verwandter Systeme I' (1931)[18]. Kreisel fails, however, to recognize Wittgenstein's specific interest in the topic and looks to question his understanding of the proof.

According to Kreisel, Wittgenstein did not appreciate that Gödel's proof involves an assumption of consistency[19]. His criticism is directed at remarks 11-14 (and 17) in *RFM*, I, App. III, where Wittgenstein discusses contradiction in connection with the interpretation of Gödel's theorem. Wittgenstein here supposes (11) that the proof of the unprovability of the Gödelian proposition, P, is a proof in Russell's system, so that it makes sense to say that we have proved both P and -P. Having introduced the topic of contradiction, Wittgenstein embarks on a brief digression. In sections 11-13, he makes points, similar to those made in earlier writings and conversations, before and during 1931, questioning the attitude of mathematicians towards contradiction.[20] In section 14, he describes a proof of unprovability as 'a proof concerning the geometry of

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[18] Reprinted in an English translation in *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931* (Cambridge, Harvard University Press, 1967).

[19] Bernays, similarly, doubts whether Wittgenstein 'is sufficiently well aware of the rôle played by the condition of consistency in the reasoning of proof-theory'. He says: 'the discussion of Gödel's theorem of non-derivability in particular suffers from the defect that Gödel's quite explicit premiss of the consistency of the considered formal system is ignored', *Philosophy of Mathematics*, pp. 522-23.

[20] See, in particular, the sections headed 'Consistency' in *Ludwig Wittgenstein and the Vienna Circle*.

proofs' comparable to a proof concerning the impossibility of a geometrical construction. He points out that such a proof contains an element of prediction which cannot be expressed by a contradiction[21]. None of these remarks is about the notion of formal consistency involved in Gödel's proof.

Wittgenstein's remarks (11-14, 17) are made relevant by Gödel's failure in the interpretation of his incompleteness theorem to distinguish rigorously between the different sorts of proposition involved in its proof: propositions of the formal system are understood simultaneously as propositions of mathematics and propositions about the formal system. Gödel thus states that he has constructed a true but unprovable proposition[22]; and it is because of this that Wittgenstein can assume for the purposes of his discussion that the situation is one in which there is a genuine logical contradiction; and, in general, it allows him to outline the proof in the crudest possible way, using a single

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[21] Bernays finds this remark strange and points out that: 'Such proofs of impossibility always proceed by the deduction of a contradiction' (*Philosophy of Mathematics*, p. 523). The *result* of a proof of impossibility is not, however, a contradiction; one of the assumptions which led to the contradiction is rejected.

[22] *From Frege to Gödel*, pp. 598-9.

letter,'P', to stand for all the different sorts of proposition involved[23].

Wittgenstein's remarks thus bring out Gödel's failure to understand his own assumption of consistency when attempting to explain the philosophical interpretation of his first incompleteness theorem, and they do not show that Wittgenstein failed to recognize the assumption of formal consistency involved in the proof of that theorem.

Kreisel also criticizes Wittgenstein for suggesting that the correctness of the translation of the arithmetic formula (n)A(n) might be questioned and for asking what it means to suppose that a formula is provable.

Concerning the first point, Wittgenstein considers, in the section to which Kreisel refers, and in more detail in *RFM*, VII, 22, what should be said if P were proved in Russell's system. To consider such a possibility is obviously to consider our having made a mistake in the construction of P, which we believed ensured its formal unprovability. Now, Wittgenstein was well aware of the rigorous nature of Gödel's demonstration and of the 'geometric' notion of proof involved in the construction of

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[23] In his statement of Gödel's result, Kreisel also confuses different types of proposition, identifying the 'true formula of the form (n)A(n)', a proposition of mathematics, with the proposition which states that 'for every n, n is not the number of a proof of the formula with number q', a proposition about the formal system.

the Gödelian formula, and he wasn't suggesting that a mistake had been made in the construction of P, but he did find it instructive to consider what ought to be said if P were proved in Russell's system, or proved 'directly', as he would put it. (*RFM*, I, App. III, 17). This forces us to consider how the *two* propositions, 'the proof concerning the geometry of proofs' and the constructed formula, acquire their meanings and how these meanings are related. Ultimately he wants to question whether it is appropriate to talk of a proposition having been proved at all; hence the preliminary and final remarks of the sequence on Gödel.

Concerning the second point, what does Wittgenstein mean when he says (*RFM*, VII, 22g) that his 'task as far as concerns Gödel's proof seems merely to consist in making clear what such a proposition as: "Suppose this could be proved" means in mathematics'? Kreisel complains that in the case of the Gödelian proposition a perfectly clear geometric notion of proof is involved and indeed that this was one of the very reasons why Gödel considered formal systems.

Wittgenstein obviously thought that Gödel's argumentation illustrated certain philosophical confusions about the meaning of mathematical supposition. Again, one has to look at the informal introductory argument that prefaces the main proof. Gödel argues that P is not provable in Russell's system and as part of the argument he says that to suppose P's provability is to suppose its truth; but here the meaning of what is supposed to have been proved shifts from the proposition of the formal system to the proposition about the formal system, i.e. to the statement that P is not provable. Wittgenstein wants to insist, however, that the meaning of a mathematical supposition depends on what one supposes to be its method of proof just as much as the meaning of a mathematical theorem depends on its proof; so that this shift in meaning is illegitimate. Again he is attempting to bring out the incoherence of Gödel's interpretation of P as at one time a statement about a formal arithmetical calculus at another as a formal proposition and at yet another as a statement of number theory.

Rather than misunderstanding Gödel's theorem, I should say that Wittgenstein succeeded brilliantly in exposing some of the incoherences in its contemporary philosophical interpretation.

(11). Kreisel describes Wittgenstein's 'criticism of the consistency problem' as ranging from 'a proposal to use the double negation as an enforced negation' (referring to *RFM*, App. III, 18) 'to the proposal of not drawing conclusions from a contradiction' (referring to *RFM*, VII, 15n) 'to modifications in our arguments after we reach a contradiction'.

Kreisel's first reference is particularly careless[24], and in neither passage mentioned can Wittgenstein seriously be interpreted as making a technical proposal for the solution of the consistency problem, as Kreisel implies. Wittgenstein wanted rather to suggest ways in which the prevailing attitude to contradiction might be altered. He believed that mathematicians had become obsessed with contradiction and he wanted to show that its importance had been exaggerated to the point of incoherence; he wanted to overcome the attitude that a contradiction necessarily 'destroys' the calculus in which it appears (*RFM*, VII, 15i).

Meaning to correct Wittgenstein, however, Kreisel refers to the 'specific point' of Hilbert's programme:

> '...if one can prove, by limited ("understood") methods, the consistency of a system S, then if a universal recursive formula can be proved in S, then the same formula can be proved by the limited methods. Since he considered more elaborate formulae as "ideal" elements, with no hope of assigning a clear constructive meaning to them, this was the most he could expect from foundations of arithmetic.'

Here, Kreisel demonstrates an inclination to say what Hilbert really meant, ignoring what seems to him inept or

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[24] In *RFM*, I, App. III, 18, Wittgenstein is surely making the point that the Gödelian proposition, P, can be supposed to be false in a way that is totally irrelevant; he is thus highlighting the fact that just how it is supposed false is of crucial importance. Similarly, the case where we use double negation as a strengthening of the negation is a trivial exception to --p = p.

foolish in Hilbert's writings[25]. Hilbert's main goal was 'to establish once and for all the certitude of mathematical methods'[26], and the 'specific point' of Hilbert's programme, summarized here by Kreisel, was supposed to be understood to achieve this. In examining Hilbert's claims one has, therefore, to address questions about the relationship between the particular formal system in which the consistency proof is carried out and actual arithmetic, and so questions about the nature of formal arithmetical statements and the status of 'metamathematics' are crucial. How else are we to understand what Hilbert meant by a foundation of arithmetic? Without this understanding, one certainly will not appreciate Wittgenstein's subtle investigations.

Kreisel does acknowledge the one-sidedness of the consistency problem (11.1), which Wittgenstein felt ought to arouse our suspicion, but he concentrates on it from the point of view of its mathematical interest, and so misses Wittgenstein's point.

What Kreisel states in the next paragraph (11.2), however, shows some good sense: an inconsistency in mathematics can simply be removed by modifying the

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[25] See also his article "Hilbert's Programme", *Dialectica*, 12 (1958), which is reprinted with revisions in *Philosophy of Mathematics*.

[26] 'On the Infinite', *Philosophy of Mathematics*, p. 184.

calculus: 'Cantor's proofs still stand although they can be embedded in Frege's inconsistent system', and in applied mathematics 'there are ambiguities and uncertainties in the physical assumptions, so why not put up with similar features in the mathematical manipulation?' He also refers appropriately to *RFM*, III, 82e:

> 'My aim is to alter the attitude to contradiction and to consistency proofs. (*Not* to shew that this proof shews something unimportant. How *could* that be so?)'

Unfortunately, Kreisel understands this as agreeing with the claim that 'proofs of consistency and, more generally, of independence, yield, perhaps, a better control over a calculus than anything else'; whereas Wittgenstein simply wants to insist that he is not denying anything.

(12). Contrary to Kreisel's claim that Wittgenstein 'did not know what to say about the paradoxes', Wittgenstein was always clear on the fundamental point that the paradoxes discovered in logic and the foundations of mathematics at around the beginning of the century had none of the calamitous consequences suggested in the writings of Frege and Hilbert. Also, a genuinely mathematical interest in contradictions is something which Wittgenstein had foreseen[27]; and he would have had no objection, of course, to mathematicians 'getting something out of them'. That, however, was not part of his critical task; he merely encouraged a new attitude towards the discovery of

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[27] *Ludwig Wittgenstein and the Vienna Circle*, 'Hilbert's Proof', p. 139.

contradiction. Thus, he suggested, vaguely, that Russell's contradiction might be conceived as a Janus head standing like a monument over the propositions of logic. The mathematics is left for mathematicians to work out[28].

(13). Kreisel's review ends with a personal note, which contains, besides a further misunderstanding[29], the following statement:

> 'I did not enjoy reading the present book. Of course I do not know what I should have thought of it fifteen years ago; now it seems to me to be a surprisingly insignificant product of a sparkling mind.'

Though repeated by some as an authoritative evaluation of Wittgenstein's remarks on the foundations of mathematics, it is clear that the author had no real understanding of Wittgenstein's philosophy.

This conclusion prompts the question: Why did Wittgenstein have such a high opinion of Kreisel's ability?

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[28] Kreisel suggests, interestingly, that 'the Russell paradox involves the same argument as the theorem that there is no greatest integer' (integer understood according to a set-theoretical definition).

[29] When Wittgenstein talks of 'a solid core to all these glistening concept formations' (*RFM*, V, 16f), he is not referring to such things as seeing 'the mathematical core of the Chinese remainder theorem in a certain result of cohomology theory', as Kreisel believes, but to the 'methods of proof' opposed to 'conceptual confusion' of the sort which he regarded as typically philosophical (*PI* (1953), Part II, final remark.) That is to say, the Chinese remainder theorem already has an intelligible application, whereas, according to Wittgenstein, the interpretation of Cantorian set theory among his contemporaries was quite fantastic (*RFM*, V, 5a).

Why did he say to Rhees that Kreisel was 'the most able philosopher he had ever met who was also a mathematician'?[30] It is known that Kreisel, unlike Rhees, was not shown Wittgenstein's current work on the *Philosophical Investigations*, but he was shown the *Blue Book*[31]; so it might be in Kreisel's next article, the review of *The Blue and Brown Books*, that the answer to this puzzle is to be found.

In the introductory comments, Kreisel states that *The Blue and Brown Books* made a 'vastly superior impression'[32] on him than the *Remarks on the Foundations of Mathematics*. His approach to their interpretation, however, is no better here than it was in the previous case. He does not try to relate the contents of the books to Wittgenstein's philosophical aims, 'because...they are bad ones'.[33] Instead, opposing himself to Wittgenstein's view that philosophy is a *method*, which can be applied to various disciplines, he tries to see 'what emerges from these books

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30 See section 1.2, p. 80.

31 See section 1.2, p. 72.

32 'Wittgenstein's Theory and Practice of Philosophy', p. 238.

33 Ibid. Bernays and Dummett dismissed Wittgenstein's own stated aims in a similar fashion and with similar consequences for the quality of their interpretations. Bernays chose, for example, to reject Wittgenstein's claim that he was not 'a behaviourist in disguise', (*Philosophy of Mathematics*, p. 511), while Dummett insisted that Wittgenstein's claim not to be interfering with the mathematicians was 'not to be taken too seriously' (Ibid., p.493).

for philosophy as a discipline in its own right, or, better, as several such disciplines'.

Before undertaking this task, however, Kreisel attempts to 'convey something of the first impression'[34]. He examines the purpose and value of Wittgenstein's clarification of concepts and of his use of 'language-games', and he also attempts a general characterization of Wittgenstein's practice of philosophy. Here Kreisel is strongly influenced by Bernays' review of *Remarks on the Foundations of Mathematics*, which had been published in the previous year.

Kreisel accepts that many of Wittgenstein's language-games, and 'other imagined situations', 'illustrate vividly (i) possibilities that may easily be overlooked when one hears for the first time traditional philosophical problems and views or (ii) associations which they perhaps unconsciously, arouse'[35], but he doubts the scientific value of such illustrations. He says that 'when the subject is developed and more is known, the puzzle takes care of itself'[36], and that when the subject is not developed Wittgenstein's method is likely to be merely confusing.

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[34] 'Wittgenstein's Theory', p. 238.

[35] Ibid.

[36] Ibid., p. 240.

Wittgenstein's clearest and most explicit rejection of such a view comes not in the *Blue Book*, but later, in the *Philosophical Investigations*:

> 'The confusion and barrenness of psychology is not to be explained by calling it a "young science"; its state is not comparable with that of physics, for instance, in its beginnings. (Rather with that of certain branches of mathematics. Set theory.)'[37]

Wittgenstein was not concerned with puzzles arising in the early development of a science nor indeed with any puzzles involving mathematical technicalities; these did not have the peculiarly 'tenacious'[38] character of the sort which interested him. The puzzles which Wittgenstein investigated, mainly, but not exclusively, in psychology and mathematics, were not thought by him to result merely from ignorance, they were each thought to have a deeper cause, which was to be found in the grammar of our language.

Kreisel admits that Wittgenstein's examples show that 'often, when we say "I mean X" no particular mental act of meaning is involved' and 'that sometimes a substantive does not denote a material object or a sensation', but he complains that 'Wittgenstein does not tell us for which further study this clarification is to make room'.[39] The

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[37] *Philosophical Investigations* (1953), Part II, final remark.

[38] *Wittgenstein's Lectures*, Lecture I, pp.14-15.

[39] 'Wittgenstein's Theory', p. 240.

following passage from the *Blue Book* on Frege and the formalists is instructive here:

> 'Frege ridiculed the formalist conception of mathematics by saying that the formalists confused the unimportant thing, the sign, with the important, the meaning. Surely, one wishes to say, mathematics does not treat of dashes on a bit of paper. Frege's idea could be expressed thus: the propositions of mathematics, if they were just complexes of dashes, would be dead and utterly uninteresting, whereas they obviously have a kind of life. And the same, of course, could be said of any proposition: Without a sense, or without the thought, a proposition would be an utterly dead and trivial thing. And further it seems clear that no adding of inorganic signs can make the proposition live. And the conclusion which one draws from this is that what must be added to the dead signs in order to make a live proposition is something immaterial, with properties different from all mere signs.
>
> But if we had to name anything which is the life of the sign, we should have to say that it was its *use*.'[40]

Wittgenstein does not have an alternative theory of meaning to oppose, say, to Frege's theory: there is no Wittgensteinian *theory* of meaning as use. Wittgenstein would direct our attention towards the use of certain words in order to expose a particular philosophical confusion: in this case, it is the idea that for a sign to have meaning it is essential for there to be a specific mental act of meaning. There is no 'further study', following philosophical clarification, which, Wittgenstein believed, it was ever the philosopher's business to develop.

Kreisel, nonetheless, rejects Wittgenstein's claim that his philosophy is 'purely descriptive'[41], arguing

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[40] *The Blue and Brown Books*, p. 4.

[41] Ibid., p.18.

that, because his descriptions of language use are 'patently of a limited kind'[42], there is a reduction involved; and, as in the earlier review, he looks to see what further study Wittgenstein ought to have developed. Kreisel understands Wittgenstein to be advancing a modification of 'crude nominalism' in the philosophy of mathematics, which identifies numbers with number signs; and, parallel to this, in the philosophy of psychology, Wittgenstein is understood to be advancing a modification of 'crude behaviourism', which identifies mental acts with physiological processes and larynx movements[43]. According to Kreisel, rather than identifying numbers with number signs, Wittgenstein describes what we 'do' with them, i.e. he describes their role in language in 'concrete' terms, eliminating any reference to abstract objects:

> 'Both his examples and the studies in the foundations of mathematics show clearly that *we have a general tendency to describe language*, and, in particular, mathematical practice, *by means of*

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[42] 'Wittgenstein's Theory', p. 241.

[43] Ibid. Bernays' interpretation (*Philosophy of Mathematics*, p. 511), which Kreisel follows here, was based, in part at least, on a simple misreading of the following passage from *Remarks on the Foundations of Mathematics* (II, 61):

> 'Finitism and behaviourism are quite similar trends. Both say, but surely, all we have here is... Both deny the existence of something, both with a view to escaping from a confusion.'

Wittgenstein was not, however, describing his own views here, as Bernays believed, but views which he opposed. See *Wittgenstein's Lectures*, Lecture XII, p. 111, and also *Remarks on the Foundations of Mathematics*, V, 36 and *Philosophical Investigations* (1953), Part I, §§ 307-308.

concepts whose level of abstraction is higher than the minimum actually needed.'[44]

This characterization of Wittgenstein's view forms the basis for the criticism of Wittgenstein's 'theoretical positions', which is contained in the second half of Kreisel's article.

Wittgenstein's 'theoretical positions' are defined by Kreisel as those which 'constitute a basis or at least directives for a "descriptive philosophy"'. They are: '(i) negative assertions on what cannot be said..., such as what is common or essential to those cases which he describes as families of concepts, (ii) assertions on what should be accepted as a decisive criterion for (equality or difference in) meaning, such as the actual use of a term, (iii) the identification of metaphysical distinctions with grammatical ones'.[45]

Kreisel's criticism mainly concerns Wittgenstein's notion of a family resemblance concept.[46] In the *Blue Book*, Wittgenstein describes the philosophers' 'craving for generality', which makes it difficult to enjoy the 'great advantage' of a piecemeal investigation of language by

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[44] 'Wittgenstein's Theory', p. 242. Kreisel says that his own investigations in the foundations of mathematics have been influenced by this view of Wittgenstein's work.

[45] Ibid., p. 244.

[46] This notion seems to have been of particular interest to Kreisel already in 1942, when he first read the *Blue Book*. See 'Zu Einigen Gesprächen', Section 2.

means of language-games.[47] 'The tendency to look for something in common to all the entities which we commonly subsume under a general term' is introduced as one cause of this craving.[48] Nowhere does Wittgenstein state, however, that a general term never subsumes under itself entities which all have something in common. Nor does Wittgenstein state that a family resemblance concept cannot be an abstract concept. This, nevertheless, is how Kreisel chooses to interpret Wittgenstein. He seems to confuse Wittgenstein's insistence that philosophers consider 'concrete cases'[49] with the insistence that they consider cases which can be described in concrete rather than abstract terms.

Having settled for this interpretation, Wittgenstein's examples of family resemblance concepts now seem inappropriate; so Kreisel tests the idea in more 'vivid' applications in mathematics. The application is 'vivid' in these cases: 'when we ask for what is common to certain formal properties or relations, for instance what abstract structure is common, for example to rotations in the plane and multiplication of complex numbers'; and also 'when it is evident that the property could not conceivably be

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47 *The Blue and Brown Books*, p. 17.

48 Ibid.

49 Ibid. p. 19.

expressed in concrete terms'.[50] See the discussion of Gödel's theorem below.[51]

About Wittgenstein on meaning and use, Kreisel has this to say:

> '"Actual use" may refer to the words spoken: this has the attraction that here we have a subject matter for philosophy comparable to the "hard" experimental facts of the natural sciences or the combinatorial facts of mathematics to which one can refer when the theoretical framework creaks. Furthermore, since it includes everything that is said, it would seem to leave room for all things between Heaven and Earth. But "actual use" may also mean the *real* rôle of the word (as Wittgenstein puts it), undistorted by the vagaries of linguistic expression from which philosophy should free us: this has the attraction that now we are getting real knowledge. But, unfortunately, the latter is, in general, achieved at the cost of the former.'[52]

'Actual use' does, of course, mean the real role of a word, or sentence. It was not, however, from 'the vagaries of linguistic expression' that Wittgenstein believed philosophy ought to free us, but from distorted pictures of the use of a word, or sentence. Kreisel has something right when he says that 'actual use' has some of the features of '"hard" experimental facts'; Wittgenstein wanted to solve philosophical puzzles by referring to the most evident features of language on which, he believed,

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[50] 'Wittgenstein's Theory', p. 245.

[51] Pp. 151-2.

[52] 'Wittgenstein's Theory', p. 247.

everyone ought to be able to agree.[53] This is, perhaps, the most characteristic feature of his critical philosophy.

Kreisel also questions the value of Wittgenstein's '"reduction" of metaphysics to grammar'[54]. He says:

> '...while it seems to me perfectly apt to speak, as Wittgenstein does, of the grammatical rôle of a word in a language, the difficulty of formulating this seems to be of an entirely different order from school grammar where one classifies words into categories often even independently of their position in a sentence.'[55]

Kreisel's objections, including this one, assume that Wittgenstein sought to replace metaphysics with a new subject devoted to the task of providing a comprehensive description of 'the logical grammar of language'. Wittgenstein was clear, however, that a piecemeal description of different grammatical features was all that a philosopher could reasonably hope to achieve. Also, he did not believe that this was an easy undertaking. In fact, he later appealed to the difficulty and complexity of grammatical investigations when justifying his reluctance to interfere with the mathematicians:

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[53] In 1939, he said: 'The investigation is to draw your attention to facts you know quite as well as I, but which you have forgotten, or at least which are not immediately in your field of vision. They will all be quite trivial facts. I won't say anything which anyone can dispute. Or if anyone does dispute it, I will let that point drop and pass on to say something else.' *Wittgenstein's Lectures*, Lecture I, p. 22. Cf. *Philosophical Investigations* (1953), Part I, § 129.

[54] 'Wittgenstein's Theory', p. 247.

[55] Ibid., p. 248.

'We certainly see bits of concepts, but we don't clearly see the declivities by which one passes into others.

That is why it is of no use in the philosophy of mathematics to recast proofs in new forms. Although there is a strong temptation here.'[56]

Towards the end of his review, Kreisel makes some comments which clarify his earlier discussion of Wittgenstein and Gödel. Kreisel says that the significance of impossibility or underivability assertions is connected with the fact that Wittgenstein's 'theoretical positions' are wrong, and he says that this may be a partial explanation of the 'wildness' of Wittgenstein's remarks on Gödel's theorem.

'Suppose the impossibility of a characterisation by certain means (e.g. mechanical procedures[57]) is to be shown, where the means considered form a family of concepts in Wittgenstein's sense. Suppose further we find an abstractly formulated property (here: recursiveness) which is certainly satisfied by all members of the family, and possibly by things outside it. *If we now establish the impossibility of achieving the required end by all methods which have the abstractly defined property, then we have a negative result which is unaffected by uncertainties* about the exact extent of the family of concepts considered. (It is of course not even required that the "family" should have an exact extent.) I believe the epoch making character of the work initiated by Gödel rests largely on satisfying all these suppositions... The possible need for an abstractly defined property is also apparent here; namely if all the well-defined properties which are common to such a family (and sufficient to derive the required conclusion) are definable only on a higher level of

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[56] *Remarks on the Foundations of Mathematics*, V, 52.

[57] Kreisel is referring here to Turing's notion of 'mechanical procedure' described in his 'On Computable Numbers, with a note on the Entscheidungsproblem', *Proceedings of the London Mathematical Society*, xlii (1937).

> abstraction. Now, if one rejects the use of abstract concepts as means of philosophic analysis (or, at least, considers an explanation in concrete terms more fruitful) one will tend to reject the particular interpretation just discussed. Since this interpretation is certainly natural it seems understandable why Wittgenstein objects so strongly to attributing philosophic significance to the impossibility results of mathematical logic.'[58]

As we have seen, Wittgenstein neither rejected the use of abstract concepts as a means of philosophical analysis nor considered an analysis of mathematics in concrete terms more fruitful; so it is clear that this is not the explanation for his rejection of the philosophical significance of impossibility results; Wittgenstein rejected the philosophical significance of *any* mathematical result[59]. This is not, of course, to say that the acceptance of a recursive definition of 'mechanical procedure' does not underlie the *mathematical* significance of Gödel's theorem; although Wittgenstein would probably have preferred to describe this case by saying that one concept had been replaced by another one with more rigid limits[60].

Kreisel concludes his review of *The Blue and Brown Books* by expressing his opinion that: 'As an introduction to the significant problems or traditional philosophy the

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[58] 'Wittgenstein's Theory', pp. 248-9.

[59] *Philosophical Investigations* (1953), Part I, § 124 (cf. *The Blue and Brown Books*, p. 18).

[60] *Philosophical Investigations* (1953), Part I, § 68. See Rockingham Gill's *Deducibility and Decidability* (London, Routledge, 1990), Chapter IV.2.

books are deplorable'[61]. He explains, in a footnote, that this is 'largely based on a personal reaction':

> 'I believe that early contact with Wittgenstein's outlook has hindered rather than helped me to establish a fruitful perspective on philosophy as a discipline in its own right,...'

It ought to be clear that Kreisel's review of *The Blue and Brown Books* shows no more understanding of Wittgenstein's philosophy than does his earlier review of *Remarks on the Foundations of Mathematics*. Besides the 'personal reaction', Kreisel is clearly hindered here, as he was in the previous case, by his attempt to understand Wittgenstein's writings as a contribution to the philosophy of mathematics considered as a scientific discipline.

Before the publication of Kreisel's next article on Wittgenstein there was a gap of some 16 years, during which time his feeling of reaction against Wittgenstein's early influence on him seems to have abated. In 1970 he added a postscript to his review of *Remarks on the Foundations of Mathematics*, explaining how his views had changed. He says that now 'it seems more useful to concentrate on (what seems to me) the positive aspects of Wittgenstein's ideas', and here his thinking shows a definite advance:

> 'Wittgenstein recognizes and formulates in an acceptable manner the *objectivity* of (certain) mathematical notions... Even if the phrase "mathematical object" (as referring to such objectivity) seems quite apt, the fact remains that it has been given ludicrous interpretations in terms of a Platonic Heaven. Wittgenstein's formulations avoid this temptation.'

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[61] 'Wittgenstein's Theory', p. 251.

However, Kreisel still believes that Wittgenstein is a 'strict finitist'; although he again alters his view of what this amounts to: it is 'best thought of as an interest in theoretical (introspective) psychology of mathematics'. And, he is still inclined to look for a technical justification for 'Wittgenstein's persistent stress on the question of equivalence of proofs and not only equivalence of results...'.

'Der unheilvolle Einbruch der Logik in die Mathematik', is the first of four articles on Wittgenstein published by Kreisel in the late 1970s.[62] The title is a quotation taken from *Remarks on the Foundations of Mathematics*[63], whose 'plausibility' Kreisel had stressed in his earlier review of that work.[64] The main topic is the truth or falsity of the quotation under various interpretations, i.e. the nature and the extent of the influence of logic on mathematics. An Appendix discusses the contrast between Wittgenstein's two major works, *Tractatus Logico-Philosophicus* and *Philosophical Investigations* in the light of the preceding discussion.[65]

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[62] See pp. 104-5 above.

[63] V, 24a: '"The disastrous invasion" of mathematics by logic'.

[64] 'Der unheilvolle Einbruch', p. 168.

[65] Because of an overlap in their contents this appendix is examined in conjunction with the main body of Kreisel's next article.

Kreisel comments, first of all, on the 'literal'[66] or 'statistical'[67] interpretation of Wittgenstein's remark, and he helps to clarify its sense, in a preliminary way, by pointing out certain facts about the logical education of mathematicians. Kreisel says that 'most mathematicians know precious little of logic anyway except (some) symbols for the logical particles', but there are important exceptions:

> 'Above all, a few gifted mathematicians have found quite excellent applications of logic, using in an essential way concepts which were, patently, developed in specifically logical investigations. For example, in algebra and number theory and on their border (theory of *p*-adic numbers), both recursion theory and model theory have been used successfully. ... Other gifted mathematicians, like N. Wiener or J. von Neumann, who originally specialized in logic, but did not find it particularly suited to their talents, later seem to have used their familiarity with the "new" logic to good effect, in work on cybernetics and above all on programming computers.'[68]

It cannot be doubted that from a scientific point of view the characterization of the nature and the extent of the influence of logic on mathematics is, as Kreisel says, a 'delicate statistical matter'[69]. Kreisel concludes, sensibly, that the quotation must concern 'specific parts of mathematics or stages of its development...; perhaps not

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[66] 'Der unheilvolle Einbruch', p. 87.

[67] Ibid., p. 88.

[68] Ibid., p. 167.

[69] Ibid., p. 167.

even mathematics itself, but rather the analysis of mathematics'.[70]

Kreisel thus proceeds to apply Wittgenstein's remark, appropriately, to Hilbert's proof theory and its development. He notes that Wittgenstein knew Hilbert's essay 'Über das Unendliche'[71] and that Wittgenstein was acquainted with Turing, who attended some of his lectures[72]. In Kreisel's opinion:

> 'The claims of proof theory to have uncovered the true, in particular, formal nature of mathematical reasoning surpass in pretentiousness the claims of most traditional philosophers.'[73]

However, Wittgenstein's 'critique of proof theory and its principal problems', for example, in *Remarks on the Foundations of Mathematics*, is, according to Kreisel, 'wildly exaggerated, and therefore quite unconvincing'.

> 'Even where his critique applies to Hilbert's own formulations, it rarely applies to the more reasonable formulations which any average logician will find for himself...'[74]

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[70] Ibid., p. 168. It ought to be remembered, also, that Wittgenstein's remark was written in 1942 or 1943.

[71] Kreisel remembers that in the margin of Wittgenstein's copy of Hilbert's essay, 'opposite one of Hilbert's particularly thoughtless passages', Wittgenstein had written: 'Heiliger Frege!'. Wittgenstein also referred to Frege when discussing Hilbert's view in conversations with Waismann in 1930. See *Ludwig Wittgenstein and the Vienna Circle*, IV, 'Consistency II'.

[72] See section 1.2, p. 63.

[73] 'Der unheilvolle Einbruch', p. 169.

[74] Ibid.

This objection shows that Kreisel has still not understood the precise nature of Wittgenstein's critique, which is not meant to apply to 'more reasonable reformulations', i.e. it is not meant to apply to reformulations of proof theory which do not share the original philosophical confusions:

> 'It is not that a new building has to be erected, or that a new bridge has to be built, but that the geography, *as it now is*, has to be described.'[75]

The same misunderstanding is also apparent in Kreisel's complaint that 'Wittgenstein's own attempts to characterize what is essential to proofs aren't much better (than Hilbert's)'. This complaint refers specifically to Wittgenstein's remarks on the creation of concepts during mathematical proof and to his remarks on surveyability.

Kreisel says Wittgenstein 'stresses that proofs *create* - or at least use! - *new concepts*'.

> 'This is surely true, and much stressed by mathematicians, especially when those new concepts seem to have nothing to do with the theorem proved. This occurs not only in analytical number theory, but also in quite elementary (witty) proofs, for example, of the irrationality of $\sqrt{2}$, that is, $p^2 \neq 2q^2$ for natural numbers p and q. Here, the largest divisors, of the form 2^n, of p^2 and $2q^2$ are considered; n is clearly even, resp. odd in the two cases, and so $p^2 \neq 2q^2$...'[76]

Kreisel says that here the 'concept of *power* (of 2) used in the proof is "new" inasmuch as, apparently, it has nothing to do with the proposition $p^2 \neq 2q^2$ (which we understand as soon as we can multiply)'. His objection is that

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[75] *Remarks on the Foundations of Mathematics*, V, 53.

[76] 'Der unheilvolle Einbruch', p. 170.

'*propositions concerning these new concepts have to be proved too*'. Kreisel explains that Wittgenstein 'never quite faces this fact' because of his '*reductionist aim* of analyzing the meaning of mathematical and other abstract notions in terms of what we "do" with them (and *bien entendu*, this "doing" was not meant to include proving propositions about these notions)'.

Kreisel has clearly misunderstood what Wittgenstein meant when he said that mathematical proofs create or introduce new concepts. Consider the following passage from *Remarks on the Foundations of Mathematics*, for example:

> 'When I said that a proof introduces a new concept, I meant something like: the proof puts a new paradigm among the paradigms of the language; like when someone mixes a special reddish blue, somehow settles the special mixture of the colours and gives it a name.
>
> But even if we are inclined to call a proof such a new paradigm - what is the exact similarity of the proof to such a concept-model?
>
> One would like to say: the proof changes the grammar of our language, changes our concepts. It makes new connections, and it creates the concept of these connections. (It does not establish that they are there; they do not exist until it makes them.)'[77]

If the development of Wittgenstein's thought on this topic is traced in the *Remarks on the Foundations of Mathematics*[78], it can be seen that he holds on to this basic insight, while adding various qualifications as he tests his analysis against different examples of proof,

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[77] III, 31.

[78] I, 42-46; III, 24, 31-32, 41,; IV, 30-31, 47.

'jumping about all round the problem', as he would put it. However, in each example the new concept is introduced by the proof and not, as Kreisel would have it, introduced to the proof.

Kreisel's example *is* similar, however, to some of those which were used by Wittgenstein in his critique of Russell's supposed logicist reduction of arithmetic.[79] Wittgenstein suggested that here we were, in fact, 'introducing new concepts into the Russellian logic without knowing it'[80]. Typically, his discussion employed elementary examples:

> 'Tell me: have I discovered a new kind of calculation if, having once learnt to multiply, I am struck by multiplications with all the factors the same, as a special branch of these calculations, and so I introduce the notation "a^n =..."?'[81]

Finding a 'new aspect' in this way could well involve the introduction of a concept which already existed in mathematics. However, it would then be a new connection that was created and not, somehow, the concept itself. Kreisel's equivocation on 'create' and 'use' seems to be essential to his argument.

Kreisel's complaint about Wittgenstein's analysis of mathematical proof refers, secondly, to Wittgenstein's

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[79] See, for example, *Remarks on the Foundations of Mathematics*, III, 46-47.

[80] Ibid., III, 46g.

[81] Ibid., III, 47c.

notion of 'surveyability'[82], which Kreisel equates with 'some kind of simplicity'[83]. However, according to Kreisel, 'all this is clearly secondary, as long as there are (genuine) doubts about the *principles of proof* that are used'[84].

Kreisel's view on this last question is that: (1) 'There are no realistic doubts concerning the concepts used in current mathematical practice and concerning their basic properties formulated in the usual axiomatic systems. This applies both to (the usual) theories of sets and to current intuitionist mathematics including the theory of lawless sequences' and (2) 'There is...no hint of evidence for the assumption that any *analysis* of the usual concepts or of their basic properties could improve in any *general* way on the mere recognition of their *validity*.' Kreisel says that these views 'accord, in effect if not in intention, with the general aims of *RFM* (if it is remembered that Wittgenstein's specific criticisms of set theory apply to those early expositions he knew, which were really either defective or shallow)'.[85] He says:

> 'Many readers will of course object to (1) and (2) as ignoring rather than solving their problems; forgetting that, at one time, similar problems *were* genuine; we can now assert (1) and (2) just *because* those problems were solved. Perhaps the principal

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[82] See pp. 122-3 above.

[83] 'Der unheilvolle Einbruch', p. 170.

[84] Ibid.

[85] Ibid., p. 171.

> reason for the objection is this: one is tempted to feel that "doubts", for example, concerning validity show a higher philosophical sensibility (or responsibility!) than acceptance - as if doubts and questions could not be equally THOUGHTLESS or UNCRITICAL as acceptance and assertions resp.. Of course, this point is utterly banal: Wittgenstein's literary skill made it memorable (and useful).'[86]

If it was not already apparent after the examination of his two earlier reviews, it ought to be apparent by now that Kreisel's opinions differ far less from Wittgenstein's opinions than Kreisel realises. This is important, because it is frequently true of other mathematicians too: it is often only because they have misunderstood Wittgenstein that his remarks seem so irreconcilable with their own understanding of mathematical practice. Kreisel and Wittgenstein differ primarily in their opinions about the nature of philosophical questions and about the proper role of philosophy in mathematics. It could not, for example, be said that the doubts about the validity of set theory which concerned Wittgenstein had been 'solved', even if those doubts, or similar ones, had largely disappeared from the writings of mathematicians and philosophers.

Having resolved the issue of validity to his own satisfaction, Kreisel returns to the question of surveyability or, as he understands it, 'complexity'. In his own research, and he connects this with his '"conversion" to the view of the silent majority'[87],

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[86] Ibid., pp. 172-3.

[87] Footnote 5 to the Appendix 'Proofs and Rules' in 'The Motto'.

Kreisel had begun to look deliberately for a measure of complexity involving 'methods of proof and properties of proofs which are *trivial for proof theory*, but *essential for mathematical practice*'[88]. Kreisel's specific technical considerations do not interest us; these were certainly no part of Wittgenstein's interest in surveyability. Kreisel does, however, make some interesting remarks on the subject of 'explicit definitions'. He says:

> 'As is well-known, this way of introducing a new concept is trivial for Hilbert's proof theory, because such concepts are in an obvious way *eliminable*. On the other hand, for mathematical practice they are not only useful; but as it were typical - at least for modern mathematics which is dominated by the *axiomatic method*. This proceeds as follows. A structure is defined explicitly in set theoretic or number theoretic terms, and then is shown to be, say, a unitary group: the axioms for unitary groups then constitute the supplementary "list of properties" (of the structure or concept) mentioned above. The choice of such properties - or, as one says, of the proper *cadre* - is often the key to solving mathematical problems.'[89]

Summarizing his discussion, he says:

> '...let us call (mathematical) "theory of proofs" the study of those properties of and relations between proofs which strike the ordinary mathematician when he reflects on his activity by the light of nature; and let us take Hilbert's proof theory as an example of the "logical view" of that activity. Certainly - in terms of the title of this article - there has been an invasion of (this part of) mathematics by logic. Was it disastrous?'[90]

Kreisel answers this question by disagreeing with Wittgenstein's view that some of the questions posed in proof theory are meaningless. They are, rather, '*too*

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88 'Der unheilvolle Einbruch', p. 174.

89 Ibid., pp. 174-5.

90 Ibid., p. 176.

banal, incorporating too little of the properties of proofs...which we have learnt to appreciate (only) after experience of modern mathematics.'[91]

Kreisel considers next 'a more subtle "invasion" by logic, namely, a somewhat exaggerated idea of the role of so-called logical languages, for example, of predicate logic of first order'[92]. He concludes that here an unrewarding choice of problems resulted from the influence of logic in the field of the analysis of efficient decision procedures. Later, he expressed himself in this way:

> 'Some 25 years ago logicians considered themselves to be misunderstood martyrs when the silent majority of mathematicians showed little interest in the compactness theorem and other generalities about first order logic. ... The logicians involved presented their general results as *obviously* significant because they concerned arbitrary axiomatic systems. But this left open the possibility that, for any particular (familiar) axiomatic system, those results have only superficial consequences. In fact, 25 years ago, practically all the applications of the general results consisted of trivial theorems which had clumsy proofs in the literature.'[93]

Kreisel's opinions act as a useful counterweight to the views of other expert authorities who claim that Wittgenstein's hostility to mathematical logic led him to give an absurd account of its influence. Wittgenstein's objections might occasionally be overstated, but they are

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[91] Ibid., p. 177.

[92] Ibid.

[93] 'The Motto', p. 30. Kreisel's remarks here recall the metaphor which he related to Wittgenstein after the war. See section 1.2, pp. 80-1.

never 'plainly silly'[94]. In general, Kreisel's articles are a valuable source of advanced examples for the exposition of Wittgenstein's view.

Finally, Kreisel applies Wittgenstein's remark, taken in its literal sense, to computer science:

> 'There seems to be a fairly widely accepted ideal in computer science, of a *universal programming language*, preferably together with a so-called *universal semantics*. This ideal surely comes from the corresponding ideal of a pretentious logic, for example, the universal semantics of *Tractatus*...'[95]

Such schemes have 'diverted attention from less grandiose, but much more effective "local" schemes'. For example, 'the language of set theory as a "universal" language for mathematics which, for a long time, diverted attention away from the much more useful enterprise of finding a *few* concepts, the so-called "structures - mère" (not: the *one* concept of "set") in terms of which *many* mathematical concepts are built up in a genuinely manageable way.' Kreisel is here describing some consequences of what Wittgenstein would refer to as the 'craving for generality'. The 'more useful enterprise' to which Kreisel refers was set out by Bourbaki, whose views are discussed in Kreisel's final review article.

Kreisel's next article, 'The Motto of *Philosophical Investigations* and the Philosophy of Proofs and Rules', has

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[94] Dummett in *Philosophy of Mathematics*, p. 496.

[95] 'Der unheilvolle Einbruch', pp. 180-1.

a similar structure to the previous one. A quotation from Wittgenstein, in this case the motto from *Philosophical Investigations*, is used to provide the main theme, and there is a substantial appendix: 'Proofs and Rules'[96].

The motto which Wittgenstein finally chose for his major work comes from Johann Nestroy[97]. Its meaning is clarified by the immediate context:

> 'There are so many means of extirpating and eradicating, and nevertheless so little evil has yet been extirpated, so little wickedness eradicated from this world, that one clearly sees that people invent a lot of things, but not the right one. And yet we live in an era of progress, don't we? I s'pose progress is like a newly discovered land; a flourishing colonial system on the coast, the interior still wilderness, steppe, prairie. It is in the nature of all progress that it looks much greater than it really is.'[98]

Kreisel interprets the motto as saying 'in effect that the ratio of *actual progress*, as judged by mature reflection, to *apparent progress*, measured by expectations after a few initial successes, is generally poor'[99]. He applies Wittgenstein's remark by comparing progress in 'traditional philosophy' to progress in 'some of the younger "heirs" of

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[96] This appendix, which seems to have been written after the original lecture, overlaps in its content with Kreisel's next article on Wittgenstein, and for that reason the two are examined together.

[97] *Der Schützling*, Act IV, scene 10. The quotation, under the heading 'Motto', is entered in MS **134** on 25 April 1947. It was subsequently added to TS **227** in place of the motto from Hertz. See 'The Motto' in Baker and Hacker's *Analytical Commentary*.

[98] Ibid.

[99] 'The Motto', p. 13.

philosophy such as the natural and mathematical sciences'[100]. He believes that Wittgenstein meant to acknowledge that the progress of the *Philosophical Investigations* over the *Tractatus Logico-Philosophicus* is smaller than it might seem.

Baker and Hacker suggest[101], and I should agree with them, that this interpretation is 'unlikely'; it is more probable that the motto is meant to echo the end of the Preface to the *Tractatus*, where Wittgenstein says:

> '...the value of this work...is that it shows how little is achieved when these problems are solved'.

There is also an obvious similarity between the sentiments expressed here and those expressed in the Preface to *Philosophical Remarks*[102], and in this connection, the following remark, from 1946, ought to be mentioned as well:

> 'The hysterical fear over the atom bomb now being experienced, or at any rate expressed, by the public almost suggests that at last something really salutary has been invented'.[103]

The bomb, Wittgenstein thought, 'offers a prospect of the end, the destruction of an evil, - our disgusting soapy water science'. It ought to be understood that Wittgenstein's distaste for contemporary attitudes towards science formed and motivated his critique of mathematics. For example, Wittgenstein's examples from higher

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[100] Ibid.

[101] *Analytical Commentary*, 'The Motto'.

[102] See section 1.1, pp. 26-7.

[103] MS **131**, p. 66.

mathematics were not chosen for their scientific interest, but they were often chosen to illustrate what Wittgenstein saw as the encroachment into mathematics of an inappropriate scientific model. This is, perhaps, most obvious in the case of Cantor's diagonal proof, whose 'charm' Wittgenstein wanted to dispel.[104]

Kreisel's article divides into three main parts. In the first part, he 'draws some general, but neglected conclusions from the obvious fact that traditional notions and questions occur to us when we know little'[105]. 'Traditional philosophy', according to Kreisel, studies superficial abstractions, rather than specific features of objects, and it cannot be expected to remain rewarding when we know more. Also, when we know little we have to rely on 'a sense of coherence and, more generally, introspection'[106], and we would do better by 'extending experience'. This is not, however, always 'economical', as Kreisel illustrates by reference to certain 'criticisms of errors':

> 'Wittgenstein himself often succeeded by paying attention to neglected experience, a much weaker kind of "extension" than the novel experiments or technological advances used in the sciences. Thus to point out an error in St Augustine's particularly narrow "theory" of language, it was sufficient to mention that language is also used for commands.'[107]

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[104] See section 1.2, p. 60.

[105] 'The Motto', p. 13.

[106] Ibid., p. 15.

[107] Ibid.

Kreisel here mistakes Wittgenstein's interest in what St Augustine wrote about language. Wittgenstein chose to quote from *The Confessions*[108] not because he wanted to criticize an early scientific theory of language, but because he believed that Augustine's words expressed a pervasive *pre-theoretical* picture of language, which was also at the root of some of the doctrines in the *Tractatus*.[109] Kreisel's lack of sympathy for such a view is made clear in the appendix to his previous article:

> 'It is easy to understand but difficult to agree with Wittgenstein's own reaction, of looking for "profound" mistakes or misconceptions that led (him) to the fiasco. He certainly felt that his view in *Tractatus* was very narrow, his paradigms extraordinarily special. He widened his view in the *Investigations*, but still staying in the range of the most familiar kind of experience; perhaps simply because the questions asked are intelligible, in fact occur to us, when we have only this kind of experience (and it would be elegant to answer them by reference to only such experience).'[110]

Speaking more generally, Kreisel says:

> 'I find it hard to have confidence in our whole "critical" philosophical tradition, with its paradoxes, its dramatic claims either to see profound errors in our ordinary views or profound misconceptions in 2000 year old questions. It all sounds like a paranoid's paradise, and forgets the most striking fact of intellectual experience: how our thoughts seem to adapt themselves to the objects concerned, as we study them and get familiar with them (in a detached way) and how, with this familiarity, comes the judgement needed to distinguish between plausible and implausible theories, substantial and superficial contributions.'[111]

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[108] *Philosophical Investigations* (1953), Part I, § 1.

[109] See Chapter 1 of Baker and Hacker's *Analytical Commentary*.

[110] 'Der unheilvolle Einbruch', p. 185.

[111] Ibid., p. 186.

These remarks recall Kant's observations about the resistance of scientists to the 'negative judgements' of critical philosophy[112]; and, following Kant, one ought to respond by referring to the success of critical philosophy in explaining just why philosophical questions persist for so long. Kreisel's example of a philosophical question which has been answered, Plato and Aristotle's: 'What is matter?', is not convincing, because the meaning of the question was clearly different for these authors than it is for a modern physicist.

The second part of Kreisel's article 'concerns the specific branch of philosophy which searches for definitions of common notions in familiar terms, after the model of Euclid's geometry...'[113]. It is in this area, Kreisel believes, that traditional and analytical philosophy have their most striking successes. These are to be found in the discovery of definitions in geometry. Euclid's definition of 'circle', for example, and the more recent definition of 'convex', 'have become an integral part of our intellectual equipment, being used constantly for advancing our knowledge of circles and other convex bodies'[114]. Kreisel uses 'Wittgenstein's slogan of "family

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[112] *Critique of Pure Reason*, A 708-9/B 736-7.

[113] 'The Motto', p. 13.

[114] Ibid., p. 18.

resemblances"'[115] to suggest various limitations to definition in the traditional style.[116] He concludes that although these might be revolutionary for the philosophical tradition, 'there is nothing here that is revolutionary for the *silent majority*, notoriously sceptical of the philosophical tradition'.[117]

The view of the 'the silent majority' in Kreisel's thought seems to correspond closely to the 'sound human understanding'[118] in Wittgenstein's philosophy. These notions allow both authors to achieve a certain level of objectivity[119] in their criticism of philosophy by reference to actual practice in everyday life and in science respectively. Russell's 'analyses of the *definite article* by contextual definition' was, as Kreisel says, less fruitful scientifically than Cauchy's 'elimination of infinitesimals and limits by contextual definitions in terms of convergence'[120] and the logicist definition of the

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[115] Ibid., p. 19. Cf. *Philosophical Investigations* (1953), Part I, §§ 66ff.

[116] The remarks in (b), on pp. 19-20, in so far as they suggest limitations to the traditional style of definition in certain general types of case are, perhaps, closest to what Wittgenstein intended. Cf. Kreisel's earlier interpretation in his review of *The Blue and Brown Books*, pp. 148-9 above.

[117] 'The Motto', p. 21. Like Russell's grandmother, Kreisel quotes from Thomas Hewitt Key: 'What is mind? - No matter. What is matter? - Never mind.'

[118] *Remarks on the Foundations of Mathematics*, V, 53.

[119] See section 1.2, p. 66.

[120] 'The Motto', p. 17.

number one might indeed be 'boring by the light of nature'[121]. Also, Frege's logical rules which were 'presented as providing a norm, a criterion of precision, allegedly superior to ordinary reasoning' are 'rarely used in practice; in contrast to the constant use of the definitions of "circle" or "convex" in geometry'[122].

In the third part of his essay, Kreisel assesses the value of Wittgenstein's philosophical method, which he calls 'intimate pedagogy'. The term 'pedagogic' is meant to indicate that no positive attempt is made to replace discredited ideas.[123] In order to show that alternatives exist to Wittgenstein's style of pedagogy, Kreisel refers to some examples from mathematical logic which he believes are useful for 'debunking some "grand" claim or notion'[124]. Gödel's first incompleteness theorem, for example, is supposed to refute 'the idea that in mathematics abstract notions are used as a *facon de parler*'[125], something

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[121] Ibid., p. 18.

[122] Ibid., p. 21.

[123] Ibid., p. 22. Kreisel's term is an apt one given the resemblances that exist and which Wittgenstein himself noticed between his own philosophical technique and Freudian psychoanalysis. See section 1.2, pp. 59 and 70-1. It is important to note that Wittgenstein applied his therapy mostly to the confusions of living individuals, rather than to those of historical figures, and, in addition, he believed that any adequate diagnosis had to be acknowledged as such by the patient (See *Philosophical Grammar* (TS **213**), section 86).

[124] Ibid., p. 23.

[125] Ibid., p. 24. See pp. 151-2 above.

which 'Hilbert had expressed explicitly and precisely in his consistency programme'. Kreisel complains that in the *Remarks on the Foundations of Mathematics*, Wittgenstein had failed to make proper use of this result:

> '...a principal *bête noire* in the *Remarks* is Hilbert's consistency criterion. Now, Gödel's incompleteness theorem shows most memorably the inadequacy of this criterion, providing formally consistent systems in which a false, purely existential arithmetic statement is formally derivable. Instead of at least attempting to use this result as support for his own objections to Hilbert's criterion, Wittgenstein tied himself up in knots talking around the subject...'[126]

Wittgenstein certainly wanted to 'by-pass' Gödel's theorem, otherwise he would be doing mathematics, and he believed that *no* mathematical discovery could dispel the lack of clarity typical of philosophical problems.[127] In response to Kreisel's specific suggestion, it must again be asked: how can an 'arithmetic statement' be formally derivable in a formal system? The examples which Kreisel uses here are unconvincing, but the objection which they were intended to support does have some force, as we shall see.

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[126] 'The Motto', p. 25. Similarly, Alan Anderson says:

> 'It is hard to avoid the conclusion that Wittgenstein failed to understand the problems with which workers in the foundations of mathematics have been concerned. Nor, I think, did he appreciate the real bearing of the results on the logicist thesis. For Gödel's theorem, and the line of development it culminates, have more often been cited in *favor* of Wittgenstein's position concerning the relation of logic to mathematics...' (*Philosophy of Mathematics*, p. 489.)

[127] *Philosophical Investigations* (1953), Part I, § 123-24.

Kreisel's next article, which is the last one to be examined here, is his review of *Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939*[128]. Kreisel here criticizes Wittgenstein's views by comparing them with the views of Bourbaki[129]; and he examines the style in which they each present their conclusions in the light of discoveries in mathematical logic, which both are said to neglect[130].

The review begins with a brief survey of Wittgenstein's career, which locates the *Tractatus* in the 'heroic tradition of Western philosophy, with its questions about the general structure of knowledge or the correct analysis of (all meaningful) propositions'[131]. According to Kreisel, the questions answered in the *Tractatus* - and this is true of philosophical questions in general - occur in advance of 'detailed intellectual experience'. The

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[128] See section 1.2, p. 64, for a description of the origin of this book.

[129] A summary of Bourbaki's philosophy of mathematics, which might be read prior to my comments here, is contained in section 2.2.2 below.

[130] Kreisel acknowledges Wittgenstein's 'passing references to some kind of (mathematical) interest of mathematical logic', but he complains that Wittgenstein does not say specifically what that interest might be:

> 'Though this is easier to state now, by 1939 (and especially by 1948, the year of Bourbaki's manifesto), some people with their wits about them had a pretty good idea...'

For details see Kreisel's footnote [3].

[131] 'Wittgenstein's Lectures', p. 79.

general idea of the *Tractatus* is understood to be an analysis of propositions analogous to the analysis of molecules in terms of chemical bonds.[132] The main novelty of the book, he believes, is the use of truth tables for propositional logic; and the little on mathematics is described as 'a brief reference to an operational analysis - in contrast to the set-theoretic analysis in *Principia*'.[133] As Wittgenstein's thought developed towards the view expressed in *Philosophical Investigations*, he is supposed to have become disillusioned with the whole heroic tradition.[134] Kreisel says, Wittgenstein 'found that quite elementary mathematics provided excellent illustrations of weaknesses in traditional foundations'[135].

It is because Kreisel looks for novel scientific ideas in the *Tractatus* that the book seems to him to be intellectually impoverished. In order to understand the details of its philosophical doctrines one has to understand, at the very least, how the *Tractatus* borrowed

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132 Cf. *Philosophical Grammar* (1969), p. 311.

133 'Wittgenstein's Lectures', p. 79. Kreisel does not make the mistake, which some have made, of attributing to the early Wittgenstein a variety of logicism. Clarification of Wittgenstein's view is to be found in the first paragraphs of Rhees's 'Questions on Logical Inference', in *Understanding wittgenstein* (London, Macmillan, 1974) edited by G.Vesey.

134 'Wittgenstein's Lectures', p. 79. Kreisel's understanding of the development of Wittgenstein's thought is explained more fully in his Appendix to 'Der unheilvolle Einbruch'.

135 'Wittgenstein's Lectures', p. 79.

ideas from Frege, Russell and Schopenhauer; and in order to understand the overall philosophical vision, the Kantian formation of the book must also be understood[136]. There is more on mathematics too, perhaps most significantly, on the similarity between mathematical and logical propositions, which is explained in terms of the similarity between equations and tautologies.[137] Kreisel has, however, now understood one of the real reasons for Wittgenstein's choice of elementary examples. Indeed, it becomes clear that his earlier strict finitist interpretation has been revised, and this makes possible a far more fruitful criticism of Wittgenstein's views.

Wittgenstein's remarks on the foundations of mathematics are compared with the views of Bourbaki as they receive expression in their manifesto 'L'Architecture des Mathématiques'[138]. Kreisel begins by comparing the different 'strategy and tactics' of these authors. Their principal target, he says, is 'the formal deductive presentation of mathematics in a universal system': Wittgenstein was most familiar with the logicism of Frege and Russell, while Bourbaki were most familiar with a set-

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136 An interesting comparison of Kant and the early Wittgenstein is contained in *Wittgenstein's* Tractatus (Oxford, Blackwell, 1960), by E. Stenius.

137 See *Tractatus* 6.2 - 6.241.

138 In *Les Grands Courants de la Pensée Mathématique*, edited by F. Le Lionnais (Cahiers du Sud, Paris, 1948). 'The Architecture of mathematics', *American Mathematical Monthly*, 57, 1950, is an English translation.

theoretic variant originating in the work of Cantor and Zermelo.[139] Kreisel says that these systems both claim to present the fundamental analysis of mathematical notions and in Russell's case, at least, the use of a single primitive, ϵ, is supposed to reflect the unity of mathematics. The most obvious difference in Wittgenstein's and Bourbaki's tactics, according to Kreisel, is that: 'Bourbaki refer to wide experience in mathematics, while W[ittgenstein] uses very elementary examples'. These are 'elegant[140] (and popular) but leave open to what extent they are representative of wider experience'.[141]

Kreisel's criticism has some justification. It should not be expected that every problem in the philosophy of mathematics can be discussed adequately only with the aid of elementary examples nor, in general, only at an elementary level; and, if it is not aimed at a popular readership, there is no reason why a particular study in the philosophy of mathematics should have to restrict itself in this way. There is more than one reason why Wittgenstein chose both to employ mainly elementary examples and to discuss more advanced examples at an elementary level. This choice was certainly governed, in part, by the rigours of Wittgenstein's poetic style. It is

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139 'Wittgenstein's Lectures', p. 80.

140 *Pace* Wittgenstein in *Philosophical Grammar* (1969), p. 462 and *The Blue and Brown Books* (1958), pp. 18-19.

141 Ibid.

notable, for example, that in *Philosophical Grammar*, where no attempt had yet been made to arrange the remarks artistically, a fair amount of mathematical detail is included in the discussion of some topics, but in the mathematical typescript (TS **221**), which does have an artistic arrangement, a similar range of topics is discussed at an entirely elementary level. In addition, Wittgenstein wanted to lessen the perceived importance of certain proofs by placing them in the context of elementary mathematics. In the remarks in Volume XIII(2) (MS **117(2)**) on Cantor, Wittgenstein wrote:

> 'Here it is very useful to imagine the diagonal procedure for the production of a real number as having been well-known before the invention of set theory, and familiar even to school children, as indeed might very well have been the case. For this changes the aspect of Cantor's discovery. The discovery might very well have consisted *merely* in the interpretation of this long familiar elementary calculation.'[142]

Wittgenstein probably also felt that there was a danger of being distracted away from the peculiarly philosophical problems by irrelevant mathematical detail. Wittgenstein says in the introductory lecture to the series given in 1939 that he is only concerned with 'puzzles which arise from the words of our ordinary everyday language', and these 'can be exemplified by the most elementary mathematics'.[143] Puzzles arising from the technical terms employed in mathematics 'don't have the characteristic we are particularly interested in. They are not so tenacious,

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[142] *Remarks on the Foundations of Mathematics*, II, 17.

[143] *Wittgenstein's Lectures*, Lecture I, p. 14.

or difficult to get rid of'[144]. It seems unlikely, however, that every aspect of the 'charm' exercised by Cantor's proof, or by other proofs in higher mathematics, can be successfully conveyed in an elementary exposition.

The basic strategy of Wittgenstein and Bourbaki is also different, according to Kreisel. About set-theoretic foundations, Bourbaki simply state that:

> '"This is only one side of the matter, and the least interesting at that", and then go on to describe a better alternative with the same *general* aim: to exhibit, in terms of Bourbaki's basic structures, what is vaguely called the nature of mathematics.'[145]

In contrast:

> 'W[ittgenstein] attempts to convert the fundamentalists by "deflating" the notions and thus the so-called fundamental problems of t.f.[146] stated in terms of those notions. In W[ittgenstein]'s words he wants to *show the fly the way out of the fly bottle*. He does this with much ingenuity and patience, and some overkill...'[147]

Wittgenstein's style, however, is not 'efficient':

> '...current mathematical logic, which has developed several notions of t.f. (has, so to speak, given them rope), seems much better, and some of those developments have positive interest to boot.'[148]

Kreisel has seen that Wittgenstein's claim to be merely describing mathematics was no mere subterfuge, and he is thus able to appreciate the critical or, as he puts it,

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[144] Ibid., p. 15.

[145] 'Wittgenstein's Lectures', p. 80.

[146] Kreisel's abbreviation for 'traditional foundations'.

[147] Ibid.

[148] Ibid.

'pedagogic' value of Wittgenstein's work and to make the distinction between a positive and a negative contribution. He is still inclined, however, to favour the view that any philosophical problems in a scientific subject will be naturally taken care of as that subject develops. One might agree that an itch will eventually go away, but it will go away sooner rather than later if the right sort of ointment is applied.

Kreisel says that the 'general complaint' of Wittgenstein and Bourbaki is that:

> '...t.f. may be *poor philosophy*, in the broader popular sense of "philosophy", specifically, if in practice the general aims of foundations are better served by alternatives, for example, by ordinary careful scientific research and exposition.'[149]

If 'the general aims of foundations' are understood to concern mainly improvements in rigour, then this interpretation would seem to be correct.[150] Kreisel and Wittgenstein are not in disagreement over the scope of mathematical knowledge, they differ mainly in their enthusiasm for science. Philosophical clarity of the sort which Wittgenstein strived for is not valued by Kreisel nor, in general, is it valued by other mathematicians. Wittgenstein's critical philosophy is, therefore, seen to have no special advantage over anything else which tends eventually to rid mathematics of philosophical puzzles. An important difference between Kreisel and Wittgenstein is

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[149] Ibid., p. 81.

[150] See p. 115 above.

that between the scientific, progressive spirit which Wittgenstein describes in the Preface to *Philosophical Remarks* and what I should describe as the mystical spirit, which he there opposes to it. My main objection to Kreisel is that he does not appreciate how far Wittgenstein's critical methods go beyond the robust good sense of the working mathematician, and so he does not see that these methods might usefully be developed further in order to aid the development of mathematics.

Kreisel also compares Wittgenstein and Bourbaki on their 'principal complaint', which is to emphasize the significance of the choice of explicit definitions.[151] He says that for all branches of traditional foundations the choice of explicit definitions is trivial, because they can be systematically eliminated from proofs. In contrast, Bourbaki's scheme for solving problems in terms of the basic structures makes essential use of explicit definitions, and Wittgenstein's objections to logicism emphasize their role in symbolic logic:

> 'If the logical formula F_A expresses the arithmetic theorem A, knowledge of A is needed not only to recognize this fact, which goes without saying, but simply to prove F_A *convincingly*.'[152]

Kreisel's understanding of Wittgenstein's objection is expressed more fully in the appendix to his previous

[151] Cf. 'Der unheilvolle Einbruch', pp. 174-7.

[152] 'Wittgenstein's Lectures', p. 82.

article, 'Proofs and Rules'[153], where it is compared to 'the most familiar...objections'. These are, firstly, that 'the verification of the reduction needs...some numerical arithmetic such as counting', and secondly, that 'some "reduced" proofs may be too complicated to grasp (no longer *übersehbar*)'[154] Kreisel says that Wittgenstein's specific objection is this:

> 'In the particular case of numerical arithmetic, say, in the reduction of $10^2 + 12 = 12 + 10^2$ to the corresponding logical formula F, one does not merely need *some* counting (to verify the reduction), but one needs to know the arithmetic fact $10^2 + 12 = 12 + 10^2$ itself to verify that F is a *logical truth*. We use the whole paraphernalia of decimal notations in order to structure the formula F and recognize it as valid. Unquestionably, in this case, arithmetic does more for logic than logic for arithmetic.'[155]

It would be more accurate to say, I believe, that in his later writings, particularly Volumes XVIII and XIII(5) (MSS **122** & **117(5)**), Wittgenstein formulated various different but closely related objections to logicism. Each of the three objections which Kreisel mentions seems to have a counterpart in Wittgenstein's critique, for example. Wittgenstein believed, firstly, that new concepts or techniques were unwittingly introduced into the Russellian logic[156], and, secondly, that it was only by means of these techniques that long unsurveyable proofs were avoided. He concluded, thirdly, that 'it is conceivable that the

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[153] 'The Motto', pp. 26-38.

[154] Ibid., p. 28.

[155] Ibid. Cf. 'Wittgenstein's *Remarks*', p. 146.

[156] *Remarks on the Foundations of Mathematics*, III, 46g.

Russellian proof of one proposition should not be distinguishable from the Russellian proof of another and should be called different only because they are translations of two recognizably different proofs'[157].

In 'Proofs and Rules', Kreisel also objects to the idea that the formal rules for generating valid logical formulae, 'discovered by Frege and proved complete by Gödel', could be understood 'as an analysis of the *nature* of logical validity, if not as *the ultimate criterion of rigour*'[158]. He says, that 'the norm is simply not applied to the bulk of practice'. Indeed, 'mathematicians reversed the reduction of mathematics to logic, learning to use set theoretical and arithmetical principles for proving logical theorems':

> 'For reliability one needs such "mathematical" proofs in the perfectly realistic sense that "logical" proofs, say by Frege's rules, are long and likely to contain errors. To appeal to the *possibility*, the existence, of some logical derivation d of a theorem F_d is certainly not an application of a norm when we do not know *d* but are convinced by a totally different argument for F_d.'[159]

The formal rules, Kreisel notes, are also 'futile for understanding the actual processes of logical reasoning'.

> 'In mathematical proofs of logical theorems the formal logical inferences are seldom mentioned explicitly at all, and are obviously never the essential part of the argument. Thus those rules do not constitute even an *approximation* to our possibilities of recognizing

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157 Ibid., III, 14b.

158 'The Motto', pp. 30-31.

159 Ibid., p. 31.

> logical validity, of finding and following convincing proofs of logical theorems'.[160]

The same holds for proofs of mathematical theorems in general.[161]

Kreisel compares Bourbaki and Wittgenstein next on two of their 'specific complaints'. I shall consider only the first of these, which concerns the interpretation of consistency proofs.[162] Kreisel agrees with Wittgenstein that 'the familiar dramatics about consistency are unconvincing'[163] and he says that:

> '...by implication, Bourbaki too are unimpressed; treating consistency (or the existence of some model) as a by-product; for example, the model of the - theory for the field **C** of - complex numbers furnished by the Euclidean plane, which was originally hailed for "legitimizing" $\sqrt{-1}$, is reinterpreted...as a useful property of the plane.'[164]

It might be added that Bourbaki's interpretation of consistency proofs also corresponds closely to views expressed by Wittgenstein, for example, in conversation with Waismann in 1931:

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[160] Ibid.

[161] Lakatos's objections on the same theme are examined in section 2.2.1. below.

[162] The second comparison, which concerns higher cardinals, is less satisfactory because Bourbaki's views on this matter are not stated explicitly.

[163] In his previous article, Kreisel wrote: 'Wittgenstein shared what are surely almost universal misgivings: How can such a superficial property as mere consistency, which is certainly most prominent in Hilbert's programme, be central or fundamental?' ('The Motto', p. 28).

[164] 'Wittgenstein's Lectures', pp. 82-3.

'WAISMANN: But what, then, does the proof that non-Euclidean geometry is consistent mean? Let us think of the simplest case where we give a model of two-dimensional Riemannian geometry on a sphere. ...

WITTGENSTEIN: Consistency "relative to Euclidean geometry" is complete nonsense. What is going on here is the following. One rule corresponds to another rule (one configuration of a game to another configuration of a game). Here we have a mapping. That's all! Whatever else is said is everyday prose.'[165]

Kreisel also refers to Wittgenstein's 'pet complaint': 'Why not ensure consistency trivially, by modifying the rules in an obvious way?'[166] Work done by Rosser before 1939 indicates, Kreisel believes, some of the consequences of following Wittgenstein's suggestion[167]; and he uses this example to illustrate the greater efficiency of the normal style of mathematical logic over Wittgenstein's method. It seems plain, however, that although technical developments might sometimes have the effect of allowing us 'to see more'[168], and thus might help us to overcome philosophical difficulties, this is not ensured by those developments alone.

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[165] *Ludwig Wittgenstein and the Vienna Circle*, IV, 'Consistency V'.

[166] *Wittgenstein's Lectures*, Lecture XXII, p. 220. It is notable that Kreisel, in contrast to many other contemporary mathematical logicians, was never bewildered by this remark of Wittgenstein's. See 'Wittgenstein's *Remarks*', section 11.

[167] *Journal of Symbolic Logic*, 1 (1936).

[168] *Remarks on the Foundations of Mathematics*, III, 85g.

Kreisel concludes his review by attempting to describe some of the merits of traditional foundations. He begins:

> '...the weaknesses of t.f....mattered less to W[ittgenstein] than the style of t.f.: (i) the almost staggering banality of the "fundamental" notions and problems compared to the ambitious general aims, and (ii) the - basically pretentious - simple-minded language used to formulate the results of t.f.'[169]

Kreisel's formulation captures some aspects of Wittgenstein's thoughts on the 'craving for generality', but not Wittgenstein's genuine distaste for contemporary attitudes towards science. Kreisel suggests that the stylistic features of traditional foundations might have some value: 'used with much discretion and a little flair', as Gödel did, 'the ideas of t.f. provide *occasional* checks and balances on the strategy of relying on the "needs" of current practice (Bourbaki) or on current uses (W[ittgenstein]).' Kreisel's suggestion is reminiscent of Kant's 'regulative employment of the ideas of pure reason'[170].

> 'Everything that has its basis in the nature of our powers must be appropriate to, and consistent with, their right employment - if only we can guard against a certain misunderstanding and so can discover the proper direction of these powers. We are entitled, therefore, to suppose that transcendental ideas have their own good, proper and therefore *immanent* use...'[171]

Wittgenstein would probably not have been interested in such an idea, but it does seem to have a place in a balanced account of the relationship between philosophy and

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[169] 'Wittgenstein's Lectures', p. 84.

[170] *Critique of Pure reason*, A 642/ B 670ff.

[171] Ibid., A 642-3/B 670-1.

mathematics. Most mathematicians, it seems, will always be inspired by a realist picture of mathematics, and in itself this does no harm, although it might receive expression in higher and lower forms.[172]

Concluding Remarks

Kreisel's later articles each improve successively on his 'constipated and fumbling'[173] review of *Remarks on the Foundations of Mathematics* and on his review of *The Blue and Brown Books*. It cannot be said, however, that Kreisel develops into a reliable interpreter of Wittgenstein. The importance of these later articles is connected with the puzzle which I mentioned earlier[174] about Wittgenstein's great respect for Kreisel's ability. It now seems clear that, besides enjoying the liveliness of Kreisel's intellect, Wittgenstein will also have appreciated the anti-philosophical tendency in Kreisel's thinking. This tendency receives partial expression in the metaphor which Kreisel derived from his own work on hydrodynamics and which he related to Wittgenstein after the War.[175] Wittgenstein and the working mathematician *do* have common grounds for complaint against the philosopher who

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[172] See section 1.2, pp. 90-1.

[173] 'The Motto', p. 25.

[174] See pp. 141-2 and 153 above.

[175] See section 1.2, pp. 80-1.

misrepresents the nature of mathematics; and this is not difficult to see, once it is recognised that a revisionist interpretation of Wittgenstein's philosophy of mathematics is a mistake. Kreisel does come to realise this, and he is then able to make useful objections against Wittgenstein's philosophical method, which articulate the point of view of the working mathematician, i.e. a genuinely scientific point of view.

The misinterpretations of Wittgenstein in Kreisel's early reviews were unfortunate, because, when read alongside the related misinterpretations by Bernays, Dummett and others, they helped complete a picture of apparent unanimity in the reception of Wittgenstein's *Remarks on the Foundations of Mathematics* among those who seemed most competent to assess its value.[176] As a consequence, the discussion of Wittgenstein's views on the philosophy of mathematics which followed the publication of that book, often had very little to do with the real issues raised by Wittgenstein's work. In 1947, Wittgenstein wrote:

> 'Nothing seems to me less likely than that a scientist or mathematician who reads me should be seriously influenced in the way he works. ... The most

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[176] See 'Wittgenstein's *Foundations* and its Reception', *American Philosophical Quarterly* (Vol. 4, No. 4, 1967) by S. Morris Engel, p. 257. Engel's complaints about the articles chosen by Benacerraf and Putnam for inclusion in their textbook *Philosophy of Mathematics* are partially echoed by Rhees's complaint to Anscombe: 'since they are including so many reviews of the RFM, I wonder why they have included nothing from Goodstein's' (Rhees - Anscombe, 15.11.1961).

> I might expect to achieve by way of effect is that I should first stimulate the writing of a *whole lot* of garbage and that then this *perhaps* might provoke someone to write something good. I ought never to hope for more than the most indirect influence.'[177]

Wittgenstein's predictions about his influence have, in my opinion, turned out to be true. Whether 'something good' will eventually be written might just depend, however, on the extent to which Wittgenstein's thoughts *can* be explained and made palatable to mathematicians.

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[177] MS **134**, p. 149.

2.2 Philosophy for Mathematicians

2.2.0 Introduction

The discussion of Kreisel's opinions on Wittgenstein's philosophy of mathematics has raised doubts concerning the satisfactoriness of Wittgenstein's philosophical method when considered from a scientific point of view. It seems that there is a need to develop alternative philosophies of mathematics, or other legitimate heirs to philosophy[1], which, while avoiding the scientistic tendencies exposed in Wittgenstein's writings, nevertheless encourage a positive interest in science and in the advancement of scientific knowledge. Such possible new beginnings in the philosophy of mathematics are the main topic in what follows. It is assumed throughout that Wittgenstein's philosophy represents the only successful paradigm which currently exists for a comprehensive critical philosophy, in the Kantian sense.

Failure to recognize or to accept that Wittgenstein's philosophy is a type of critical philosophy and that it does not contain a theory of language of some sort has been

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1 *The Blue and Brown Books*, p. 28.

a major source of misunderstanding.[2] When, for example, Wittgenstein criticized what mathematicians are inclined to say about 'the objectivity and reality of mathematical facts'[3] he was compared to Brouwer and his belief that there are no philosophical theses[4] - perhaps the most significant and characteristic feature of his critical philosophy - was not taken seriously. Another main reason why Wittgenstein's philosophy has so often been misunderstood is that his basic attitude to philosophy, which shows itself in the poetic style of his two major works, either went unrecognised or was misconstrued. Wittgenstein's style was often considered to be unscientific, only in the sense of being chaotic and disorganised, and this was supposed to reflect his anarchism[5] and dark irrationality[6].

Both of these sources of misunderstanding have to do with features of Wittgenstein's philosophy which, I believe, would generally have least appeal to philosophers with a scientific outlook, i.e. to those philosophers who, Wittgenstein would say, are in sympathy with the spirit

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[2] See 'A Theory of Language?' by G.E.M. Anscombe, in *Perspectives on the Philosophy of Wittgenstein* (Oxford, Blackwell, 1981) edited by I. Block.

[3] *Philosophical Investigations* (1953), Part I, § 254.

[4] Ibid., § 128.

[5] Dummett in 'Reckonings', *Encounter*, 50 (1978), p. 64.

[6] Bernays in *Philosophy of Mathematics*, p. 514.

which 'informs the vast stream of European and American civilization'[7]. Kreisel, for example, could see little value in Wittgenstein's philosophical method; and when he described it as 'inefficient'[8], he did not understand that Wittgenstein never intended his method to be used by mathematicians for the advancement of their subject at the level of working detail. On the contrary, the particular topics which Wittgenstein chose to discuss from the foundations of mathematics, Russell's definition of number, the Dedekind cut, Cantor's diagonal proof, etc., were meant to illustrate confusions in mathematics connected with a pervasive form of scientism, for which Wittgenstein had a strong intellectual and moral antipathy.

The question remains, however, whether a modern critical philosophy of mathematics based on Wittgenstein's thought might be developed in a scientific spirit, and thus be brought closer to the spirit of Kant's pioneering work.

Wittgenstein did have some hope that mathematicians of the future might have a different attitude towards their subject. In Philosophical Grammar (TS **213**), for example, he writes:

> 'A philosopher feels changes in the style of a derivation which a contemporary mathematician passes over calmly with a blank face. What will distinguish the mathematicians of the future from those of today will really be greater sensitivity, and that will - as

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[7] Preface to *Philosophical Remarks* (TS **209**).

[8] 'Wittgenstein's Lectures', p. 80.

> it were prune mathematics; since people will then be more intent on absolute clarity than on the discovery of new games.'[9]

He adds:

> 'Philosophical clarity will have the same effect as light on the growth of potato shoots. (In a cellar they grow yards long.)'[10]

However, given the resistance to Wittgenstein's critical method that has been apparent among scientists in general, it is doubtful whether his work will ever have much of a direct influence on the practice of mathematicians or on our understanding of mathematical methodology. This is unfortunate, because Wittgenstein's work ought to be of interest to the working mathematician who wants to gain a broader philosophical perspective on his subject, but who has doubts about the value of traditional philosophical themes. What is needed, I believe, is a new presentation of Wittgenstein's thought in a style which has relevance for the typical scientist. The most obvious changes to be made are in the standards of scholarship and in the use of advanced mathematical examples.

Nicholas Bourbaki and Imre Lakatos are two authors whose thinking about mathematics was formed, to a significant degree, by their opposition to different philosophical attempts to understand mathematics, and both might be said to have developed a philosophy of mathematics for mathematicians. For this reason, I find it appropriate

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[9] Section 122, p. 643; (*PG*, II, p. 381).

[10] Ibid.

to compare and contrast their views with those of Wittgenstein in the search for new paradigms in the philosophy of mathematics.

2.2.1 Lakatos

In the *Philosophical Investigations*, Wittgenstein said that there are no theses in philosophy and that, therefore, there is nothing 'hidden' which ought to be of any interest to the philosopher.[11] He added that:

> 'The aspects of things that are most important for us are hidden because of their simplicity and familiarity. (One is unable to notice something - because it is always before one's eyes.) The real foundations of his inquiry do not strike a man at all. Unless *that* fact has at some time struck him. - And this means: we fail to be struck by what, once seen, is most striking and most powerful.'[12]

Following Wittgenstein's recommendation, the real foundations of mathematical inquiry are to be sought in the aspects of mathematics which generally go unnoticed because of their simplicity and familiarity; the philosopher is to attempt a 'perspicuous representation' of mathematics[13] or a description of the current geography of mathematics[14].

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[11] *Philosophical Investigations* (1953), Part I, §§ 126-128.

[12] Ibid., § 129.

[13] Ibid., § 122.

[14] *Remarks on the Foundations of Mathematics*, V, 52.

Wittgenstein's idea that a philosopher ought to be concerned with a description of grammar, which concentrates on making 'essential, fundamental distinctions'[15], is closely connected with his other idea that the philosopher's task consists in 'assembling reminders for a particular purpose'[16]. The reminders are, after all, descriptions of grammar. These two ideas can, however, be contrasted, and it is interesting to observe in Wittgenstein's writing the drift that occurs away from mere reminders when he becomes absorbed in a particular conceptual investigation.[17] Lakatos's work is examined here because it seems to represent a possible use of description in the philosophy of mathematics which goes beyond the minimal level usually found in Wittgenstein.

Lakatos devoted his career to the study of scientific methodology in general, but he was particularly interested in the methodology of mathematics. His major work in this area, *Proofs and Refutations: The Logic of Mathematical Discovery*[18], was influenced, he says, by Pólya's 'revival

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[15] See section 1.1, p. 37.

[16] *Philosophical Investigations* (1953), Part I, § 127.

[17] In Wittgenstein's last writings on epistemology, for example, reminders about the grammar of 'knowledge' and 'certainty' which Wittgenstein believes are immediately related to the philosophical problems raised by Moore, often give way, famously in the case of the river bed analogy, to further description which seems to leave those immediate philosophical problems behind.

[18] (Cambridge, Cambridge University Press, 1976).

of mathematical heuristic'[19] and by the philosophy of Karl Popper[20]. Lakatos believed that the logic of mathematical discovery parallels the logic of scientific discovery in certain important respects, and, in particular, that refutation has a heuristic function both in the natural sciences and in mathematics.

Lakatos describes the method of proofs and refutations as 'a very general heuristic pattern of mathematical discovery', which has the following principal stages[21]:

(1) **Primitive conjecture**;

(2) **Proof** (a thought experiment or argument, which decomposes the primitive conjecture into subconjectures or lemmas);

(3) **Refutation** (a counterexample to the primitive conjecture);

(4) **Re-examination of the proof:** a lemma, which may have been implicit, is added in order to exclude the counterexample.

This process, which can vary in its details and sometimes includes further stages, is investigated by means of a discussion of various case studies taken from the history of mathematics. The main case study, which forms the first

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[19] *Proofs and Refutations*, p. xii. See, in particular, *How to Solve It* (Princeton, Princeton University Press, 1945) and *Mathematics and Plausible Reasoning* (London, Oxford University Press, 1954) by G. Pólya.

[20] Author of *Logik der Forschung* (Vienna, Springer, 1934), which was translated into English as *The Logic of Scientific Discovery* (London, Hutchinson, 1959).

[21] *Proofs and Refutations*, beginning of Appendix I.

chapter of *Proofs and Refutations*[22], is an examination of the history of the problem of finding 'a relation between the number of vertices V, the number of edges E and the number of faces F of polyhedra - particularly of *regular polyhedra* - analogous to the trivial relation between the number of vertices and edges of *polygons*, namely, that there are as many edges as vertices: V = E' (§1)[23]. In particular, it concerns the conjecture, due jointly to Descartes and Euler, that: V - E + F = 2 or: All (regular) polyhedra are Eulerian. The main proof discussed (§2) is the one given by Cauchy in his 'Recherches sur les Polyèdres' (1813)[24]. Criticism of responses to the conjecture and the proof, which focusses, in particular, on the status and function of counterexamples, is achieved by means of a - heavily annotated - dialogue, which is designed to mirror the historical responses.

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[22] It existed originally as the first chapter of Lakatos's Ph.D. thesis, and then in a modified form as the essay 'Proofs and Refutations', which appeared in four parts in the *British Journal for the Philosophy of Science*, 14, 1963-4.

[23] These bracketed numbers refer to the subdivisions in Lakatos's essay.

[24] *Journal de L'École Polytechnique, 9.*

What follows is a summary of Lakatos's work[25] with indications of the points where, I believe, his views might usefully be compared to those of Wittgenstein.

Lakatos begins (§3) by making the point that, following proof, a conjecture can be criticized 'locally' through a criticism of the proof's lemmas. The following types of counterexample to the proved conjecture are therefore possible: 'global' counterexamples, concerning the main conjecture, which are also local (§4); global counterexamples which are not local (§5); and local counterexamples which are not global (§6).

Various characteristic responses to the discovery of global counterexamples, i.e. to the discovery of non-Eulerian polyhedra, are criticized (§4). Rejection of the conjecture (§4a) is not necessary, Lakatos believes, because the proof can be improved in response to counterexamples. This introduces the question, which is central to the whole dialogue: what does a proof prove?[26] The rejection of counterexamples (§4b), or 'the method of monster-barring', is not favoured, because it amounts merely to the *ad hoc* redefinition of 'polyhedron'; and it

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[25] This is not entirely straightforward, as it is not always clear which parts of the dialogue express Lakatos's views. In addition, the views which are expressed often receive subsequent qualification or development. Lakatos, like Wittgenstein, does not spare his readers the trouble of thinking.

[26] This question was, of course, also of central importance to Wittgenstein.

is regarded as a symptom of the 'dogmatist bias in the interpretation of mathematical proof', which involves the belief that a proof 'necessarily proves what it has set out to prove'. The search for a perfected definition is misconceived, Lakatos argues, because new counterexamples which might upset that definition are always possible. Another response, which is characteristic of 'the exception-barring method' (§4c), is to improve the conjecture by accepting the supposed 'counterexamples' as 'exceptions'. The conjecture is thus to be restated as: All polyhedra, except..., are Eulerian. The search for a perfected definition is considered to be analogous to and as equally misconceived as the search for a complete list of exceptions or a complete characterisation of the possible types of exception. In addition, such limitations on the domain of validity of the conjecture, besides leading to possible understatement, also make no reference to its proof. Again, we are led to the question: what does a proof prove?[27] Reinterpretation of the counterexample

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[27] Lakatos attempts to characterize the working mathematicians attitude towards proof and their puzzlement about what a proof proves:

> 'On the one hand they know from experience that proofs are fallible but the other hand they know from their dogmatist indoctrination that *genuine* proofs must be infallible. *Applied mathematicians* usually solve this dilemma by a shamefaced but firm belief that the proofs of the *pure mathematicians* are "complete", and so *really* prove. Pure mathematicians, however, know better - they have such respect only for the "complete proofs" of *logicians*. If asked what is then the use, the function, of their "incomplete proofs", most of them are at a loss. For instance, G.H. Hardy had a great respect for the logicians' demand for formal proofs, but when he wanted to

(§4d), or the method of 'monster-adjustment', is briefly dismissed by Lakatos, before the introduction of the favoured method of 'lemma incorporation'. Applying this method (§4e), a lemma in the proof is identified which restricts the domain of validity of the conjecture, and the conjecture is then restated but within the restrictions of the domain of validity of the lemma. Lakatos argues that the lemma incorporating method makes it clear that, in the traditional sense of 'prove', '*one does not prove what one has set out to prove*. Therefore no proof should conclude with the words: "*Quod erat demonstrandum*"[28]'.

An apparent objection to the method of lemma incorporation is suggested by the existence of

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> characterise mathematical proof "as we working mathematicians are familiar with it", he did it in the following way: "There is strictly speaking no such thing as mathematical proof; we can, in the last analysis, do nothing but point;...proofs are what Littlewood and I call *gas*, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils" ["Mathematical Proof", *Mind*, 38, 1928, p. 18].'

Lakatos believes that the method of proofs and refutations clarifies the situation by showing that a mathematical proof does not really prove at all. However, it is unnecessary, I believe, to state this conclusion in such a paradoxical fashion, as if denying an obvious truth. What Hardy says does not, in fact, make any real sense. If a proof only convinces us of something, then *of what* does it convince us? And, if the mathematical proposition which is pointed to is true, "what is the criterion for its being so - if not the proof?" (*Wittgenstein's Lectures*, XIV, p. 130ff).

[28] The reference is to a remark in Alice Ambrose's paper 'Proof and Theorem Proved', *Mind,* 68 (1959), p.438, which is contained in her preliminary exposition of Wittgenstein.

counterexamples which are global but not local (§5). In this case, no explicit lemma exists which can be incorporated into the conjecture, and so it is tempting to defend the theorem by refuting the counterexample (§5a). Such counterexamples are properly met, Lakatos believes, by the addition (§5b) of formerly implicit lemmas; although, he says that it would be dogmatic to insist that these were already understood prior to the refutation. In the 'method of proof and refutations' (§5c), these additional lemmas are incorporated explicitly. The rules for this method are as follows:

> 1. Attempt both to prove and refute a given conjecture. Prepare a list of non-trivial lemmas (proof-analysis); find counterexamples to the conjecture and to suspect lemmas.
>
> 2. In response to a global counterexample, replace the conjecture by one with a suitable lemma that is refuted by that counterexample and incorporate that lemma as a condition. Try to make all implicit lemmas explicit.
>
> 3. If you have a local counterexample, see if it is also a global counterexample, and if it is, apply Rule 2.

This method can be understood to have as its aim a 'rigorous proof-analysis', which would ensure that every counterexample can be refuted by incorporating lemmas which are already explicit. Lakatos concludes here (§5d) by attempting to 'show how the emergence of mathematical criticism has been the driving force in the search for the "foundations" of mathematics'. Successive revolutions in our standards of mathematical rigour have, he maintains, led to proof-analyses which penetrate ever deeper into the proofs down to the 'foundational layer', where absolute rigour of proof is assumed. He says that '*different levels of rigour differ only about where they draw the line between the rigour of proof-analysis and the rigour of proof,* i.e. *about where criticism should stop and justification should start*'. The idea that proofs should be absolutely rigorous and assume no background knowledge is taken to be a consequence of mathematicians' dogmatic belief in the certainty of proof. Lakatos recommends that we reject this belief and accept that our mathematical knowledge has no foundations.[29]

Lakatos returns (§6) to the subject of counterexamples which are local but not global. He describes (§6a) how one can increase the content of a conjecture by replacing a falsified lemma with an unfalsified one. A more radical

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[29] Wittgenstein also believed that mathematics had no need of logicist or formalist foundations (RFM, VII, 16), and the reasons which he gave correspond to those given by Lakatos. The idea that mathematics stands in need of foundations is connected by both authors with a false ideal of mathematical certainty. According to Wittgenstein, and this is one aspect of his discussion of rule-following, the foundations of mathematics are laid when we learn to calculate. In *Philosophical Grammar* (TS **213**), section 109, he had written: 'Teach it to us, and then you have laid its foundations.' Much later he wrote:

> '*This* is how one calculates. Calculating is *this*. What we learn at school, for example. Forget this transcendent certainty, which is connected with your concept of spirit.' (*OC*, § 47).

The advantage of Lakatos's account is that the extent to which the preoccupation with 'certainty' was the outcome of definite historical trends *within mathematics* is clarified. Such conclusions, which can only be arrived at by means of detailed scholarship, point in general to severe limitations in Wittgenstein's style from a scientific point of view.

alternative is to look for a substantially different proof. The different proofs of the Descartes-Euler conjecture given by Gergonne and Legendre are compared to Cauchy's proof, and the latter is recognised as having the most 'depth' or 'content'. Ideally a proof would provide not only the necessary but also the sufficient conditions for Eulerianness, and would thus achieve 'finality'. Without this finality, one might still hope to develop a number of different proofs in order to capture Eulerianness. The method of *proofs* and refutations is introduced as the method which examines any number of individual proofs of related conjectures by the method of *proof* and refutations. Lakatos recognises (§6c) that different proofs of the same primitive conjecture yield different theorems: 'The usual expression "*different proofs of the Euler theorem*" is then confusing, for it conceals the vital role of proofs in theorem-formation'[30].

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[30] This conclusion, with which readers of Wittgenstein ought to be familiar, is expressed in a far more compelling fashion when realistic examples of different proofs are employed.

I should say here also that the sense in which one can distinguish the value of different proofs, as Lakatos does, points to one part of a legitimate expression of mathematical realism. (Not 'transcendental realism', but something corresponding to Kant's 'empirical realism'.) Clear descriptions of the objectivity of mathematics, which go beyond the schematic: 'p' is true = p, are often lacking in Wittgenstein's writing. Doubtless, this has something to do with his desire to hold on to a philosophical puzzle for as long as possible (See MS **157b(2)**, p. 61) but this method can also be confusing and can leave the impression that something obvious has been denied.

Lakatos explores the problem of content further (§7). He suggests (§7a) that one should always be prepared to abandon a conjecture and replace it by a new, deeper conjecture which addresses the original problem more successfully. The means of arriving at a primitive conjecture are thus re-examined, and induction as a method is scrutinized (§7b). Lakatos shows (§7c) how the 'background knowledge' related to a problem is a source for ideas from which a proof might be constructed, and that proof might actually precede the naive conjecture. He thus distinguishes two 'heuristic patterns' which describe the manner in which we arrive at primitive conjectures: naive guessing, by means of which, for example, the Descartes-Euler conjecture was obtained, and deductive guessing, which begins with a proof-idea. He also argues that: '*Naive conjectures are not inductive conjectures: we arrive at them by trial and error, through conjectures and refutations*'. So: '*Mathematical heuristic is very like scientific heuristic - not because both are inductive, but because both are characterised by conjectures, proofs and refutations*'.[31] Lakatos introduces the method of deductive guessing, which is employed in response to counterexamples of any type in order to increase the content of the

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[31] Here Lakatos points to a similarity where Wittgenstein would be inclined to point to a difference (see section 1.2, p. 95). In *Philosophical Grammar* (TS 213), section 119, he wrote:

> 'Nothing is more fatal to philosophical understanding than the notion of proof and experience as two different but comparable methods of verification.'

theorem. This method has limitations, however, and can lead to an indefinite series of *trivial* additions to the content of a theorem. A distinction is, therefore, introduced between heuristic counterexamples and logical counterexamples, which have a limited heuristic value.

The distinction between logical and heuristic counterexamples is explained further in the next section (§8) on mathematical concept formation. The relation between the different sorts of theorem generating method, monster-barring, exception-barring, and the method of proofs and refutations, is here clarified. Properly understood, monster-barring simply restricts the conjecture to the originally intended domain. In contrast, exception-barring and the method of proofs and refutations allows for the refutation of the conjecture by means of counterexamples. This should be understood in each case, however, as the refutation of a *new* conjecture. The counterexamples are, therefore, heuristic rather than logical in character. The trivial extensions to the monster-barring definition are not genuine contributions to the growth of knowledge.[32] Lakatos also makes the point (§8b) that proof generated concepts replace naive concepts; and that, as a consequence, the lemma incorporated definition is an expansion as well as a contraction of the old idea. The proved conjecture might, therefore, have

[32] In contrast to Wittgenstein, the growth of knowledge is all important to Lakatos, and his philosophical remarks, therefore, tend to be prescriptive.

better been stated: 'All Cauchy-polyhedra are Eulerian'.[33] Naive conjectures and naive concepts are superseded by improved conjectures (theorems) and concepts (proof generated or theoretical concepts) growing out of the method of proofs and refutations.[34] (§8c) Even logical refutations expand the 'conceptual, taxonomical, linguistic, framework'. 'Every period of creation is at the same time a period in which the language changes'.[35] (§8d) Various historical patterns in the process of proofs and refutations are described. Lakatos says that taste is required to distinguish increases in content from increases in depth and failure to recognise this is connected by him with a desire for complete definition. (§8e) Limits to the capacity for growth in content are admitted: certain

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[33] It ought to be said that 'Eulerianness' changes too if the concept of edge, etc. changes.

[34] Naive classifications of polyhedra are also replaced by theoretical classifications. Lakatos comments: 'As far as naive classification is concerned, nominalists are close to the truth when claiming that the only thing that polyhedra have in common is their name. But after a few centuries of proofs and refutations, as the theory of polyhedra develops, and theoretical classification replaces naive classification, the balance changes in favour of the realist. The problem of universals ought to be reconsidered in view of the fact that as languages grow concepts change.' Lakatos points here to another legitimate sense of the objectivity of mathematics, although one which is extremely difficult to describe accurately; and, unfortunately, he seems to succumb to a form of scientism in which, for example, it would be denied that whales are in any sense fish.

[35] Lakatos quotes from *L'Aspect Moderne des Mathématiques* (1957), by L. Félix. Cf. Wittgenstein: 'However queer it sounds, the further expansion of an irrational number is a further expansion of mathematics' (*RFM*, V, 9).

freaks ought to be excluded as counterexamples, because they really belong to a different theory.

Lakatos concludes his discussion (§9) by explaining 'how criticism may turn mathematical truth into logical truth' and by expanding on his interpretation of the recent history of mathematics. He begins with the simple point (§9a) that there are limits to what can be considered rational in the way of concept stretching. Within these limits, he believes, is the radical form of concept stretching which allows any translation of the terms in a theorem which renders that theorem false (§9b). The terms which do not allow of meaningful translation are recognised as the logical constituents of the theorem. When, through the process of proofs and refutations, logical constituents are all that remain, the theorem has become a logical one. Lakatos interprets the modern development of mathematics in these terms:

> 'Nineteenth-century mathematical criticism stretched more and more concepts, and shifted the meaning-load of more and more terms onto the *logical form* of the propositions and of the few (as yet) unstretched terms.'
>
> '*This revolution in mathematical criticism changed the concept of mathematical truth, changed the standards of mathematical proof, changed the patterns of mathematical growth!*'[36]

That is the end of my summary. I shall now examine some of the wider conclusions which Lakatos derives from

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[36] See the comments on §5d above.

his study of the process of proofs and refutations. These conclusions are set out in the Introduction to his essay.

Lakatos is opposed mainly to the influence of formalism in the history and philosophy of mathematics, but particularly to Hilbert's conception of metamathematics and to variants of Hilbert's view espoused by the logical positivists[37]. Like Kreisel, Lakatos objects to formalism principally because it fails to represent actual mathematical practise, or 'live mathematics'[38]. Outside of its scope are 'all problems relating to informal (*inhaltliche*) mathematics and to its growth' and 'all problems relating to the situational logic of problem-solving'[39]. By contrast, he says, 'an investigation of *informal* mathematics will yield a rich situational logic for working mathematicians'[40]. Lakatos's particular study of the logic of mathematical discovery is meant to show up the 'bleak alternative' in formalist mathematics between dreary mechanical procedure and irrational insight. Lakatos also emphasizes the importance of definitions, which play a central role in the development of informal

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[37] Carnap's *The logical Syntax of Language* (London, Kegan & Paul, 1937) is mentioned specifically.

[38] *Proofs and Refutations*, p. 4.

[39] Ibid., p. 1.

[40] Ibid., p. 4.

mathematics and do not there have the function of mere abbreviations.[41]

In his review of Wittgenstein's *Lectures on the Foundations of Mathematics*, Dummett says that Lakatos's philosophical conclusions 'have the merit of really having been based on seeing what we actually do, as, despite his advocacy of that way of proceeding, Wittgenstein's do not'[42]. Kreisel, in his review of the same book, objects in a related fashion that Wittgenstein's elementary examples 'leave open to what extent they are representative of wider experience, too'[43]. It can certainly be doubted, as I have argued[44], whether all of the philosophical problems with which Wittgenstein was concerned can be investigated satisfactorily at an elementary level; however, the different levels of scholarship in Lakatos and Wittgenstein, reflecting their different interests, should not be allowed to obscure their fundamental agreement: both authors accept that description of the actual practice of working mathematicians is a legitimate method in the philosophy of mathematics.

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41 See section 2.1, pp. 180-2, on Kreisel's comparison of Bourbaki and Wittgenstein on the choice of explicit definitions.

42 'Reckonings', p. 68.

43 'Wittgenstein's *Lectures*', p. 80.

44 See section 2.1, p. 176.

Besides 'making excellent use of detailed observations about mathematics as it is actually practised'[45], Dummett believes that Lakatos is also in advance of Wittgenstein in his understanding of mathematical certainty. According to Dummett, Lakatos correctly exposes the idea of the absolute certainty of mathematical propositions as 'spurious', whereas Wittgenstein simply attempts to give a new account of its source. I should say, however, that Lakatos and Wittgenstein are in basic agreement on the question of the certainty of mathematical propositions, and this is because both authors have a good understanding of the relation between a mathematical proposition and its proof. As a careful reading of Lakatos's dialogue shows, mathematical propositions are not refuted in the process of proofs and refutations, they are *replaced* by other propositions.[46] In this sense, mathematical propositions are not fallible; we do 'turn our backs on them' and 'put them in the archives'. This ought not to be paradoxical, if Wittgenstein's and Lakatos's related remarks about mathematical conjecture and what it is that a proof proves have been properly understood.

In accordance with these observations, I believe that the least plausible aspect of Lakatos's interpretation of his own work is that it represents a *sceptical* response to

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[45] Dummett in his review of *Proofs and Refutations* for *Nature*.

[46] See §8.

a dogmatic belief in absolute mathematical certainty. Lakatos has done no more than criticize a philosophical picture of mathematical certainty. Properly understood, Lakatos's 'scepticism' is no more closely related to traditional scepticism than that which Malcolm finds in Wittgenstein's last writings on certainty.[47] It is preferable that the term be used to describe the philosopher who is genuinely intent on denying something.

Lakatos's work is best understood, I believe, as an illustration of the philosophical value of the detailed description of mathematical practice, which goes beyond the elementary examples employed by Wittgenstein. It ought to be recommended as a paradigm for work in a descriptive philosophy of mathematics for mathematicians.

2.2.2. Bourbaki[48]

In his review of Wittgenstein's *Remarks on the Foundations of Mathematics*, Kreisel opposed 'traditional foundations' to what he described as a 'less austere

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[47] See *Nothing is Hidden*, Chapter 11. I believe that it was unhelpful of Malcolm to describe Wittgenstein's views as a form scepticism, especially given the influence of the, obviously mistaken, sceptical interpretation by Saul Kripke in his Wittgenstein *On Rules and Private Language* (Blackwell, Oxford, 1982).

[48] 'Bourbaki' is the pseudonym for a society of French mathematicians who are the joint authors of a comprehensive treatise of modern mathematics, *Eléments de Mathématique*. The first of many volumes was published in 1939.

conception of the philosophy of mathematics':

> 'As mathematics has grown, a variety of different methods of proof, definitions, theorems have accumulated. By the light of nature we see differences, groupings within one branch, and similarities between different branches of mathematics. One may see one aim of a philosophy of mathematics in getting a clear understanding of these connections, and there is no reason in advance why this should be done only by reference to "applications", and not, e.g. by mathematical properties, by mathematical characterisations. From this point of view it is a contribution to the philosophy of mathematics if a new aspect of the methods of mathematics has been noticed...; here there is no one fundamental problem.'[49]

Wittgenstein said that philosophy might be described as that which is possible before all new discoveries and inventions[50]. Lack of clarity about the nature of mathematics is not, he believed, removed by a proof. In this sense, contra Hilbert and Ramsey, there are no 'leading problems' in the philosophy of mathematics.[51] However, Wittgenstein does not exclude the possibility that developments in mathematics might lead to greater philosophical clarity[52]; nor, I believe, does he exclude the possibility, which Kreisel mentions here, that a philosophically inspired presentation of the scope and unity of mathematics might itself be mathematical.

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[49] 'Wittgenstein's Remarks', pp. 144-45.

[50] *Philosophical Investigations* (1953), Part I, § 126.

[51] Ibid., § 124.

[52] In 'The Motto', p. 17, Kreisel recalls Wittgenstein's appreciation of 'Cauchy's *philosophical* finesse, and success in analyzing limits away'.

Wittgenstein rejected all of the contemporary mathematical philosophies of mathematics with which he was acquainted, most importantly, those of Frege, Russell, Hilbert, and Brouwer, and he spoke, in general terms, of the 'erroneous opinion that a calculus could be the mathematical foundation of mathematics'[53]. One might attempt to sum up Wittgenstein's various objections in this way: seeking to reach a greater depth in mathematics, the philosopher merely finds himself back on the old level.[54] Bourbaki, with whom Wittgenstein was not acquainted, seem, however, to be an interesting exception: their mathematical philosophy was more in the nature of an overview of mathematics of the sort which Kreisel adumbrates.

What follows is a summary of Bourbaki's famous manifesto, 'The Architecture of Mathematics'[55], with indications of the points where, I believe, their views might usefully be compared to those of Wittgenstein.

(1)[56]. Bourbaki's central question is whether a unitary conception of mathematics is possible, i.e.

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[53] *Philosophical Grammar* (1969), II, 12.

[54] *Remarks on the Foundations of Mathematics*, VI, 31.

[55] *American Mathematical Monthly*, 57 (1950), which is a translation of 'L'Architecture des Mathématiques', in *Les Grands Courants de la Pensée Mathématique* (Cahiers du Sud, 1948) ed. by F. Le Lionnais.

[56] These bracketed numbers refer to the sections in Bourbaki' article, which are the source for any quotations used in the summary of that section.

whether, given the 'extent and the varied character of the subject', there can be 'a view of the entire field of mathematical science as it exists'. Bourbaki observe that, starting with Pythagoras, the attempted integration of the subject has been undertaken by philosophers, who have started from '*a priori* views concerning the relations of mathematics with the twofold universe of the external world and the world of thought'. Bourbaki attempt to arrive at a unitary conception of mathematics by a different route:

> '...we shall not undertake to examine the relations of mathematics to reality or to the great categories of thought; we intend to remain within the field of mathematics and we shall look for an answer to the question which we have raised, by analyzing the procedures of mathematics themselves.'[57]

(2). The massive development of modern mathematics might seem to have lead to a 'progressive splintering' of the subject, but in Bourbaki's view:

> '...the internal evolution of mathematical science has, in spite of appearance, brought about a closer unity among its different parts, so as to create something like a central nucleus that is more coherent than it has ever been. The essential aspect of this evolution has been the systematic study of the relations existing between different mathematical theories, and which has led to what is generally known as the "axiomatic method"'.

This method is, they say, to be distinguished from 'logical formalism', which merely has to do with 'the language

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[57] Despite their differences, there is common to Wittgenstein, Lakatos and Bourbaki an emphasis on understanding mathematics as it is, rather than as it ought to be according to some *philosophical* ideal, and they each emphasize a type of understanding which consists in seeing connections. One might ask: 'Is this a "Weltanschauung"?' (See *Philosophical Investigations*, Part I, § 122).

suited to mathematics'.

> 'To lay down the rules of this language, to set up its vocabulary and to clarify its syntax, all that is indeed extremely useful; indeed this constitutes one aspect of the axiomatic method, the one that can properly be called logical formalism... But we emphasize that it is but one aspect of this method, indeed the least interesting one.'[58]

The axiomatic method, as Bourbaki understand it, has as its 'essential aim' something which cannot be achieved by logical formalism alone, namely 'the profound intelligibility of mathematics'. This is achieved through an understanding of the 'deep lying reasons' for the connections that are discovered between various mathematical theories.[59]

(3). The central notion of the axiomatic method is that of 'structure', which is, roughly speaking, the common

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[58] Bourbaki regard it as a 'meaningless truism' to say that formal logic is a 'unifying principle for mathematics'; it 'can certainly not account for the evident complexity of different mathematical theories, not any more than one could, for example, unite physics and biology into a single science on the ground that both use the experimental method'. Wittgenstein complains similarly that logical technique is only an 'auxiliary technique in mathematics': to say it is of fundamental significance would be like saying that 'cabinet-making consisted in glueing' (*RFM*, V, 24). Also, logical notation, he says, 'swallows the structure' of individual mathematical propositions (*RFM*, V, 25). In reaction to logicism, Wittgenstein thus wanted 'to give an account of the motley of mathematics' (*RFM*, III, 46, 48) and 'to show that we can get away from logical proofs' (*RFM*, III, 44).

Bourbaki also insist that mathematics is not 'a randomly developing concatenation of syllogisms' nor 'a collection of more or less "astute" tricks, arrived at by lucky combinations', remarks which recall Lakatos's objections to formalist accounts of mathematical methodology.

[59] See footnote 63 below.

form of two or more mathematical theories.

> 'The common character of the different concepts designated by this generic name, is that they can be applied to sets of elements whose nature has not been specified; to define a structure, one takes as given one or several relations, into which these elements enter...; then one postulates that the given relation, or relations, satisfy certain conditions (which are explicitly stated and which are the axioms of the structure under consideration.) To set up the axiomatic theory of a given structure, amounts to the deduction of the logical consequences of the axioms of the structure, excluding every other hypothesis on the elements under consideration (in particular, every hypothesis as to their own nature).'[60]

The 'metaphysical pseudo-problems concerning mathematical "beings"' disappear, they believe, when it is accepted that mathematical structures are the only 'objects' of mathematics.[61]

(4). Certain 'great types of structures' are identified, which include group structures, algebraic structures, order structures, and topological structures. It is understood that further types of structure might always be added to this list.

(5). Bourbaki hold that the axiomatic method effects an 'economy of thought':

> 'The "structures" are tools for the mathematician; as soon as he has recognized among the elements, which he is studying, relations which satisfy the axioms of a

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[60] Bourbaki's article includes an helpful illustration of this definition and some important qualifications concerning its generality.

[61] One of these 'pseudo-problems' concerns the idea, which also exercised Wittgenstein, that mathematical objects are somehow ideal abstractions from sense experience (*PG*, II, 17; *RFM*, V, 5).

> known type, he has at his disposal immediately the entire arsenal of general theorems which belong to the structures of that type.'[62]

Bourbaki also emphasize the role played in mathematical research by a special kind of intuition. The geometric representation of the imaginaries, for example, amounted to the discovery of a familiar topological structure, the Euclidean plane, in the set of complex numbers. Many recent historical advances have resulted, they believe, from similar intuitive discoveries.[63]

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[62] Wittgenstein's remark about 'the fashion of the axiomatic system', which is echoed in Lakatos's objections to formalist methodology, would seem to stand in need of some qualification, or at least clarification. There seem to be good reasons for the development of axiomatics which have to do with the standardization of mathematical technique and the efficient organization of mathematical knowledge. Also, as Lakatos attempted to show, the development of mathematical rigour is to a large degree a tendency towards axiomatics.

[63] Bourbaki describe mathematical intuition as 'a kind of direct divination (ahead of all reasoning) of the normal behaviour, which he seems to have a right to expect of mathematical beings, with whom a long acquaintance has made him as familiar as with the beings of the real world'. Belief in a special intuition of mathematical beings is, of course, widespread among mathematicians. Hardy's expression of this belief in 'Mathematical Proof', *Mind*, 38 (1929) attracted Wittgenstein's attention and Gödel's related conviction expressed first in 'Russell's Mathematical Logic', *The Philosophy of Bertrand Russell* (Northwestern University Press, 1944), became a locus for later philosophers. Deciding when this generic belief has received a particular *philosophical* expression is, however, an extremely subtle matter, i.e. it is not always clear when we are on the threshold of 'mathematical alchemy' (*RFM*, V, 16).

Wittgenstein once said, against Ramsey: 'Not empiricism and yet realism in philosophy, that is the hardest thing' (*RFM*, V, 23). One thing which, I think, Wittgenstein meant by this is that, if a calculation is not understood to express the result of an experiment, then it is difficult to see how it expresses more than what the calculator has arrived at or what he has convinced himself of, which is too subjective. Wittgenstein thus attempted

(6). The unifying nature of the axiomatic method makes possible a general survey of mathematics, replacing traditional divisions in the subject.[64]

> 'The organizing principle will be the concept of a hierarchy of structures, going from the simple to the complex, from the general to the particular.'

At the centre of this hierarchy are the great types of structure, among which there is considerable diversity: the most general structure of each type, which has the fewest number of axioms, exists alongside structures obtained by adding supplementary axioms. Beyond this central core are the 'multiple structures', which result from the

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to give an account of the objectivity of mathematics by exploring 'the limits of empiricism'. In general, however, he failed to give an adequate positive account of the objectivity of mathematics, and, in particular, he failed to investigate the legitimate basis of feelings of the sort which Bourbaki express.

[64] Bourbaki's revisionism is not philosophically objectionable, because it is based on intelligible convictions about the efficient organization of mathematical knowledge, in accord with the underlying structure of mathematical theories. No part of mathematics is deemed to be illegitimate, and so mathematics is neither divorced from its actual practice nor from its history.

Wittgenstein would, I believe, have welcomed the idea of consciously organizing mathematics in such a way as not to obscure mathematical form. He believed, for example, that finite and infinite sets were often taken together in a way which disguised their basic difference in form. His own notation for a cardinal number (positive integer) in the *Tractatus* (6.03) was meant to correct this fault. Connected with this, Wittgenstein later objected to misleading statements of the results of proofs by mathematical induction. However, in general, he was inclined to leave mathematics to the mathematicians. One of the pragmatic reasons which he gave for such abstention was the danger of becoming like 'a ham-fisted director', doing other peoples' jobs badly (PG, II, 24). This particular objection leaves open the possibility that mathematicians who have developed a nose for philosophical problems might have a special contribution to make in mathematics.

combination of two or more of the great structures by means of connecting axioms. Farther out are the truly particular structures whose objects have 'a more definitely characterized individuality'. Located here are the theories of classical mathematics, which are seen as 'crossroads, where several more general mathematical structures meet and react upon one another.'

To avoid misunderstanding, Bourbaki point out that their presentation of the axiomatic method is schematic, idealized and frozen. Mathematical accretion does not occur in the simple and systematic manner described; there are areas of mathematics in which the role of the great structures is not clearly recognized; and our understanding of the hierarchy of structures is liable to constant revision as mathematics progresses.

> 'It is like a big city, whose outlying districts and suburbs encroach incessantly, and in a somewhat chaotic manner, on the surrounding country, while the center is rebuilt from time to time, each time in accordance with a more clearly conceived plan and a more majestic order, tearing down the old sections with their labyrinths of alleys, and projecting towards the periphery new avenues, more direct, broader and more commodious.'[65]

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[65] Compare Wittgenstein:

> 'Our language can be seen as an ancient city: a maze of little streets and squares, of old and new houses, and of houses with additions from various periods; and this surrounded by a multitude of new boroughs with straight regular streets and uniform houses.' (*Philosophical Investigations*, Part I, § 18).

The similarity of the metaphors is due to the shared goal of wanting to understand mathematics in its variety.

(7). Some objections to the axiomatic method common among mathematicians have been due, Bourbaki believe, to mere historical accident: the first axiomatic theories - those of arithmetic by Peano and Dedekind and those of Euclidean Geometry by Hilbert - could only be applied to the theory from which they were extracted, unlike, for example, the theory of groups. Also, there emerged 'a whole crop of monster-structures, entirely without applications'[66] which, therefore, gave a misleading picture of what might be achieved in axiomatics.

As to the objections of philosophers, Bourbaki are most interested in 'the great problem of the relations between the empirical world and the mathematical world'. From their point of view, mathematics is a 'storehouse of abstract forms', but one whose success in providing for the description of 'empirical reality' is rather puzzling[67]. Less problematic, they believe, are the 'logical difficulties encountered in the theory of sets', which can be overcome 'in a way which leaves neither the slightest

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[66] These are part of what Wittgenstein would have referred to as 'a cancerous growth, seeming to have grown out of the normal body aimlessly and senselessly'.

[67] Like Kreisel (see 'Wittgenstein's Lectures', footnote 1) I am unmoved by Bourbaki's doubts on this point: there are no external standards by which to judge the effectiveness of mathematics in its application to the empirical world.

qualms nor any doubt as to the correctness of the reasoning' in axiomatic theories.[68]

That is the end of my summary. In his review of *Wittgenstein's Lectures on the Foundations of Mathematics*, Kreisel observed that Bourbaki had been dismissed by some philosophers as 'mere mathematicians lacking the higher sensibility needed for a true interest in t.f.!'[69] I should agree with Kreisel that Bourbaki's view, on the contrary, deserves serious consideration *as a philosophy of mathematics*. They present a natural picture of the nature and development of mathematics, which is in accord with the modern axiomatic treatment of the subject, and which, at the same time, avoids many of the philosophical confusions of their contemporaries.

A principal weakness, in my view, is that they have little to say about the broad range of mathematical activity outside of advanced mathematical research. In

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[68] For both Bourbaki and Wittgenstein mathematics provides forms for the description of the empirical world, and none of these forms is understood to have any epistemological priority. Both authors are therefore unimpressed by 'foundational crises', which are for them, at best, difficulties in one particular area of mathematics.

Bourbaki place their faith, rather, in the architecture of mathematics itself, just as, they say, the natural scientist 'starts from the *a priori* belief in the permanence of natural laws'. This, however, seems to be more an expression of optimism than a philosophical position.

[69] 'Wittgenstein's Lectures', p. 80.

that sense they do not present a comprehensive picture of the family of mathematical activities of the sort which Wittgenstein thought it would be desirable to sketch. Or, to extend Bourbaki's metaphor, they give little indication of how life in the metropolis is related to life in the surrounding country.

Concluding Remarks

One main purpose of the preceding two sections has been to help define Wittgenstein's thought on the foundations of mathematics by means of comparisons with the work of other important thinkers who reacted in a similar or related manner to the philosophical trends in mathematics with which he was concerned. These comparisons also suggest various fruitful lines of criticism.

It is unfortunate, though perhaps not surprising, that during the period of the early reception of Wittgenstein's writings, as they were first being published, few people showed much understanding of his philosophy. Most of the controversy surrounding his work had little to do with what he actually said; and, although a certain amount of friction was created, not much real work was done either in the positive development of our understanding of his views or, inevitably, in their criticism. The subtlety, originality and profundity of Wittgenstein's thought on the foundations of mathematics, in particular, went almost entirely unnoticed.

In the future, and especially with the availability of a convenient complete edition of Wittgenstein's works, it can be hoped that research into Wittgenstein's remarks on the foundations of mathematics will quickly progress to the stage where we have an accurate picture of the development of his views, at least in the major works, and of their historical background. Besides this important scholarly work, it can be hoped that developments in the philosophy of mathematics which are genuinely based on Wittgenstein's writings, either in sympathy or reaction, will begin to flourish.

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Appendix I: Tables of Correspondences

In the following pair of tables, A and B, reference is made within Wittgenstein's manuscripts by means of page numbers and within his typescripts by means of section numbers. The numbering of the sections in typescripts **222-4** follows that of *Remarks on the Foundations of Mathematics*, Part I. Blank lines in A mark the main divisions made in typescript **221** when it was revised to produce typescripts **222-4**, and in B they mark adjacent sections in typescripts **222-4** which have non-adjacent sections in typescript **221** as their sources.

A. The Manuscript Sources for Typescript **221**, and the Correspondences between Typescripts **221** and **222-4**.

MS Source(s)		TS **221**	TS **222**
-	**117**,1-2	162	1
-	**117**,3	163	2
-	**117**,3-4	164	4
-	**117**,4-5	165	5a
-	**117**,5-7	166	6
-	**117**,7	167	7
-	**117**,8	168	x
-	**117**,8-9	169	8
-	**117**,9	170	9
-	**117**,9-11	171	5b-d
118,22	**117**,11-12	172	10
-	**117**,12-13	173	12
-	**117**,13	174	13
-	**117**,13	175	14
-	**117**,13-14	176	15
118,24	**117**,14-15	177	16
118,6-7	**117**,19-20	178	3a-c
118,26-27	**117**,20-21	179	17
118,28	**117**,21	180	18
-	**117**,15-16,21-22	181	19
-	**117**,16-17	182	20
-	**117**,17	183	21
-	**117**,17-19	184	22
118,28-29	**117**,22	185	157
118,30	**117**,22-23	186	158
118,30	**117**,23	187	159
118,30-31	**117**,23	188	160
118,31-33	**117**,23-24	189	99
118,33-34	**117**,24-25	190	100
118,36	**117**,25	191	101
118,36-37	**117**,25	192	102
118,37	**117**,25-26	193	x
118,37-38	**117**,26	194	113
118,38	**117**,26	195	115
118,38-39	**117**,27	196	117
118,40	**117**,27-28	197	161a-b
118,41-42	**117**,28	198	155
118,42-43,43-44	**117**,29	199	156

118,44-46	**117**,29-32	200	116
118,46	**117**,32	201	162
118,46-47	**117**,32	202	163
118,47	**117**,32-33	203	103
118,47	**117**,33	204	104
118,47-50	**117**,33-35	205	105a-e
118,50	**117**,35	206	114
118,50	**117**,35	207	23
118,51	**117**,35-36	208	25
118,51-52	**117**,36	209	26
118,52-54	**117**,36-38	210	27
118,54-55	**117**,38	211	28
118,55	**117**,39	212	29
118,57	**117**,39	213	30
118,57-59	**117**,39-40	214	31
118,59	**117**,40-41	215	32a-c
118,59-60	**117**,41	216	33
118,60-62	**117**,41-42	217	34
118,62	**117**,42	218	35
118,62-64	**117**,42-43	219	36
118,64-66	**117**,43-44	220	37
118,66	**117**,44	221	x
118,66-67	**117**,44-45	222	38
118,67-68	**117**,45-46	223	143
118,68	**117**,46	224	144
118,69	**117**,46	225	145
118,69	**117**,47	226	146
118,69-70	**117**,47	227	147
118,70	**117**,47	228	x
118,70	**117**,47	229	134
118,70	**117**,47	230	24
118,70-71	**117**,47-48	231	148
118,71-72	**117**,48	232	149
118,72	**117**,48-49	233	150
118,73	**117**,49	234	152
-	**117**,49	235	151
118,74-76	**117**,50-51	236	153
118,76-77	**117**,51-52	237	154

118,77-78	**117**,52	238	136
118,78-79	**117**,52-53	239	137
118,79	**117**,53	240	138
118,79	**117**,53	241	139
118,79-80	**117**,53-54	242	140
118,80-81	**117**,54	243	39
118,81-82	**117**,54-55	244	40
118,82-85	**117**,55-57	245	41
118,85-87	**117**,57-58	246	42
118,88	**117**,59-60	247	43
118,88-89	**117**,60-61	248	44
118,89-90	**117**,61	249	45
118,90	**117**,61	250	46
118,90	**117**,61	251	47
118,97-100	**117**,64-66	252	118
118,100	**117**,67	253	48
118,100-101	**117**,67	254	49
118,106-108	**117**,71-72	255	50
118,108-110	**117**,72-74	256	106
118,110-111	**117**,74	257	107
118,111	**117**,74-75	258	108
118,111-112	**117**,75	259	109
-	**117**,75-76	260	110
-	**117**,76-77	261	111
-	**117**,77-78	262	112
118,113-114	**117**,73	263	51
118,114	**117**,78	264	52
118,114	**117**,78-79	265	53
118,114-115	**117**,79	266	54
118,115	**117**,79	267	55
118,116	**117**,79-80	268	56
118,116-117	**117**,80-81	269	57
118,119	**117**,81	270	x
118,120	**117**,81-82	271	x
118,120	**117**,81	272	x
118,120	**117**,81	273	x
118,120-121	**117**,82	274	58
118,122-123	**117**,82	275	x
118,123-124	**117**,83	276	59
118,125	**117**,83	277	60
118,125-126	**117**,84	278	61
118,130-31	**117**,84	279	62
118,131	**117**,84-85	280	63
118,131-32	**117**,85	281	64
118,132-33	**117**,85-86	282	65

118,133-35	**117**,86-87	283	96
-	**117**,88	284	97
118,137-38	**117**,87-88	285	98
-	**117**,88-89	286	66
118,139-40	**117**,89-90	287	67
118,140-41	**117**,90	288	68
118,141	**117**,90	289	69
118,141	**117**,90-91	290	70
118,142	**117**,91	291	71
118,142,143-44	**117**,91	292	x
118,144-47	**117**,92-94	293	72
-	**117**,94	294	73
-	**117**,95-96	295	74
118,150-151	**117**,96-97	296	11
118,151-152	**117**,97	297	161c
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118,152-154		298	II,2b-d
118,155-156		299	II,3
118,156-157		300	II,4
118,157		301	II,5
118,157-158		302	II,6
118,158		303	II,7
118,158-160		304	II,8
118,160		305	131
118,160-162		306	132
118,162		307	x
118,162-164		308	119b-e
118,164-166		309	120
118,168-169		310	3d-e
118,169-170		311	II,9
118,170-171		312	II,10
118,171		313	II,11
118,172		314	II,12
118,172-173		315	II,13
118,174		316	133
118,175-177		317a-c	II,1
		317d	II,2a
118,181		318	119a

118,191-192	319	x
118,192-193	320	x
118,193-195	321	x
118,195	322	x
118,195-196	323	x
118,196	324	x
118,197-200	325a-c	75
118,201-206	326a-e	76
118,206	327	77
119,1	328	142
119,5	329	78
119,6-8	330	79
119,8-9	331	80
119,12	332a	81
119,13	333	82
119,13-15	334	83
119,15	335	84
119,15-17	336	x
119,18	337	x
119,18-19	338	x
119,119	339	85
119,119	340	x
119,20-21	341	86
119,26-27	342	121
119,28-31	343	122
119,35	344	123
119,36	345	124
119,36-41	346	125
119,41-42	347	126
119,42-43	348	127
119,43-44	349	128
119,44	350	129
119,44-46	351	130
119,46-47	352	-
119,47-49	353	-
119,49-51	354	-
119,51-54	355	-
119,54	356	-
119,54-56	357	-
119,56-59	358	-
119,50	359	-
119,61-63	360	141
119,65	361	-

119,94	362	164
119,94-95	363	165
119,95	364	166
119,95	365	x
119,95-96	366	167
119,96-97	367	169
119,97-98	368	170
119,99	369	135
118,206	370	-
118,207	371	-
118,208-209	372	-
118,209	373	-
118,209-210	374	-
118,210	375	-
118,211-212	376	III,1
118,212-213	377	III,2
118,213	378	III,3
118,213-215	379	III,4
118,215	380	-
118,216	381	III,5
118,216-217	382	III,6
118,217-219	383	III,7
118,219-221	384	III,8
118,222	385	III,9
118,222-223	386	III,10
118,223	387	III,11
118,224	388	III,12
118,225	389	III,13
118,225-226	390	III,14a
118,228	391	III,15a
118,229	392	III,16
118,229-232	393	III,17
118,232	394	III,18
118,232-233	395	III,19
-	396	III,20

115,59-61	397	I,1
115,61	398	I,2
115,61-62	399	I,3,4
115,62	400	I,5
115,62-63	401	I,6
115,63-64	402	I,10
115,64	403	I,7
115,64-65	404	I,8
115,65	405	I,9
115,65	406	I,11
115,65-66	407	I,12
115,66	408	I,13
115,66-67	409	I,14

115,67	410	I,15
115,67-68	411a	I,16a
	411b	I,16b
115,68	412	I,17
115,68-69	413	I,18
115,69	414	I,19
115,69	415	I,20
115,69	416	I,21
115,69-70	417	I,22
115,70	418	I,23
115,70	419	I,24
115,69	420	I,25
115,71	421	I,26
121,54	422	87
121,55	423	x
121,55	424	x
121,55	425	x
121,55-56	426	x
121,56	427	x
121,56	428	88
121,57	429	89
121,57	430	90
121,56,58	431	x
121,58	432	x
162a,1-2	433	91
162a,2	434	92
162a,2-3	435	x
162a,3-4	436	93
162a,4-5	437	94
162a,5	438	x
162a,5	439	x
162a,6	440	95
162a,6-8	441	x
162a,8-9	442	x

B. The Correspondences between Typescripts **221**, **222-4** and **227**.

220	**239**	**227**
158	202	185
159	203	186
160	204	187
161a-b	205	188
161c	206	189a
221	**222**	
162b-d	1	189b-c
163	2	190
178	3a-c	-
310	3d-e	-
164	4	-
165	5a	-
171	5b-d	-
166	6	-
167	7	-
169	8	-
170	9	-
172	10	-
296	11	-
173	12	-
174	13	-
175	14	-
176	15	-
177	16	-
179	17	-
180	18	-
181	19	-
182	20	-
183	21	-
184	22	-
207	23	-
230	24	-

208	25	-
209	26	-
210	27	-
211	28	-
212	29	-
213	30	-
214	31	-
215	32a-b	-
-	32c	-
216	33	-
217	34	-
218	35	-
219	36	-
220	37	-
222	38	-
243	39	-
244	40	-
245	41	-
246	42	-
247	43	-
248	44	-
249	45	-
250	46	-
251	47	-
253	48	-
254	49	-
255	50	-
263	51	-
264	52	-
265	53	-
266	54	-
267	55	-
268	56	-
269	57	-
274	58	-
276	59	-
277	60	-
278	61	-
279	62	-
280	63	-
281	64	-
282	65	-

286	66	-
287	67	-
288	68	-
289	69	-
290	70	-
291	71	-
293	72	-
294	73	-
295	74	-
325a-c	75	-
326a-e	76	-
327	77	-
329	78	-
330	79	-
331	80	-
332a	81	-
333	82	-
334	83	-
335	84	-
339	85	-
341	86	-
422	87	-
428	88	-
429	89	-
430	90	-
433	91	-
434	92	-
436	93	-
437	94	-
440	95	-
283	96	-
284	97	-
285	98	-
189	99	-
190	100	-
191	101	-
192	102	-

203	103	-
204	104	-
205	105a-e	-
-	105f	-
256	106	-
257	107	-
258	108	-
259	109	-
260	110	-
261	111	-
262	112	-
194	113	-
206	114	-
195	115	-
200	116	-
196	117	-
252	118	-
318	119a	-
308	119b-e	-
309	120	-
342	121	-
343	122	193
344	123	191
345	124	192
346	125	194
347	126	195
348	127	196
349	128	-
350	129	-
351	130	197
305	131	-
306	132	-
316	133	-
229	134	-
369	135	-

238	136	-
239	137	-
240	138	-
241	139	-
242	140	-
360	141	-
328	142	-
223	143	-
224	144	-
225	145	-
226	146	-
227	147	-
231	148	-
232	149	-
233	150	-
235	151	-
234	152	-
236	153	-
237	154	-
198	155	-
199	156	-
185	157	-
186	158	-
187	159	-
188	160	-
197	161a-b	-
297	161c	-
201	162	-
202	163	-
362	164	-
363	165	-
364	166	-
366	167	-
317d	168	-
367	169	-
368	170	-

-	171	-	
397	I,1	-	
398	I,2	-	(Z 314)
399a	I,3	-	
399b-d	I,4	-	
400	I,5	557	
401	I,6	-	
403	I,7	-	
404	I,8	554	
405	I,9	555	
402	I,10	552	
406	I,11	-	
407	I,12	-	(Z 141)
408	I,13	-	(Z 142)
409	I,14	-	
410	I,15	-	
411	I,16	559	(Z 140)
412	I,17	561	
413	I,18	562	
414	I,19	563	
415	I,20	564	
416	I,21	565	
417	I,22	566	
418	I,23	567	
419	I,24	568a	
420	I,25	-	
421	I,26	-	
-	I,27	-	
	224		
317a-c	II,1	-	
317d	II,2a	-	
298	II,2b-d	-	
299	II,3	-	
300	II,4	-	
301	II,5	-	
302	II,6	-	
303	II,7	-	
304	II,8	-	
311	II,9	-	
312	II,10	-	
313	II,11	-	
314	II,12	-	
315	II,13	-	

223

376	III,1	-
377	III,2	-
378	III,3	-
379	III,4	-
381	III,5	-
382	III,6	-
383	III,7	-
384	III,8	-
385	III,9	-
386	III,10	-
387	III,11	-
388	III,12	-
389	III,13	-
390	III,14a	-
-	III,14b	-
391	III,15a	-
-	III,15b	-
392	III,16	-
393	III,17	-
394	III,18	-
395	III,19	-
396	III,20	-

Appendix II: A Chronological Catalogue of Wittgenstein's Works

The items in von Wright's catalogue of Wittgenstein's papers are represented in this appendix as follows:

First date	Last date
Pages **vW no.** Language Type	
Number	Name

The different *types* of item are volume (V), large notebook (Ln), notebook (N), pocket notebook (Pn) and loose sheets (Ls). The language is German unless otherwise indicated. A chronologically displaced source item is represented by its von Wright number and an inner rectangle only. (Dashes indicate spaces where information has yet to be entered.)