Carnap Brought Home

The View from Jena

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A Quasi-analytical Constitution of Physical Space

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1. Attacks on Carnap's Constitutional Theory

Carnap's quasi-analytical constitution theory of the *Aufbau* has been subjected to many criticisms on quite different levels. Let us mention just three: (i) Goodman attacked the roots of Carnap's account claiming that the constitutional method of the *Aufbau* is doomed to fail from its very beginning, since the constitution of qualities from elementary experiences is fatally flawed (Goodman 1951); (ii) Quine attacked the constitutional theory of the *Aufbau* on an intermediate level contending that, when it came to spacetime, Carnap was not able to constitute it. Instead, Quine objected, Carnap changed the method of constitution without clearly announcing it, introducing a new undefined connective "is at" (Quine 1951); (iii) recently Friedman contended that even if the quasi-analytical method of constitutions worked properly throughout, it would not deliver what Carnap expected from it, to wit, a complete structuralization of empirical knowledge (Friedman 1999).

I think there are good reasons to assume that Goodman's criticisms can be defused (cf. Proust 1989, Mormann 2003). Thus, I will say nothing about them, nor will I treat Friedman's objections dealing with difficulties concerning Carnap's notion of foundedness (cf. *Aufbau* §154–55, Friedman 1999). In this paper I only want to deal with Quine's criticism concerning the constitution of spacetime. Quine maintained that the *Aufbau*'s account of the constitution of the physical world is principally flawed:

> Statements of the form "Quality q is at point-instant x;y;z;t" were, according to [Carnap's] canons, to be apportioned truth values in

> such a way as to maximize and minimize certain over-all features, and with growth of experience the truth values were to be progressively revised in the same spirit. I think this is a good schematization . . . of what science really does; but it provides no indication, not even the sketchiest, of how a statement of the form "Quality q is at x;y,z;t" could ever be translated into Carnap's initial language of sense data and logic. The connective "is at" remains an added undefined connective; the canons counsel us in its use but not in its elimination. (Quine 1951, p. 40)

In this paper I'd like to provide such a sketch. Admittedly, something like this cannot be found in Carnap's original account, at least not explicitly. Nevertheless, I claim that my constitution of the "is at"-connective is fully in line with the spirit of Carnap's approach. That is to say, Carnap *could* have constituted physical space by quasi-analytical methods alone. He wasn't forced to fall back on other, conventionalist constitutional methods. Although Quine's empiricist interpretation of the *Aufbau* has recently been critized from many quarters, his thesis of the alleged break in the methodology of the *Aufbau* and the resulting unreducibility of the coordinating connective "is at" has remained unchallenged up to now (cf. Richardson 1998). Against this common wisdom, I'd like to show how the notorious connective "is at" may indeed be eliminated.[1] The outline of this paper is as follows:

In section 2 we sketch the geometric background of the constitutional theory of the *Aufbau*. Relying on the rather neglected relationship between *Aufbau* and *Der Raum* (Carnap 1922) it is shown that the basic intuition for the constitutional theory is to be found in the realm of synthetic geometry. In section 3 it is argued that the constitutional method of quasi-analysis may be interpreted as a genuine geometric method designed to treat appropriate relational structures (similarity structures) by the methods of synthetic geometry. The aim of section 4 is to show that the affine Euclidean plane may be conceived as a similarity structure for which a quasi-analysis may be set up that yields the original affine incidence relation. Applying a fundamental theorem of synthetic geometry, the so-called coordinatization theorem, this implies that an appropriate quasi-analysis of an affine similarity structure yields a sort of auto-coordinatization of this structure. In section 5 it is argued that this auto-coordinatization may be interpreted in Carnapian terms as the desired constitution of physical space. We close with some remarks on the general relevance of this result in section 6.

1. In order not to overburden the paper with mathematical technicalities I will prove only a simplified version of this contention, to wit, how a statement of the form "Quality q is at (x,y)" $(x,y) \in \mathbb{R}^2$, can be translated into a statement using only the basic terms of a constitutional system. Here, of course, $\mathbb{R}^2$ is the 2-dimensional vector space over the real numbers $\mathbb{R}$. The generalization to 4-dimensional spacetime is not too difficult.

2. The Geometric Background of the *Aufbau*

As has been shown in (Mormann 2003), geometry was an important source of inspiration of Carnap's philosophical thought. To a large extent, the constitutional theory of the *Aufbau* was inspired by the relational systems of synthetic geometry. The first traces of a constitutional theory of conceptual systems can already be found in *Der Raum*. Since the *Aufbau* program may be said to have been decisive for Carnap's philosophy *überhaupt*, one may say that geometry had a substantial influence on his philosophy in its entirety. In this paper I do not want to dwell upon the historical details of this "geometric" interpretation of the *Aufbau*, rather I'd like to recall with as little fuss as possible the basic conceptual ideas of synthetic geometry necessary to understand the quasi-analytical constitution of physical space that Carnap could have carried out, if only he had paid more attention to the expressive power of geometry.[2]

The *leitmotif* of synthetic geometry is order. As Carnap put it, geometry is a general theory of *Ordnungsgefüge* (complexes of order stipulations). Carnap understood *Ordnungsgefüge* in a semi-technical sense intended to mean something like "relational structure" or "structured set." By conceiving a domain as a possible application for the theory of *Ordnungsgefüge* one imposes some order on it. This is achieved by certain *Ordnungssetzungen* (stipulations of order). Hence, as a theory of *Ordnungsgefüge*, synthetic geometry has a strong applicative dimension. In *Der Raum* Carnap explains this fact for projective geometries at great length. According to him, synthetic geometry is designed to offer an arsenal of possible conceptual schemes applicable to many domains. This leads to the following two characteristic features of geometry:

(1) Space (and other geometric notions such as points and lines) are conceived as general notions having many different instantiations. Geometry studies them all without blinders. It does not aim to single out one geometric system as the "true" one.

(2) Synthetic geometry is relational: the objects of geometric systems are determined by a net of implicit relational definitions. The ontological status of a geometric object is determined by its relational position within a certain relational system.

Understanding geometry as a general theory of *Ordnungsgefüge* gives *Ordnungssetzungen* a crucial role. The most important *Ordnungssetzungen* are lines. To put it bluntly, lines are the entities that establish geometric order. It is surprising that this simple idea, the imposition of order by lines, is sufficient to constitute *all* concepts of geometry. That is to say, points and lines are the basic building blocks for all other geometrical concepts. Of course, one has to subscribe to a general concept of line in order that lines can play this almost uni-

2. For a fuller account, see (Mormann 2003).

versal role of *Ordnungssetzungen*. Lines in the sense of synthetic geometry need not look like the lines we are accustumed to. For instance, in projective geometry "lines" have the structure of "circles." The point is that the "lines" of geometric systems *function as* lines.

The upshot of this is the following: a geometric system in the sense of synthetic geometry may be defined as a triple (V, L, I): V is the set of points, L the set of lines and I is the so-called incidence relation $I \subseteq V \times L$. Obviously, the incidence relation I fully characterizes the geometric system (V, L, I). Synthetic geometry, then, is the theory of incidence relations (cf. Buekenhout 1995). More specifically, the incidence relation I determines which points are related to which lines. Intuitively stated, it determines which points are on which lines: $(x, m) \in I$ is to be interpreted as the fact that in the geometric system defined by the incidence relation I "the point x is on the line m." In the same vein, two lines m and k are said to intersect if and only if there is a point x that belongs to both of them, i.e., there are ordered pairs $(x, m), (x, k) \in I$; two lines m and k are parallel ($m \parallel k$) if and only if they do not have a common point; two points x and y are collinear if and only if there is a line m such that $(x, m), (y, m) \in I$. Depending on the axioms imposed on I, different types of geometric systems are obtained. Traditionally, the most important ones are *affine* and *projective* systems, but in contemporary geometry many other systems are studied as well (cf. ibid.).

For later use let us note that the systems (V, L, I) of synthetic geometry are *extensional* in the following sense:

(2.1) *Definition (Extensional Geometric Systems).* Let S = (V, L, I) be a system of synthetic geometry. For $m \in L$ denote by $V(m) := \{x; (x, m) \in I\}$ the set of points of the line m. Then S is an *extensional* system iff two lines m and n are equal iff their point sets V(m) and V(n) coincide.

Extensional systems (V, L, I) may be cast in a canonical form that eliminates lines as primitive: denote the power set of V by PV. Then an extensional geometric system is isomorphic to a system of the form (V, L, I) where $L \subseteq PV$ by identifying lines with their sets of points. In the following it is assumed throughout that all geometric systems are extensional, although we do not always explicitly denote this.

Now let us begin to connect synthetic geometry as the theory of incidence structures with the constitutional theory of the *Aufbau*. First let us note a rather curious piece of evidence for such a connection. In *Der Raum* Carnap considers some geometric systems (P, C, I) that may be considered as primitive forerunners of the constitutional systems of the *Aufbau* (for details see Mormann [2003]): P is a set of objects (*Gegenstände*), C a set of concepts (*Begriffe*) such that $(p, c) \in I$ iff the object p can be subsumed under c. Or, in other words, $(p, c) \in I$ if and only if p is a case of c or p can be subsumed under c. For systems of this kind, which he calls "conceptual geometries," Carnap requires the following axioms to hold:

(2.2) *Conceptual Geometries (Der Raum, p. 14).* "Let us assume that the objects P_1, P_2, . . . fall under the concept P such that the following conditions are satisfied: there is a concept G, under which not objects are subsumed but concepts g_1, g_2, g_3, . . . such that the following requirements are satisfied:

(1) At least three P-objects fall under any g-concept, but not all P-objects can be subsumed under one g-concept.

(2) For two different P-objects there is always one and only one g-concept under which they fall, their "common" concept.

(3) If P_1, P_3, P'_2 fall under g_1, P_2, P_3, P'_1 under g_2, and g_1 and g_2 are different, then there exists an object P_4 that falls under the common concept of P_1 and P'_1 and under the common concept of P_2 and P'_2; moreover there is a concept g_3, which subsumes P_1, but no object of g_2."

The first two axioms of this system of conceptual geometry are easily understood, even if they may not appear very plausible for concepts. The third, as it stands appears hopelessly abstruse. In geometric terms it essentially tells us that the space of the conceptual geometry has (at least) three dimensions. In sum, what Carnap is doing here is just taking the familiar axiom system of 3-dimensional projective space and replacing the standard geometric interpretations "point," "line," and "is incident with" by the expressions "object," "concept," and "falls under" ("is subsumed"), respectively. At first glance, the "projective conceptual geometry" obtained by this procedure may appear to be nothing but an amusing idea devoid of any deeper meaning. This, however, would be a misunderstanding. As is shown in (Mormann 2003), the conceptual geometries are the primitive precursors of the constitutional systems of the *Aufbau*. They may be considered as powerful intuition pumps for the constitutional theory of the *Aufbau* (cf. §70, 72). One may say that Carnap took the constitutional systems of the *Aufbau* and the conceptual geometries of *Der Raum* as being of the same ilk.

In this paper I want to show that synthetic geometry not only belongs to the *Aufbau's* prehistory, but should also be regarded as its conceptual core. This helps elucidate the true nature of the *Aufbau* program. In order to render plausible this contention, we have to recall an important result of 19th-century mathematics, to wit, the coordinatization theorem of affine geometries. We will use this theorem to show that the real number space $\mathbb{R}^2$ can be constituted from a purely qualitative base, or, to use Quine's wording, from Carnap's initial language of sense data and logic. In the context of the present paper, this base will be a relational structure defined by a set of *Elementarerlebnisse* endowed with a purely qualitative similarity relation (cf. *Aufbau* §108ff). In other words, we may consider qualitative, non-metrical geometry as a foundational theory for number systems. Today, philosophers of mathematics do not attribute any philosophical relevance to this fact. Rather unanimously they consider geometry as reducible to linear algebra in such a way that geometry as a mathematical discipline loses any genuine philosophical interest. It is remarkable, and mathematically non-trivial, that the direction of reduction may be reversed. The proof of this crucial result is long and involved, and need not concern us here. It may suffice to present an axiom system which shows, at least in principle, how number systems such as the real numbers $\mathbb{R}$ can be reconstructed in the framework of synthetic geometry:

(2.3) *Affine Incidence Structures.* Let A = (V, L, I) be a system of synthetic geometry. A is called an *affine Pappus incidence structure* (AP-structure) iff the following conditions are satisfied:

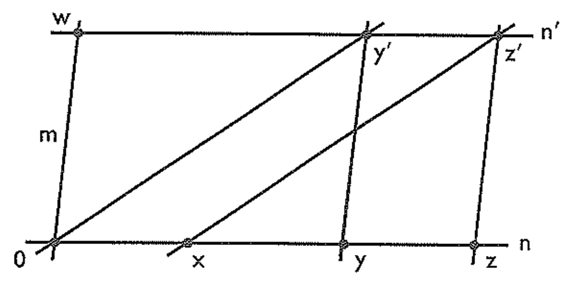
(PA1) There exist at least three non-collinear points. (Nontriviality)

(PA2) Any two distinct points x and y lie on exactly one line. Hence, we may denote the line determined by x and y with xy. (Linearity)

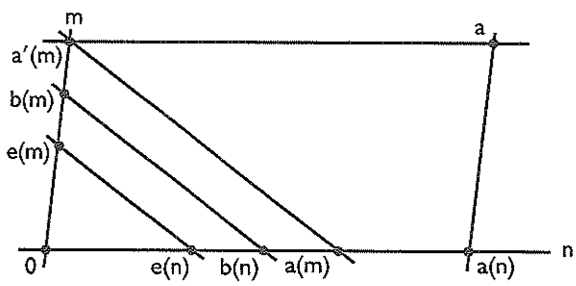
(PA3) Given a point x and a line m, there is exactly one line k that passes through x and is parallel to m. (Parallel axiom)

(PA4) If x, y, z is a triple of points on m, and x′, y′, z′ points of m′ such that xy′ || x′y and xz′ || x′z then yz′ || y′z. (Pappus's axiom)

The axioms (PA1)-(PA4) suffice to ensure that the lines of an AP-incidence structure have a quite rich algebraic structure; they are fields in the sense of mathematics (cf. Goldblatt 1987). That is, for the points of a line n ∈ L, operations of addition and multiplication can be defined which obey the laws of associativity, commutativity, distributivity etc., thereby rendering n a field $\mathbb{K}$. We need not study these operations in detail; their definition can be found in any textbook of synthetic geometry (cf. Buekenhout 1995, Coppel 1998). Rather, we are content to recall the addition of collinear points in the special case of the Euclidean plane for which the field $\mathbb{K}$ is the familiar field $\mathbb{R}$ of real numbers:



Choose two distinct lines m and n which intersect in a point 0. Fix some point w on m different from 0. The line through w parallel to n is denoted by n′. Let x, y ∈ n. Let the line through y parallel to m meet n′ at y′, and then let the line through x parallel to 0y′ meet n′ at z′. Then the line parallel to m through z′ meets n at z. Declare x + y = z. It can be shown that this operation + renders n a commutative group, i.e., addition on n is associative, commutative, has a neutral element 0, etc. In a similar way one can define a commutative multiplication • on n and show that it obeys the laws a multiplication of a field has to satisfy. In sum, these geometrically defined operations + and • render (n, +, Σ) a field. This construction may be expanded to a coordinatization of the AP-structure A as follows:



Choose distinct points e(m) and e(n) on m and n respectively, both different from 0. The assignment of b(m) to b(n) (defined by the line b(m)b(n) parallel to e(m)e(n)) establishes a bijective correspondence between the points of n and of m. Then, given a point a in the plane, let the lines through a parallel to n and m meet m at a′(m) and n at a(n). Let a(m) on n correspond to a′(m) on m. Then the ordered pair (a(n), a(m)) are the coordinates of a in the coordinatization of A defined by the lines n, m, and the points e(m), e(n). Of course, choosing different basic lines and points, one obtains a different coordinatization. But, as is well known, all coordinatizations obtained in this way are isomorphic (cf. Goldblatt 1987, ch. 2).

In the rest of this paper, I intend to show that this coordinatization provides the base for a quasi-analytical constitution of spacetime in the sense of the *Aufbau.* A first bit of evidence that the constitutional theory of the *Aufbau* is related to the systems of synthetic geometry are the "conceptual geometries" of *Der Raum*. In order to substantiate this relation we have to delve deeper into the technicalities of the *Aufbau*.

The result will be that in the constitutional theory, incidence structures (E, Q, I) will play an important role. Here, E is an already constituted domain, and the next higher level of constitution can be described in terms of the system (E, Q, I). This is strong evidence that something like a coordinatization of AP-planes may be carried out in the conceptual framework of the constitutional theory of the *Aufbau*. The aim of the following sections is to do exactly this. More precisely, it will be shown that Carnap's quasi-analysis may be interpreted as a method whose task is to construct appropriate incidence structures (E, Q, I) for certain "inhomogeneous" sets E, as Carnap called them.

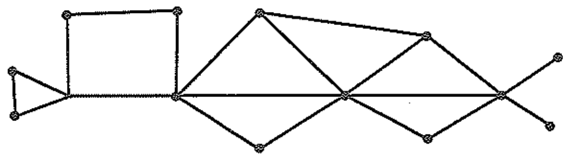
3. Quasi-analysis and Synthetic Geometry

The *Aufbau* is a complex work that has been interpreted in many different, sometimes incompatible ways. In this article I assume that the main aim of the *Aufbau* is *not* to present a full-fledged constitutional system based on *Elementarerlebnisse*. Rather I assume that the *Aufbau* was intended to exemplify a new general philosophical discipline, called *Konstitutionstheorie* (constitutional theory), which had the task of "investigating all possible forms of stepwise definitional systems of concepts" (Friedman 1999, p. 115, *Aufbau* §46). Carnap intended to create a scientific successor to traditional epistemology and philosophy of science that remained neutral with respect to the futile metaphysical quarrels that had plagued the traditional accounts.[3] As the core of this new kind of philosophy of science, Carnap considered not the constitutional system he had sketched in the *Aufbau* but the *method* by which it was constituted. This constitutional method is the method of quasi-analysis. Hence, if the thesis of the geometric origins of the *Aufbau* is to be taken seriously, quasi-analysis should fit into the conceptual framework of synthetic geometry. In this section I want to show that this is indeed the case. More precisely I contend the following:

(3.1) *Quasi-analysis in the Framework of Synthetic Geometry.* Let E be an inhomogeneous set. Quasi-analyzing E is taking the elements of E as points of a geometric system (E, Q, I). The system (E, Q, I) may be considered as a complex of *Ordnungssetzungen*, i.e., as a geometrical *Ordnungsgefüge*. The *Ordnungsgefüge* set up by (E, Q, I) may be interpreted as an externalisation of the inhomogeneities of E.

In order to unfold this succinct characterization of quasi-analysis, the following remarks may be in order. The key term of (3.1) is "non-homogeneous set." Hence, first we have to explain what is to be understood by that term. Then we will explain what is meant by embedding an inhomogeneous set E in a geometric frame (E, Q, I). For Carnap, the foremost examples of non-homogeneous sets E are similarity structures. A similarity structure, denoted by (E, ~), is a set E endowed with a binary similarity relation ~. Two elements e and e′ of E related by the relation ~ are said to be similar to each other (e ~ e′). The relation ~ is assumed to be reflexive and symmetric, i.e. each element is assumed to be similar to itself (e ~ e) and e ~ e′ implies e′ ~ e. The relation need not be transitive, however. This is plausible, since if e is similar to e′, and e′ is similar to e″ then e need not be similar to e″. A non-homogeneous set (E, ~) may be represented more perspicuously as a graph:

3. In this paper, I do not want to argue for this sweeping claim, rather I'd like to rely on the interpretative efforts of authors such as Friedman, Proust, and Richardson who have supplied ample evidence for this contention (cf. Friedman 1999, Proust 1989, Richardson 1998). According to their interpretations, the topics of phenomenalism, gestalt theory, or reductionism do *not* lie at the heart of the *Aufbau*.



Here, two distinct similar elements of E are connected by a straight line, and two elements that are not similar are not directly connected by a straight line. Actually, the concepts of simple graph and similarity structure are strictly equivalent. Hence, according to the *Aufbau*, the world (or some part of it) may be conceived of as a huge graph: the vertices of this graph are the "elementary experiences" and the edges are formed by the pairs of similar elements.

In an unpublished manuscript with the programmatic title *Quasizerlegung – Ein Verfahren zur Ordnung nichthomogener Mengen mit den Mitteln der Beziehungslehre* (Quasi-analysis – A Method to Order Non-Homogeneous Sets by Means of the Theory of Relations [Carnap 1922/23]), the task of *Quasizerlegung*, i.e., quasi-analysis for similarity structures, is described as follows:

> Suppose there is given a set of elements, and for each element the specification to which it is similar. We aim at a description of the set which only uses this information but ascribes to these elements quasicomponents or quasiproperties in such a way that it is possible to deal with each element separately using only the quasiproperties, without reference to other elements. (*Quasizerlegung*, p. 4)

As *Quasizerlegung* makes clear, the method of quasi-analysis is a purely formal method. It may be applied to any nonhomogeneous set (E, ~), not just to *Elementarerlebnisse* as in the *Aufbau*. Submitting (E, ~) to quasi-analysis means imposing certain *Ordnungssetzungen* on it in order to unfold its structure. Thereby appropriate invariants may be found which characterize its structure in a succinct manner. As modern mathematics teaches us, the finding of characteristic invariants is an unending task. Even apparently simple structures give rise to a profusion of invariants. Although Carnap concentrates on the quasi-analysis of similarity structures, he mentions the possibility of submitting *all* kinds of relational structures to a quasi-analytical constitution process (*Aufbau* §104). Hence, I think it would not be totally off to interpret quasi-analysis in a generalized sense as mathematical constitution *überhaupt*. Be that as it may, in the following we will concentrate on the quasi-analysis of similarity structures. In *Quasizerlegung* (p. 4) Carnap describes this kind of quasi-analysis axiomatically as the assignation of quasiproperties to the objects to be quasi-analysed, whereby these quasiproperties function as *Ordnungssetzungen* such that the similarity structure of E is taken into account:

(3.2) *Definition.* Let (E, ~) be a non-homogeneous set. A set of quasiproperties of the elements of E is a set of entities which satisfy the following requirements[4]:

(C1) If two elements are similar they share at least one quasiproperty.

(C2) If two elements are not similar they do not share any quasiproperty.

(C3) If two elements are similar to exactly the same elements they have the same quasiproperties.

(C4) No quasiproperty can be removed without violating (C1)-(C3).

The general intention of these requirements is to regulate the relation between "being similar" on the one hand and "sharing a property" on the other. (C1)-(C3) express characteristics of this relation that may be considered as more or less common sensical asserting that similarity and properties covary. The requirement (C4) is a principle of parsimony which intends to bar superfluous properties not needed for an economic description of the similarity relation.

Let us postpone for the moment the question of what quasiproperties "really are," simply assuming that there is a set Q := {q; q quasiproperty of (E, ~)}. With the help of Q we may define an incidence relation $I \subseteq E \times Q$ by

(3.3) $(e, q) \in I$:= the element e has the quasiproperty q.

In this way the relation between elements of E and their quasiproperties q may be succinctly described by the incidence relation I. If the resulting system (E, Q, I) of incidence geometry satisfies (C1)-(C4) it is called a quasi-analysis of the non-homogeneous set (E, ~). This construction of a geometric system is analogous to that of conceptual geometries Carnap discussed in *Der Raum*. In contrast to the axioms for "conceptual geometries" in *Der Raum* (which are simply copied from the axioms for projective spaces) the requirements (C1)-(C4) for (E, Q, I) are much better adapted to the intuitive requirements one entertains for property distributions. Hence, we are still in the realm of incidence structure, although Carnap no longer maintains the rather absurd thesis that conceptual geometries are to be modelled after the patterns of projective geometry.

It will be expedient to define a quasi-analysis in still another but equivalent way. As is evident, every relation $I \subseteq E \times Q$ gives rise to a mapping

(3.4) $r_I: E \rightarrow PQ$ defined by $r_I(e) := \{q; (e, q) \in I\}$

Obviously, I and r_I determine each other uniquely. Hence we may characterize a quasi-analysis either by (3.3) as an incidence relation, or by (3.4) as a suitable map (cf. Mormann 1994). The version (3.4) is particularly convenient if one wants to check if conditions such as (C1)-(C4) are satisfied. Summarizing we

4. As is shown in Mormann (1994) there are indeed similarity structures that have essentially different sets of quasiproperties.

end up with the following characterization of a quasi-analysis of similarity structures (or, in terms of *Quasizerlegung*, "non-homogeneous sets"):

(3.5) *Definition of Quasi-analysis.* Let (E, ~) be a similarity structure. A *quasi-analysis* for E is an incidence structure (E, Q, I) as defined in (3.3), or equivalently, as a representation r_I: E → PQ satisfying the requirements (C1)-(C4) as defined in (3.3).

Now let us come back to the question of what quasiproperties "really are." Since the constitutional theory of the *Aufbau* is extensional, we may assume that the quasi-analytical systems (E, Q, I) are extensional systems in the sense of (2.1) where the quasiproperty q is identified with the set {e; (e, q) ∈ I}. A set Q of quasiproperties may thus be considered as a subset Q ⊆ PE of subsets of E. Hence the general format of a quasi-analysis of a similarity structure (E, ~) is

(3.6) $I \subseteq E \times PE$ or $r_I: E \rightarrow PPE$

such that (C1)-(C4) are satisfied. This description of quasiproperties is still rather vague. In the following we will give a more specific characterization of quasiproperties for the so called quasi-analysis of the first kind, i.e., we will characterize quasiproperties of (E, ~) as special subsets of E by taking into account the similiarity structure of (E, ~). For this purpose we need the following preparatory definition:

(3.7) *Definition.* Let (E, ~) be a similarity structure. A similarity circle T *(Ähnlichkeitskreis)* is a subset T ⊆ E which satisfies the requirements

(i) $(x)\,(y)\,(x,y \in T \rightarrow x \sim y)$

(ii) $(x)\,\exists\, y\,(x \notin T \rightarrow y \in T \text{ and NOT}(x \sim y))$

Denote the set of similarity circles of (E, ~) by SCE. Conceiving a similarity structure (E, ~) as a graph, similiarity circles T may be characterized as maximal subgraphs of (E, ~) all of whose elements are similar to each other. Hence, for all elements x not belonging to T there is a y of T such that Not $(x \sim y)$ obtains. The concept of similarity circles gives rise to the following definition:

(3.8) *Definition (Aufbau §69, Mormann 1994).* A quasi-analysis (E, Q, I) of the similarity structure (E, ~) is *of the first kind* if and only if all its quasiproperties are elements of SCE, i.e. Q ⊆ PSCE. In mapping form this is expressed by the requirement that a quasi-analysis of the first kind has the form

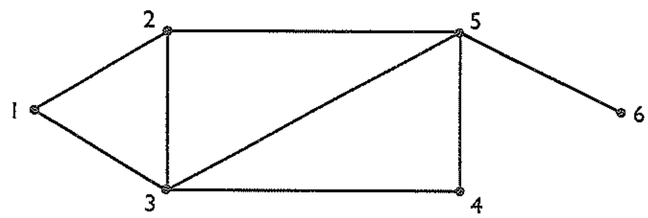
$$r_I: E \Rightarrow PSCE \text{ rather than } r_I: E \rightarrow PPE$$

In the following we will only consider quasi-analysis of the first kind.[5] Similarity circles may be used to define a "standard pseudo-quasi-analysis" in the following way:

5. In the *Aufbau*, Carnap also considers a quasianalysis of the second kind which has formally less satisfying properties.

(3.9) *Proposition.* Let (E, ~) be a similarity structure. Define a map r_I: E → PSCE by $r_I(x)$: = {T : T ∈ SCS(E) and x ∈ T}. Then r_I satisfies the conditions (C1)-(C3).[6]

In order to make the preceding chain of abstract definitions more vivid, let us consider the following elementary example (Goodman 1951):



The similarity structure defined by this graph has four similarity circles: a = {1, 2, 3}, b = {2, 3, 5}, c = {3, 4, 5}, and d = {5, 6}. Thus for this graph we get the following property list:

	1.	a	4.	c
(3.10)	2.	ab	5.	bcd
	3.	abc	6.	d

A list of this kind is to be read as "1 has the property a," "2 has the properties a and b," etc. In this way we see that 1 and 2 share the property a, 2 and 3 share the properties a and b etc. As is easily seen the property distribution provided by the list (3.11) satisfies Carnap's requirements (C1)-(C4).

In the tradition of Carnap and Goodman, the virtues and vices of the quasi-analytical approach have been discussed almost exclusively in terms of small examples such as (3.10) (cf. Goodman 1951, Mormann 1994). The geometric reinterpretation of this method shows that the domain of applications of the quasi-analytic approach is not exhausted by these rather elementary cases. This will be shown in the following section.

4. The Affine Plane as a Similarity Structure

A new field of applications for quasi-analytical constitutional theory is opened when we take seriously the fact that quasi-analysis of similarity structures are incidence structures (E, Q, I) of synthetic geometry. A particularly important example is the incidence structure of the familiar plane of Euclidean geometry. This

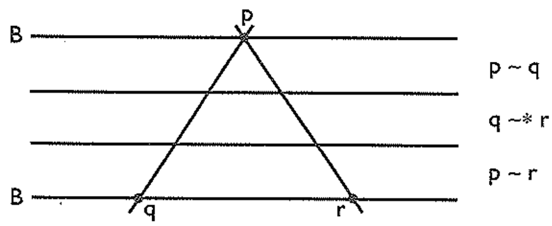
6. It may be called "pseudo-quasianalysis," since, as is shown in (Mormann 1994) it does not necessarily satisfy (C4).

example will bring us close to our ultimate destination, the quasi-analytical constitution of physical space time. The details are as follows. Let A be the Euclidean plane. Denote its points by x,y, z, and its lines by k, m, n, etc. As is well known, the geometric structure of A may be codified in terms of an incidence relation $I \subseteq A \times PA$. Since in A any two different points x and y determine exactly one line, a line m may be denoted by xy, x and y being two different points of m.

Now, our task is to characterize the Euclidean plane A (endowed with its standard incidence structure) as a similarity structure (A, ~). First, let us define an appropriate similarity relation. For this purpose, choose a class B of parallel lines of A. Depending on B we will define a similarity relation ~ B on A. Hence, the resulting similarity structure should be denoted by (A, ~ B). In order not to overload denotation, however, we will denote it simply by (A, ~). This is justified since for different B the resulting similarity structures turn out to be canonically isomorphic. For reasons of intuitive vividness we may refer to the lines of B as horizontal lines. Having chosen B two points x and y are defined to be similar iff they are equal or are on a line m *not* belonging to B:

(4.1) $$x \sim y := (x \neq y \text{ and } xy \notin B) \text{ or } x = y$$

The following diagram exhibits the geometrical meaning of this definition:



Obviously, the relation ~ is reflexive and symmetric, but not transitive. Hence, (A, ~) is a similarity structure. Define the complementary similarity structure (A, ~*) by $x \sim^* y := (xy \in B \text{ or } x = y)$). (A, ~*) is a very special similarity structure, to wit, it is an equivalence structure, whose equivalence classes are just the lines of B. Obviously, (A, ~) and (A, ~*) determine each other, and all considerations dealing with ~* could be formulated in terms of ~, and vice versa. Hence, dealing with (A, ~) and (A, ~*) (instead of (A, ~) or (A, ~*) alone) does not add anything new. After these preparations we are able to construct the following quasi-analysis of A:

(4.2) Lemma. Let (A, ~) be the similarity structure defined by (4.1). Then define $Q: A \rightarrow PPA$ by $Q(x) := \{xy;\ xy \notin B\}$. Then the map Q is a quasi-analysis of A of the first kind.

Proof: Geometrically, the quasiproperties attributed to x by Q are just the lines of A through x not belonging to B. First we show that Q satisfies (C1) and (C2).

For $x \neq y$ we have $x \sim y$ iff $xy \notin B$. Hence $xy \in Q(x) \cap Q(y) \neq \emptyset$. On the other hand, if $m \in Q(x) \cap Q(y)$ we have $m \notin B$. Since x and y are on m we may write $m = xy$, hence $x \sim y$. Moreover, since any non-horizontal line m may be characterized as $m = xy$ for some points x and y satisfying $x \sim y$, removing m would amount to a violation of C1. Hence, Q satisfies (C2). The conditions (C3) and (C4) are obvious. In order to prove that Q is of the first kind we have to show that any line q not belonging to B is a similarity circle of the similarity structure (A, ~). Let $m \in Q(x)$. For any $y \in m$ and $x \neq y$ we have $xy = m$. Hence, all points of m are similar to x. Suppose $z \in m$ and $z \sim x$. Then there is a unique horizontal line k through z which meets m at, say, z'. Hence, by definition, $z \sim^* z'$. Hence $m \in SC(A, \sim)$ and Q is of the first kind.

The representation Q has some properties which deserve to be singled out, since they will be crucial for the following (cf. (2.3)):

(4.3) *Lemma.* Let (A, ~) be the Euclidean plane endowed with the similarity relation defined by (4.1) endowed with the quasi-analysis defined by (4.2). Then the following holds:

(PA1)' There exist at least three non-collinear points.

(PA2)' Any two distinct similar points x and y lie on exactly one line. Hence, we may denote the line determined by x and y with xy.

(PA3)' Given a point x and a line m, there is exactly one line k that passes x and is parallel to m.

(PA4)' Suppose that x, y, z is a triple of points on m, and x', y', z' a triple of points of m' such that $x \sim y'$, $x' \sim y$, $x \sim z'$, $x' \sim z$, $y \sim z'$, $y' \sim z$. If $xy' \parallel x'y$ and $xz' \parallel x'z$ then $yz' \parallel y'z$.

Lemma (4.3) asserts that the quasi-analysis Q of (A, ~) renders (A, ~) essentially an AP-plane in the sense of (2.3) changing (P1)-(P4) to (P1)'-(P4)' since the lines of B have to be left out.

The incidence relation I_Q defined by Q is not quite the incidence relation of the affine Euclidean plane we are looking for, since the lines of B are missing. In order to include them we proceed as follows. First note that the complementary similarity relation $\sim^*$ is an equivalence relation whose equivalence classes are just the lines of B. Hence, for the complement similarity structure $(A, \sim^*)$ we have a canonical mapping $Q^*: A \to PPA$ which maps x to the singleton {q}, q being the unique line of B with $x \in q$. Q^* satisfies the condition (C1), (C2), and (C4). Denote the incidence relation defined by Q^* by I_{Q^*}. Then we define the set theoretical union $I_{QQ^*} \subseteq A \times PA$ of I_Q and I_{Q^*} by

(4.4) $$(x, m) \in I_{QQ^*} := (x, m) \in I_Q \text{ or } (x, m) \in I_{Q^*}$$

This is the relation we need for the construction of a coordination mapping $r_{QQ^*}: A \to \mathbb{R}^2$. As is easily seen, I_{QQ^*} indeed defines an AP-plane in the sense of

(2.3). Imposing some further axioms on $I_{QQ}*$ as we will do in the next section, one can ensure that the affine plane defined by $I_{QQ}*$ is an Euclidean plane. Then the coordinatization procedure sketched in section 2 ensures that A can be mapped onto $\mathbb{R}^2$ in such a way that the incidence structure $I_{QQ}*$ on A is isomorphically mapped onto the standard real affine structure $I \subseteq \mathbb{R}^2 \times P\mathbb{R}^2$.

Before we come to this task let us observe that the quasi-analytical construction of an affine plane achieved so far is unique up to isomorphism. This is seen as follows: if we had chosen another family B′ of parallels, we would have obtained a different similarity structure (A, ~′). But then the similarity structures (A, ~) and (A, ~′) are isomorphic, since for any pair B and B′ of parallels one can find an affine map which maps B onto B′ preserving the affine structure, to wit, incidence relation and parallelism. This map defines an isomorphism between the similarity structures (A, ~) and (A, ~′).

The last step to get the real numbers is to impose some further axioms on the incidence relation I in order to ensure that the field is indeed $\mathbb{R}$. The crucial point in the proof that the field of the plane A is indeed $\mathbb{R}$ is the observation that $\mathbb{R}$ is distinguished from other fields in that it is a *Dedekind complete ordered* field. That is to say, the elements of a line of the real affine plane can be ordered in such a way that we may talk about positive and negative elements in a sense to be specified. In particular, this order allows us to define a triadic relation of betweenness for collinear points x, y, z. Thus, in order to construct the real affine plane from a similarity structure (S, ~) one has to assume the existence of an order on the similarity circles $T \in Q(SC(A,\sim)) \cup Q^*(SC(A, \sim^*))$. This leads to the following definition:

(4.5) *Definition.* Let (E, ~) be a similarity structure. A quasi-analysis $Q: E \Rightarrow PSCE$ is an ordered quasi-analysis if and only if the similarity circles $T \in Q(E) \subseteq SCE$ are endowed with a linear order.

It can be shown that the real plane $\mathbb{R}^2$ (conceived as a similarity structure (A, ~) via its standard affine structure) has an ordered quasi-analysis that is compatible with the field structure defined on the lines m. On every m we may distinguish between positive elements ($0 < x$) and negative elements ($x < 0$) in such a way that addition and multiplication are compatible with the relation <. Hence we may assume that the similarity circles of a similarity structure (E, ~) having an ordered quasi-analysis in the sense of (4.1) are ordered fields. Now we are almost done. The last requirement we need to obtain the real affine plane is to stipulate that the ordered fields of our lines are *Dedekind complete* in the standard sense (e.g. Goldblatt 1987, pp. 69, 70):

(4.6) *Definition.* Let $\mathbb{K}$ be an ordered field. Assume that $\mathbb{K}$ is the union of two non-empty sets C and D such that $x < y$ for all $x \in C$ and $y \in D$. $\mathbb{K}$ is Dedekind complete if and only if there is some $z \in \mathbb{K}$ such that $x \leq z \leq y$.

As is well known, the structure of a Dedekind ordered complete field is categorical, i.e., up to isomorphism, there is only one Dedekind complete ordered field, to wit, the field of real numbers $\mathbb{R}$. Summarizing we have obtained the result that we may describe the affine Euclidean plane as a similarity structure (E, ~), which has a Dedekind complete ordered AP-quasi-analysis as defined by (4.2)-(4.6). In the next section we will show that this result may be "read backwards" leading to a quasi-analytical coordinatization of a physical domain $\mathbf{P}$ that may be interpreted as physical space in the sense of Carnap.

5. The Constitution of Physical Space

Now we have gathered all the pieces we need to tackle our main task, the constitution of physical space along the lines of the quasi-analytical constitution theory of the *Aufbau*. To make clear what is going on, let us recall what, according to the canons of the *Aufbau*, is to be constituted and what are the assumptions under which this constitution is carried out.

To begin with, let us note that Carnap had a rather peculiar conception of physical space which essentially differs from that of common sense. Before one can embark on the task of constituting it by a quasi-analysis it has to be explained what we are after in the constitutional endeavor. For this purpose, we have to begin with Carnap's own constitution system.

Carnap started the constitution of physical space with the presupposition that we already have the four-dimensional Minkowski vector space $\mathbb{R}^4$ as a purely mathematical (or logical) object. He was entitled to do so, since the constitution theory is based on the assumption that the *Aufbauer* has available for his purposes the full resources of logic and mathematics (cf. also Quine 1951). The mathematical object $\mathbb{R}^4$ cannot be regarded as physical space, of course, and Carnap does not make this assertion. Rather, according to him, the mathematical object $\mathbb{R}^4$ acquires its status as physical space through the coordinatization of physical qualities such as colours by the points of this heretofore purely logical object. So we may say that physical space is the physically interpreted mathematical space $\mathbb{R}^4$. This interpretation has to satisfy certain requirements, for example, stability conditions, but this need not concern us for the moment. Our question is whether this physical interpretation of the purely mathematical object $\mathbb{R}^4$ can be carried out in the constitutional system of the *Aufbau* by using the method of quasi-analytical constitution only, without the introduction of new undefined primitives such as the notorious "is at" relation. Quine claimed that Carnap's constitutional sketch failed to do this and went on to contend that the quasi-analytical constitution of spacetime is principally impossible. He took this as a conclusive argument against the feasibility of empiricist reductionism. Up to this day, Quine's verdict has been accepted almost unanimously. That is to say, even those who maintain that

Quine's empiricist interpretation of the *Aufbau* is untenable agree with him, not only that Carnap did not provide a sketch of the constitution of spacetime, but also that this is actually impossible to do in the framework of the quasi-analytical program. It is this second thesis that I want to refute in the following. I want to show that, although Carnap has failed to provide a quasi-analytical constitution, he *could* have done it. A quasi-analytical constitution *is* feasible. For this purpose it is expedient first to recall Quine's full argument, in which he spotted a change of method in Carnap's constitutional enterprise. He describes Carnap's procedure as follows:

> He [Carnap] explained spatio-temporal point-instants as quadruples of real numbers and envisaged assignment of sense qualities to point-instants according to certain canons. Roughly summarized, the plan was that qualities should be assigned to point-instants in such a way as to achieve the laziest world compatible with our experience. The principle of least action was to be our guide in constructing a world from experience.
>
> Carnap did not seem to recognize, however, that his treatment of physical objects fell short of reduction not merely through sketchiness, but in principle. Statements of the form "Quality q is at point-instant x; y; z; t" were, according to his (Carnap's) canons, to be apportioned truth values in such a way as to maximize certain over-all features, and with growth of experience the truth values were to be progressively revised in the same spirit. I think this is a good schematization (deliberately oversimplified, to be sure) of what science really does; but it provides no indication, not even the sketchiest, of how a statement of the form "Quality q is at x; y; z; t" could ever be translated into Carnap's initial language of sense data and logic. The connective "is at" remains an added undefined connective; the canons counsel us in its use but not in its elimination. (Quine 1951, p.40)

As I have said, even those who reject Quine's empiricist interpretation of the *Aufbau* concede him this point. I think, this is not necessary. Carnap can be saved from Quine's attack, or so I want to argue. Strictly speaking, Quine does not offer any argument why it may be impossible to provide a translation of the desired kind. Rather, he is content to correctly point out that Carnap does not provide such a translation. This, however, does not imply that such a translation is impossible.

Bringing to bear the apparatus of synthetic geometry developed in the preceding sections, this can be done as follows: The fatal flaw of Carnap's attempt to constitute physical space quasi-analytically resides in the fact that he separates what should not be separated. That is to say, he starts with the ready-made mathematical object $\mathbb{R}^4$, on the one hand, and the physical object, i.e.

the quasi-analytically constituted domain of physical qualities $\mathbb{P}$, on the other. Then, the problem is to bring together these two separated domains. This can, obviously, be done only by fiat, i.e., one has to stipulate that there is an 1-1 assignment between quadruples of real numbers $r \in \mathbb{R}^4$ and physical qualities $q \in \mathbb{P}$. Even if we grant that this is possible following certain acceptable "canons" of empiricism, this assignment of number quadruples to qualities cannot be considered as a quasi-analytical constitution. That is to say, we have a fatal gap between the physical and the mathematical that cannot be bridged by quasi-analytical constitution. Rather, as Quine correctly observes, one has to rely on the un-quasi-analytical extra primitive relation "is at" to assert statements of the type "Quality q *is at* (x,y,z;t)." So far, so good. The question is whether Carnap's deviation from the path of true quasi-analytical constitution was unavoidable. Indeed, it is possible to save Carnap from himself and Quine and his followers. It is not necessary to assume that the ready-made mathematical structure $\mathbb{R}^4$ is already there, waiting to be related or interpreted empirically. Rather, we may constitute physical space, i.e., a physically interpreted $\mathbb{R}^4$, in one fell swoop, so to speak. This amounts to a sort of auto-coordinatization of the already constituted domain $\mathbb{P}$ of qualities along the lines sketched in the previous section.

As already announced in the introduction, to avoid unnecessary technicalities let us replace $\mathbb{R}^4$ by $\mathbb{R}^2$. Hence, the problem is to provide a quasi-analytically acceptable coordinatization of the already constituted domain $\mathbb{P}$. This can be done as follows: The system $\mathbb{P}$ is assumed to be a similarity structure (E, ~). It does not matter whether the elements of E are intuitively interpreted as *Elementarerlebnisse* or qualities or whatever, since the "nature" of the elements does not play a role in constitution theory. According to (3.5) a quasi-analysis of (E, ~) is a geometric system (E, PE, I) which satisfies the requirements (C1)-(C4). Using the results of section 4 we will construct a quasi-analysis of the first kind that gives rise to a coordinatization of E in the sense that each element e of E can be uniquely named by an ordered pair (a, b) of elements of a similarity circle of E. Then the coordinatization theorem tells us that E is the 2-dimensional real number space $\mathbb{R}^2$. That is to say, in contrast to Carnap's flawed coordinatization, which first separated the physical and the mathematical, and later attempted to bring them together again by the notorious "is at" relation, our approach constitutes the coordinating numerical structure directly from the physical structure. Let us start from the following definition:

(5.1) *Affine Pappian Similarity Structure.* Let (E, ~) be a similarity structure. E is called an *affine Pappian similiarity* structure (AP-similarity structure) iff the following holds:

(1) E has a quasi-analysis (E, SCE, I) of the first kind satisfying (C1)-(C4).

(2) The complementary similarity structure (E, ~*) is an equivalence relation.

(3) I satisfies the requirements (P1)′–(P4)′.

(4) For J : = I ∪ I* the system (E, SCE ∪ SCE*, J) is an affine Pappian plane.[7]

As has been shown in the previous section, AP-similarity structures exist. In particular, it should be noted that the definition of an AP-similarity structure is defined fully in terms of what Quine called "Carnap's initial language of sense data and logic" (Quine 1951, p. 40).

Now a quasi-analytical coordinatization of $\mathbb{P}$ = (E, ~) is at hand: choose three non-collinear points r,s, and t. With the help of the intersecting lines rs and rt one may construct an internal coordinatization of E which renders it isomorphic to the 2-dimensional plane $\mathbb{K}^2$ of some commutative field $\mathbb{K}$. Thus, the statement "quality point x is at (p,q)" has the meaning "with respect to the coordinatization based on r, s, and t, the quality point a is represented by the ordered pair of (p,q) of elements p and q of $\mathbb{K}$." This may not be a coordinatization by real numbers, since we cannot be be sure that the field $\mathbb{K}$ is the field $\mathbb{R}$ of real numbers. In order to ensure this some further axiomatic requirements has to be imposed on E. This can be done in several ways. Perhaps the simplest is the following (cf. Goldblatt 1987):

(5.2) *Definition.* An *ordered field* $\mathbb{K}$ is a field with a distinguished subset $\mathbb{K}^+$ closed under addition and multiplication such that for each $x \in \mathbb{K}$, exactly one of the conditions $x \in \mathbb{K}$, $-x \in \mathbb{K}$, $x = 0$ is true. $\mathbb{K}^+$ is to be thought of as the set of positive elements of $\mathbb{K}$. With the help of $\mathbb{K}^+$ one can define an order on $\mathbb{K}$ by $x < y := (y - x) \in \mathbb{K}^+$.

The last step to ensure that $\mathbb{K}$ is the real number field $\mathbb{R}$ is done by imposing the further axiom on $\mathbb{K}$ that it also satisfies the Dedekind completeness axiom (4.6). Noting that the Dedekind axiom can be expressed in "Carnap's initial language of sense data and logic," we are done: it can be proved that $\mathbb{K}$ is (up to isomorphism) just $\mathbb{R}$. Hence, the coordinatization as described in section 2 yields that $\mathbb{P}$ can be identified with $\mathbb{R}^2$.[8]

Summarizing, we may say that for AP-similarity structures (E, ~) whose lines are Dedekind-complete linearly ordered point sets, there exists a

7. Here, of course, SCE* is the class of similarity circles of the similarity structure (E, ~), intuitively to be interpreted as the class of deleted parallels.

8. Another, possibly more elegant way to cope with the problem of endowing the lines with an order structure would be to pursue the approach by Robb (1936) (cf. also Goldblatt 1987). Robb's reconstruction of spacetime is based on a single primitive "y is after z" which formally (but not intuitively) corresponds to Carnap's *Ähnlichkeitserinnerung* (cf. *Aufbau* § 110) whose symmetrization is the similarity relation ~. Robb's original presentation is difficult, and a more accessible modern account is to be found in (Goldblatt 1987, Appendix B). The constructions are still complicated and I cannot go into the details. Nevertheless it should be said that the constructions are quite compatible with the spirit of quasianalytical constitution. As it seems Carnap did not know Robb's work. Although Robb's account has been treated by some authors in recent years, nobody seems to have studied more closely its possible relations with the *Aufbau*-approach.

quasi-analytically constituted coordinatization, which assigns real number coordinates to the elements of E. Of course, this coordinatization is not unique: Given an AP-quasi-analysis (E, SCE, I) of E one may choose another triple of non-collinear elements of r', s', t' yielding another coordinatization (E, SCE, I') related to the former by a unique linear isomorphism. Moreover, given (E, ~) the choice of an AP-quasi-analysis (E, SCE, I) may not be unique. It may happen that for a given (E, ~) there exist several different quasi-analyses (E, SCE, I) and (E, SCE, I').[9] This gives ample leeway for conventional choices. That is to say, for some reason or other the *Aufbauer* may prefer one coordinatization over another, since one allows maximizing certain desirable over-all features while the other does not. Thereby the quasi-analytical constitution of physical space, i.e. the constitution of a spatio-temporal coordinatization of the physical domain **P** = (E, ~), is seen to follow, at least schematically, what science actually does (cf. *Aufbau* §§135, 136, Richardson 1998, pp.70ff).

6. Concluding Remarks

We conclude that the program of quasi-analytical constitution is not bound to break down when it comes to the quasi-analytical constitution of physical space. It may founder at other points, or may be considered to be unattractive for other reasons, but there is no deep reason why it has to fail at the foundations of physical space, as Carnap understood this notion.

One may well wonder why Quine and so many philosophers following him were confident that Carnap's failure to provide a sketch of a quasi-analytical constitution of spacetime was tantamount to the impossibility of achieving this task in general. Quine never gave any indication, even the sketchiest one, why this constitution should be impossible in principle. A not unplausible answer seems to be that he and many philosophers underestimated the expressive power of synthetic geometry of the 19th century, to say nothing of its modern achievements (cf. Buekenhout 1995, Coppel 1998). It is nothing but a commonsense prejudice that a qualitative language like Carnap's initial language is principally unable to cope with the quantitative as it crops up in the real number coordinatization of physical space.

In a general vein one may say that modern synthetic geometry, which in this paper we used only in a very elementary way, may well have the potential to support rational reconstruction programs such as the *Aufbau*'s. Hence, some basic knowledge of synthetic geometry may still be useful also for philosophers of the 21st century.

9. For an argument (contra Goodman) that the non-uniqueness of quasianalysis should not be considered as a fatal flaw, see (Mormann 2003).

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