The Argument from Collections

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Very broadly, an *argument from collections* is an argument that purports to show that our beliefs about sets imply — in some sense — the existence of God. **?** first sketched such an argument in "Two Dozen" and filled it out somewhat in his **?** monograph *Where the Conflict Really Lies: Religion, Science, and Naturalism*.[1] In this paper I reconstruct what strikes me as the most plausible version of Plantinga's argument. While it is a good argument in at least a fairly weak sense, it doesn't initially appear to have any explanatory advantages over a non-theistic understanding of sets — what I call set theoretic *realism*. However, I go on to argue that the theist can avoid an important dilemma faced by the realist and, hence, that Plantinga's argument from collections has explanatory advantages that realism does not have.

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[1] I developed a *model* on which sets, together with numbers, properties, and propositions, are the products of God's intellective activity in **?** but did not attempt to turn it into a positive theistic *argument*.

1 Plantinga's Original Argument

The first premise of Plantinga's argument from collections is two-fold: first, that there *are* such things as sets[2] and, second, that they have a *nature*, which includes at least the following properties: (i) they are non-self-membered; (ii) they have their members essentially; and (iii) they collectively form an "iterated structure" that, therefore, yields a well-known explanation of Russell's paradox.[3]

The second premise is that the existence of sets with these distinctive properties is explained by the fact that sets are quite naturally thought of as the products of "a certain sort of intellective activity — a collecting or 'thinking together' ". Thus, as **?**, 282 famously wrote in the *Beiträge*: "By a 'set' we understand any collecting M of well-distinguished objects m of our intuition or our thought ... into a whole."[4] Hao **?**, 182 expresses the idea more explicitly still:

> It is a basic feature of reality that there are many things. When a multitude of given objects can be collected together, we arrive at a set. For example, there are two tables in this room. We are ready to view them as given both separately and as a unity, and justify this by pointing to them or looking at them or thinking about them either one after the other or simultaneously. Somehow the viewing of certain given objects together suggests a loose link which ties the objects together in our intuition.[5]

[2]Plantinga doesn't emphasize this first part of the premise, but it is of course implicit and I believe it enhances the argument if it is drawn out explicitly.

[3]The so-called iterative conception of the set theoretic universe will be discussed in some detail in §4 below. The explanation it yields of Russell's paradox is spelled out in the first paragraph of §5.

[4]The German: "Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschienden Objekten m unsrer Anshauung oder unseres Denkens ... zu einem Ganzen." 'Zusammenfassung ... zu einem Ganzen' is usually translated 'collection into a whole' (as in **?**) but the use of the present participle 'collecting' seems to me better to convey the active connotation of the phrase, that it is something *done* by an agent. Wang also translates 'Zusammenfassung' as a "collecting together" in a 1967 letter to Gödel; see **?**, 401-2.

[5]In **?** (370) I attempted to illustrated Wang's idea in a way that might be helpful:

> Consider the following array:
>
> * * *
> * * *
> * * *
>
> Think of the asterisks as being numbered left to right from 1 to 9, beginning at the upper left hand corner. While focussing on the middle dot 5, it is possible to vary at will which

As to how this conception of sets explains their existence and the properties mentioned in the first premise, Plantinga writes:

> First, if sets were collections, the result of a collecting activity, the elements collected would have to be present before the collecting; hence no set is a member of itself. Second, a collection could not have existed but been a collection of items different from the ones actually collected, and a collection can't exist unless the elements collected exist; hence collections have their members essentially, and can't exist unless those members do. And third, clearly there are noncollections, then first level collections whose only members are noncollections, then second level collections whose members are noncollections or first level collections, et cetera.)

Third premise: obviously, however, there are far too many sets, many with far too many members, for them to be the product of any sort of (finite and limited) human collecting activity; only an infinite mind — for simplicity, let's call it *God* — has the power to collect the vast infinity of sets that exist in the set theoretic universe according to our best theories. Hence, Plantinga appears to conclude, God exists.

Adopting the terminology in **?** for theories that explain abstract entities in terms of divine intellective activity, call this the *activist* conception of sets; and call the view that sets exist independently of any mind *set theoretic realism*, or *realism* for short. On its face, it might appear that Plantinga's activist argument is meant to be of a piece with the classical theistic arguments — a deductive argument to God's existence from clearly true or, at least, plausible, premises. So conceived, however, there is an obvious gap, viz., an intermediate inference from the second premise — that the existence and nature of sets is explained by their being the products of some kind of intellective "collecting" activity — to the proposition that sets *are* indeed the products of such activity. But that follows only if (a) the existence and nature of sets *requires* an explanation and (b) the proposed explanation is the *only* explanation. Some realists might reject (a) but I suspect most would accept (a) and

dots in the array stand out in one's visual field (with perhaps the exception of 5 itself), e.g., [1,5,9], [2,4,5,6,8], or even [1,5,8,9]. The dots thus picked out, I take Wang to be saying, are to be understood as the elements of a small "set" existing in the mind of the perceiver.

reject (b) on the grounds that the existence of a set is fully explained simply by the existence of its members; a set, that is, exists *because* its members do. The members of a set are thus *logically prior* to the set and, hence, a set cannot contain itself (property (i)). Moreover, the realist can continue, a set has exactly the members it does *essentially* (property (ii)) because that is simply (part of) what it *is* to be a set, viz., a "collective" object whose identity is wholly determined by the things it contains. Finally, membership generally is *well-founded*. Hence, while the existence of every set is explained, in the first instance, by its members, and their existence in turn by their members, and so on, ultimately, it is explained by the existence of the initial non-sets, or *urelements*, from which the set was built up. Well-foundedness is thus simply a reflection, from the top down, so to say, of the structure of the sets that are given from the bottom up in the iterative conception (property (iii)).[6] The metaphor of "collecting" that motivates the argument from collections, realists will insist, is at best just a useful but unnecessary heuristic for describing the well-founded/iterative structure of the sets and is not to be taken literally.

However, I think the appeal to *explanation* in the second premise is meant to indicate that Plantinga intended his argument to be *abductive* rather than deductive, that is, to be an argument to the best explanation. So understood, the missing intermediate inference that sets are the products of an intellective collecting activity should be replaced by a further premise: the explanation for the properties of sets noted in premise 1 yielded by the hypothesis of premise 2 — that sets are the products of such an intellective activity — is the *best* explanation of their possession of those properties. The former third premise now becomes the fourth: only an infinite mind — which we're calling *God* — is capable of producing *enough* collections to account for the sets that exist according to our best theories. Given that the best explanation of a phenomenon is confirmed by that phenomenon and that God is the best explanation of the existence of sets, the conclusion is now revised accordingly: the existence of God is confirmed by the existence of sets.[7].

Thus the argument. But is it a good one? In the preface to "Two Dozen", Plantinga provides a trenchant exploration of the senses in which a theistic ar-

[6]More exactly, well-foundedness rules out the possibility of infinite descending membership chains, so that every set "bottoms out" in some initial (possibly empty) set of urelements. For more see the discussion of the Foundation axiom **Fnd** in §4.

[7]My thanks to Nevin Climenhaga for this sentence, which is a much better expression of what I'd wanted to say here. I should note that he has mild reservations even about the revision.

gument can be considered *good*. And the abductive argument here surely seems to be good in at least one particularly important sense, viz., that the argument "contributes to the broader project of Christian philosophy [and theistic philosophy generally] by showing good ways to think about a certain topic or area from a theistic perspective" (**?**, 209).[8] For in shifting the focus to activism as the best *explanation* for our beliefs rather than a (dubiously derived) consequence of them, the abductive version of the argument, at the least, presents "a good way to think about" set theory and the existence of sets from a theistic perspective.[9]

That said, realists will not likely find this abductive version of the argument any more persuasive than the deductive version, and for the same reason: they will claim that the existence of sets is explained at least as well, and with a great deal less ontological commitment, by the existence of its members and, hence, that activism provides no explanatory advantage. However, in the remainder of the paper I will argue that, in fact, a well-known tension in the conceptual foundations of set theoretic realism puts the realist's explanation of set existence, but not the activist's, in peril and, hence, that activism does enjoy an explanatory advantage over realism. However, the argument requires quite a lot of stage-setting, to which I now turn.

2 Preliminaries: Plurals and Plural Quantification

It is illuminating to express the issues surrounding the argument from collections in terms of plural quantification. Note that natural language can express quantification in both singular and plural forms. As understood in first-order logic, singular quantifiers range over individuals — when we say "Some F is G", we mean that there is at least one F that is also a G; when we say "every F is G", we mean that each F individually is also G. Typically, of course, propositions of these forms can also be expressed using plurals: "Some/All Fs are Gs". However, the converse does not appear to be true; that is, not all plural quantifications can be equivalently expressed using singular quantifiers. Perhaps the best known example of this is the

[8]Plantinga in fact specifically identifies the argument from collections (sets) as a good one in this sense.

[9]The argument also seems to be good in the other two senses that Plantinga mentions (ibid.): it might move some people closer to theism, and it might have the consequence of strengthening and confirming some theists' beliefs.

so-called *Geach-Kaplan* sentence:

GK Some critics admire only one another.

Kaplan himself (as reported by **?**, 293) took the existential quantifier here to be ranging over properties of individuals and, hence, took the logical form of **GK** to be second-order:

GK2 $\exists X[\exists y Xy \wedge \forall y(Xy \rightarrow (Cy \wedge \forall z(Ayz \rightarrow (Xz \wedge y \neq z))))]$,[10]

that is, on the usual semantics of second-order languages, there is a nonempty set (or, perhaps more generally, class) of critics who only admire other critics in the set. Importantly, Kaplan showed **GK2** to be *essentially* second-order.[11] Assuming, therefore, that **GK2** is an accurate representation of **GK**'s logical form, it follows that **GK** itself has no logically equivalent sentence involving only singular first-order quantifiers.[12]

Now, as is well-known, **??** argued persuasively that, while **GK2** is in fact the correct logical form for **GK**, its second-order quantifier — insofar as it is meant to represent plural quantification — should not be understood in terms of the standard semantics of second-order quantification as ranging over sets of individuals.[13] Rather, it should be understood as ranging over exactly the same things as the first-order quantifiers, albeit "plurally":

> It is not as though there were two sorts of things in the world, individuals and collections of them, which our first- and second-order vari-

[10]Boolos's (**?**, 432) formalization of **GK** differs from **GK2** because he assumes a domain containing only critics.

[11]The simple, elegant proof is that, if one substitutes $y = y$ for Cy and $(y = 0 \vee y = z + 1)$ for Ayz in **GK2**, the resulting sentence — expressing (in the language of arithmetic) essentially "Some positive natural numbers succeed only one another" — is true in all and only nonstandard models of arithmetic, a property that no sentence of first-order arithmetic can have (lest it be possible to construct a categorical axiomatization of first-order arithmetic, contrary to Gödel's incompleteness theorem). See **?**, 432-3, esp. fn 7.

[12]*Strictly* speaking, as **?**, 5 points out, this only follows on the assumption that we are restricting ourselves to sentences containing identity and the predicates C and A. Thus **?**, 293 argued that one could formalize **GK** just as well in the first-order language of set theory (hence with the addition of the membership predicate $\in$) if we simply replace $\exists X$ with $\exists x$, Xy with $y \in x$, and Xz with $z \in x$ in **GK2**. However, **?**, 470 argues convincingly that the English rendering of the result of doing so is not logically equivalent to **GK** and, hence, that Quine's first-order formalization is inadequate. See also Yi's fn 29 for more on **GK** and some of the surrounding issues.

[13]Or, for that matter, in terms of so-called "general", or "Henkin", semantics, on which the range of the second-order quantifiers needn't include all sets of individuals. See **?** Ch. 4, esp. §1 and §4.

ables, respectively, range over and which our singular and plural forms, respectively, denote. There are, rather, two (at least) different ways of referring to the same things ... . (1984, 449)

To clarify, Boolos offers up an alternative way of paraphrasing **GK** and its like that spells out its meaning without any obvious reference to sets of individuals: “There are some critics such that each one of them is such that she admires a person only if that person is also one of them (but not her)”. And, indeed, taking those plural expressions at face value in this way involves no obvious ontological commitments beyond the critics themselves that, together, make it true.

Although Boolos himself suggested that (monadic) second-order quantification in general can be understood as plural quantification,[14] the view has not found wide acceptance and second-order quantification in most contexts is still given its usual semantics; confusion is inevitable if this ambiguity in second-order languages were to persist. Moreover, the use of the usual syntactic representation of predication ‘Xy’ to indicate that y is *among* the things X is misleading, insofar as it suggests that y and the Xs are of different types instead of simply all being individuals, albeit referred to in different ways. For these reasons, contemporary discussions formalize plural quantification by introducing a new class of variables ‘xx’, ‘yy’, etc that behave much more like ordinary first-order variables. In particular, to express that an individual y is among some things xx, a distinguished 2-place predicate ‘$\prec$’ is introduced, $y \prec xx$, that takes the variable ‘xx’ as an argument. In this framework, then, **GK** is represented as:

GKP $\exists xx[\exists y\, y \prec xx \wedge \forall y(y \prec xx \rightarrow (Cy \wedge \forall z(Ayz \rightarrow (z \prec xx \wedge y \neq z))))]$.[15]

We will follow Boolos in adopting the “ontologically innocent” understanding of plural quantification in this discussion. This understanding is not uncontroversial. However, we do so for convenience only; nothing of substance hangs on it for our purposes here. It simply provides us with a very convenient framework

[14] See Boolos 1984, 449: “The lesson to be drawn from the foregoing reflections ... is that neither the use of plurals nor the employment of second-order logic commits us to the existence of extra items beyond those to which we are already committed. We need not construe second-order quantifiers as ranging over anything than the objects over which our first-order quantifiers range ... .”

[15] In many plural logics, it is is a logical truth that there are no empty pluralities, that $\forall xx \exists y\, y \prec xx$ (see, e.g., **?**, §1.2). In such contexts, the first conjunct inside the plural quantifier of **GKP** is superfluous. However, for purposes here it is convenient to allow for empty pluralities (following **?** and **?**).

in which to state and discuss the issues. Likewise, as we've already assumed with the plural pronoun "them" in Boolos's take on **GK**, we will take plural demonstrative/anaphoric expressions like "those objects", as well as terms like "plurality", "the plurality of Fs" and "The Fs", to be in themselves ontologically innocent, i.e., simply to refer "plurally" to the indicated objects and not to a separate class or collection containing them.

3 Pluralities, Sets, and Russell's Paradox

So-called "naive" set theory can be traced back to Gottlob Frege, particularly his great **?** work *Grundgesetze der Arithmetik* (though, strictly speaking, it can only be considered a reconstruction of a fragment of Frege's system). Naive set theory is based on two simple, intuitive principles. The first is that sets are *extensional*, that the identity of a set is *wholly determined by* its members:

Ext Sets a and b are identical if they have the same members. Formally: $\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b$.

The second principle is that every plurality constitutes a set, i.e., that, for any given things, there is a set containing exactly *them*:

Collapse $\forall xx \exists y \forall z(z \in y \leftrightarrow z \prec xx)$

In itself, though, **Collapse** tells us nothing about what sets there are until we have some principle that tells more exactly what pluralities there are. And here there is an obvious principle, the principle of *Plural Comprehension*:

PC For any property P of things, there are exactly the things that have P.

Unfortunately, as it stands, **PC** appeals to properties, which are perhaps even more controversial than sets. Intuitively, however, a property is just the meaning of a description, or *predicate*, like 'is human' or 'is a prime number less than 1000'. Thus, a more tractable way of expressing the idea behind **PC** is to say that, for any description, there is a set consisting of exactly the things satisfying that description. So, for example, given the predicates above, by this principle there are all the humans and all the prime numbers less than 1000. The notion of a predicate is captured formally in first-order logic by means of formulas φ (typically containing free variables) in a given formal language L designed to describe whatever piece of the world we're interested in. Our more tractable take on **PC**, then, is expressed in first-order logic by means of a *Plural Comprehension schema* that generates a distinct axiom for each predicate φ (of L):

P-Comp For any formula φ containing no free occurrences of the variable 'yy', there are the things satisfying φ; formally: $\exists yy \forall x(x \prec yy \leftrightarrow \varphi(x))$.

Given **Collapse** and **P-Comp** the more familiar Comprehension schema of naive set theory follows immediately:

Comp For any formula φ of L containing no free occurrences of the variable 'y', there is a set consisting of exactly the things satisfying φ; formally: $\exists y \forall x(x \in y \leftrightarrow \varphi(x))$.

For all its apparent simplicity, naive set theory is extraordinarily powerful and enables one to prove a great many interesting theorems about sets.[16] Alas, as the familiar story goes, while studying Frege's *Grundgesetze*, Russell discovered the famous paradox that showed that naive set theory (more exactly, a basic principle of the *Grundgesetze* that is more or less equivalent to Comprehension) is inconsistent. To see the problem, consider the property *non-self-membership*, that is, the property expressed by the predicate '$x \notin x$'. By **Comp**, there is a set r consisting of exactly the things satisfying this predicate, that is, the things (in particular, the sets) that are not members of themselves; formally, $\forall x(x \in r \leftrightarrow x \notin x)$. Instantiating to r, we have that $r \in r$ if and only if $r \notin r$, contradiction.

The discovery of Russell's Paradox led to the development of much more rigorously conceived set theories, most notably, Ernst Zermelo's set theory Z, most of whose axioms Zermelo initially proposed in a famous **?** paper. Russell's Paradox showed that not all pluralities, in particular, not all those determined by a well-defined predicate, are "safe"; some things, on pain of contradiction, cannot jointly form a mathematically well-behaved set. At the same time, there *are* some pluralities that seem clearly safe, that can clearly be assumed to constitute a set; and, of course, conversely, any plurality we've shown independently to constitute a set is safe. Thus, we need to replace **Collapse** with:

[16] Without, of course, simply exploiting its inconsistency and proving anything one wants by contradiction. Notably, it is an easy exercise to derive all the axioms of Zermelo-Fraenkel set theory ZF (other than **Ext**) as instances of Comprehension. Hence, every theorem of ZF can be proved in naive set theory exactly as in ZF once the axioms used in the proof are derived.

Safe $\forall xx(Safe(xx) \leftrightarrow \exists y \forall z(z \in y \leftrightarrow z \prec xx))$.

The general problem, then, is to distinguish the safe pluralities from those that are not. Zermelo's brilliant, and brilliantly executed, idea was to introduce, via carefully chosen axioms, a well-circumscribed class of intuitively safe pluralities to get things a-going along with a variety of safe "set-building" operations introduced by further axioms that lead safely from given sets to further sets. In this way, Zermelo hoped to have a theory that was powerful enough to yield the many important results that had already been proved in naive set theory but not so powerful as to collapse into logical contradiction.

4 Zermelo's Axioms for Safe Pluralities[17]

Although Zermelo himself did not formulate his axioms explicitly in terms of pluralities, it will be useful to do so for our purposes here. Toward that end, note first that, for any expressible condition $\varphi(x)$, our principle **P-Comp** warrants the introduction of a term $[x : \varphi(x)]$ to refer (plurally, hence innocently) to the things x that satisfy φ.[18] Hence, for something to be among the φs is simply for it to satisfy φ:

P-Abs $z \prec [x : \varphi(x)] \leftrightarrow \varphi(z)$.

Likewise, when a plurality $[x : \varphi]$ has been deemed safe and, hence, constitutes a set containing exactly the things that are φ, we can switch to traditional set abstraction notation $\{x : \varphi\}$. More exactly, and more generally, whenever we can show that the φs constitute a set y (i.e., that $[x : \varphi]$ is safe) and that y satisfies ψ, then we can (by definition) express this as $\psi(\{x : \varphi\})$; formally:

Sets $\psi(\{x : \varphi(x)\}) \equiv_{df} \exists y(\forall x(x \in y \leftrightarrow \varphi(x)) \wedge \psi(y))$.

Finally, for finitely many terms $t_1, ..., t_n$, the expression $[t_1, ..., t_n]$ will as usual just be shorthand for $[x : x = t_1 \vee ... \vee x = t_n]$; analogously for $\{t_1, ..., t_n\}$.

[17] The exposition in this section and smaller portions of the following two sections is similar to that found in Bueno and Shalkowski (**?**), which was written largely in parallel with the present paper.

[18] Strictly speaking, by using the definite article here I'm assuming a plural extensionality principle which, to express, requires a dedicated plural identity predicate $\approx$: $\forall z(z \prec xx \leftrightarrow z \prec yy) \rightarrow xx \approx yy$. As nothing hinges upon this for purposes here, I'll not bother to introduce the machinery needed make this explicit.

The most pressing order of business in light of Russell's paradox is a replacement for the set comprehension principle **Comp** which, of course, no longer follows from **P-Comp** when **Collapse** is replaced by **Safe**. On reflection, **Comp** makes clear just how wildly profligate **Collapse** is (together with the ontologically innocent **P-Comp**): given *any* description φ whatever, no matter how obscure, complex, or logically dubious, the principle generates *ex nihilo* a new thing, viz., the set of things that satisfy φ. Zermelo tames **Comp** by declaring, not that the plurality of *all* the things satisfying φ is safe but, rather, only those satisfying it that are *already* the members of some antecedently given set s. Since, by **Safe**, the members of a set are jointly safe, any of those members, as part of a larger safe plurality, should be jointly safe as well. This is formalized in the axiom schema of *Separation*:

Sep $\mathit{Safe}([x : x \in s \wedge \varphi(x)])$.

Safe and **Sep** together, then, yield (in place of **Comp**) the more familiar set theoretic form of the principle:

S-Sep For any formula φ containing no free occurrences of the variable 'y', given a set s, there is a set consisting of exactly the members of s satisfying φ; formally: $\exists y \forall x(x \in y \leftrightarrow x \in s \wedge \varphi(x))$.

Thus, given only **S-Sep**, it is no longer possible to generate sets *ex nihilo* from any given plurality; one can only carve them out of sets that one has already proved to exist. One cannot, in particular, prove the existence *ex nihilo* of a set of all non-self-membered sets but, rather, only the set $a = \{x : x \in s \wedge x \notin x\}$ of non-self-membered sets in some *given* set s. Running Russell's argument on a yields only the harmless conclusion that $a \notin s$ (and, moreover, that $a \notin a$); that there is no universal set $\{x : x = x\}$ — as there is under **Comp** — is an immediate corollary.[19]

This, of course, leaves the question of what sets we *can* prove to exist — of itself, **Sep** gives us nothing, since we need to have a set in hand to apply it. The simplest safe pluralities that Zermelo postulates — via the axiom of *Pairing* — are

[19] In more detail: Suppose s is a set. By **S-Sep**, there is a set a containing all the members of s satisfying the predicate '$x \notin x$', that is, the members of s that are not members of themselves; formally $\forall x(x \in a \leftrightarrow (x \in s \wedge x \notin x)$. Instantiating to a we have that $a \in a$ if and only if $(a \in s \wedge a \notin a)$. Assuming $a \in a$, it follows that $a \notin a$. So $a \notin a$. And assuming $a \notin a$, we have that either $a \notin s$ or $a \in a$. But we already know $a \notin a$. So $a \notin s$. Otherwise put, with **S-Sep**, all we're able to prove is that, for any set s, there is some (non-self-membered) set not in s and, hence, as an immediate corollary, that there is no universal set, no set that contains everything.

those consisting of one or two antecedently given things. That is, the axiom tells us that, given any (not necessarily distinct) objects a and b, they are (jointly) safe and, hence, by **Safe**, form a set:

Pr $\mathit{Safe}([a,b])$.

Pairing already gives us important insights into Zermelo's conception of set. Note first that we get at least one thing simply by logic alone (as '$\exists x\, x = x$' is a logical truth). This, together with **Sep**, is enough to yield the empty set $\varnothing$ and, hence, an infinite number of "pure" sets built up from it.[20] But, as matters of empirical and, perhaps, mathematical fact, we know that there are urelements, things that are not themselves sets — persons, planets, natural numbers, etc. Hence, we know by Pairing that any pair of (not necessarily distinct) urelements is a safe plurality and hence that, together, they constitute a set. But then, once we know we have a set or two, we can apply the axiom again to these "new" sets and our initial urelements to prove the existence of yet further sets containing them. And, given *those* sets, together with the ones previously shown to exist, along with our urelements, it follows from Pairing that yet further sets exist, and so on. Moreover, given a further Zermelian axiom, *Union*, which stipulates that the members of the sets in a safe plurality together are a safe plurality, it follows that, not just pairs, but any finite plurality is safe:

Un $\mathit{Safe}([x{:}\exists y(y \in z \wedge x \in y)])$.[21]

Pairing and Union together thus yield with a rudimentary version of the set theoretic universe according to Zermelo's iterative conception: the sets have a *cumulative*, *hierarchical* structure, advancing "upwards" in an expanding series of levels. We "begin" at the first level with some urelements. At the next level we have everything in the first level together with all the finite sets that can be formed

[20]Specifically, it is a truth of first-order logic that something exists; call it a. Hence, by **Sep** we have that the "empty plurality" $[x{:}x \in a \wedge x \neq x]$ (see fn 15) is safe, and so, by **Safe**, we have the empty *set* $\varnothing = \{x : x \in a \wedge x \neq x\}$. By iterated applications of Pairing, then, we also have, e.g., the series of singletons $\{\varnothing\}$, $\{\{\varnothing\}\}$, $\{\{\{\varnothing\}\}\}$, ... , which are all distinct from one another by the extensionality axiom **Ext**.

[21]This is an easy induction: The case $n = 0$ is given by **Sep**, as seen in fn 20. Suppose then that any n things are a safe plurality and that we have a plurality $[a_1,...,a_n,a_{n+1}]$ of $n+1$ things. By our induction hypothesis, $[a_1,...,a_n]$ is safe and hence, by **Safe**, constitutes a set $\{a_1,...,a_n\}$. By **Pr**, $\{a_{n+1}\}$ is a set and hence also $\{\{a_1,...,a_n\},\{a_{n+1}\}\}$. By **Un**, the plurality $[a_1,...,a_n,a_{n+1}]$ is safe and, hence, constitutes a set $\{a_1,...,a_n,a_{n+1}\}$. We conclude by induction that any finite plurality constitutes a set.

from them; and at subsequent levels, we have everything in all the preceding levels together with all the finite sets that can be formed them.

However, three elements of Zermelo's full iterative conception are missing. First, the urelements themselves should be jointly safe.[22] After all, they are all *there* to begin with, "prior" to any sets, and, on the face of it, unlike the problematic principle **Collapse** that *any* plurality constitutes a corresponding set — the plurality of non-self-membered sets, for example! — there seems to be no reason not to consider *just the urelements* jointly to be the initial, base level of our hierarchy and, subsequently, to constitute a set U in the *next* level of the hierarchy, the first level in which sets are formed. Thus, since U is provably not a member of itself,[23] and that fact does not in turn lead to the conclusion that, like the contradictory set r, it also *is* a member of itself, there appears to be no Russell-style paradox looming anywhere in the vicinity. We therefore add the joint safety of the urelements as an explicit axiom, where $Set(x) \equiv_{df} x = \varnothing \vee \exists y\, y \in x$:

Ur $Safe([x{:}\neg Set(x)])$.

Second, the restriction to finite pluralities in the "construction" of a given level from pluralities of things in the preceding level seems unwarranted. Regardless of how many entities there might be in a given level, whether finite or infinite in number, the next level should include all the sets constituted by *any* plurality of things in the preceding level. Finally, each level after U, consisting of the things in the preceding level and all the sets constituted by any or all of them, should itself be safe and, hence, should itself constitute a set.

The addition of the *Powerset* axiom enables us to capture these aspects of the iterative conception precisely. Specifically, *Powerset* says that, given a set s, all of its

[22]It's actually not clear that Zermelo thought that absolutely all urelements form a set, though he is clear in his later works (notably **?**) that one needs to assume they do in any particular application of the theory (though it is consistent with his theory that they do not — see, e.g., **?**). Several recent papers (notably, **?** and **?**) have argued that (especially in certain modal metaphysical frameworks) there might well be "too many" urelements to constitute a set and that, under certain assumptions, assuming there is a set U of urelements leads to contradiction. But (as argued in **?**) these arguments depends on theoretically unnecessary restrictions built into the axiom schema of Replacement (discussed briefly below), not on the assumption *per se* that the urelements constitute a set.

[23]That U is not a member of itself follows formally from **Sep**. For, since no urelement has any members, the predicate '$x \notin x$' is true of all of them. Hence, the set y such that $\forall x(x \in y \leftrightarrow x \in U \wedge x \notin x\}$ that we get from **Sep** is exactly U. Instantiating to U, following the reasoning detailed in 19, it follows directly that $U \notin U$.

subsets are jointly safe and, hence, constitute a set $\wp(s)$, the *powerset of s*. Formally, letting '$x \subseteq s$' mean, as usual, that x is a subset of s:

Pow *Safe*$([x : x \subseteq s])$.

Given the safety of the urelements and our axioms **Un**, **Ur**, and **Pow**, the iterative conception can be expressed in a very clear, mathematically rigorous way (albeit, for the time being, only in our metalanguage[24]) *via* an inductive definition on the natural numbers:

D1
$$\begin{aligned} U_0 &= U \\ U_{n+1} &= U_n \cup \wp(U_n) \end{aligned}$$

That is, as depicted in Figure 1, the first level U_0 of the universe is just the set of urelements U (which exists by **Ur**); and each subsequent level U_{n+1} consists of everything in the preceding level U_n together with all the new sets $\wp(U_n)$ that can be formed from members of U_n (and hence from all preceding levels U_m, for $m \leq n$, since the levels are cumulative). Each "disk" at each level in Figure 1 thus signifies that all the things in earlier levels plus all the new sets they constitute together constitute a new, determinate set.

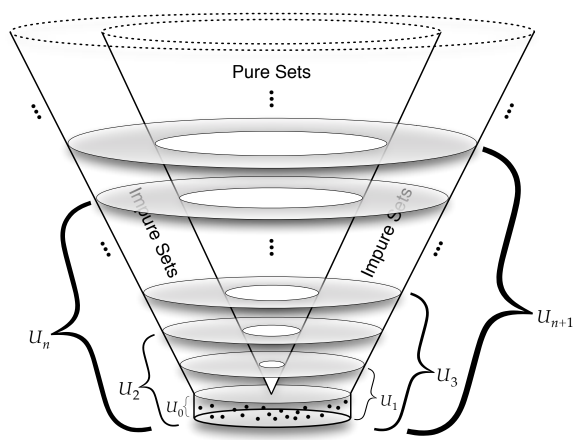

Figure 1: The Cumulative Hierarchy (so far)

As we first noted in §1, and as we see clearly here, the iterative conception yields a cumulative hierarchy of *well-founded* sets: the fact that we start with a set

[24]The reason for this is that the existence of inductively defined functions like this one here in general require both the axioms of Infinity and Replacement, which are only first introduced below.

of urelements and build each subsequent level only from sets constructed from entities in the preceding level means that are no self-membered sets and, more generally, no infinite descending membership chains of the form: $\ldots \in a_{n+1} \in a_n \in \ldots \in a_2 \in a_1 \in a_0$. As this feature of the cumulative hierarchy does not follow from the other axioms, it requires an axiom of its own, the axiom of *Foundation*. As this is just a structural axiom about sets, it does not involve the notion of safety:

Fnd $\forall x(x \neq \varnothing \rightarrow \exists y(y \in x \wedge \neg\exists z(z \in y \wedge z \in x)))$.

That is, every nonempty set x contains a member that has no members in common with x.[25]

Unbounded Hierarchies: Infinity and Replacement. Note that, while we have shown that the members of each level U_n of the hierarchy are a safe plurality (i.e., each U_n is a set), there is nothing in the axioms thus far to guarantee that the plurality $[x : \exists n\, x \in U_n]$ of *all* the members of *all* the levels is safe. In fact, we need two new principles to guarantee this, both of which are exceptionally important to the modern theory of sets but also perhaps the most problematic for set theoretic realism, as we'll see.

This first of these is the axiom of *Infinity*. What is particularly important to note about this axiom is that it is not *merely* the assertion that an infinite set exists. Rather, in addition, it asserts that pluralities with a certain "unbounded" character are safe. To get at the idea, define (recursively) the *rank* $\rho(a)$ of an entity a to be 0 if it has no members (i.e., it is either $\varnothing$ or an urelement) and, otherwise, to be the smallest number n greater than the ranks of its members. Thus, for example, for an urelement a, $\rho(\{\varnothing, a\}) = 1$, $\rho(\{a, \{\varnothing, a\}\}) = 2$, and $\rho(\{\{\{\{\varnothing, a\}\}\}\}) = 4$. Intuitively, the rank of an object is a measure of how "high up" it first occurs in the hierarchy of levels U_n; the two are correlated as follows:

Fact For finite n, a set with rank n will first occur in either U_n or U_{n+1}.[26]

[25] It's not obvious how Foundation rules out infinite descending membership chains because it in fact does so only given some of the other axioms of set theory — in particular, the axioms of Infinity and Replacement, which are discussed in the following paragraphs.

[26] In particular, every urelement has a rank of 0 and first occurs in U_0. By contrast, $\rho(\varnothing) = 0$ but $\varnothing$ first occurs in U_1. (By the definition of the hierarchy above, $U_1 = U_0 \cup \wp(U_0)$; since $\varnothing$ is not an urelement, $\varnothing \notin U_0$. But $\varnothing \subseteq U_0$ and hence, $\varnothing \in \wp(U_0)$). More generally, let a be a set of rank n. If a is pure, i.e., "built up" solely from the empty set, and, hence, contains no urelements in its transitive

Now, suppose that the objects satisfying a predicate $\varphi(x)$ constitute a (finite or infinite) plurality $[b_0, b_1, b_2, ...]$ of objects in the hierarchy of Figure 1 all of which have ranks $< m$ for some natural number m.[27] Then $[b_0, b_1, b_2, ...]$ is obviously in a clear sense "bounded" in the hierarchy — by the preceding **Fact**, each b_i will have first occurred in some level U_i, for $i \leq m$; the plurality, so to say, "runs out" by U_m; U_m thus represents a bound beyond which the plurality does not extend. Moreover, given the cumulative nature of the hierarchy, the b_i will all *exist* in U_m. Hence, $[b_0, b_1, b_2 ...]$ is just the plurality $[x : x \in U_m \wedge \varphi(x)]$ and so, by **Sep**, it is safe and constitutes a set.

By contrast, consider the plurality $[\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, ...]$ given by the following inductive definition:

D2
$$\begin{aligned} \varnothing_0 &= \varnothing \\ \varnothing_{n+1} &= \varnothing_n \cup \{\varnothing_n\} \end{aligned}$$

Then $\rho(\varnothing_0) = \rho(\varnothing) = 0$, $\rho(\varnothing_1) = \rho(\{\varnothing\}) = 1$, $\rho(\varnothing_2) = \rho(\{\varnothing, \{\varnothing\}\}) = 2$, and so on, and each $\varnothing_i \in U_{i+1}$.[28] Hence, the plurality $[\varnothing_0, \varnothing_1, \varnothing_2, ...]$ is *unbounded* in the hierarchy of finite levels depicted in Figure 1: unlike our bounded plurality above, there is no natural number m such that $[\varnothing_0, \varnothing_1, \varnothing_2, ...]$ "runs out" by level U_m; rather for every number m, there is some $c \prec [\varnothing_0, \varnothing_1, \varnothing_2, ...]$ such that c first occurs in some higher level U_m (in fact, every $\varnothing_i$ for $i \geq m$). Thus, no proof of its sethood like the one above for $[b_0, b_1, b_2, ...]$ is forthcoming.

Such unbounded pluralities, then, are of a rather different ilk structurally than any we've seen so far, and the Zermelian axioms laid out to this point do not guarantee their safety. What grounds might there be for trusting them? Pragmatically speaking, we need an infinite set in order to reconstruct classical mathematics in set theory and, ideally, we'd like to be able to do so in the context of pure set theory wherein the set U of urelements is assumed to be empty. But every level U_n of the hierarchy of pure sets that arises under that assumption is finite. Hence, to guar-

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closure (that is, it has no urelement as a member, or a member of a member, or a member of a member of a member, ...) then it first occurs in U_{n+1}. If, by contrast, a is built up solely from urelements and does not contain the empty set in its transitive closure, then it first occurs in U_n. If a is of "mixed" origins, i.e., if it is built up from both urelements and the empty set, then whether it first occurs in U_n or U_{n+1} will depend upon the exact manner in which it is constructed.

[27] The discerning reader will note that such a plurality could be infinite only if there are infinitely many urelements.

[28] See fn 26.

antee the existence of an infinite pure set, the only option is to postulate the safety of some unbounded plurality.

But really, the philosophical justification for the safety of unbounded pluralities ultimately seems no different than when we postulated the safety of our initial plurality of urelements in our axiom **Ur**. Recall there were two aspects to the justification. First, like the urelements, the entities in an unbounded plurality are all *there* — we are not talking about some sort of merely potential plurality; the things in question are fully determinate entities. Second, just as the set U containing the urelements, being a set, is not a member of itself, so too a set s constituted by an unbounded plurality, and hence containing only entities of arbitrarily high finite rank, by definition cannot itself have a finite rank and, hence, cannot be a member of itself; like U, s (if it is to exist) must first occur in a level "above" those of its members. And just as U's non-self-membership does not seem to lead to contradiction, neither, it appears, does s's; even the faintest specter of paradox is nowhere to be seen on the assumption that an unbounded plurality like $[\varnothing_0, \varnothing_1, \varnothing_2, ...]$ constitutes a set.

Accordingly, the Infinity axiom simply vouches for the safety of our unbounded plurality $[\varnothing_0, \varnothing_1, \varnothing_2, ...]$ the way **Ur** does for the urelements. However, we obviously cannot express this as "$\mathit{Safe}([\varnothing_0, \varnothing_1, \varnothing_2, ...])$", since "$[\varnothing_0, \varnothing_1, \varnothing_2, ...]$" does not represent a proper term of our language; the ellipsis is shorthand for infinitely many terms and the terms and formulas of our language are all of finite length. A more promising possibility is "$\mathit{Safe}([x : \exists i(i \in \mathbb{N} \wedge x = \varnothing_i])$", where $\mathbb{N}$ is the set of natural numbers. However, while that set is available to us in the metalanguage we are using for our exposition and in which we formulated definition **D2**, we do not yet have it available to us in our theory proper; indeed, as noted above, to prove the existence of a set that can serve as the set of natural numbers is exactly why (among other things) we need an axiom of Infinity.

So we need to find some other way to pick out our plurality $[\varnothing_0, \varnothing_1, \varnothing_2, ...]$, i.e., recall, the plurality $[\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, ..]$. And the key to doing so is to identify the *structural* property responsible for its unboundedness, namely: the property of being a plurality yy of things such that (a) the empty set $\varnothing$ is one of them and (b) $z \cup \{z\}$ is one of them whenever z is; formally, the property (call it *UB*) $\varnothing \prec yy \wedge \forall z(z \prec yy \to z \cup \{z\} \prec yy)$. Of course, *UB* is true of many other pluralities as well — it's true, in particular, of the Russell plurality of non-self-membered things, $[x : x \notin x]$, so we obviously cannot count on UB guarantee a plurality's safety. However, note that our desired plurality is the *smallest* plurality with *UB*, in the sense that it is made up of exactly the things that are in *every* plurality that has *UB*; that is, it consists of the things x such that $\forall yy(UB(yy) \to x \prec yy)$. And that is the property we will use in our axiom of Infinity to pick out our chosen plurality (without appealing to an antecedently existing infinite set) and declare it safe:

Inf $Safe([x : \forall yy(\varnothing \prec yy \wedge \forall z(z \prec yy \to z \cup \{z\} \prec y) \to x \prec yy)])$.[29]

Hence, by **Safe**, our unbounded plurality constitutes a set $\{\varnothing_0, \varnothing_1, \varnothing_2, \ldots\}$. This is of course is the set ω of *finite von Neumann ordinals* $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}, \ldots$ which, in modern set theory, are usually identified with the natural numbers. ω thus serves both as a convenient representation of the set of natural numbers and as the first *transfinite* number, the first (ordinal) number greater than all the natural numbers — we will let it so serve for us henceforth.[30]

Now, the entire iterative hierarchy of finite levels $[U_0, U_1, U_2, \ldots]$ is of course also an unbounded plurality. However, although obviously structurally similar to $[\varnothing_0, \varnothing_1, \varnothing_2, \ldots]$, we have as yet nothing to guarantee its safety. A further principle, therefore, is needed, one brought to light by Abraham Fraenkel: the axiom schema of *Replacement* (the addition of which to Zermelo's set theory Z gives us ZF). The idea is quite simple. As with **Sep**, we need a set s to start with.[31] If you then have some mapping defined on s's members — that is, some description $\psi(x, y)$ that associates each member x of s with a single corresponding entity y — then the range of the mapping, the plurality of things that the members of s are mapped to via $\psi(x, y)$, is safe. (More figuratively put, one can "replace" the members of

[29] The usual ZF Infinity axiom **Inf**′, expressed in terms of plurals, says only that *some* plurality with *UB* — hence some plurality containing all of the $\varnothing_i$ — is safe. Hence, by **Safe**, there is a set s containing all of the $\varnothing_i$. Hence, our unbounded plurality $[\varnothing_0, \varnothing_1, \varnothing_2, \ldots]$ is exactly those members of s that are in every plurality that has *UB* and, hence, it can be declared safe by means of an instance of **S-Sep**: $Safe([x : x \in s \wedge \forall yy(UB(yy) \to x \prec yy)]$. So **Inf**′ implies **Inf**. But **Inf** obviously implies **Inf**′ — if $[\varnothing_0, \varnothing_1, \varnothing_2, \ldots]$ is safe, then some plurality with *UB* is safe — so, in the context of the other axioms, the two axioms are equivalent.

[30] Zermelo's own axiom of infinity postulates a set that includes the somewhat different unbounded plurality $[\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \ldots]$. These are often referred to as the *Zermelo numbers* because they were the surrogates that Zermelo used to represent the natural numbers.

[31] Separation can in fact be derived from Replacement in ZF but set theorists often appreciate knowing what can be proved without it, that is, what can be proved in Zermelo's original set theory Z, so most presentations of ZF include both schemas.

the original set s with the things they are mapped to in order to derive a new set.) Formally, then (where, as usual, $\exists! x\varphi \equiv_{df} \exists x \forall y(\varphi(y) \leftrightarrow x = y)$):

Rep $\forall x(x \in s \rightarrow \exists! y\psi(x,y)) \rightarrow Safe([y : \exists x(x \in s \wedge \psi(x,y))])$.

Taking our von Neumann ordinals to serve as the natural numbers, then, we can show that $[U_0, U_1, U_2, ...]$ is safe by simply mapping each such ordinal n to U_n; that is, we simply let $\psi(x,y)$ be the formula '$x \in \omega \wedge y = U_x$'.[32] Since ω is a set, the range of this mapping $[U_0, U_1, U_2, ...]$ (i.e., is safe and, hence, constitutes a set $\{U_0, U_1, U_2, ...\}$ ($= \{y : \exists x(x \in \omega \wedge y = U_x)\}$). But now that we have shown that the levels constitute a set $\{U_0, U_1, U_2, ...\}$, it follows immediately by the Union axiom **Un** that there is set $U_\omega = \bigcup\{U_0, U_1, U_2, ...\}$ consisting of all of the members of all of the levels.

This is the final critical missing piece from Zermelo's full iterative conception: not only are the pluralities resulting from any finite number of iterations of the Powerset and Union operations safe, as per our definition of the finite levels U_n above, so to is the plurality the results from *infinitely many* iterations of those operations. We can, that is, figuratively speaking, put a "disk" at the top of the hierarchy of finite levels depicted in Figure 1. However, we of course can't leave at that, as if the hierarchy, so capped off, comes to an end. For once we have determined that all the entities in the finite levels U_n constitute a set U_ω, the iterative construction begins anew, and all the finitely unbounded pluralities drawn from U_ω form the basis of a series of *new*, transfinite levels $U_{\omega+1}, U_{\omega+2}, U_{\omega+3}, ...$ (new members x

[32]Note that at this point, with both Infinity and Replacement at our disposal, the existence of the inductively defined function U_x in Definition **D1** above can be demonstrated *within* our theory, so the proof sketch here is legitimate. However, it is also redundant, as it will follow simply from the existence of the function U_x that its range $\{U_0, U_1, U_2, ...\}$ exists as a set. The point of the example here, of course, is simply to illustrate how the axiom works. In fact, however, it is possible to define the plurality of finite levels $[U_0, U_1, U_2, ...]$, and hence an appropriate mapping $\psi(x,y)$ on ω, without any appeal to the function U_x. For note that this plurality has the following structural property P: (a) the set U of urelements is one of them and (b) $z \cup \wp(z)$ is one of them if z is; formally: $U \in xx \wedge \forall y(y \prec xx \rightarrow y \cup \wp(y) \prec xx)$. Similar to how we defined $[\varnothing_0, \varnothing_1, \varnothing_2, ...]$ without any appeal to the natural numbers, then, we can define our desired plurality $[U_0, U_1, U_2, ...]$ of finite levels of the hierarchy without any appeal to the function U_x as the *smallest* plurality with property P, that is, as the sets s such that, for *every* plurality with P, s is one of them. And that in turn enables us to define the mapping $\psi(x,y)$ on ω "$x \in \omega \wedge \theta(x,y)$", where $\theta(x,y)$ expresses "y is the set in the smallest plurality with P such that $x \in y$ and, if $z \subset y$ is also in that plurality, then $x \notin z$"; that is, roughly put, "y is the first level of the hierarchy in which x occurs". This maps each $i \in \omega$ to U_{i+1} and, hence, by **Rep**, the plurality $[U_1, U_2, ...]$ (i.e., $[y : \exists x(x \in \omega \wedge \theta(x,y))]$) is safe and, hence, constitutes a set $\{U_1, U_2, ...\}$. Our desired set U_ω of all the members of all the levels, of course, is now simply $U \cup \bigcup\{U_1, U_2, ...\}$.

of which have ranks $\rho(x)$ of ω, $\omega+1$, $\omega+2$, ... respectively) as depicted in Figure 2. The members of these transfinite levels then constitute further pluralities yy that

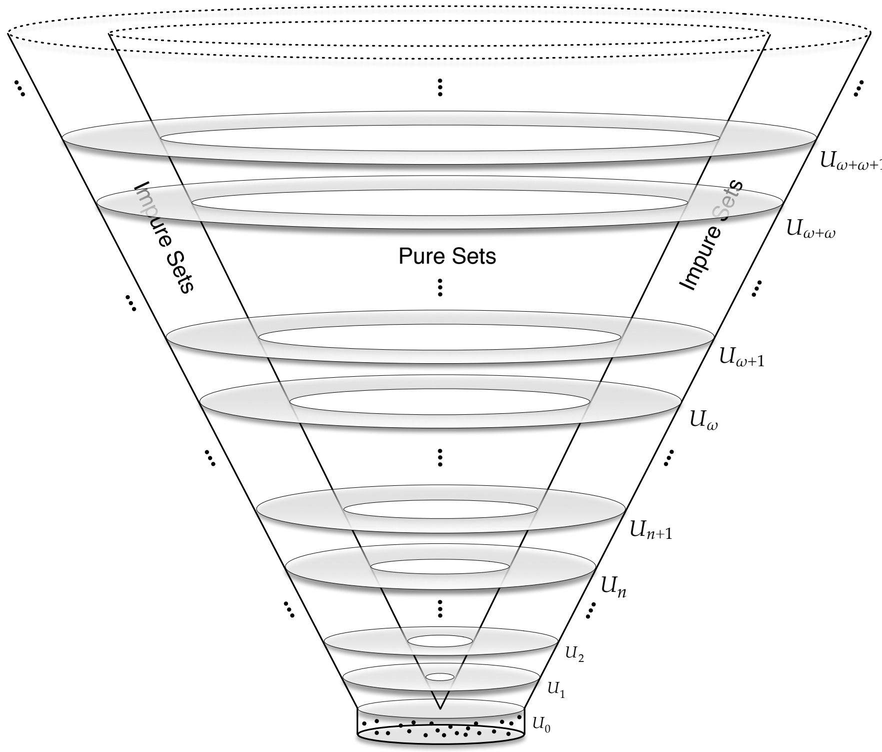

Figure 2: The Cumulative Hierarchy

are also unbounded, albeit in the slightly different sense that there is no natural number n such that, for all $x \prec yy$, $\rho(x) < \omega + n$. Using Replacement once again we can show that $[U_{\omega+1}, U_{\omega+2}, U_{\omega+3}, ...]$ is safe and, hence, by the Union axiom, that all of *their* members are jointly safe and, hence, constitute a new "limit" level $U_{\omega+\omega}$, whence the hierarchy begins anew yet again. With the notion of a "limit" level — more exactly, that of a limit *ordinal* — reasonably well in hand, we can state the general definition of Zermelo's full cumulative hierarchy, for all ordinal numbers α:

D3
$$\begin{aligned} U_0 &= U \\ U_{\alpha+1} &= U_\alpha \cup \wp(U_\alpha) \\ U_\lambda &= \bigcup_{\beta<\lambda} U_\beta, \text{ for limit ordinals } \lambda \end{aligned}$$

It is a fairly straightforward exercise to show that, so long as we choose an ordinal

κ that is big enough (at least what set theorists call *inaccessible*[33]) and keep the number of urelements smaller than κ, the set U_κ is a natural model of our ZF axioms.[34]

5 The Realist's Impasse

The cumulative hierarchy is undoubtedly a compelling picture of the set theoretic universe that seems to capture something deep and intuitive about the structure of sets and, moreover, seems to provide a convincing explanation of what goes wrong in Russell's paradox. More specifically, it seems to explain exactly what makes certain pluralities $[x\!:\!\varphi(x)]$ unsafe: some predicates $\varphi(x)$ pick out pluralities that are *absolutely* unbounded. That is, some pluralities yy — the Russell plurality $[x : x \notin x]$ in particular — are such that they have no "cap"; that is, for every level U_α of the hierarchy, there are things x among the yy that first appear in levels above U_α; equivalently put, given the **Fact** above (generalized to all ordinal numbers), for every ordinal α, there are things x among the yy whose ranks are greater than α.[35] (In the case of the Russell plurality, since in fact *nothing* in the cumulative hierarchy is self-membered, the Russell plurality is identical to the entire universe $[x : x = x]$ which is obviously absolutely unbounded.) Hence, there is no level at which such pluralities "run out" and, hence, no level at which they are "available"

[33]Inaccessibility is usually ascribed to *cardinal* numbers, which, in set theory, are identified with certain ordinals, viz., those that are larger in size than any of their predecessors. ω, being the first infinite ordinal, is a paradigm here; in its cardinal guise is known as $\aleph_0$. A cardinal κ is *inaccessible* if (a) it is uncountable (i.e., it is $> \aleph_0$), (b) it is not the sum of fewer than κ smaller cardinals, and, (iii) for cardinals $\lambda < \kappa$, $2^\lambda < \kappa$. Thus, if s is an uncountable set of size κ, where κ is inaccessible, by property (b) it will not be possible to partition s (i.e., divide it up into subsets such that no two of them have a common member) into fewer than κ cells all of which are smaller in size than s; otherwise put, there either have to be as many cells in the partition as there are members of s, or one of the cells must already be as large as s. And by property (c), the powerset of any set smaller than s will also be smaller than s. To get a sense of how big inaccessibles are, note that $\aleph_0$ has properties (b) and (c). So the jump from the "accessible" infinite cardinals to the first inaccessible is, in a sense, as enormous as that from the natural numbers to $\aleph_0$. For a bit more on inaccessibles and "large" cardinals generally, see fns 46 and 50.

[34]**?** himself was the first to define the cumulative hierarchy as above and prove that U_κ was a model of ZF (formulated to allow urelements) for κ inaccessible. For a good introduction to ZF set theory and the cumulative hierarchy, see **?**, ch. 2, esp. §2.3. For a philosophically rich, illuminating study that axiomatizes the cumulative hierarchy directly, see **?**.

[35]The correlation between ranks and levels once we move into the transfinite is more definite: for infinite ordinals α and sets x, $\rho(x) = \alpha$ if and only if x first occurs in $U_{\alpha+1}$.

for collection into a set at the next level.

But here's the problem: the cumulative hierarchy purportedly contains all of the sets there could possibly be that are built up from our initial set U of urelements. But why does the construction of the hierarchy not continue further still? The hierarchy is a definite plurality $[x : x = x]$ containing everything, all the urelements and all the sets that can eventually be constructed from them. Why is *it* not safe? Note, importantly, the question here is *not*: Why is there no universal set? We know from Russell's paradox that there can be no set that contains everything. Rather, the question is why the "process" of constructing new sets does not *continue* with the sets that, in fact, there are. Why can we not imagine a further "disk" atop the entire hierarchy depicted in Figure 2 — just as we proposed one for the hierarchy of finite levels in Figure 1 — and that the hierarchy continues on with further levels still?[36] For note that it appears we can give precisely the same justification for doing so that we gave for declaring the finitely unbounded plurality $[U_0 \cup U_1 \cup U_2 \cup ...]$ consisting of all of the members of all of the finite levels safe. First, the things constituting $[x : x = x]$ are all *there*. We are not talking about some sort of merely potential plurality; the things in question — the urelements and all the sets in all the levels — are fully determinate, actually existing entities. Second, just as the set U_ω formed from the plurality $[U_0 \cup U_1 \cup U_2 \cup ...]$ contains entities of arbitrarily high finite rank and, hence, cannot itself have a finite rank and, hence, cannot be a member of itself, so a new set — call it U_Ω — containing the things consituting the (*de facto*) absolutely unbounded plurality $[x : x = x]$ ($= [U_0 \cup U_1 \cup ... \cup U_\alpha \cup U_{\alpha+1} \cup ...]$, for all ordinals α) would exist in a new level "above" those of all of its members and, hence, would not be a member of itself. Hence, for the same reasons again, no paradox appears to be in the offing on these assumptions. And, given U_Ω, the axioms of Powerset, Union, and **Safe** together would then seem to yield yet another unbounded extension of the hierarchy.

In a nutshell: For the set theoretic realist for whom (a) all the sets there could possibly be (built up from the actual urelements) are already *there* and (b) the iterative conception is the correct conception of set, there is no clear answer to the

[36]It is important to emphasize that the term "$[x : x = x]$" is being used "non-rigidly" here to pick out whatever happens to exist. If (*per impossibile*, according to the realist) the plurality of *actually existing* things (that is the things *actually* picked out by "$[x : x = x]$") were safe, then they would constitute a set and, hence, the plurality that *would be* picked out by "$[x : x = x]$" would be properly include the plurality that is *in fact* picked out by "$[x : x = x]$".

question of why the plurality $[x : x = x]$ of everything (i.e., everything that *actually* exists) is not safe, why the cumulative hierarchy is necessarily only as "high" as it is in fact and could not be extended. Call this the *realist's impasse*.

Importantly, note also that activism as it stands seems to be at the same impasse: even if we identify sets with products of the divine intellect, so that the cumulative hierarchy necessarily has its existence in the mind of God, the question arises: why has God only collected exactly the sets that God has? The assumption for the activist no less than the realist is that all the sets there could possibly be (built up from the actual urelements) exist in the divine intellect; they are all *there*. But why, then, is the plurality $[x : x = x]$ not safe? Why can God not turn God's intellect on pluralities like that one that are absolutely unbounded and collect them into further sets?

6 Pluralities, Sets, and Modality

Recent work by **??** (which, in turn, draws on seminal work by **?**) suggests an answer to the realist's impasse, one that I will follow quite closely. The heart of the idea idea is that the axioms of set theory are implicitly *modal*; in particular, to say that some things are jointly safe is *not* to say that they *actually* constitute a set, but that they *could*. Thus, in the case of the cumulative hierarchy, regardless how "high" the hierarchy *actually* extends, the process of constructing new sets *could* always be extended further still; the *actual* absolutely unbounded pluralities there are, while not *in fact* constituting sets, *could* constitute sets in a different possible world. Set theory is thus the study, not of the sets there actually are — indeed, how many there are is irrelevant — but, rather, the study of the various set theoretic universes there *could* be.

Now, the actual details involved in working out this deeply interesting and important idea are rather complex (see, in particular, **?**). Fortunately, for purpose here, it will suffice only to get a flavor of the broader picture. To begin, the modal operators '$\Box$' and '$\Diamond$', in the context of set theory, will be interpreted such that $\ulcorner\Diamond\varphi\urcorner$, intuitively, means that "it is possible that sets exist that make φ true" and $\ulcorner\Box\varphi\urcorner$ means that "no matter what sets come to be (in addition to those there are), it will remain the case that φ". The interpretation of '$\Box$' here is particularly important. For, expressed in terms of possible worlds, it is not to be thought of as ranging over all worlds absolutely but, rather, at any given world w, over those worlds that include everything in w — all of its urelements and all of its sets. Thus, in particular, at a given world w, the axioms always describe what *further* sets there *could* beyond those there happen to be at w. Hence, the modality characterizes, for any given world w, every possible way the universe of sets that exist in w could be *extended* by sets whose members exist as mere pluralities in w or in worlds w' accessible from w. As we are thinking about the possibilities of extending the existing sets there might happen in fact to be, this seems like the right modality.[37]

On this understanding, the accessibility relation will be a partial order (reflexive, anti-symmetric, and transitive), since there are many different ways in which the universe of sets in a given world w could be extended to a world w' depending on which unbounded pluralities of w are taken to constitute sets in w'. Hence, our corresponding modal logic will include the system S4.[38] Moreover, when there are alternative possibilities about how the sets in w could be extended (as there always will be), it is reasonable to assume that, regardless which possibility is chosen, the other alternatives do not go away. In terms of accessibility, this means that for any worlds w_1 and w_2 accessible from a given world w, there will always be a common world w_3 accessible from both w_1 and w_2. This condition, known as *convergence*, when added to our partial order, yields an accessibility relation characteristic of the system known as S4.2, the result of adding the following axiom schema to S4:

4.2 $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$.[39]

Now, Linnebo (**?**, 156ff) goes on to note that the modal character of the sentences of set theory is captured chiefly by analyzing the universal quantifier '$\forall$' as '$\Box\forall$' and the existential quantifier '$\exists$' as '$\Diamond\exists$'; accordingly for a given sentence φ of our non-modal set theoretic language, let $\varphi^{\Diamond}$ be the result of replacing every quantifier occurrence in φ with its modalized counterpart. With this distinction between modal and non-modal readings of the quantifiers in hand, Linnebo points out that

[37]**?** develops an alternative bi-modal theory that builds instead on a linear temporal metaphor on which (roughly) the stages of the iterative hierarchy are constructed through time; one modality thus can "look ahead" to future (hence, larger) stages and the other can "look back" at past (hence, smaller) stages. This makes for a rather more elegant axiomatization than on Linnebo's approach.

[38]That is, the system of propositional modal logic whose axioms are **K** ($\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$), **T** ($\Box\varphi \rightarrow \varphi$), and **4** ($\Box\varphi \rightarrow \Box\Box\varphi$).

[39]S4.2 is sound and complete for convergent partially ordered frames. See **?**, 134-6 for more on S4.2.

Russell's paradox can be given a cogent and compelling analysis. Specifically, the paradox depends on the non-modal readings of the two principles **Collapse** and **P-Comp**, and our initial solution was to accept the latter and place the blame on the former, which led us ultimately to the realist's impasse. However, under our now preferred modal interpretation, the situation is reversed. So interpreted, **Collapse** is true and, indeed, expresses quite precisely the modal intuition underlying the iterative conception that the universe of sets is always *extensible*:

Collapse$^{\Diamond}$ $\Box\forall xx\Diamond\exists y\Box\forall z(z \in y \leftrightarrow z \prec xx)$.

That is, necessarily (i.e., in worlds that contain our urelements and sets, and perhaps more sets as well built from them), for any things whatever, it is *possible* that they be collected into a set, that is, it is possible that there be a set containing (necessarily) exactly them. By contrast, while the principle **P-Comp** is unproblematically true on the non-modal interpretation,[40] its modalized counterpart

P-Comp$^{\Diamond}$ $\Diamond\exists yy\Box\forall x(x \prec yy \leftrightarrow \varphi(x))$, where φ contains no free occurrences of the variable 'yy'

is false in general. For, while it is trivially the case that there is a world w in which there are the things satisfying φ — the actual world is such a world, for any φ — for some predicates φ, there is not always a world w such that that there are, *in* w, all the things that *could* satisfy φ, i.e., all the things that satisfy φ in any accessible world. For, on our understanding of the modalities, for any world w, there are worlds w' accessible from w in which there are sets that don't exist in w — and hence are not among any plurality in w — that satisfy φ in w'. This is true in particular of the predicate '$x \notin x$' since, on the iterative conception, nothing in any world is a member of itself; necessarily, everything satisfies '$x \notin x$'. Hence, the modal interpretation **P-Comp**$^{\Diamond}$ of **P-Comp** for the predicate at issue in Russell's paradox is false, and the paradox dissolves.

Now, as **?** shows, given only the system of quantified S4.2, the iterative principle **Collapse**$^{\Diamond}$, the axioms of Extensionality and Foundation, and a few other intuitive principles concerning the modal properties of sets and pluralities,[41] it

[40] But see **?**.

[41] Particularly important among these are principles $\exists yy\Box\forall u(u \prec yy \leftrightarrow u \in x)$ and $\exists yy\Box\forall u(u \prec yy \leftrightarrow u \subseteq x)$ expressing that sets have their members and their subsets essentially. See **?**, §6.

is possible to derive the modalized versions of all of the standard axioms of pure ZF.[42] Indeed, since the notion of a safe plurality reduces to the possibility of a certain set, plurals and the notion of safety can simply drop out of the picture in the statement of the axioms proper. Thus, to get a sense of how things look, instead of **Pr** we have:

Pr$^\Diamond$ $\Box\forall a\forall b\Diamond\exists y\Box\forall z(z \in y \leftrightarrow (z = a \vee z = b)$

That is, for any things a and b, there *could* be a set that (necessarily) contains exactly them. Likewise, instead of **Pow** we have a modalized counterpart telling us that the subsets of any given set s *could* themselves constitute a set:

Pow$^\Diamond$ $\Box\forall s\Diamond\exists y\Box\forall z(z \in y \leftrightarrow z \subseteq s)$.

And Infinity now says that there *could* be a set containing the von Neumann ordinals $\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \ldots$:

Inf$^\Diamond$ $\Diamond\exists u(\varnothing \in y \wedge \Box\forall z(z \in y \rightarrow z \cup \{z\} \in y))]$.[43]

Moreover, Linnebo shows that, in the context of (pure) modal set theory, the modalized quantifiers '$\Box\forall$' and '$\Diamond\exists$' "behave proof-theoretically very much like ordinary quantifiers" (2013, 213) and hence can also largely drop out of the picture — thus explaining why the explicit modalities "do not surface in ordinary set-theoretic practice" (2010, 164).

Succinctly (and somewhat crudely) put then: Linnebo solves the realist's impasse by proposing that set theory is not the study of the sets there *are* — indeed, it is irrelevant which sets actually exist — but, rather, of what sets there *could be*. Linnebo's solution is thus able to fully embrace what had been an embarrassment for the realist — the fact that the hierarchy of sets must necessarily come to an arbitrary end. But because the hierarchy is always extensible — as embodied in **Collapse**$^\Diamond$ — the impasse is resolved.

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[42]ZF with urelements would simply be a somewhat fussier affair that would call for a few further principles.

[43]We use (the modalization of) the weaker but equivalent form **Inf**′ of the Infinity axiom (see fn 29), as the modalized version of **Inf** is needlessly complicated. We also assume for convenience (apparently along with Linnebo (2013, 223-4)) that $\varnothing$ exists (and hence exists in all accessible worlds). Without this assumption we would either have to adopt some variant of free logic (in which constants might not denote) or complicate the Infinity axiom considerably.

Note also that the impasse is equally well resolved for the activist: the answer to the question of why absolutely unbounded pluralities like $[x : x = x]$ are not safe, i.e., why God could not turn the divine intellect on such pluralities and collect them into further sets, is that God *could* do exactly that. As a matter of contingent fact God has not; but God could. Understood in activist terms, **Collapse**$^\diamond$ says that, necessarily, every plurality *could* be (but might in fact not have been) collected into a set by God. And the impasse again is thereby resolved.

7 Realism and the Set Theoretic Modality

All in all, this modal take on the nature of sets, especially its analysis of Russell's paradox, seems quite compelling. But a critical question remains for the realist: what, exactly, is the nature of the modality in modal set theory? **?**, 207 himself is quite clear that the modality in question

> ...*is not metaphysical modality in the usual post-Kripkean sense*. Rather, the modality employed in this article is related to that involved in the ancient distinction between a potential and an actual infinity. This modality is tied to a process of building up larger and larger domains of mathematical objects. A claim is possible, in this sense, if it can be made to hold by a permissible extension of the mathematical ontology; and it is necessary if it holds under any permissible such extension. Metaphysical modality would be unsuitable for our present purposes because pure sets are taken to exist of metaphysical necessity if at all.[44]

And here things seem to go a bit off the rails, metaphysically. For it appears that pure sets at least, for Linnebo, exist necessarily after all; the possibility of the modalized ZF axioms are not the possibility of existence but, well, something else — something having to do with "permissible extension(s) of the mathematical ontology". But what is the notion of permissibility here? The questions at issue here are deep and difficult (see, e.g., **?**) but I take it that a permissible extension of a given model of a theory is a further model that (subject to certain constraints, perhaps) preserves all the relations of the original model. So Linnebo (and perhaps even more so, **?**, 10-11) seems to have something like the following (semi-fictional)

[44] See also **?**, §V, where he spells this idea out in a bit more detail.

story in mind.[45] Let κ be the least inaccessible cardinal, i.e., the smallest of the so-called "large" cardinals.[46] Prior to the discovery of inaccessibles, U_κ was our entire set theoretic universe, albeit unrecognized as a set. From our perspective "inside" that universe, κ was unknown and the plurality picked out by "$[x : x = x]$" was unbounded relative to κ and, hence, from our limited perspective, absolutely unbounded. But upon discovering inaccessibles, our conception of the extent of the universe expanded to include a hierarchy of inaccessibles as well. However, suppose next a new large cardinal property is discovered — measurability, say — and it is proved that the smallest cardinal μ with this property is much larger than κ. Then we will have discovered that what we'd thought was the universe prior to this discovery was really U_μ. With each new discoveries of a new class of cardinals, our conception of the extent of the set theoretic universe grows. As there seems no limit to the new large cardinal properties we *could* discover, it seems that is it always possible that the our conception of the extent of the universe of sets could grow.

It seems clear, however, that this conception permissible extensibility is not a metaphysical one but an epistemological one; it is not the case that new sets could come to be but, rather that our *knowledge* of the nature and extent of the universe can grow. If that is right, then indeed, as Linnebo appears to acknowledge, the (pure) sets exist necessarily after all, in the usual post-Kripkean sense — in any world, the sets that *could be* constructed from the urelements of that world *already exist*; news sets do not really come to be in other worlds that contain the same urelements. If so, however, then it is simply *not* possible to that the universe of sets be extended at all; if a plurality is not a set in a given world, it is not a set in any world.

By my lights, then, if the modal gambit is to be a truly successful way around the realist's impasse, the modality in question has to be the post-Kripkean metaphysical modality and it really does have to be a contingent matter what sets exist.

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[45]The rest of this paragraph assumes a bit more set theory than has been assumed hitherto, but the details are not essential to the argument.

[46]"Large" cardinals are so-called because they are provably larger than any cardinal whose existence can be proved in ZF. The reason such cardinals cannot be proved to exist is because they are all inaccessible and, as just noted, for inaccessible κ, U_κ is a model of ZF. By Gödel's second incompleteness theorem, ZF (if consistent) cannot prove its own consistency and, hence (by the soundness theorem for first-order logic), cannot prove that it has any models; so it can't prove the existence of an inaccessible.

For the impasse is skirted only if there *really could be* more sets than there actually are (constructed from the same urelements), only if there *really could be* sets whose members are, as a matter of contingent fact, mere pluralities of actually existing sets and urelements. If that is *not* what is meant — if, after all, all the sets there could be already exist — then the impasse recurs: there is no explanation for why the hierarchy is only as high as it is; no explanation for why the plurality $[x : x = x]$ cannot form a further set.

8 Advantage: Activism

But notice that the realist has in fact simply traded one metaphysical problem for another of the same sort and severity. For the contingency of set existence required by the modal gambit entails a sort of vicious metaphysical capriciousness for the realist: it entails that there are possible worlds that are identical in every respect but for the fact that in one, inexplicably, there are sets built up from the urelements of the world that in the other, inexplicably, do not exist, despite the existence of those very same urelements. And this brings us full circle back to the argument from collections. For recall that a critical element of the realist's response to the argument is that the existence of a set needs no more explanation than the existence of its members and hence, ultimately, given the well-foundedness of the membership relation, the existence of the urelements it is built up from. A set exists *because* its members do, and no further explanation is needed. But this explanation is utterly undercut if its a radically contingent matter which sets exists given the urelements; if a set s exists in one world w but not in another w' despite the existence of its members in w', then its existence in w is metaphysically capricious; since s might not have existed even if its members had, the existence of its members does *not* after all explain its existence in w. Thus what we might call the realists' dilemma: they seem forced to choose between the original impasse and the metaphysical capriciousness of the modal gambit. Either way, the realist seems stuck with pluralities that either inexplicably do not form a set or inexplicably do.

The activist, by contrast, happily avoids the realist's impasse by accepting the modal gambit, but avoids capriciousness because she has a satisfying explanation for the existence of the sets in any given world: they are the ones whose members God in fact chose to collect. As the products of the intellective activity of a free agent, there is no mystery whatever as to how it could be that certain pluralities don't form sets and how it could be that an existing set might not have existed even if its members had. It is no more capricious that God in fact chose not to collect certain pluralities than that God chose not to create certain biological species. For it is always possible for God to have created more things, or other kinds of things, than God actually has. It is no knock against God's creative power that God has in fact chosen not to have done so.

But perhaps it isn't quite that easy. Several objections suggest themselves that I will address briefly here but which no doubt deserve further exploration. First, one might question whether it makes sense to hold that God only *could* form a set from a plurality, i.e., whether God's thinking of the plurality *just is* to think of it as a whole. I think that we are misled by language a bit here, specifically, that the use of the substantive "a plurality" (and the singular anaphoric pronoun "it") is misleading. It is less so, I think, if we frame the question using plurals: is it the case that God's thinking (*de re*) of some things *just is* for God to think of them as a whole? So expressed, it is not at all clear on the face of it. Notably, God's thinking of some things does not of itself seem to be sufficient to account for the iterative structure of the cumulative hierarchy. For in thinking of some things xx it does not follow that there is something — namely, the set $\{y : \forall z(z \in y \leftrightarrow z \prec xx)\}$ such that something is a member of it just in case it is one of them — that is capable of being a member of *further* pluralities that, in turn, can be collected in the divine mind. God's simply thinking of some things is not of itself sufficient for that; thinking of them *as a whole*, collecting those things into a set, has to be a separate intellective act if we are to account for the iterative structure of the sets along activist lines.

But this answer might lead to a related worry about omniscience.[47] If God *could* collect some things into a set, God *knows* that this is possible and, hence, knows that there would be such a set. But this, the worrier continues, would seem to entail that God *actually* know that God could collect *that set*. For how could God know that he *could* create a set from some things without God's actually apprehending that very set?

But the reply above seems to apply here as well: the move from the *de dicto* to the *de re*, at the least, does not follow. We have just seen that God's thinking of some

[47]Versions of which are due to Michelle Panchuk and Mike Rea.

things as a whole is a separate intellective act and, hence, that his thinking of some things is not of itself to think of them jointly as a set. Hence, when God knows that God *could* collect some things into a set that God has not in fact collected, it neither follows, nor does there seem to be any compelling to think, that God must have *de re* knowledge of a set that *would* exist that contains those things — and indeed, as we've gone to some lengths to show, on the activist's account, it *cannot* always be that such a set would exist; there must *necessarily* be pluralities that are not collected. And, of course, since, there *are* no such sets for such pluralities, there are no propositions about any such sets and, hence, no true propositions that God fails to know. Hence, there is no clear problem for God's omniscience here.[48]

A further concern is that, while the activist doesn't have the problem of metaphysical capriciousness that the realist seems to have, a vicious sort of capriciousness nonetheless might remain. For if, necessarily, all the existing sets are contingent, then perhaps God chose, capriciously, not to collect, e.g., Michelle and Barack Obama into a set $\{m, b\}$. This might seem particularly odd, indeed inexplicable, given that we ourselves seem perfectly able to collect them together in our *own* intellects.

But it seems the activist, as part of her overall explanation of the connection between God and sets, can reasonably make a case for the actual existence of a very large cumulative hierarchy in the divine intellect. To elaborate, it seems impossible that God, being omniscient, wouldn't know that I, say, had collected the Obamas in my own intellect without God also doing so. Moreover, God knows what finite

[48]Thanks to Brian Cutter for making this point in discussion. There is perhaps an inkling of a difficulty here though. For, insofar as what is *possible* is determinate, one might argue that there have to be propositions that *actually* have an internal structure that corresponds to all the collecting God could possibly do beginning with a certain plurality of urelements To make things simple, suppose we just have the single urelement a. Then, as a is a plurality of one, God knows there could be a set s_0 containing just a; and hence that there could be a set s_1 containing a and s_0 and all the sets that can be formed from them; and hence that there could be a set s_2 containing all those things and all the sets that can be formed from them; and hence ... ; and at limit stages, that there could be a set s_λ containing a and, for $\beta < \lambda$, and all the sets that can be built up from the members of all the s_β; and so on. But it seems that there can't really *be* any such proposition, for any initial plurality xx, since it would *actually* have to include embedded conjuncts for all the collecting that is *possible* for God to do beginning with xx. And the resulting structured proposition would seem to be as problematic as the realist's assumption that all the sets there could be are already actual.

I will leave this objection unanswered, except for noting that it seems open to the activist to reject the idea that propositions have the sort of "internal structure" that would seem to be required to get this objection off the ground. That said, I am not confident that there aren't other cardinality worries along these lines lurking in the nearby bushes (as Plantinga might put it).

pluralities we, hence arbitrarily powerful finite minds, are simply *capable* of collecting and, hence, has in fact collected all the finite levels U_n. And, of course, *we* have grasped the idea of levels extending beyond the finite, so it is reasonable to think God has collected all the elements of the finite levels into the first infinite level $U_\omega = \bigcup_{n<\omega} U_n$. Indeed, as God knows how many sets there must be in a cumulative hierarchy to make the axioms of ZF true — the axiom system that most of us think best characterizes the basic structure of the universe of sets — it seems reasonable to think that God in fact collects at least up to the first inaccessible cardinal κ such that the cumulative hierarchy $U_\kappa = \bigcup_{\alpha<\kappa} U_\alpha$ is a model of ZF.[49] And, indeed, since God of course knows what inaccessibles are, it is reasonable to suppose God forms sets of inaccessible size and, hence, collects to the first Mahlo cardinal (the "next smallest" type of large cardinal); however, for the same reason it is reasonable to suppose God collects up to the first weakly compact cardinal; and so on.[50] Indeed, given that God knows all of the large cardinal properties — certainly at least those we are in principle capable of formulating — it is reasonable to suppose that God does enough collecting to satisfy all of them.[51]

Aside from whatever intrinsic philosophical merit it might have, an advantage of this hypothesis is that, if correct, ZF and all conceivable large cardinal extensions of it are all *actually* true. Hence, the modal axioms of ZF are unnecessary *except* to characterize possible extensions of the cumulative hierarchy that God *could* construct beyond the massively large hierarchy that God has actually constructed.

In sum, then. The argument from collections is a good one in at least the sense that it provides an illuminating way of thinking about set theory from a theistic

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[49]This reasoning is actually a bit quick. For by a remarkable theorem of **?**, if κ is the smallest inaccessible, there is a cardinal $\mu < \kappa$ (hence, "small") such that U_μ is a model of ZF. By Gödel's second theorem, once again, the existence of μ cannot be proved in ZF alone, even though it is provably "accessible". So, assuming the existence of an inaccessible, the smallest inaccessible is not actually the first cardinal κ such that U_κ is a model of ZF.

[50]The intuitive idea behind the postulation of this hierarchy of ever larger large cardinals, on the realist's picture of the cumulative hierarchy, at least, is that the universe of sets is so rich that any property of it that we can think of must already be "reflected" in some set. Large cardinal axioms typically arise, therefore, through the identification of some interesting property of the universe that is not true of any set whose existence can be proved in ZF (perhaps supplemented by existing large cardinal axioms) and postulating the existence of a set with that property. For an excellent discussion of the ZF axioms, the iterative conception, and large cardinals, see **?**, esp. 501-508.

[51]This is a bit simplistic as, currently, the known large cardinal properties, ordered by strength, do not quite form a linear hierarchy (see **?**, 472), so some of those properties might not actually be true of any genuinely possible set.

perspective. Additionally, however, activism fills a rather dramatic — arguably irresolvable — explanatory gap in set theoretic realism and, hence, provides a significantly better explanation than realism for the existence and nature of sets. To that extent, at least, then, insofar as one wishes to hold on to the idea that the axioms of ZF (at least in their modal guises) tell us something robustly true about the sets and what they are like, the argument from collections provides positive grounds for believing that God exists.