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Reference to numbers in natural language

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Abstract A common view is that natural language treats numbers as abstract objects, with expressions like *the number of planets*, *eight*, as well as *the number eight* acting as referential terms referring to numbers. In this paper I will argue that this view about reference to numbers in natural language is fundamentally mistaken. A more thorough look at natural language reveals a very different view of the ontological status of natural numbers. On this view, numbers are not primarily treated abstract objects, but rather 'aspects' of pluralities of ordinary objects, namely number tropes, a view that in fact appears to have been the Aristotelian view of numbers. Natural language moreover provides support for another view of the ontological status of numbers, on which natural numbers do not act as entities, but rather have the status of plural properties, the meaning of numerals when acting like adjectives. This view matches contemporary approaches in the philosophy of mathematics of what Dummett called the Adjectival Strategy, the view on which number terms in arithmetical sentences are not terms referring to numbers, but rather make contributions to generalizations about ordinary (and possible) objects. It is only with complex expressions somewhat at the periphery of language such as *the number eight* that reference to pure numbers is permitted.

Keywords Numbers · Abstract objects · Tropes · Frege · Referential terms · Adjectival Strategy · Abstraction

The question whether numbers are objects and, if so, how such objects can be understood is a central question in the philosophy of mathematics. In addressing this question, philosophers have often made appeal to natural language, either to clarify

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our intuitions about numbers in general or to obtain supporting evidence for a particular philosophical view about numbers. Frege is a good case in point. For Frege, numbers are objects because there are singular terms that stand for them, and not just singular terms in some formal language, but in natural language in particular. Thus, Frege thought that both noun phrases like *the number of planets* and simple numerals like *eight* as in (1) are singular terms referring to numbers as abstract objects:

(1) The number of planets is eight.

Thus, it appeared that natural language gives clear support for numbers having the status of abstract objects or what I call *pure numbers*. The view that our ordinary discourse contains pervasive reference to pure numbers, by numerals and terms like *the number of planets*, is in fact not only Frege's view, but is currently the most common view about the role of number expressions in natural language.

In this paper, I will argue that this view about reference to numbers in natural language is fundamentally mistaken. Natural language presents a very different view of the ontological status of natural numbers. On this view, numbers are not primarily abstract objects, but rather 'aspects' of pluralities of ordinary objects, namely number tropes, a view that in fact appears to have been the Aristotelian view of numbers. Natural language moreover provides support for another view of the ontological status of numbers, on which natural numbers do not act as entities, but rather have the status of plural properties, the meaning of numerals when acting like adjectives. This view matches contemporary approaches in the philosophy of mathematics of what Dummett called the *Adjectival Strategy*, the view on which number terms in arithmetical sentences are not terms referring to numbers, but rather make contributions to generalizations about ordinary (and possible) objects. It is only with complex expressions somewhat at the periphery of language that reference to pure numbers is permitted.

More specifically, I will argue that while *the number of planets* is in general a referential term, it is not a term referring to a number—and in fact in the particular context of (1) it is not a referential term at all. When occurring as a referential term, *the number of planets* does not refer to an abstract object, but rather to a number trope, the instantiation of a number property in a plurality of objects such as the property of being eight in the plurality of the planets.

I will also present some semantic and syntactic evidence that numerals like *eight* are not really referential terms at all, even though they may occur in apparently referential position and fulfil standard (Fregean or Neo-Fregean) criteria for referential terms. Numerals behave in a number of respects differently from truly referential terms such as *the number eight*, that is, *explicit number-referring terms*. Numerals are better considered *quasi-referential terms*. As a quasi-referential term, a numeral has the syntactic status of a full noun phrase, but it retains the semantic value it has when acting as a determiner or noun modifier. While this view coincides with that of Hofweber (2005a), the motivations are new, and unlike in Hofweber (2005a), numerals are sharply distinguished semantically from explicit number-referring terms like *the number eight* as well as terms like *the number of planets*.

English does have terms referring to pure numbers, namely explicit number-referring terms like *the number eight*. The entity that *the number eight* refers to appears to have a derivative ontological status, though, going along with the compositional semantics of the complex term. In *the number eight*, *eight* occurs nonreferentially and *number* itself, I will argue, is not a mere sortal but has a reifying function, introducing a pure number on the basis of the role of the numeral in arithmetical contexts.

This paper not only argues against the Fregean view according to which terms like *the number of planets* and numerals are number-referring terms; it also undermines the linguistic motivation for Frege's specific view of pure numbers. Frege took the form of number-referring terms like *the number of planets* to be indicative of the nature of pure numbers. For Frege, *the number of* in the construction *the number of planets* expresses a function that applies to a concept (denoted by *planets*) and maps it onto a pure number. It appears that natural language reflects a very different view of the nature of numbers. On the one hand, natural language reflects an older notion of abstraction, according to which numbers are abstractions from pluralities of objects (number tropes). At the same time, natural language treats numerals, with their adjectival meanings, as quasi-referential terms and thus reflects the Adjectival Strategy.

I will first focus on number tropes as referents of terms like *the number of planets* and discuss certain other, question-related uses of such terms, as in fact in (1). I then present some new linguistic observations about simple numerals and indicate how the 'Adjectival Strategy' can be applied to the semantics of natural language. Finally, I will outline an account of explicit number-referring terms like *the number eight* and the ontology that they, as I will argue, go along with.

1 *The number of planets* and number tropes

1.1 *The number of planets* as a referential, but not a number-referring term

At least since Frege (1884), it has been taken for granted that terms like *the number of planets* are referential terms referring to numbers. Let me call such terms '*the number of*-terms'. It was Frege's view that since *the number of*-terms are referential terms, they must have the function of standing for objects, and since Frege thought that only numbers could be the right objects of reference, numbers are objects.[1]

I will argue that in many (though not all) contexts, *the number of planets* has indeed the status of a referential term, but it does not refer to a pure number. Rather it refers to a number trope, a particularized property which is the instantiation of a number property (the property of being eight) in a plurality (the plurality of the planets).

Let me make a few remarks about tropes in general. Tropes are generally characterized as 'particularized properties'. They are particulars just like the individuals that are their bearers; yet they also play the role of properties: they constitute one aspect of an individual, in abstraction from all the other properties the individual may have (hence they are 'abstract particulars', to use Campbell's (1990)

[1] This is a consequence of Frege's context principle, on one reading of it (Wright 1983; Hale 1987). If an expression has a meaning ('Bedeutung') only in the context of a sentence and the contribution of a singular terms to the truth conditions of a sentence is that of reference to an object, then there is no further question that if an expression is in fact a singular term, it refers to an object.

term).[2] Tropes have also been viewed as the instances of properties. Given that view, two tropes that instantiate the same property are similar or exactly similar—the latter in case the property is 'natural'.[3] Standard examples of tropes are those that are instances of qualitative properties, such as 'Socrates' wisdom' or 'the redness of the apple'. If the apple and the tomato are both red, then there are two tropes, 'the redness of the apple' and 'the redness of the tomato', which are similar; if the apple exhibits the very same shade of redness as the tomato, then the redness of the apple and the redness of the tomato are exactly similar. Alternatively, properties themselves may be identified with classes of similar or exactly similar tropes (Williams 1953; Campbell 1990). Tropes are ontologically dependent on their bearer: in general two tropes are identical only if they have the same bearer, and a trope can exist in a world at a time only if its bearer exists in the world at the time. Two tropes are identical just in case they are exactly similar and have the same bearer. There are also relational tropes, such as the love of John toward Mary. Moreover there are tropes that have pluralities as bearers, for example the love among the people or the similarity among the proposals.

A number trope differs from standard examples of tropes in that it is a quantitative trope, in fact a quantitative trope whose bearer is a plurality. Other quantitative tropes are, for example, John's weight, Mary's height, the amount of water in the container, or the direction of the line.[4] Psychologically speaking, a number trope involves abstracting from all the qualitative aspects of a plurality and attending to just how many individuals the plurality consists in. Ontologically speaking, a number trope is that particular feature of a plurality that exactly matches that of any equally numbered plurality.

A range of semantic evidence indicates that noun phrases of the sort *the number of planets* (*the number of*-terms for short) do not refer to pure numbers, but rather to number tropes. Not all occurrences of *the number of*-terms are referential terms referring to tropes, though.

First of all, *the number of*-terms can be used as terms referring to entities with 'variable manifestations' (as different tropes in different circumstances).

Second, *the number of* can be used as a replacement of a numeral. *The number of* is then on a par with *three, several*, and *many* as well as *a great number of* or *a small number of*, as below[5, 6]:

[2] The term 'trope' is due to Williams (1953). Other recent references on tropes are Bacon (1995), Campbell (1990), Lowe (2006), and Woltersdorff (1970). Tropes already played an important role as 'accidents' or 'modes' in ancient philosophy (Aristotle) as well medieval Aristotelian philosophy. They also play an important role in early modern philosophy (Locke, Berkeley, Hume).

[3] For the notion of a 'natural property' or a 'sparse' conception of properties, see Armstrong (1978) and Lewis (1983).

[4] For the notion of a quantitative trope see Campbell (1990), who calls such tropes 'instances of quantities'. On the topic of reference to quantitative tropes in natural language in general see Moltmann (2009).

[5] See Kayne (2007) for a recent discussion of phrases like *a large number of people*. Kayne in fact assumes that the plural determiners *several, few*, and *many* are modifiers of an unpronounced noun *number* (of course on the amount-specification reading).

[6] The use of *the number of* as a numeral replacement is also indicated by the possibility of plural agreement rather than singular agreement in English:

(i) A great number of women were arrested.

(2) a. John kissed a great number of babies.
b. John made a small number of mistakes.

The number of appears to be used as a numeral replacement in that sense below in (3) (which certainly does not mean that Mary counted a single number trope):

(3) Mary counted the number of mistakes she had made.

I will later argue that the use as a numeral replacement is also involved when *the number of*-terms occur as 'concealed questions' (standing for a question about a cardinality rather than a number trope) or a 'concealed fact' (standing for a fact about a cardinality rather than a number trope). Furthermore, it is involved when a *the number of*-term occurs as the subject of a so-called specificational sentence (arguably a particular case of a concealed question), which is the case in (1).

I will come to these uses later. First, I will focus on the use of *the number of*-terms as referential terms (and in this context I will talk about such occurrences simply as '*the number of*-terms', not distinguishing expressions from expressions in a particular syntactic function in a sentence). *The number of*-terms clearly occur as referential terms in a range of contexts. For example, in contexts such as (4a) below, *the number of women* satisfies any tests of referentiality, occurring as subject of a sentence whose predicate generally acts as a predicate of individuals, just as with the explicit number-referring term *the number eight* in (4b):

(4) a. The number of women is small.
b. The number eight is small.

Explicit number-referring terms and *the number of*-terms display a range of semantic differences with various classes of predicates, which indicate that the two kinds of terms refer to fundamentally different kinds of entities: *the number of*-terms refer to number tropes, whereas explicit number-referring terms refer to pure numbers.

1.2 Predicates of causation and perception

One class of predicates that distinguishes referents of *the number of*-terms from pure numbers are predicates of causation and perception. As long as the plurality in question consists of concrete entities, perceptual and causal predicates make sense with *the number of*-terms, but under normal circumstances not with explicit number-referring terms:

(4) John noticed the number of the women / ?? the number fifty.[7]
(5) a. The number of infections / ?? The number ten caused Mary's immune system to break down.

[7] Of course *John noticed the number fifty* is possible on a non-perceptual reading of *notice* or else when the object of the noticing is in fact an inscription of the numeral. Unlike that sentence, though, *John noticed the number of women* can naturally describe a perceptual situation of John looking at the women and counting them.

b. John's outbursts of anger were the cause of the high number of complaints / ?? the number twenty.

Here and throughout this paper, '??' (as well as the weaker '?' and the stronger '???') means 'semantically unacceptable' (which in turn generally means 'cannot be true except in implausible circumstances').

Predicates of perception and causation also apply to other quantitative tropes:

(6) a. John noticed Mary's height.
b. Mary's weight caused the bream to break.

Being perceivable and causally efficacious is characteristic of concrete as opposed to abstract entities, and thus, unlike pure numbers, the referents of *the number of*-terms with a concrete plurality qualify as concrete.

One might try to maintain the view that *the number of*-terms stand for pure numbers by appealing to a secondary use of such terms as 'concealed facts', descriptions that instead of referring to their usual referent refer to suitable facts corresponding to the description (Grimshaw 1997). Facts after all can be 'noticed' and can enter relations of causal explanation (if not perhaps relations of causal explanation). However, such an account could not give justice to *the number of*-terms in general, as I will discuss in the next section, in particular the way the referents of *the number of*-terms can be evaluated quantitatively, display particular identity or similarity conditions, and can undergo mathematical operations or have mathematical properties.

Another common criterion for concreteness is that of being located in space and time. Spatio-temporal location is a problematic criterion of concreteness for number tropes, however—as it is in fact for tropes in general.[8] A location in time as well as a temporal extension can perhaps be attributed to qualitative tropes (as in *John's happiness today, John's happiness lasted only two years*). But attributing a spatial extension or location to tropes is quite problematic. *The length of the table's blackness* is hardy possible and so for *the table's blackness in the room* or *the stone's heaviness on the table*. Similarly, *the location of the number of Mary's children* is impossible, as is *the number of children in the garden* on an understanding on which *in the garden* modifies *number*, rather than *children*. Also other predicates involving spatial relations are hardly applicable to number tropes (??? *the parents surrounded the number of children*). Tropes are ontologically dependent on a bearer which itself may be located and extended in space, but it appears that tropes are not themselves located or extended in space.

1.3 Evaluative predicates

Predicates that express a quantitative or 'neutral' evaluation of an object generally behave differently with *the number of*-terms and with explicit number-referring

[8] This is despite the fact that it is a standard assumption in the recent philosophical literature on tropes that tropes come with two fundamental relations: similarity and co-location (Williams 1953; Campbell 1990).

terms, for example the predicates of neutral comparison *exceed* and *equal*. *Exceed* and *equal* are perfectly acceptable with *the number of*-terms, but not with explicit number-referring terms (with which they display at least a less natural reading):

(7) a. The number of women exceeds / equals the number of men.
b. ?? The number fifty exceeds / equals the number forty.

The same predicates are also acceptable with the corresponding plural when adding the modifier *in number*:

(7) c. The women exceed / equal the men in number.

One-place evaluative predicates such as *negligible*, *significant*, *high*, and *low* display the same pattern with *the number of*-terms, the corresponding plurals, and explicit number-referring terms:

(8) a. The number of animals is negligible / significant.
b. The animals are negligible / significant in number.
c. ?? The number 10 is negligible / significant. (different understanding of the predicate).
(9) a. The number of deaths is high / low.
b. The deaths are high / low in number.
c. ??? The number ten is high / low.

Other evaluative predicates such as *unusual* or *compare* display different readings with the *the number of*-terms and explicit number-referring terms. With explicit number-referring terms, they evaluate an abstract object (a pure number); with *the number of*-terms, they evaluate a plurality, namely in just one particular respect: with respect to how many individuals it consists in[9]:

(10) a. The number of women is unusual.
b. The number fifty is unusual.
(11) a. John compared the number of women to the number of men.
b. John compared the number fifty to the number forty.

(10a) has quite a different reading from (10b), and (11a) from (11b). The readings that (10a) and (11a) display can be made transparent by the near-equivalence with sentences just about the corresponding plurality such as (12a) and (12b), with the predicate modifier *in number*:

(12) a. The women are unusual in number.
b. John compared the women to the men in number.

Thus, the referent of a *the number of*-term shares evaluative predicates with the corresponding plurality as long as the predicates in the latter case are modified by *in number*. That is, it shares those predicates that apply to the plurality just with

[9] Another pair of examples making the same point is the following:

(i) a. The number of bathrooms in this house is average.
b. ?? Three is average.

respect to how many individuals it consists in. Unlike the plurality, the referent of a *the number of*-term requires no restriction to a numerical respect.[10, 11]

This behaviour of evaluative predicates manifests itself with other quantitative tropes as well:

(13) a. John's height exceeds Mary's height.
b. John exceeds Mary in height.
c. ? Two meters exceed one and a half meters.

The sorts of properties that the referents of *the number of*-terms share with the corresponding pluralities are indicative of the kinds of entities that *the number of*-terms refer to: they constitute just one aspect of a plurality, concerning just how many individuals the plurality consists in.

1.4 Number tropes

The number of-terms thus refer to entities that have two characteristics:

[1] They may have properties of perception and causation as long as the corresponding plurality consists of concrete entities.
[2] They share properties of neutral evaluation with the corresponding plurality when ascribed to the latter with the qualification 'in number'.

Number tropes, tropes whose bearer is a plurality and which attend to just the numerical aspect of the plurality, are just the kinds of entities that fulfil those two conditions. A trope in general is concrete, and thus perceivable and causally efficacious, just in case its bearer is. Moreover, a quantitative evaluation of a number trope naturally consists in evaluating the corresponding plurality with respect to its numerical aspect.

The referents of *the number of*-terms also display the similarity relations expected for number tropes. Recall that tropes that instantiate the same property are similar, and

[10] Some evaluative predicates seem to receive a more intensional interpretation than the trope analysis would predict. If in a particular context, the women are in fact the students, then (ia) and (ib) can still have distinct truth conditions:

(i) a. The number of women is unusually high.
b. The number of students is unusually high.

Even though on the present account *the number of women* would refer to the same trope as *the number of students*, the difference between (ia) and (ib) appears to be independent of the view of what the subject terms stand for. Relative adjectives such as *high* generally may depend for their evaluation on the description used, not just the referent. Thus the same contrast appears in (ii), supposing that the gymnast and the basketball player in question are in fact the same person:

(ii) a. This gymnast is unusually tall.
b. This basketball player is unusually tall.

[11] There are some predicates that can be attributed to tropes, but cannot apply with the same reading to the corresponding plurality. An example is *large*. *Large* when applied to a plurality has a very different reading than when applied to a number trope:

(i) a. The women are large.
b. The number of women is large.

in fact exactly similar if the property is natural. The similarity relations among tropes are reflected in the application of identity predicates in natural language. Even though John's anger is a distinct entity from Mary's anger, if they resemble each other sufficiently, then the identity predicate *is the same as* can apply to them:

(14) John's anger is the same as Mary's anger.

The same identity predicate applies to distinct quantitative tropes that are qualitatively (and thus quantitatively) exactly similar, that is, 'equal'. Thus, if John weighs as much as Mary and if there are as many women as men, the following identity statements are acceptable:

(15) a. John's weight is the same as Mary's.
b. The number of women is the same as the number of men.

Also the predicate *is identical to* can apply to qualitatively exactly similar but numerically distinct tropes:

(16) a. John's weight is identical to Mary's weight.
b. The number of women is identical to the number of men.

The *is* of identity is considerably worse however (in particular when it is focused):

(17) a. ?? John's anger is Mary's anger.
b. ?? John's weight is Mary's weight.
c. ?? The number of women is the number of men.

(17a), (17b) and (17c) just do not sound right. This indicates that the *is* of identity expresses in fact numerical identity, which thus confirms that the terms in question refer indeed to numerically distinct if very similar tropes.

1.5 The semantics of number trope terms

For the semantics of number trope terms, it is important to distinguish a relational noun *number* as it occurs in *the number of*-terms from the homophonic nonrelational noun as it occurs in explicit number-referring terms such as *the number eight*.[12]

[12] The distinction is reflected in some languages, for example German. In German, *Zahl* is used relationally or non-relationally, whereas *Anzahl* is used only relationally (though, in fact, non-relational uses such as *die Anzahlen* 'the numbers'—which to my ears are entirely deviant—are found in Frege (1884)). This means in particular that *Anzahl* can occur only in the *the number of*-term construction and does not allow for the plural:

(i) a. Die Zahl/Anzahl der Planeten ist acht.
'The number of the planets is eight'.
b. die Zahl/* Anzahl acht
'the number/number eight'
c. diese Zahl/* diese Anzahl
'this number/number'
d. Er addierte die Zahlen/* die Anzahlen.
'He added the numbers/the numbers'.

There are two fundamentally different approaches to plurals, which go along with two different ways of conceiving of number tropes. On the first approach, plurals are taken to refer to single entities which are collections of some sort, for example mereological sums or sets. Let me call this *the reference-to-a-plurality view*. The alternative to taking pluralities to be single objects of reference is to analyse plural terms as referring plurally to various individuals at once (Booles 1984; Yi 1998, 1999, 2006; Oliver and Smiley 2004). Let me call this *the plural-reference view*. On the plural-reference view, *the children* plurally refers to the various individual children at once.

On the reference-to-a plurality view, there are no entities that are 'pluralities'; thus when using the term 'plurality' in discussing that view, I will simply mean all the referents of a plurally referring term.

What is the semantic status of numerals like *two*? *Two* can occur as what looks like a quantifier as in *two children* as well as in the function of an adjectival modifier as in *the two children*. In the latter case, *two* obviously expresses a property of pluralities, and this property can also be used to analyse *two* in the other function, by adding an existential quantifier ranging over pluralities of 'two'.

On the reference-to-a-plurality view, *two* will then be a predicate that applies to single collective objects, whereas on the plural-reference view, it will be a predicate that applies to several individuals at once to give a truth value. In the latter case, *two* is true of several individuals just in case among them are two individuals such that each of the several is identical with one of the two[13]:

(18) For individuals ww,
$two(ww) = 1 \text{ iff } \exists x \exists y(x \leq ww \ \& \ y \leq ww \ \& \ x \neq y \rightarrow \forall z(z \leq ww \rightarrow z = x \vee z = y))$.

In (18), 'ww' is a plural variable, that is, a variable that can stand for more than one individual at once, $\leq$ is the relation 'is one of', and 'x' and 'y' are singular variables, variables that can stand for only single individuals.

The two views of plural terms, reference to a plurality and plural reference, correspond to two different ways of conceiving of number tropes. On the first conception, number tropes have a single collective entity as their bearer and are instances of properties of collective entities. On such a view, *the number of planets* stands for the instantiation of the property of having eight members in the collection that consists of all the planets. On the second conception, number tropes have one or more bearers and instantiate a property that can hold of several individuals at once, that is, a plural property. On that view, *the number of planets* stands for the instantiation of the plural property that holds of eight and only of eight individuals at once.

[13] Bigelow (1988) takes numerals to express relations of distinctness. Thus *two* expresses the relation that holds of two entities *x* and *y* in case *x* and *y* are distinct. The most plausible version of such an approach would take numerals to be multigrade predicates (since a numeral like *two* can also take three distinct arguments of which it would then be false). The multigrade predicate view as such is discussed at length in Oliver and Smiley (2004). In general there are two criteria for taking a predicate to be multigrade rather than a plural predicate (a predicate that can be true of several individuals at one). If the order of the arguments matters and if an individual can occur as an argument more than once, then the predicate must be multigrade rather than a plural predicate. This is obviously not the case for numerals, and thus numerals should better be regarded plural predicates. For further discussion of the two views and a defence of the plural-predicate view of numerals, see Yi (1998).

While the reference-to-a-plurality view is the more common one in linguistic semantics, the plural-reference view has been pursued more recently by a number of philosophers and logicians, and it is the view I will adopt. There are various arguments against the reference-to-a-plurality view which I will not go into.[14] I want to mention just one problem for the view because it manifests itself also with number trope terms. The problem is that if a plural term were to refer to a single collective entity, the term should be replaceable by a singular term referring to just that entity. But with many predicates such a replacement is impossible, resulting either in unacceptability or in a different reading of the predicate, as seen in (19) and (20):

(19) a. John compared his children.
b. ?? John compared the set (sum / collection) of his children.
(20) a. John counted his children.
b. ?? John counted the set (sum / collection) of his children.

Unlike with standard 'category mistakes' or semantic selectional requirements, no semantic type shift or coercion can make (19b) and (20b) acceptable with the reading that is available in (19a) and (20a). On the plural-reference view, *his children* and *the set of children* differ fundamentally in their semantic function: *his children* refers to several children at once, whereas *the set of his children* refers to a single entity. This substitution problem manifests itself in just the same way with terms referring to number tropes. Thus replacing the plural in (21a) by a singular collective NP as in (21b) results in unacceptability:

(21) a. the number of children
b. ?? the number of the set (sum / collection) of children

Number trope terms are formed with the unspecific functional relational noun *number*. *Number* in trope-referring terms expresses a plural function, a function which maps *n* individuals simultaneously to the trope that is the instantiation of the property of being *n* in those individuals:

(22) For entities dd, *number*(dd) = f(P, dd) for some number property P such that P(dd).

Here 'f' stands for the function mapping a property and an individual or several individuals to the instantiation of the property in the individual or the individuals (in case the individual(s) instantiate(s) the property; it will be undefined otherwise).

In some languages, 'the number of' is followed by a definite plural (or a specific indefinite), for example in German (*die Anzahl der Planeten / * von Planeten* 'the number of the planets / of planets'). In English, though, *the number of* is not followed by a standard plural term, that is, a definite plural NP, but rather by a bare (that is, determiner-less) plural. While there are different views about the semantic function of bare plurals, it is generally agreed that bare plurals can act as kind-referring terms (as in *lions are rare*) (Carlson 1977). Kind terms in fact might themselves be considered plurally referring terms, referring plurally to the various

[14] See for example Booles (1984) and Yi (2005, 2006) for discussion.

instances in the various possible circumstances at once. Some predicates such as *rare* will then take into account actual and possible instances, while in other contexts (such as that of the functor *the number of* or the predicate *kill*) only actual instances will matter (that is, the same entities that a definite plural noun phrase such as *the lions* would refer to).

The semantics of *the number of planets* can thus be given as follows, where $[]^{w,i}$ is the restriction of the plurally referring term *planets* to the actual circumstances, the actual world *w*, and the present time *i*:

(23) $[\textit{the number of planets}]^{w,i} = [\textit{number}](P, [\textit{planets}]^{w,i})$, for some number property P that holds of $[\textit{planets}]^{w,i}$.

1.6 Functional number trope terms

Terms that can refer to tropes in general have an alternative use as terms referring to function-like entities, entities that have manifestations as possibly different tropes in different circumstances. This can be enforced by using certain kinds of predicates, as below:

(24) a. The beauty of the landscape has changed.
b. The amount of corruption in this administration has become more noticeable.

Change in (24a) compares tropes that have the landscape as bearer and are manifestations of 'the beauty of the landscape' at different times. Similarly, *become more noticeable* in (24b) compares quantitative tropes to each other, with respect to the extent to which they are noticeable.

In just the same way, *the number of*-terms may refer to entities that have variable manifestations as number tropes, a reading that can be enforced by suitable predicates or noun modifiers[15]:

(25) a. The number of students has changed.
b. the increasing number of students

Clearly, the functional use of *the number of*-terms is not a replacement for trope reference, but rather involves itself the notion of a number trope. The trope manifestations involved in the functional use of trope terms behave just like tropes with respect to relevant predicates, accepting *high* and *exceed* and allowing for the application of causal predicates:

[15] Functional uses of NPs are not generally clearly distinguishable from referential uses. Functional and referential uses can be involved in antecedent-anaphora relations, as in (i), and one and the same term appears to be used both referentially and functionally in cases of co-predication, as in (ii):

(i) a. The number of students is high. It had increased a lot over the last years.
b. The president of the US is a democrat. He is not always a democrat of course.
(ii) a. The number of students, which has increased a lot in the last few years, is very high.
b. The president of the US, who is elected every four years, is currently a democrat.

In general, neither anaphora nor co-predication requires strict identity of an object, as is familiar from theories of discourse and lexical semantics (see, for example, Pustejovsky 1991).

(26) a. The number of students is getting higher and higher.
b. The number of teachers sometimes exceeds the number of students.
c. The increasing number of students causes problems for the current availability of class rooms.

The functional use of trope terms is in fact just a case of the functional use of functional descriptive NPs more generally. Thus, *the president of the US* can be used to refer to a function-like entity that manifests itself as different individuals at different times (in sentences such as *the president of the US is elected every four years*).

There are two possible views regarding the ontological status of the semantic values of such functionally used terms. First, following Montague (1973), they may be viewed as higher-level semantic values, intensions of the non-functional use of the term or individual concepts, and thus in the case of (24a, b) and (25a, b) functions from times (and perhaps worlds) to tropes.[16] Alternatively, they may be viewed as entities *sui generis*, entities that manifest themselves as possibly different tropes at different times, just as individuals have different material manifestations at different times.

There is evidence in favour of the latter view, and that is the possibility of forming terms for tropes whose bearers are just what functionally used terms stand for:

(27) a. the constancy of the number of students
b. the impact of the increasing number of students

As bearers of tropes, the semantic values of functionally used terms must have objectual status, as entities with variable manifestations in different circumstances.

1.7 Intensional number trope terms

There is yet another type of *the number of*-term that needs to be mentioned. It involves an intensional verb and appears to refer to a trope that fails to have a particular actual bearer:

(28) a. the number of people that fit into the car
b. the number of books John has to write
c. the number of assistants John needs
d. the number of screws that are missing

Let me call such terms intensional *the number of*-terms, and distinguish them from extensional *the number of*-terms, the terms discussed so far, which involved extensional verbs.

One might think that *the number of* in these terms acts just as a numeral replacement. This appears the case in the examples below (where Mary counted the screws, not a single trope):

[16] Montague (1973) took *the temperature* in *the temperature is rising* to stand for a function from times to numbers (which on the present view would rather be temperature tropes).

(29) Mary counted the number of screws that are missing.

But intensional *the number of*-terms can also be used to refer to tropes. This is supported by the same evidence for *the number of*-terms with extensional verbs acting as trope-referring terms.

First, predicates of neutral comparison are applicable to intensional *the number of*-terms in just the same way as to extensional *the number of*-terms, and like the latter they alternate with plurals and the modifier *in number* (though now the plural will have an intensional restriction):

(30) a. The number of people that fit into the bus exceeds / equals the number of people that fit into the car.
b. The people that fit into the bus exceed / equal the people that fit into the car in number.

(31) a. John compared the number of books Mary wants to write to the number of books Sue wants to write.
b. John compared the books Mary wants to write to the books Sue wants to write in number.

Also one-place predicates of quantitative evaluation are applicable with intensional *the number of*-terms:

(32) a. The number of people that fit into the bus is high.
b. The number of screws that are missing is negligible.

Even predicates of perception and causation can apply with such terms:

(33) a. Mary noticed the number of screws that are missing.
b. The number of people that would fit into the car astonished Mary.
c. The number of screws that were missing caused the door to fall off.

Finally, predicates of similarity and numerical identity apply just as they do with extensional *the number of*-terms:

(34) a. The number of women in the room is the same as the number of men in the room.
b. ?? The number of women in the room is the number of men in the room.

(35) a. The number of books Mary wants to write is the same as the number of books Sue wants to write
b. ?? The number of books Mary wants to write is the number of books Sue wants to write.

That *the number of*-terms with intensional verbs can refer to tropes is also plausible in view of the fact that there are corresponding intensional terms for other quantitative tropes and for qualitative tropes:

(36) a. the height of the desk John needs
b. the length of the time John might be away

(37) a. the originality of the book John wants to write
b. the simplicity of the dress Mary needs for the occasion

The predicates in the examples below confirm the ability of such terms to refer to tropes:

(38) a. Mary was astonished at the length of the time John might be away.
b. The height of the desk John needs exceeds by far the height of the desk John is using right now.

(39) The elegance of the dress that the bridesmaid needs should not exceed the elegance of the dress that the bride will wear.

Note that in (38b) and (39) the predicate compares a trope described by an intensional *the number of*-term to a trope described by an extensional *the number of*-term.

Intensional *the number of*-terms also permit a functional use:

(40) a. The number of books John wants to write is constantly changing.
b. The number of books that we need is increasing every day.
c. The number of screws that are missing becomes more and more noticeable.

But how can intensional terms refer to tropes when there is no actual bearer? Note that the bearer could not be an intentional object (or a collection of them). Intentional objects may be the arguments of verbs like *think about, plan,* or *want*, but no intentionality is involved with predicates like *is missing* or *need*. Intensional verbs arguably take intensional quantifiers arguments (Montague 1973; Moltmann 1997), not intentional 'objects of thought'.

The ability of intensional verbs to characterize the bearer of a trope can be related to their ability to form definite descriptions of the sort below (Moltmann 2008):

(41) a. The book John has to write will have to be 200 pages long.
b. The desk John needs does not have to be very high.

Such NPs appear to make reference to individual concepts, partial functions from circumstances (worlds, situations, times) to individuals or pluralities of individuals, or perhaps better 'functional objects' correlated with such individual concepts. Thus in (41a), this would be the function mapping any world in which John does what he needs to do to the entity that is a book John writes in that world. The entities such definite descriptions take as their semantic values appear to act precisely as the bearers of the tropes described by intensional *the number of*-terms. They will thus have objectual status, with the trope representing an actual feature of such functional objects.

1.8 Mathematical operations on number tropes

Number tropes have not only the kinds of properties that are characteristic of tropes in general. They also have certain kinds of mathematical properties, even though they do not display the full range of mathematical behavior that pure numbers display. I will argue that the more limited range of mathematical properties of number tropes (in contrast to pure numbers) follows from the nature of number tropes, as tropes that have pluralities as bearers.[17]

Let us first look at predicates that classify numbers according to their mathematical properties. Predicates such as *even, uneven, finite,* and *infinite* are possible not only with pure numbers but also with number tropes:

(42) a. Mary was puzzled by the uneven / even number of guests.
b. Given the only finite number of possibilities,…
c. John pointed out the infinite number of possibilities.

There are other predicates, however, that are acceptable only with pure numbers, and not with number tropes. They include *natural, rational*, and *real*:

(43) ??? the natural / rational / real number of women

Furthermore, many mathematical functions are inapplicable to number tropes. These include one-place functions such as the successor, predecessor, root, and exponent functions as well as some two-place functions such as 'smallest common denominator':

(44) a. ??? the successor / predecessor / root / exponent of the number of planets
b. ??? the smallest common denominator of the number of men and the number of women

By contrast, the two-place functors *sum* and *plus* are applicable to number tropes:

(45) a. the sum of the number of men and the number of women
b. The number of children plus the number of adults is more than a hundred.

What distinguishes the mathematical predicates or functors that are applicable to number tropes from those that are not? To answer this question, it suffices to think of the kinds of mathematical properties pluralities of objects can have and the kinds of operations that can apply to them.

First of all, there is a sense in which pluralities can be even or uneven: to see whether a plurality is even or uneven, it just needs to be checked whether or not the plurality can be divided into two equal subpluralities. Similarly, in order to see whether a plurality is finite or infinite it simply needs to be seen whether or not a 1–1 mapping can be established from the members of the plurality onto the members of a proper sub-plurality of the plurality. A number trope will then be even, uneven, finite, or infinite simply because the plurality that is its bearer is. We can then state the following generalization: a mathematical predicate is applicable to one or more number tropes just in case its application conditions can be formulated in terms of hypothetical operations on the pluralities that are the bearers of the number tropes.

[17] I have become aware that the intuitions discussed in this section do not hold in the same way for speakers accustomed to the way *the number of* is used in parts of mathematics. As an anonymous referee has pointed out, in a number of areas of mathematics, such as elementary cominatorics, 'the number of Xs', is explicitly defined as a pure number, thus not as a number trope. It is likely that that use influences the way the data in this section are judged.

Such a condition also explains the applicability of the functor *sum*. Addition is applicable to two number tropes because it can be defined in terms of an operation on the two pluralities that are the bearers of those number tropes[18]:

(46) Addition of Number Tropes
For two number tropes t and t′, sum(t, t′) = f(P, dd) for individuals dd such that $\forall d(d \leq dd \leftrightarrow d \leq ee \vee d \leq ff)$ and some number property P such that P(dd), provided that $\neg\exists d(d \leq ee \ \& \ d \leq ff)$ for individuals ee such that t = f(P_1, ee) and individuals ff such that t′ = f(P_2, ff), for number properties P_1 and P_2.

Why isn't the successor function applicable to number tropes? The reason is simply that the successor function cannot be viewed as an operation on concrete pluralities. The successor function as a function applying to a concrete plurality would require adding an entity to the plurality. However, given a 'normal' universe, there is not just one single object that could be added, but rather there are many choices as to what object could be added to the plurality to yield its successor. Thus, no uniqueness is guaranteed, which means as an operation on pluralities, the successor function is just not a function. Similar considerations rule out the predecessor, root, and exponent functions as operations on number tropes.

Thus, arithmetical properties and operations regarding number tropes must satisfy the following conditions:

(47) Condition on arithmetical properties of and functions on number tropes

a. If P is an n-place arithmetical property of number tropes, then for some n-place property Q of pluralities, for any number tropes $t_1,\ldots, t_n$: $P(t_1,\ldots, t_n)$ iff $Q(pp_1,\ldots, pp_n)$ for the bearers $pp_1,\ldots, pp_n$ of $t_1,\ldots, t_n$.[19]

b. If f is an n-place function on number tropes, then for some n-place function on pluralities g, for any number tropes $t_1,\ldots,t_n$: $g(pp_1,\ldots,pp_n) = f(t_1,\ldots,t_n)$ for the bearers $pp_1,\ldots,pp_n$ of $t_1,\ldots,t_n$.

Again, pp_1, pp_2,… are plural variables standing for several objects at once.

What about the predicates *natural*, *rational*, and *real*? These are technical predicates that already at the outset are defined just for the domain of all numbers,

[18] (46) presupposes that the bearers of two number tropes to which addition applies are non-overlapping pluralities. This may not seem entirely adequate since it seems not impossible to add 'the number of students' to 'the number of girls', with the students including some of the girls. What happens in this case, I suggest, is the use of a certain operation that is available to facilitate the application of arithmetical operations to number tropes. This is an operation of copying which creates hypothetical tropes that are distinct from, though quantitatively equivalent to a given actual trope. That is, in a case in which tropes with overlapping bearers are added, addition actually applies to hypothetical copies of the tropes. I will return to this operation of copying again later.

[19] 'Property' in (47a) must be understood in a sufficiently restricted, intensional sense: if there is a corresponding property for every set of sets, (47a) would be trivial. It would be fulfilled by any property of number tropes, since there would always a plural property that holds just of the bearers of a trope of which a number trope property is true.

rather than only the natural numbers. They will therefore not be applicable to number tropes, which are outside the domain of their application.[20]

The possibility of some mathematical properties and functions being applicable to number tropes on the basis of operations on concrete pluralities is also reflected in the acceptability of descriptions of agent-related mathematical operations on number tropes:

(48) a. John added the number of children to the number of adults, and found there were too many people to fit into the bus.
b. John subtracted the number of children from the number of invited guests.

Addition as a mathematical operation performed by an agent, as in (48a), is possible with number tropes for the same reason as addition as a mathematical function. What matters is that the operation as an operation on number tropes is definable in terms of an operation on the underlying pluralities. This does not necessarily mean that when John added the number of children to the number of adults he first mentally put together the plurality of children with the plurality of adults and then counted the result. But it means that if he obtained the correct result, he might as well have obtained it by performing an operation on the corresponding pluralities first.

Subtraction of a number trope t from a number trope t' as in (48b) is possible in case the plurality that is the bearer of t' includes the plurality that is the bearer of t; otherwise it is hard to get, for example in (49a):

(49) a. ?? John subtracted the number of planets from the number of invited guests.

It is possible to make sense of (49a), though, on a reading more naturally available in a case like (49b):

(49) b. John subtracted the number of passports from the number of applicants.

The reason why (49b) is possible is that it presupposes a natural 1–1 association between passports and applicants. Subtraction can then be viewed as an operation on pluralities as well: start with the applicants, associate them with their passports, take away the passports together with their associated applicants, and the number of the remaining applicants will the result of the subtraction.

Division of one number trope by another, as in (50a), is strange too:

(50) a. ?? John divided the number of invited guests by the number of planets.

Division is possible, though, when the second term is a numeral, as in (50b), but not when the first term is a numeral, as in (50c) (unless the sentence is understood with

[20] In fact, it is not clear that predicates such as *natural* in *natural number* have a meaning that is independent of the content of *number*, since such predicates can occur only as noun modifiers and not as predicates:

(i) * This number is natural.

This means that *natural* with the relevant meaning could not apply to entities other than numbers anyway.

respect to a particular context in which the numeral already presents a concrete plurality, let's say a plurality of tables):

(50) b. John divided the number of invited guests by two.
c. ?? John divided eighteen by the number of invited guests.

Divide by two is a complex predicate that involves an arithmetical operation definable as an operation on a plurality. By contrast *divide eighteen by* is not such a predicate: eighteen is not associated with a particular plurality that a division could target, and the plurality that is the bearer of a number trope is not something by which it could be divided.

As with subtraction, with some effort it is in fact possible to get a reading of a sentence like (50a), a reading that is easier to get in the case of (51):

(51) John divided the number of invited guests by the number of tables.

(51) is possible, obviously, because the circumstances may invite an association of the same number of guests with each table. In such circumstances, John's mathematical operation naturally goes along with an operation on the underlying pluralities (namely associating each table with a number of guests that allow a 1–1 mapping onto the guests of any other table). Thus, again, division is possible because it corresponds naturally to an operation on concrete pluralities.

Multiplication with number tropes is available as well, in certain circumstances. (52a) is a simple case:

(52) a. John doubled the number of invited guests.

(52a) describes an operation on a number trope that obviously goes along with an operation on the corresponding plurality. In (52a), John's act of 'doubling' consists in the replacement of one number trope (the number of invited guests at a time *t*) by another (the number of invited guests at a t'), for example by adding as many names as there already are on the list of invited guests.

(52b) is a slightly more complicated case:

(52) b. Three times the number of children can fit into the bus.

(52b) does not just describe a mathematical operation of multiplying the number of children by three, but involves a comparison of the actual number of children with the maximal number of children that fit into the bus, that is, of the actual number of children with a hypothetical number trope with three times as many children as bearers.

How is multiplication of a particular number trope by three achieved such that it results again in a particular, if hypothetical, number trope? It appears that there is a particular way in which this is done in the context of natural language and which is indicated by the actual linguistic form for expressing number trope multiplication. With multiplication applied to number tropes, the occurrence of the expression *times* is crucial: in (52b), *times* first applies to the number trope term *the number of children*; then the numeral *three* applies to the expression *times the number of children*. The expression *times* in general, when applying to trope-denoting terms, appears to 'generate' quantitative copies of tropes. That is, it maps a trope *t* (the 'standard') to copies of *t* that are quantitatively equivalent to *t*. This is illustrated nicely also by (53), with a qualitative trope:

(53) John has three times the strength of Mary.

In (53), *times* maps a trope *t* that is the trope of Mary's strength to 'concatenations' of quantitative copies of *t*. (53) thus says that John is the bearer of a strength trope that 'measures' three quantitative copies of Mary's strength.

Multiplication of tropes thus proceeds by adding quantitative copies of tropes, an operation that is itself indicated by the expression *times*. There are two kinds of quantitative copies of tropes: quantitatively equivalent copies of a qualitative trope, as in (53), and copies of a number trope (which is inherently quantitative).

The use of *times* for the purpose of multiplying number tropes gives further support for the view that *the number of*-terms do in fact refer to number tropes: number tropes require an additional operation of abstracting measuring units (quantitative copies of a given trope) in order to allow for multiplication.[21]

Given that arithmetical operations are possible with number tropes only if they can be defined as operations on the underlying pluralities, it is expected that 'mixed operations' involving both number tropes and pure numbers are excluded. This is indeed the case:

(54) a. ??? John subtracted the number ten from the number of children.
b. ??? John added the number twenty to the number of children.

The reason obviously is that pure numbers have no pluralities as bearers on which corresponding operations could be performed.

Number tropes can have only those mathematical properties that are derivative of operations on the underlying pluralities. In addition, number tropes have empirical properties tied to the particular nature of their bearers, properties that pure numbers do not have. The difference in the range of properties that number tropes and pure numbers may have also shows up in the way general property-related expressions are understood with number trope terms and with explicit number-referring terms. Such expressions include *investigate*, *property*, and *behaviour*:

[21] There is in principle a different way in which multiplication could be defined on the basis of particular pluralities, namely by making use of quantification over higher-order pluralities. Mayberry (2000), who also pursues a reduction of arithmetical operations to operations on pluralities, proposes hat '$2 \times 3 = 6$' be analysed as 'two (non-overlapping) pluralities of three (distinct) entities is six entities'. On the present account, quantification over pluralities is avoided because *two* in *two times the number of* N is taken to range over number tropes that 'measure' two copies of the number trope 'the number of N'. Measurement of tropes thus replaces higher-order plural quantification. Natural language appears to give evidence for that way of doing multiplication rather than the way Mayberry suggests.

More generally, even though the trope-copying operation appears rather 'theoretical', it gains significant plausibility from the fact that it appears to correspond to a natural language construction. There should in fact be a general constraint on what one might posit as 'theoretical' operations on number tropes. Certainly, in the context of 'natural language ontology', only those operations should be posited for which some form of evidence can be obtained from natural language itself.

(55) a. John investigated the number 888.
b. John investigated the number of women.

(56) a. the properties / behaviour of the number 888
b. the properties / behaviour of the number of women

Whereas (55a) can only mean that John investigated the mathematical properties of 888, (55b) implies that John's investigation was also an empirical one regarding the women in question, namely how many women there actually were. Similarly, whereas (56a) can only refer to the mathematical properties or the mathematical behaviour of a number, (56b) also refers to nonmathematical, empirical properties of the plurality of women.

Reference to number tropes with their particular arithmetical behavior thus plays a rather important role in natural language. Number tropes hardly figure in contemporary mathematics or philosophy of mathematics; but they recall pre-twentieth-century mathematical writing about quantities (Schubrig 2005). Moreover, if Mayberry (2000, Chap. 2) is correct, then number tropes also correspond well to ancient views of numbers, in particular that of Aristotle (Metaphysics M). For Aristotle, numbers were not special abstract objects, but pluralities viewed 'in the manner appropriate to mathematicians', pluralities consisting of normal objects but viewed only 'qua single countable things'.

Number tropes are particulars that approximate natural numbers. In fact, there is a 1–1 correspondence between natural numbers and classes of number tropes, namely classes of exactly similar number tropes. Thus, 2 corresponds to the class of number tropes instantiating the property of being two. While number tropes are abstractions from pluralities of particulars, the result is still a particular, but one that establishes a new resemblance class of particulars matching a natural number.

One might then ask whether natural language allows for reference to such a resemblance class as a way of referring to a pure number. A class of exactly similar number tropes would hardly be a plausible object of reference for a natural language term, though a kind, that is, a kind of number trope, would. However, while natural language does allow reference to kinds (with terms like *lions*) and kinds of tropes in particular (with terms like *wisdom* (Moltmann 2003b)), at least English does not seem to have a device for referring to a kind of number trope.

The same holds for numbers viewed as kinds of pluralities of ordinary objects. Numerals might be candidates for terms referring to kinds of pluralities, so that *two* would refer to the kind whose instances are all the pluralities of two things (as proposed in Mayberry 2000). However, numerals just do not act like kind-referring terms, that is, terms like *lions* or *wisdom*. This is illustrated in (57a, b):

(57) a. ?? Two is rare.
b. ?? John likes two.

(57a) cannot mean 'pluralities of two things are rare', and (57b) cannot mean 'John likes any plurality of two things' (*two* here is rather followed by a silent discourse-related anaphoric noun as in 'speaking of lions, John likes two').

There is another kind of abstraction conceivable by which a natural number could be obtained from a plurality of individuals. It would consist in abstraction from the particularity of the individuals making up the plurality so that the result would be a plurality of 'pure units'. This would be Cantorian abstraction (see Fine 1998 for a reconstruction and discussion of this notion). Setting aside the problems and limits of Cantorian abstraction itself, one might ask the question whether natural language makes use of Cantorian abstraction to refer to pure numbers. Given Fine's reconstruction, this would mean that we should find reference to number tropes with 'arbitrary' pluralities as bearers. Natural language does not seem to provide any such construction, though, that would make reference to a pure number. The closest one might get is a construction like *the number of any three children* (where *any three children* should stand for an arbitrary plurality of three objects), but this does not get close to what *the number three* refers to. We will see later that the strategy that natural language uses to obtain pure numbers as objects of reference is rather different. This strategy involves a quasi-referential use of numerals, namely in explicit number-referring terms such as *the number eight*.

1.9 *The number of*-terms as concealed questions and as subjects of specificational sentences

As mentioned, *the number of*-terms can also act as concealed questions. A concealed question is a definite noun phrase that rather than denoting its usual referent, stands for a closely related question (Grimshaw 1997). A standard example of a concealed-question use of a definite NP is that in (58a); a concealed question use of a *the number of*-term is given in (58b):

(58) a. I do not know John's address.
b. I do not know the number of women.

In (58a), *John's address* stands for a question of the sort 'where does John live?' and in (58b) *the number of women* for a question of the sort 'how many women are there?'—a question which has as its answer a sentence like 'there are *n* women' for a numeral *n*.

With the notion of a concealed question, we can address the problem of apparent identity statements like (1), repeated below:

(59) a. The number of planets is eight.

I would like to show that (59a) is neither an identity statement nor a subject-predicate sentence, but rather is of a third sort, namely what linguists call a *specificational sentence*.

A first indication that (59a) is not a statement of identity is that substituting the simple numeral *eight* in (59a) by an explicit number-referring term results in a sentence that is much less acceptable, or rather a sentence that would in fact not be true[22]:

[22] This is despite Frege's own claim to the contrary (Frege 1884). The example is equally unacceptable in German. In fact, also Frege's other German example below, in which the numeral occurs with a definite determiner, is unacceptable to my ears:

(i) ?? Die Anzahl der Planeten ist die Acht.
'The number of planets is the eight.'

(59) b. ?? The number of planets is the number eight.

(59c) and (59d) are equally unacceptable:

(59) c. ?? Which number is the number of planets?
d. ?? The number of planets is the same number as eight.

In case (59a–d) are not as such convincing (intuitions are arguably tainted by the Fregean tradition itself), there is further linguistic evidence (from German) that (59a) is not an identity statement. I will come to that shortly.

A specificational sentence is a copula sentence characterized by a range of syntactic particularities (Higgins 1973; Heycock and Kroch 1990). Specificational sentences come in two forms: with a wh-phrase and with a definite NP in subject position as in (60a, b) and (60c, d) respectively:

(60) a. What John did is kiss himself.
b. What every man is is proud of his son.
c. John's weight is a hundred kilo.
d. The best player is John.

Crucially, in specificational sentences, the expression in postcopula position need not be a referential NP, as in (60a), (60b), and, arguably, (60c) and (60d). On one common view about the semantics of specificational sentences, specificational sentences express a relation between a question and an (elliptical) answer (den Dikken et al. 2000; Schlenker 2003; Romero 2005).[23] On that view, the subject in (60a) expresses the question 'what did John do?' and the partial answer 'kiss himself' is given by the postcopula expression. A definite NP in subject position, as in (60c), would express a question in a more indirect way (let's say, in (60c), a question of the sort 'how much does John weigh?'), thus acting as a *concealed question*.

(59a) clearly can qualify as a specificational sentence, and in fact Higgins (1973), one of the earliest major works on specificational sentences, classifies sentences like (59a) as sentences of this sort.[24] Given the question–answer analysis of specificational sentences, *the number of planets* in (59a) would then not be a referential term standing for an object, but rather would in this context stand for the question 'how many planets are there?' or perhaps for the question 'how many are the planets?' In

[23] There is an alternative view pursued in the literature, using higher-order equations (Jacobson 1994; Sharvit 1999). On that view, specificational sentences express an identity of semantic values, possibly of higher type if nonreferential expressions are involved.

[24] See Higgins (1973, p. 199). Higgins is worth citing in the present context: 'Many philosophers have tended to treat sentences of the specificational variety as if they were identity sentences, and have then proceeded to build theories that rest on shaky foundations. The most impressive of such a misconstrual, which has spawned an enormous literature, is the following sentences: (1) The number of planets is nine' (Higgins 1973, p. 199). Higgins then remarks '[…] it is doubtful whether *the number of planets* has any […] referential use at all—it seems rather to be akin to nouns such as *defect* and to have at most a kind of obscure referentiality associated with indirect questions' (Higgins 1973, p. 200). Higgins then goes on citing substitution issues as a crucial argument against the view of (1) as an identity statements, such as the non-equivalence of sentences such 'I counted up to nine' with (the unacceptable) 'I counted up to the number nine', or 'nine is the square root of eighty-one' with (the unacceptable) 'the number of planets is the square root of the number of eighty-one'.

both forms of answers, *eight* can be taken to express a plural property (so that the two answers would have the logical forms '∃xx(eight(xx) & planets(xx))' and 'eight([*the planets*])' respectively). Also *eight* in (59a) would not be a referential term standing for an object, but rather it would be elliptical for an answer of the sort 'there are eight planets', or perhaps 'The planets are eight'.[25, 26]

The possibility of using a *the number of*-term as a concealed question in the subject position of a specificational sentence explains a potential problem for the view that *the number of*-terms stand for tropes, namely the possibility that the number in question may be zero, as in the following example:

(61) I am surprised at the number of women. It was zero.

Surprise, like *know*, takes questions rather than tropes as arguments; and a question such as 'how many women were there?' can of course have as its answer 'zero'. The second sentence in (61) is a specificational sentence whose subject relates anaphorically to a concealed question and *zero* here specifies the answer to a question (and not a number in an identity statement).

The next question to address then is, how is the NP *the number of planets* able to denote a question? It is not obvious how this is possible if *the number of planets* is a term whose literal denotation is a number trope. Recall, however, that *the number of* can also have the function of a numeral replacement. With *the number of* acting as a numeral replacement, a concealed-question analysis becomes straightforwardly available. *The number of* in *the number of planets* in (59a) will then the same semantic function as *how many*, thus naturally allowing for a question-denotation of the right type. (Of course *the number of* and *how many* have different syntactic functions, the former serving to form a complex definite NP, the latter a wh-clause.)

Not only can (59a) be regarded as a specificational sentence, but it appears that it *must* be a specificational sentence. One piece of evidence comes from the choice of pronouns in the subject position of specificational sentences in languages that require formal gender agreement for pronouns, such as German. Specificational sentences in English may contain the neutral pronouns *that* and *it* in subject

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[25] If specificational sentences are not analysed as question–answer structures, but as involving higher-order equations (footnote 23), the numeral in postcopula position would come out as nonreferential only as long the subject is taken to be non-referential. The latter is hard to maintain for the case of (59a), though; that is, it is hard to maintain that *the number of planets* is of the semantic type of a numeral.

[26] Brogaard (2007) argues against treating the sentence 'The number of moons of Jupiter is four' as a specificational sentence. She argues that analysing such sentences as specificational sentences by deriving them from sentences like 'Jupiter has four moons' or by considering them as expressing question–answer pairs is problematic in general since such analyses could not in fact account for what they originally aimed to account for, namely connectivity, a characteristic feature of specificational sentences (Higgins 1973). Connectivity consists, for example, in the unusual binding of the anaphor *himself* by *John* in *The person John admires most is himself*, where *himself* is not c-commanded by *John*. Brogaard argues that the failure of a syntactic treatment of connectivity (and certain other wrong predictions that such treatments would make) requires viewing The number of moons of Jupiter is four 'as is', that is, as an identity statement involving two referential terms, and thus treating numbers as objects. The problem with Brogaard's argument is that she disregards the various other criteria for specificational sentences besides connectivity that have been established in the literature since Higgins (1973). What is crucial about specificational sentences is that neither the subject nor the postcopula NP need to be referential, whatever the right syntactic or semantic treatment of such sentences may turn out to be.

position, for example when they are anaphoric to a preceding question (Mikkelsen 2004):

(62) What is the biggest problem? *That* / *It* certainly is John.

Pronouns that are not neutral such as *he* or *she* are excluded as subjects of specificational sentences (Mikkelsen 2004). Similarly, in German, pronouns in the subject position of specificational sentences can only be the neutral pronouns *das* 'that' or *es* 'it', not, for example, a feminine pronoun such as *sie* 'she'. German *die Zahl der Planeten* 'the number of planets' is feminine, but the only pronoun that can replace it is *es* (neutral) or (more colloquial) *das*, as seen in (63a, b), unlike in ordinary identity statements where a feminine pronoun is required, as in (63c):

(63) a. Die Zahl der Planeten ist acht. Frueher dachte man, es waeren neun.
'The number of planets is eight. Before it was thought that it was (plural) nine'.
b. * Die Zahl der Planeten ist acht. Frueher dachte man, sie waere neun.
'The number of planets (fem) is eight. Before it was thought that she was nine.'
c. Maria ist nicht Susanne, ? sie / *es ist Anna.
'Mary is not Sue, she / * it is Ann.'

One potential alternative analysis of (59a) to that of an identity statement is as a subject-predicate sentence, with the subject referring to a trope and the numeral acting as a predicate of tropes. *Eight* would then be true of a trope *t* in case *t* is the instantiation of being eight in some plurality. One problem with such an analysis is that subject-predicate sentences generally do not allow for inversion, as seen in (64) (Heycock and Kroch 1990). By contrast, (59a) does, as does a specificational sentence in general, as seen in (65):

(64) a. John is honest.
b. * Honest is John.
(65) a. Eight is the number of planets.
b. Kiss Mary is what John did.

Hofweber (2005a) gives a very different account of (59a), which also tries to avoid taking numbers to be objects of reference. Hofweber argues that (59a) is derived from (66) by a syntactic operation of focus movement, just as (67a) would be derived from (67b):

(66) There are eight planets.
(67) a. The number of bagels I have is five.
b. I have five bagels.

More generally, Hofweber maintains that *the number of* generally has the function of a place-holder for a focus-moved numeral. There does not seem to be any independent syntactic evidence for the alleged focus-movement, however, and in fact the kind of movement required is very implausible: to derive (59a) from (66) would require inversion of *there* and *eight planets*, lowering of the numeral to postcopula position, and spelling out the remaining trace as *the number of*—all operations that would not be acceptable within current syntactic theory and have no parallel elsewhere. In fact, Hofweber does not give a syntactic justification or exact description of the required syntactic operations.[27] Another problem for Hofweber's account of (59a) is that *the number of*-terms can occur in many other contexts than specificational sentences, as we have seen—contexts in which *the number of* could not possibly act as a place-holder for a focus-moved numeral.

2 Numerals and the Adjectival Strategy

2.1 The Adjectival Strategy

In this section, I will turn somewhat briefly to simple numerals. I will present both theoretical and empirical support for the view that simple numerals retain their meaning as determiners or noun modifiers, rather than referring to abstract objects.

Simple numerals obviously occur primarily as determiners or modifiers of nouns, as in (68a), and only secondarily as (apparent) referential terms, as in (68b):

(68) a. Eight women were invited.
b. Eight is divisible by two.

Eight in (68b) has the (apparent) status of a referential term both because it occupies the subject position of a sentence and because it occurs with a predicate that ordinarily applies to the referent of a referential term, such as *the number eight*.

The common view about occurrences of numerals as in (68b) is that they are ordinary referential terms referring to pure numbers. Standard tests for referential terms used in the philosophical literature seem to support this view, such as Frege's criterion of being able to occur on both sides of the identity symbol, as in (69a, b), and the possibility of replacing the numeral by a quantifier such as *something*, as in (69c) (cf. Hale 1987)[28]:

(69) a. The sum of two and six is eight.
b. Eight is the sum of two and six.
c. If eight is divisible by two, then there is something that is divisible by two.

There is another view, though, of simple numerals, and that is that they retain their meaning of determiners or noun modifiers, namely as quantifiers or plural properties. Thus, Hofweber (2005a) takes numerals to always have quantifiers as their meaning.[29] Hofweber's motivation is to account for the linguistic fact that numerals occur both as determiners and as singular terms. He argues that in arithmetical statements like (70a) the numeral retains its meaning as a quantifier and that such sentences are generic sentences, with a meaning as in the paraphrase in (70b):

[27] See also Brogaard (2007) for further criticism of Hofweber's view.

[28] Hale's (1987) criterion involving quantifiers actually is considerably more complex.

[29] Hofweber (2005a) takes numerals to have a quantifier meaning in all contexts. This would not account for 'adjectival' occurrences though, as in *the eight planets*.

(70) a. Two and two is four.
b. Two things and two things are four things.

On Hofweber's account, for numerals to occur in arithmetical sentences they must undergo 'cognitive type shift', which means they retain their meaning as (generalized) quantifiers, but syntactically they become singular terms, for the purpose of facilitating arithmetical calculation.

The view that numerals do not semantically act as singular terms referring to numbers has also been explored for purely philosophical reasons by philosophers of mathematics such as Bostock (1974), Gottlieb (1980), and Hodes (1984, 1990). As such the view has become known as the *Adjectival Strategy*, following Dummett's (1995) terminology. Thus, Hodes (1990) has shown how a formal language of arithmetic can be interpreted within the Adjectival Strategy, so that number terms do not designate numbers, but 'encode' quantifiers. The Adjectival Strategy must make use of modality to account for a domain that is too small to make the relevant sentences true or rather false. Thus, (70a), on the Adjectival Strategy, is best paraphrased as below:

(70) c. If there are (were) two things and two other things, then there would be four things.

Using plural logic, (70a) can be formalized as in (71), translating numerals as plural predicates. That is, P_1 symbolizes the predicate 'two' and P_2 the predicate 'four', 'xx' and 'yy' are plural variables, and '$\leq$' symbolizes the 'is / are some of' relation:

(71) $\Box\ (\exists xx \exists yy(P_1(xx)\ \&\ P_1(yy)\ \&\ \neg\exists z(z \leq xx\ \&\ z \leq yy) \rightarrow \forall xx \forall yy(P_1(xx)\ \&\ P_1(yy)\ \&\ \neg\exists z(z \leq xx\ \&\ z \leq yy) \rightarrow \exists ww(P_2(ww)\ \&\ xx \leq ww\ \&\ yy \leq ww\ \&\ \neg\exists z(z \leq ww\ \&\ \neg\ z \leq xx\ \&\ \neg\ z < yy))))$

This is to be read as: 'in any world in which there are two things and a different two things, for any two things and a different two things, there are four things consisting of just them'.[30]

One issue with (71) as the logical form of (70a) is that the connective *and* and the copula *is* need to be viewed as syncategorematic expressions that have a meaning only together. More precisely, in (70a) *and* and *is* need to be treated as a semantic unit expressing the following three-place relation among number properties:

(72) $\text{trans}(\textit{and, is}) = \lambda PP'P''[\Box\ (\exists xx \exists yy(P(xx)\ \&\ P'(yy)\ \&\ \neg\exists z(z \leq xx\ \&\ z \leq yy) \rightarrow \forall xx \forall yy(P(xx)\ \&\ P'(yy)\ \&\ \neg\exists z(z \leq xx\ \&\ z \leq yy) \rightarrow \exists ww(P''(ww)\ \&\ xx \leq ww\ \&\ yy \leq ww)\ \&\ \exists z(z \leq ww\ \&\ \neg\ z \leq xx\ \&\ \neg\ z \leq yy)))]$

Here, *trans* is the translation function mapping natural language expressions onto expressions of the formal language. Note that on this analysis, the connective and

[30] The common formalization of (70a) within approaches using the Adjectival Strategy is by making use of quantification over concepts (Hodes 1984, 1990):

(i) $\forall F\ \forall G\ (\exists_2 x\ Fx\ \&\ \exists_2 x\ Gx\ \&\ \neg\ \exists x(Fx\ \&\ Gx) \rightarrow \exists_4 x\ (Gx\ v\ F\ x))$

This presupposes the view that counting means counting objects that fall under a concept. See Bigelow (1988) for a critique of that view.

the copula apply to plural properties rather than pure numbers. If the connective were to apply to pure numbers, this would require a different semantics.

Within the Adjectival Strategy, also one-place functors must be treated syncategorematically, namely as expressing a plural property together with the number term to which they apply. For example, *successor of t*, for a numeral t as in *the successor of two* can be defined as follows:

(73) trans(*successor of t*) = λxx[$\exists$yy $\exists$z(trans(t)(yy) & $\neg$z $\leq$ yy & z < xx & $\forall$w(w $\leq$ xx $\rightarrow$ w = z v w $\leq$ yy))]

That is, the successor of *t* is the property that holds of a plurality *xx* in case among the *xx* are only entities that are among some *yy* of which the plural property holds that *t* expresses or that are identical to a single entity that is not among the *yy*.[31]

It is clear that if the Adjectival Strategy is applied to natural language, a predicate or expression acting as a functor (or functor + predicate) will have a different meaning with numerals than with referential noun phrases. Thus, natural language predicates and functional expressions that can apply to simple numerals as well as referential noun phrases will be polysemous and have a special, derivative meaning in arithmetical statements involving simple numerals.

2.2 Linguistic evidence for the Adjectival Strategy

The Adjectival Strategy provides a way of analysing simple numerals as contributing not an abstract object to the meaning of a sentence, but rather a plural property, the kind of meaning numerals have when they occur as noun modifiers. In this section, I will present some linguistic support for the Adjectival Strategy applied to simple numerals in natural language—apart from the fact that simple numerals occur both as noun modifiers and as singular terms. One kind of evidence is semantic and concerns restrictions on the predicates with which simple numerals can occur; another kind of evidence, from German, is syntactic, involving differences in the syntactic properties of simple numerals and singular terms. They indicate that the (Neo-)Fregean syntactic criteria for singular terms are not conclusive as to the semantic role of such terms and that a distinction needs to be drawn between ordinary referential terms and what I call *quasi-referential terms*.

[31] There is a potential problem for the Adjectival Strategy, namely quantifiers that can replace numerals in referential position, such as *something* or *the same thing*:

(i) a. John added something to something else, namely he added ten to twenty.
b. John added something to the number of children: he added two.

This is not evidence against the Adjectival Strategy, however. Rather quantifiers like *something* are special quantifiers. Such quantifiers are well-known to be able to replace various kinds of nonreferential complements, for example predicative and clausal complements (Moltmann 2003a). In fact, it has been argued that quantifiers like *something* have a substitutional or quasi-substitutional status, their role just being that of enabling inferences (Hofweber 2005b), or, on an alternative analysis, that they are nominalising quantifier, introducing new entities into the semantic structure of the sentence that would not have been present in the absence of the quantifier (Moltmann 2003a). Either analysis is compatible with the Adjectival Strategy concerning simple numerals.

2.2.1 Predicates selecting simple numerals, explicit number-referring terms, or both

The Adjectival Strategy obviously can apply only to mathematical predicates and functors and not to non-mathematical ones: non-mathematical predicates, for example *think of* or *write about*, cannot be analysed as contributing logical notions to a formula involving only plural properties and no abstract objects. In fact, it appears that in natural language simple numerals are much less tolerated with non-mathematical predicates. Moreover, the 'syncategorematic' meaning of the connective + copula combination is as such not applicable to pure numbers, which again matches the linguistic intuitions.

Before turning to that evidence, let us note that there are two kinds of predicates that are applicable to both kinds of terms, simple numerals and explicit number-referring terms. These are one-place mathematical predicates such as *even* and predicates describing agent-related mathematical operations:

(74) a. Four is even.
b. The number four is even.
(75) a. John added two to four.
b. John added the number two to the number four.

However, such predicates constitute rather special cases and I will later suggest explanations for why they allow for both simple numerals and explicit number-referring terms.

Simple numerals and explicit number-referring terms differ with respect to nonmathematical predicates. The difference is most apparent with relative clause constructions[32]:

(76) a. ?? Twelve, which interests me a lot, is an important number in religious and cultural contexts.
b. The number twelve, which interests me a lot, is an important number in religious and cultural contexts.
(77) a. ?? twelve, which I thought about a lot / which God created on the third day / which I like to write my dissertation about / whose significance in geometry is well-known,…
b. the number twelve, which interests me a lot / which God created on the third day / which I like to write my dissertation about / whose significance in geometry is well-known,…

[32] For some reason, the difference is less apparent in main clauses:

(i) a. Twelve interests me more than eleven.
b. Twelve is very interesting; five is not.

There are two possible explanations. First, the numerals here are actually focused, which may mean that the sentences, with respect to their subject position, presuppose a domain of pre-individuated objects and the numeral just serves to pick out one of those objects. Second, there may be an implicit sortal *number* in subject position, which for some reason may not be present in object position. In that case, the numeral in (ia, b) may in fact be in topic position and linked to a silent NP in subject position containing *number* as a sortal, as more explicitly in (ii):

(ii) Twelve, that number is very interesting.

The same construction with the relative clauses modifying a simple numeral is fine when the predicate expresses a purely mathematical property:

(78) a. Twelve, which is divisible by two, is not a prime number.
b. Twelve, which is smaller than fifteen, is greater than ten.

The unacceptability of nonmathematical predicates with simple numerals remains when a mathematical and a nonmathematical predicate are conjoined[33]:

(79) a. ?? twelve, which is divisible by two, three, four, and six and is very interesting
b. the number twelve, which is divisible by two, three, four, and six and is very interesting

If the sortal *number* introduces a complex predicate, though, non-mathematical predicates can follow:

(80) twelve, which is a number that interests me a lot

I will sketch an account of this particular construction later in Sect. 3.

Let us then turn to mathematical statements. With mathematical operations on two or more numbers there is a noticeable difference between the acceptability of simple numerals and of explicit number-referring terms. Whereas (81a) is the natural way of expressing addition, (81b) is very strange:

(81) a. Two and two is four.
b. ?? The number two and the number two is the number four.

The examples below display similar if less striking contrasts:

(82) a. Two plus two is four.
b. ? The number two plus the number two is the number four.
(83) a. Two times two is four.
b. ? The number two times the number two is the number four.
(84) a. Four minus two is two.
b. ? The number four minus the number two is the number two.

[33] One might suggest that simple numerals might be unacceptable with non-mathematical predicates simply because simple numerals primarily act as determiners and non-mathematical (but not mathematical) predicates give rise to the expectation of the numeral being followed by a noun. This is problematic, however, because the unacceptability of non-mathematical predicates remains in a conjunction in which the preceding conjunct is a mathematical predicate, as in (79a). Here the mathematical predicate should suffice to licence the simple numeral, which it does not. The suggestion is problematic for another reason. In some languages, simple numerals bear a different morphology from numerals occurring as determiners. For example, the German simple numeral for 'one' is *eins*, whereas the determiner is *einer, eine, or ein*. In German, *eins* is just as unacceptable with mathematical predicates as numerals with the morphology of a determiner:

(i) a. ?? eins, was den Mathematiker Hans sehr interessiert.
'one which interests the mathematician John a lot
b. eins, was eine Primzahl ist
'one which is a prime number'
c. die Zahl eins, die den Mathematiker Hans sehr interessiert
'the number one which interests the mathematician John a lot'

Also explicit number-referring terms which do not contain a numeral are not very good:

(85) a. ?? The first number and the second number is the third number.
b. ? The first number times the second number is the third number.
c. ? The first number minus the second number is the third number.

Simple numerals are also unacceptable with certain expressions of identity that are fine with explicit number-referring terms:

(86) a. ?? Two plus two is the same number as four.
b. The number four is the same number as the number four.

Moreover, with simple numerals no inversion of what occurs in subject position and what occurs in postcopula position is possible:

(87) a. ?? Four is two and two.
b. ?? Four is two times two.

The way the content of mathematical formulae like '$2 + 2 = 4$' is expressed in natural language thus does not match the syntax of the mathematical formula. In the mathematical formula, '2' and '4' are singular terms that stand for pure numbers, '+' is a two-place functor, and '=' the two-place predicate expressing identity among pure numbers. In natural language, *and* and even *plus* and *times* do not act as functors taking number-referring terms. Moreover, *is* in arithmetical statements does not act as the identity predicate expressing a symmetric relation among objects.[34] Yet, syntactically, in (81a), *two and two* occurs as a normal subject, and *is* is a predicate taking *four* as its complement.

The conditions on arithmetical statements are in fact the same as are found in explicit descriptions of calculations, as below[35]:

(88) a. Two plus two makes four.
b. ?? The number two plus the number two makes the number four.
c. ??? Four makes two plus two.

This suggests that *is* in (81a) has in fact the meaning of *makes*, as the '*is* of calculation'. As such, it will take not take pure numbers as arguments, but specifies an operation involving number properties.

Predicates expressing agent-related mathematical operations, as mentioned earlier, accept both simple numerals and explicit number-referring terms:

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[34] Note that *plus* and *times* can easily act as functors with number tropes and other quantitative tropes:

(i) a. The number of women plus the number of men is greater than the number of children.
b. The number of women times the number of dresses is over a hundred.
(ii) a. Sue's weight plus Mary's weight is more than that of Joe.
b. John's height is greater than three times the height of Mary.

Plus obviously has a special meaning when applying to two quantitative tropes *t* and *t*', forming a trope of the same kind as *t* and *t*' whose bearer consists in the plurality of the bearer(s) of *t* and the bearer(s) of *t*'. *Times* in (iib) involves the formation of hypothetical copies of tropes.

[35] I would like to thank Per Martin-Loef for pointing out the connection to statements of calculation to me.

(89) a. John added two to four.
b. John added the number two to the number four.
(90) a. Mary subtracted four from ten.
b. Mary subtracted the number four from the number ten.

Predicates expressing agent-related mathematical activities are partly mathematical (involving mathematical operations) and partly non-mathematical (involving agenthood). It appears that it is the former that licences simple numerals, and the latter that licences explicit number-referring terms. I will turn to an explanation why one-place mathematical predicates such as *even* allow for both kinds of terms later in Sect. 3.

To conclude, the acceptability of simple numerals and explicit number-referring terms with the various predicates and functors in natural language appears just like the Adjectival Strategy predicts.

2.2.2 Syntactic evidence for the quasi-referential status of simple numerals

There is also syntactic linguistic evidence for the non-referential status of simple numerals. One piece of evidence comes from the choice of relative pronouns in German. In German, there are two kinds of relative pronouns. The first kind, let's call them 'w-pronouns', consists in the pronoun *was*. *Was* occurs with sortal-free quantifiers as in *alles, was*; *nichts, was*; *vieles, was* ('everything that', 'nothing that', 'many things that'). The second kind, let's call them 'd-pronouns', consists in *das, der, die* (singular feminine), and *die* (plural). D-pronouns cannot occur with sortal-free quantifiers (*alles, vieles, nichts*), but must, it appears, modify an NP with a sortal as head, as in *der Mann, der* ('the man who'); *die Frau, die* ('the woman who'); *das Kind, das* ('the child that'); *die Leute, die* ('the people that'), or else a proper name as in *Hans, der* ('John, who'). What is of interest in the present context is that the two kinds of relative pronoun show a systematic difference with respect to referential and non-referential expressions. Predicative complements generally are considered non-referential; that is, they are generally not taken to refer to a property that is to act as an argument of the embedding copula verb. The relevant observation is that only w-pronouns can be used with predicative complements, whereas explicit property-referring terms (with the sortal *property*) require d-pronouns:

(91) a. Hans wurde weise, was / * das Maria bereits war.
'John became wise, which Mary already was.'
b. Hans hat die Eigenschaft, weise zu sein, die / * was Maria auch hat.
'John has the property of being wise, which Mary has too.'

The crucial observation is that simple numerals accept only w-pronouns, not d-pronouns, as opposed to explicit number-referring terms, which require d-pronouns:

(92) a. zwoelf, was / * das / *? die eine Zahl ist, die mich sehr interessiert, …
'twelve, which is a number that interests me a lot, …'
b. zwoelf, was / * das / * die durch zwei teilbar ist, …
'twelve, which is divisible by two, …'
c. die Zahl zwoelf, die / * was durch zwei teilbar ist, …
'the number twelve, which divisible by two, …'

Thus, simple numerals are not treated as referential, but they have the same non-referential status as predicative complements.[36]

German provides another test for the non-referentiality of simple numerals, and that is support of plural anaphora (for some reason the data are much less clear in English).

The generalization is that conjoined simple numerals generally do not support plural anaphora, as opposed to conjoined explicit number-referring terms[37]:

(93) a. Hans addierte zehn und zwanzig. Maria addierte diese Zahlen / ?? sie auch.
'John added ten and twenty. Mary added them / those numbers too.'
b. Hans addierte die Zahlen zehn und zwanzig. Maria addierte sie auch.
'John added the numbers ten and twenty. Mary added them too.'

The same observation can be made about conjoined *that*-clauses, which, as has often been argued, have a non-referential status as well: conjoined *that*-clauses do not support plural anaphora, but the conjunction of explicit proposition- or fact-referring terms does:

(94) a. John erwähnte, dass Maria weinte und dass Anna lachte. Bill bemerkte das / *sie / diese Tatsachen auch.
'John mentioned that Mary cried and that Ann laughed. Bill noticed that / them / those facts too.'
b. John erwahnte die Tatsache, dass Maria weinte, und die Tatsache, dass Anna lachte. Bill erwahnte sie auch.
'John mentioned the fact that Mary cried and the fact that Ann laughed. Bill mentioned them too.'

Plural anaphora (in German) obviously must refer to pluralities of *objects*, which are not what conjunctions of non-referential terms, such as simple numerals or *that*-clauses, stand for.[38]

[36] One might think of a syntactic explanation why numerals and predicative complements require w-pronouns in German. Arguably, predicative complements and numerals are not gender-marked and w-pronouns are selected whenever the expression modified by the relative clause lacks gender-marking. However, when anaphoric pronouns are chosen that need to agree with numerals in gender, the pronouns must be neutral, which indicates the neutral gender of numerals:

(i) Zehn ist nicht groesser als die Summe seines Vorgaengers mit eins.
'Ten is not greater than the sum of its (neut.) predecessor with one.'

[37] Again a number sortal in the predicate has the same effect as a number sortal in an explicit number-referring term, rendering a plural anaphor acceptable:

(i) a. Drei und fuenf sind beides Primzahlen. Sie sind nicht durch zwei teilbar.
'Three and five are both prime numbers. They are not divisible by two.'
b. Drei und fuenf sind nicht durch zwei teilbar. ?? Sie sind beide(s) Primzahlen.
'Three and five are not divisible by two. They are both prime numbers.'

[38] It remains, of course, to be explained why the generalization seems much less strong in English than in German.

3 Explicit number-referring terms

Explicit number-referring terms such as *the number eight* are truly referential NPs referring to objects. Therefore, there is no reason to expect that they should be acceptable with predicates that require non-referential subjects or objects. One question then is: why are at least some mathematical predicates (such as *even*) acceptable with explicit number-referring terms? More important even is the question, given the non-referential status of simple numerals, what is the ontological status of the pure numbers that explicit number-referring terms make reference to? In what follows, I will give a mere outline of a semantic account of explicit number-referring terms. What guides that account is the linguistic structure and the linguistic properties of explicit number-referring terms, which is taken to be indicative of the nature of pure numbers itself.

The generalizations established in the preceding sections present the following picture of the status of natural numbers in the ontology of natural language. At a primary level, that of basic arithmetical operations on natural numbers, numbers have 'adjectival status', as properties of pluralities; they do not occur as objects. At a next level, that of certain one-place mathematical properties as well as agent-related mathematical operations, numbers may still have adjectival status, but also reference to numbers as objects is permitted. Finally, at a third level, that of nonmathematical properties, numbers have the status of objects of reference only.

This picture suggests an account that treats pure numbers on a par with fictional characters. The parallels with fictional characters are in fact striking, given a view of fictional entities such as that of Kripke (1973), Searle (1979), or Van Inwagen (2000). On that view, within the story the author engages only in pretend reference with a given referential term. The story itself attributes certain properties, 'nuclear' properties, to the individual the author pretends to refer to with that term. However, reference to a fictional character takes place as soon as properties are predicated externally of the individual the author pretends to refer to within the story. These are properties predicated from outside the context of the story or 'extranuclear' properties. 'Living on Baker Street' and 'being a detective' are nuclear properties of Sherlock Holmes; properties such as 'being a frequently cited fictional character' or 'being created by Sir Arthur Conan Doyle' are extranuclear properties.[39] Nuclear properties can be attributed also to fictional characters, but fictional characters do not really 'have' the properties attributed to them in the story (otherwise conflicts may arise with certain extranuclear properties they may have), but rather they 'hold' the properties, as Van Inwagen (2000) puts it.

Given the Adjectival Strategy, purely mathematical contexts involve neither reference nor pretend reference to numbers. But the nuclear—extranuclear distinction makes sense for numbers as well: mathematical properties certainly side with nuclear properties on the nuclear—extranuclear distinction, and

[39] The distinction between nuclear and extranuclear predicates is generally considered problematic as a distinction between types of predicates: the relevant distinction does not so much reside in a difference between types of predicates, but in a difference between a predicate predicated of an entity internally (within the story) and a predicate predicated of it externally. Some predicates, for example *influential*, can be predicated both internally and externally of an entity.

nonmathematical properties with extranuclear properties. Pure numbers as objects of reference, and only they, allow the attribution of nonmathematical predicates—just as only reference to fictional characters allowed the attribution of extranuclear properties. Pure numbers also can have (or rather 'hold') at least certain mathematical properties, namely certain one-place properties as well as agent-related properties.

Fictional characters depend entirely for their existence and identity on the story and its context. However, this does not mean that fictional characters themselves are 'created'. Rather, as Schiffer (1996) has argued, once the story exists in a world, the fictional character exists there as well, whether or not anyone has conceived of it or referred to it. The same would apply to numbers: once there are the mathematical contexts in which numbers have 'adjectival status', numbers as objects can be 'read off' those contexts and can act as objects of reference of explicit number-referring terms.[40]

The parallel with fictional characters also has linguistic plausibility, since the following terms are of exactly the same type:

(95) a. the fictional character Hamlet
b. the number ten

In this construction, a sortal (*fictional character* or *number*) as head nominal is followed by a nonreferential occurrence of an expression, a name as used in a story or a simple numeral.

Further examples of the same construction are given below, where the sortal is followed by an adjective, again a nonreferential expression[41]:

(96) a. the color red
b. the direction north
c. the truth value true

Given the general view of the introduction of entities on the basis of fictional or mathematical contexts just outlined, the semantics of this construction is evident. The sortal here does not just have the function of characterizing the kind to which an entity belongs. Rather it serves to characterize the strategy on the basis of which an entity can be 'read off' non-referential occurrences of an expression in a context, such as a context of fiction or a context of arithmetics. The sortal *fictional character* characterises one such strategy; the sortal *number* another.

Such a reifying use of a sortal arguably is involved also when the sortal occurs in predicative position, as below:

(97) a. Twelve is a number that interests me a lot.
b. Red is a color that I like.

[40] This account faces important challenges that a proper development needs to address, such as clarifying and justifying the strategy by which a context, fictional or mathematical, individuates objects of the relevant sort.

[41] See Dummett (1973, pp. 72–73) for a discussion of parallels between expression for colors and for numbers with their adjectival and substantival uses.

c. North is a direction that we should not take.
d. True is a truth value.

Here the expressions in subject position occur in the very same nonreferential way as they occur after the sortal in (95) and (96). The sortal with the indefinite pronoun occurs predicatively, and the relative clause following it can express any property that could be true of the 'reified' pure number, color, direction, or truth value. The semantic relation between the expression in subject position and the sortal in predicate position is arguably the very same as the one between the expression following the head noun in (95) and (96) and the sortal head noun. Thus, in (97a) *twelve* (as always) occurs nonreferentially and it is the sortal *number* in predicate position that introduces on its basis a pure number of which *interests me a lot* is then predicated.

The logical form of *the number twelve* will then roughly be as in (98a), where C is the context in which *twelve* occurs nonreferentially. Similarly, the logical form of (97a) can be given as in (98b):

(98) a. $\iota y\ [y = \text{trans}(\textit{number})(\text{trans}(\textit{twelve}), C)]$
b. $\lambda x C[\exists y\ (y = \text{trans}(\textit{number})(x, C)\ \&\ \text{trans}(\textit{interests me a lot})(y)]$ $(\text{trans}(\textit{twelve}), C)$

The remaining question concerning explicit number-referring terms is, why do the pure numbers they refer to have (or rather 'hold') some mathematical properties at all, such as 'being even'? Such properties generally are defined in terms of a mathematical operation, requiring number properties. But since their content (unlike that of *plus* or *times*) is derivative with respect to that operation, the definition can equally well be given for pure numbers: the pure number *n* has (or 'holds') a property *P* just in case the number property corresponding to *n* plays such and such a role in a particular mathematical operation in terms of which *P* is defined.

Pure numbers will not obtain all their properties from the corresponding numeral, though. Rather, like fictional characters, they have properties of their own. Some of them may be definable by using the corresponding numeral; others are simply supervenient on the role that the content of the numeral plays in various possible mathematical contexts and in various mathematical and nonmathematical uses of it.

4 Conclusion

The approach of this paper was to take a closer look at the linguistic data involving natural numbers and to see what sort of ontological view about them natural language presents. The view it presents in fact differs greatly from what philosophers usually take for granted. Natural language does not easily allow for reference to numbers as objects. Rather it allows primarily for reference to number tropes and for a quasi-referential use of numerals, a use on which numerals in 'referential position' retain their meaning as noun modifiers. Only the use of the complex syntactic construction of explicit number-referring terms makes reference to pure numbers possible. The latter, I suggested, naturally goes along with an account of numbers treating them like fictional characters, as objects of reference derived from the nonreferential use of numerals in arithmetical contexts.

The paper also has pointed at directions that are of independent interest. First, it has shown the importance of reference to tropes in the semantics of natural language. Second, it has introduced a notion of quasi-referentiality which will have wider applications than just to number-referring terms. Quasi-referential terms require not only a revision of the standard semantics of 'singular terms' (or rather the equivalent category in natural language), but also of lexical meaning. With a quasi-referential term a predicate cannot just express a property of objects, but rather it will need to interact in more complex ways with the semantic contribution of the quasi-referential term.

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