A Topological Completeness Theorem for a weak Version of Stalnaker's Logic of Knowledge and Belief

Abstract. The aim of this paper is to prove a topological completeness theorem for a weak version Stalnaker's logic KB of knowledge and belief. In contrast to the strong version, the weak version of KB does not require that the belief operator B satisfies the contentious axiom (NI) of negative introspection. This has the consequence that B is not uniquely definable in terms of K. Rather, for the weak version, the knowledge operator K is compatible with a whole (partially ordered) family of belief operators B in the sense that all pairs of operators (K, B) satisfy the rules and axioms of Stalnaker's logic KB (except (NI) of course). In other words, instead of a one-one-relation between knowledge and belief as in the strong version of KB, the weak version leads to a one-many-relation between these concepts. This has considerable philosophical advantages. The appropriate formal framework for this pluralistic logic of knowledge and belief heavily uses concepts of modern point-free topology. Particularly, the concept of a nucleus turns out to be useful for the formalization of a pluralist theory of belief operators B that (more or less closely) approximate a given knowledge operator K. Further, a canonical topological model for weak KB can be constructed. For this canonical topological model of (weak) KB a truth lemma holds for the operators K and B such that a topological completeness theorem can be proved in an analogous way as this has been done for the unimodal logic of knowledge S4.

Keywords: Logic of knowledge and belief, completeness, nuclei, topology, sublocales, Canonical topological model.

1. Introduction. The aim of this paper is to prove a topological completeness theorem for a weak version of Stalnaker's logic KB of knowledge and belief. In contrast to the strong version, the weak version of KB does not require that the belief operator B satisfies the contentious

axiom (NI) of negative introspection.[1] Abandoning (NI) has the consequence that the operator B is no longer uniquely definable in terms of the knowledge operator K. Rather, for the weak version of KB, every knowledge operator K is compatible with a partially ordered family of belief operators B in the sense that all pairs (K, B) satisfy all rules and axioms of Stalnaker's logic KB (except (NI) of course). In other words, instead of a one-one-relation between knowledge and belief that characterizes the strong version of KB, the weak version leads to a one-many-relation between K and B. From an epistemological perspective, this is considerably more realistic and plausible than a unique determination of belief by knowledge, or so I want to argue in this paper.

The topological epistemology of this paper is a knowledge first epistemology: Knowledge is modeled by a topological interior kernel operator K operating on a set of possible worlds X endowed with a topological structure (X, OX) and a belief operator B. More precisely, if subsets A of X are conceived as propositions, then a proposition int(A) $\subseteq$ X is to be epistemologically interpreted as the proposition (denoted by K(A)) "A is known" or "A cognitive agent that relies on the epistemic operator int knows that A", or similarly.

The Kuratowski axioms of topology entail that the interior kernel operator int has several properties that are intuitively appealing for a knowledge operator. For instance, one immediately obtains from the Kuratowski axioms that knowledge is factive, i.e., that only facts can be known. Formally this is expressed as int(A) $\subseteq$ A. Similarly, a topological knowledge operator satisfies the famous (or notorious) "KK-principle", asserting that knowing a proposition A entails that one knows that one knows A. This is expressed formally by the assertion that the interior kernel operator int satisfies the inclusion int(A) $\subseteq$ int(int(A)). Other

[1] The strong version of KB is, as it should be, a special case of the weak version. As is well known, a topological completeness theorem holds for the strong version of KB: Such a theorem is just the topological completeness theorem of S4.2 with respect to the class of extremally disconnected spaces (cf. Baltag et al. (2019, Theorem 2, p. 215)).

intuitively plausible results for a topological model of knowledge can be derived similarly from elementary properties of the Kuratowski axioms. In sum, topological epistemology seems to be promising starting point for a formal version of a knowledge first epistemology. Not so clear, however, is how the concept of belief fits into a topological framework. As will be shown in this paper, an appropriate formal framework for a pluralistic logic of knowledge and belief heavily uses concepts of modern point-free topology. In particular the concept of nuclei turns out to be essential for the formalization of a pluralist theory of belief operators that (more or less closely) deviate from a given knowledge operator (cf. Johnstone (1980), Borceux (1994), Picado and Pultr (2012)). Thereby, in a somewhat indirect way the doxastic concept of belief depends in epistemic concept of knowledge. That is to say, the topological epistemology presented in this paper is indeed a knowledge first epistemology.

In the framework of point-free topology, a canonical topological model for weak KB logic can be constructed for which a truth lemma for the operators K and B holds such that a topological completeness theorem for weak Stalnaker logic KB can be proved in the familiar way.

The plurality of belief operators B that are compatible with a given knowledge operator K has been virtually unobserved so far. There are different reasons for this blind spot. One reason is that for the familiar Kripke relational semantics this plurality is difficult to see. Another reason for neglecting this plurality may have been the fact that for Stalnaker's original version of KB (that assumes the axiom (NI) of negative introspection to be valid) the belief operator B turns out to be uniquely defined by the knowledge operator K (Baltag et al. (2019), Stalnaker (2006)). This determination of belief by knowledge has been considered by Stalnaker, Baltag and al. and other authors as a particular virtue of KB, since thereby the bimodal logic KB of knowledge K and belief B is actually shown to be a unimodal logic of knowledge K, since B turns out to be uniquely definable by K.[2] One may doubt, however, whether conceptual economy is the only criterion for a "good" comprehensive epistemological logic.

Thus, from the perspective of weak KB, the widely discussed issue whether belief can be (uniquely) defined in terms of knowledge or, the other way round, whether a knowledge first approach is preferable, is too simple. A more flexible formalism should be pursued according to which the operator int defines a topological structure (X, OX) which gives rise to variety of belief operators.

In sum, the topological approach of this paper is a "knowledge first" epistemology, but in a novel flexible sense: belief operators are defined in terms of knowledge, but in an open, not fully determined in form. Thus, the maxim "knowledge first" in epistemology need not mean that other epistemic concepts are to be defined uniquely in terms of knowledge. It may be that knowledge only provides a framework that can be filled in various ways.

The organization of this paper is as follows. To set the stage, in the next section we recapitulate the rules and axioms of Stalnaker's combined logic KB of knowledge and belief (cf. Stalnaker (2006)). In section 3 a topological semantics for KB is defined. The semantics of the knowledge operator K is the familiar topological semantics that conceives the knowledge operator as the interior kernel operator K of a topological space (X, OX). The topological semantics for the belief operators is defined with the concept of dense nuclei B of the Heyting algebra OX. Nuclei may be understood as a kind of derivation of the topological structure that is encapsulated in the interior kernel operator that defines the of topology. More precisely, a nucleus is a map of OX into OX with certain structural properties such that it gives rise to a uniquely defined belief

[2] In the terminology of Baltag et alii (2019) this unique determination is given by $B\varphi \longleftrightarrow \neg K \neg K\varphi$, i.e., belief is just the possibility of knowledge (ibid., Proposition 5, p. 221). Topologically, for extremally disconnected spaces this is rendered $B\varphi \longleftrightarrow \mathrm{clint}(\varphi)$. For these spaces, however, clintφ is equivalent to intclint(φ). In this paper, we will show that for <u>all</u> topological spaces the pair of operators (int, intclint) satisfies all rules and axioms of Stalnaker's logic KB except (NI). Hence, it seems appropriate to call the operator intclint Stalnaker's belief operator. It will be denoted by B_S. In contrast to the strong version of KB, there are many other belief operators B that in tandem with int satisfy the axioms of weak KB. Hence, for the weak version of KB, the knowledge operator K no longer determines B.

operator, also denoted by B, such that the pair (K, B) satisfies all the rules and axioms of Stalnaker's logic KB. It is easily shown that the logic KB is sound with respect to the class of topological models (X, OX, B) of topological space (X, OX) endowed with a dense nucleus B. Nuclei can be conceived as generalized subspaces ("sublocales") of the topological spaces (X, OX). The bijection between nuclei and sublocales is used in section 4 to construct a canonical topological model of KB that is endowed with a canonical nucleus B such that for K and B a truth lemma holds. Thereby a topological completeness theorem for KB is proved in the usual way. Some concluding remarks are offered in section 5.

2. Stalnaker's Logic KB of Knowledge and Belief. Now let us recall the basics of the grammar and syntax of the bimodal logic KB of knowledge and belief put forward by Stalnaker (2006). In recent years Baltag, Bezhanishvili, Özgün, and Smets in various recent publications proposed a topological semantics for KB (cf. Baltag et al. (2014, 2019)). This semantics will be also be the basis of the used in this paper. The main formal novelty of this paper is the extension of this semantics to a semantics for belief operators B of KB. This is necessary, since, in contrast to Stalnaker's original version of strong KB the belief operators B are no longer uniquely definable in terms of the knowledge operator K. The main ingredient for the more flexible semantics of B is the concept of a nucleus, introduced in the 1980s in modern point-free topology (cf. Johnstone (1982), Picado and Pultr (2012)).

We start with a standard unimodal language L_K with a countable set PROP of propositional letters, Boolean operators $\neg$, $\wedge$, and a modal operator K to be interpreted as a knowledge operator. The formulas of L_K are defined as usual by the grammar

$$\varphi ::= p \mid \neg p \mid \phi \wedge \psi \mid K\varphi \quad , \qquad p \in \mathrm{PROP}.$$

The abbreviations for the Boolean connectives $\vee$, $\rightarrow$, and $\longleftrightarrow$ are standard. Then, analogously to L_K, a bimodal epistemological language L_{KB} for operators K and B is defined accepting $B\varphi$ as another type of well-formed formula, to be interpreted as "φ is believed" or similarly.[3]

Now, for the sake of definiteness, let us recall the axioms and the inference rules of Stalnaker's KB-systems (cf. Stalnaker (2006), Baltag et al. (2014, 2019)): The language of the KB-systems is an extension of classical (Boolean) propositional language by two modal operators K and B that have to fulfil the following axioms and rules:

(2.1) Definition (Stalnaker's axioms and inference rules for knowledge and belief).

(CL)	All tautologies of classical propositional logic CLP.	
(K)	$K(\phi \rightarrow \psi) \rightarrow (K\phi \rightarrow K\psi)$	(Knowledge is additive).
(T)	$K\phi \rightarrow \phi$	(Knowledge implies truth).
(KK)	$K\phi \rightarrow KK\phi$	(Positive introspection for K).
(CB)	$B\phi \rightarrow \neg B \neg \phi$	(Consistency of B).
(PI)	$B\phi \rightarrow KB\phi$	(Positive introspection of B).
(NI)	$\neg B\phi \rightarrow K\neg B\phi$	(Negative introspection of B).
(KB)	$K\phi \rightarrow B\phi$	(Knowledge implies belief).
(FB)	$B\phi \rightarrow BK\phi$	(Full belief).

Inference Rules:

(MP)	From ϕ and $\phi \rightarrow \psi$, infer ψ.	(Modus Ponens).
(NEC)	From ϕ, infer $K\phi$.	(Necessitation).♦

For the topological approach to knowledge and belief, the axiom (NI) plays a special role. It is easily shown that (NI) holds only for topological models of a very special kind, namely, models

[3] For a more detailed presentation of topological semantics, the reader may consult the recent papers of Baltag et alii mentioned above.

that are based on extremally disconnected spaces. For the systems of knowledge and belief considered in this paper we will only require that they are <u>weak</u> Stalnaker systems in the following sense:

<u>(2.2) Definition</u>. A bimodal system based on the bimodal language L_{KB} is a weak Stalnaker system iff it satisfies all of Stalnaker's axioms and rules given in (2.1) except possibly the axiom (NI) of negative introspection.♦

There are various reasons for abandoning (NI): first, (NI) is an intuitively a rather implausible requirement for belief. Second, from a topological perspective, the axiom (NI) is very restrictive. Only the topological models based on the restricted class of extremally disconnected spaces (X; OX) satisfy (NI). Most spaces that "occur in nature", do not belong to this class. For instance, the familiar Euclidean spaces and their relatives are far from being extremally disconnected. Finally, the axiom (NI) leads to a 1-1-relation between knowledge K and belief B. This is an implausible and too simplistic understanding of the complex relation between knowledge and belief. If one conceives a belief system as a kind of extension of a knowledge system it is not very plausible to assume that only one such extension exists. Rather, a variety of belief systems should be compatible with one knowledge system. This idea of a one-many-relation between knowledge and belief is rendered precise by the concept of a weak Stalnaker logic KB.

More precisely, the situation is as follows: If all rules and axioms of KB are assumed to be valid (included (NI)), then the belief operator B is uniquely determined by K (cf. Stalnaker (2006), Baltag et al. (2014, 2019). This implies that KB can actually be interpreted as a unimodal logic defined as the extension of classical Boolean propositional logic CL by the modal operator K. As is well known, Aiello et al. (2003) proved a topological completeness theorem for this logic relying on the classical work of McKinsey and Tarski (1944). Further, the validity of (NI) entails that the topological spaces underlying the models of strong KB are extremally disconnected spaces (cf. Baltag et al. (2014, 2019)).

If (NI) is no longer assumed to be valid the belief operator B is no longer uniquely determined by K. Instead, for a given K a whole family of belief operators B exists that are compatible with K in the sense that the pairs (K, B) satisfy the axioms of KB (except (NI) of course). Nevertheless, this weak Stalnaker logic of K and B is also topologically complete (and sound, of course). This topological completeness theorem will be proved by constructing a canonical topological model for the bimodal language of KB for which a truth lemma holds. Abandoning (NI) has two advantages:

(i) The strict determination of belief B by knowledge K is replaced in the weak version of KB by a more flexible relation. K defines a kind of conceptual space, namely, its family of dense nuclei where the belief operators B live that are compatible with K.

(ii) The restriction of the topological universes of possibilities to extremally disconnected spaces is given up in favor of a more flexible account that allows all kinds of topological spaces for the construction of universes of possibilities.

3. The topological Semantics of Knowledge and Belief Operators. Now let us recall the basics of the interior semantics for epistemic logic of knowledge and belief as presented by Baltag et al. (2014, 2019)). This semantics will be used throughout the rest of this paper. In the second part of this section this semantics will be extended to a topological semantics of the belief operator B. First of all, recall the definition of a topological space:

(3.1) Definition. Let X be a set with power set PX. A topological space is an ordered pair (X, OX) with OX $\subseteq$ PX that satisfies the following conditions:

(i) $\emptyset, X \in OX$.

(ii) OX is closed under finite set-theoretical intersections $\cap$ and arbitrary unions $\cup$. ♦

The elements of OX are called the open sets of the topological space (X, OX). The set-theoretical complement **C**A of an open set A is called a closed set. The set of closed subsets of (X, OX) is denoted by CX. The interior kernel operator int and the closure operator cl of (X, OX) are defined as usual: The interior kernel int(A) of a set $A \in PX$ is the largest open set that is contained in A; the closure cl(A) of A is the smallest closed set containing A. For details, see Willard (2004), Steen and Seebach Jr. (1982), or any other textbook on set-theoretical topology). The operators int and cl are well-known to satisfy the Kuratowski axioms:

<u>(3.2) Proposition (Kuratowski Axioms)</u>. Let (X, OX) be a topological space, $A, B \in PX$. The interior kernel operator int and the closure operator of (X, OX) satisfy the following (in)equalities

(i)	$\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$.	$\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$.
(ii)	$\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.	$\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.
(iii)	$\mathrm{int}(A) \subseteq A$.	$A \subseteq \mathrm{cl}(A)$.
(iv)	$\mathrm{int}(X) = X$.	$\emptyset = \mathrm{cl}(\emptyset)$. ♦

In the following these axioms and its elementary consequences are often used without mentioning them explicitly. Moreover, we will use freely the fact that the operators int and cl are inter-definable: int(A) = **C**cl(**C**A) and cl(A) = **C**int(**C**A).

Often it is expedient to conceive the operators int and cl as operating on PX. This is possible in two (slightly different, but equivalent) ways: One may conceive int either as an operator PX—int—>OX or as an operator PX—int—>PX (using implicitly the canonical inclusion OX —i—>PX). An analogous assertion holds for the closure operator cl that may be conceived as

$PX —cl—> CX$ or as an operator $PX —cl—> PX$. On the latter interpretation of int and cl the concatenation of these operators makes perfect sense. In the following, concatenations such as intcl, intclint will play an important role. In the following both interpretations of these operators will be used not distinguishing between them.

For the definition of consistent belief operators B the concept of dense subsets of a topological space will play an important role:

<u>(3.3) Definition</u>. Let (X, OX) be a topological space with interior operator int and closure operator cl, let $Y, Z \in PX$.

(i) Y is a dense subset of X iff cl(Y) = X.

(ii) Z is a nowhere dense in X iff int(cl(Z)) = Ø.♦

After these preparations, a topological semantics for the modal languages L_K and L_{KB} and L_B can be defined. First, we recall the semantics for L_K:

<u>(3.4) Definition</u>. Given a topological space (X, OX), we define a topo(logical) model for L_K as M = (X, OX, v), where $PROP —\mu—> PX$ is a valuation function from the set PROP of propositional letters into PX. The interior semantics for the Boolean connectives $\wedge$ and $\neg$ is defined as usual. If a formula φ of L is interpreted as $\mu(\varphi) = A \in PX$, then the formula $K\varphi$ of L_K is interpreted as $\mu(K\varphi) := \mathrm{int}(A)$. ♦

Usually, it is not necessary to explicitly mention the interpretation v of a model (X, OX, v). Hence, in order to simplify denotation we write A, K(A) or int(A), instead of $\mu(\varphi)$, $K\mu(\varphi)$, for $A = \mu(\varphi)$, $\mu(K\varphi)$ etc.

Now we going to extend the topological semantics to formulas that contain belief operators B. For this purpose, we introduce the concept of (topological) nuclei (cf. Johnstone (1982), Borceux (1994), Picado and Pultr (2012). Nuclei are the essential ingredient for the definition of belief operators related to a topological knowledge operator K. The concept of a (topological) nucleus is basic for the rest of this paper.[4]

(3.5) Definition. Let (X, OX) be a space, and let $A \in OX$. An operator $OX \xrightarrow{B} OX$ is called a nucleus of (X, OX) if it satisfies the following properties:

(i) $A \subseteq B(A)$. (Inflation)

(ii) $B(B(A)) \subseteq B(A)$. (Idempotence)

(iii) $B(A \cap D) = j(A) \cap j(D)$. (Distributivity)

The set of nuclei of a topological space (X, OX) is denoted by NUC(OX). ♦

(3.6) Definition. The set of nuclei NUC(OX) is partially ordered by the relation ≤ defined by

$$B \leq B' := B(A) \subseteq B'(A) \text{ for all } A \in OX. ♦$$

As is easily proved, this partial order ≤ renders (NUC(OX), ≤) a complete lattice. Even more, (NUC(OX), ≤) can be shown to be a complete Heyting algebra (Johnstone (1982 (II, 2.4, Lemma), Borceux 1994 (Theorem 1.5.7)).

(3.7) Definition. A nucleus $B \in$ NUC(OX) is called a dense nucleus iff $B(\emptyset) = \emptyset$. The subset of dense nuclei of NUC(OX) is denoted by $NUC(OX)_d$.[5] ♦

Dense nuclei will play a central role for the definition of consistent belief operators compatible with the knowledge operator int. The following proposition shows that usually there are many different dense nuclei for a given knowledge operator:

[4] This paper does not aim to give a full-fledged introduction into the theory of nuclei. Instead, we intend to provide the basic definitions and facts so that the reader can understand that this theory has interesting applications regarding the modal theory of belief and knowledge. For a fuller account, the reader may consult Johnstone (1982), Borceux (1994), or Picado and Pultr (2012) and the extensive bibliographies mentioned there.

[5] It is not difficult to show that $NUC(OX)_d$ also has the structure of a complete Heyting algebra.

(3.8) Proposition. Let $F \subseteq X$ be a dense subset of a topological space (X, OX), $D \in OX$.

(i) Define $B_F(D) := \mathrm{int}(\complement F \cup D)$. Then B_F is a dense nucleus.

(ii) The Stalnaker nucleus $B_S(D) := \mathrm{intcl}(D)$ is a dense nucleus.[6]

Proof. An elementary calculation using the Kuratowski axioms (3.2). ♦

Almost all topological spaces (X, OX) have many dense subsets. For the purposes of the present paper the essential point of (3.8) is that it ensures the existence of many dense nuclei. This is equivalent with the existence of many consistent belief operators.

Even for familiar spaces like Euclidean spaces the precise structure of $NUC(OX)_d$ is, however, extremally complicated and very often not completely known. This renders the logic of belief operators B that are compatible with K (encapsulated in $NUC(OX)_d$), the more interesting. This issue cannot be pursued further in the present paper. Rather, we are content to state the following fact that is generally considered as the most important single fact of nuclei and related concepts. It will also play a central role in the topological epistemology of knowledge and belief and is encapsulated in the following mathematically highly non-trivial theorem:

(3.9) Theorem (Isbell's Density Theorem, (cf. Johnstone (1982), Picado and Pultr (2012)). For all topological spaces (X, OX) the Stalnaker nucleus B_S (defined by (3.8)(ii)) is the largest dense nucleus of (X, OX), i.e., for all $B \in NUC(OX)_d$ one has $B \leq B_S$. ♦

The proof of (3.9) goes well beyond the horizon of this paper and cannot be given here. The reader is recommended to consult the excellent references (Johnstone (1982), Picado and Pultr (2012)).[7]

[6] In the topological literature B_S is usually called the regular nucleus, since $B_S(OX) \subseteq OX$ are just the regular open sets of (X, OX).

In the last decades, the investigation of NUC(OX) has turned out to be a fruitful research programme for studying topological problems of various kinds, particularly problems related to point-free topology (cf. Johnstone (1982), Borceux (1994), Picado and Pultr (2012)).

Now, we will show that dense nuclei define dense belief operators in the following canonical way. For a topological space (X, OX) let $PX \xrightarrow{int} OX$ the interior kernel operator and $OX \xrightarrow{i} PX$ the canonical inclusion. For a nucleus $OX \xrightarrow{B} OX$ the concatenation $PX \xrightarrow{int} OX \xrightarrow{B} OX \xrightarrow{i} PX$ is well defined. Clearly, B determines i B int uniquely. Thus, the following definition makes sense:

<u>(3.10) Definition</u>. Let $B \in NUC(OX)_d$. The concatenation $PX \xrightarrow{iBint} PX$ is called the belief operator defined by the nucleus B (related to the knowledge operator int).♦

Since the nucleus B uniquely determines the belief operator iBint, the belief operator iBint may be also denoted by B. Thus, committing a harmless abuse of language we may say that a (dense) nucleus $B \in NUC(OX)_d$ is a belief operator.

The definition (3.10) allows us makes to extend the familiar topological semantics of the unimodal language L_K to a bimodal language L_{KB} of modal operators K and B:

<u>(3.11) Definition</u>. Let (X, OX) be a topological space with interior operator int and B a belief operator in the sense of (3.10). Then a topo(logical) model for the bimodal logic L_{KB} is given by M = (X, OX, B, v), where $PROP \xrightarrow{v} PX$ is a valuation function from the set PROP of propositional letters to PX. The interior semantics for the Boolean connectives $\wedge$, $\neg$, and formulas $K\varphi$ are interpreted as before for L_K, formulas $B\varphi$ are interpreted by $v(B\varphi) := B(v(\varphi))$.♦

The following theorem shows that (3.11) is a reasonable and fruitful definition that defines a numerous family of well-behaved belief operators B for a knowledge operator K that enjoy all properties that one intuitively expects from "good" belief operators.

(3.12) Theorem. Let (X, OX) be a topological space with an interior kernel operator K, and B be any dense belief operator B∈ $NUC(OX)_d$. Then for any valuation PROP—v—>PX the model (X, OX, B, μ) defines a weak Stalnaker system for which all rules and axioms of (2.2) are valid.

Proof. This is easily proved by checking the definitions using the Kuratowski axioms (3.2).♦

(3.13) Corollary. Let (X, OX, B, μ) be a topo-model of the weak Stalnaker logic KB (2.2). Then the belief operator B defines a KD4 logic[8].

Proof. This is easily calculated from the fact that K is a KT4-logic and K(A) ⟶B(A) for all A ∈ OX.♦

(3.14) Corollary. The weak KB logic (2.1) is sound with respect to the class of models (X, OX, B) of topological spaces (X, OX) endowed with a dense nucleus B ∈ $NUC(OX)_d$.♦

Usually, the set $NUC(OX)_d$ has many elements. Succinctly, we have shown that a topological knowledge operator K is always accompanied by a multiplicity of compatible belief operators B in the sense that the all the pairs (K, B) satisfy the rules and axioms of a weak Stalnaker system (2.2).[9] This pluralism of belief operators renders a combined topological logic of knowledge and belief more complex than one may have previously thought, but, on the other hand, this pluralist aspects render this logic more interesting. Moreover, the pluralism of different coexisting belief operators renders epistemological logic more realist and flexible. After all, it is simply not plausible to assume that different cognitive agents who rely on the

[8] A list of the most common normal modal logics can be found in Chagrov and Zakhayaschev (1997, Table 4.2., p. 116).
[9] The only exception are topological spaces (X, OX) for which OX is a Boolean algebra. This corresponds to the peculiar epistemic logic S5. In this case, the only belief operator compatible with int is int itself.

same knowledge operator, have to use the very same belief operator as well. Rather, conceiving beliefs as hypotheses or conjectures that go beyond established knowledge, it cannot be expected that all cognitive agents subscribe to the same hypotheses.

This does not mean that all beliefs how extravagant they may be, are equally reasonable. The very minimum of reasonableness that a belief system should satisfy in order to be acknowledged as acceptable is that it is not inconsistent, i.e., that no contradictions are believed. In topological terms, non-contradictoriness of belief operators is expressed in terms of density: Only dense operators define consistent belief operators. Although usually a knowledge operator K is accompanied with many compatible belief operators B, Isbell's density theorem guarantees that consistent belief operators B cannot deviate arbitrarily far from knowledge K. Rather, for all topological spaces Stalnaker's operator B_S is always the uniquely determined consistent belief operator that most diverges from knowledge and still being consistent.

Different types of belief operators may be distinguished. For instance, Stalnaker's operator B_S differs topologically considerably from the profusion of operators B_F that are defined with the help of dense subsets F of (X, OX).[10] Still other types of operators exist, but we cannot treat this issue in this paper in any greater depth. Be it sufficient to say that for most spaces (X, OX), the determination of NUC(OX) is difficult and for many spaces NUC(OX) is only partially known. This holds even for familiar spaces such as Euclidean spaces.

For the purposes of this paper, it sufficient to know that nuclei provide the appropriate formalism to deal with issues of the semantics of belief operators:

[10] Indeed, it can be shown that for most spaces the operator B_S is not of the form B_F for any dense subset F of (X, OX).

(3.15) Definition. Let (X, OX, μ, B) a topological model for KB, i.e., (X, OX) a topological space, and B the belief operator i • B • int defined by a dense nucleus B $\in$NUC$(OX)_d$. A formula φ of L_{KB} is defined to be true at a point w of X by induction on the length of φ:

- $w \models$ iff $w \in \mu(p)$;
- $w \models \neg \varphi$ iff not $\models \varphi$;
- $w \models \varphi \wedge \psi$ iff $w \models \varphi$ and $w \models \psi$;
- $w \models K\varphi$ iff $(\exists U \in OX$ (w$\in$ U and $(\forall v \in U)(v \models \varphi)$;
- $w \models B\varphi$ iff $(\exists U \in OX$ (w$\in$ B(U) and $(\forall v\ (v \in U \Rightarrow (v \models \text{int}\varphi)$. ♦

In this definition the only new clause is the last one. Thus, it may be expedient to give the following comment. Note, that the belief operator for the special case of the nucleus B = id boils down to i id int = int, i.e., the last clause of (3.16) coincides with the penultimate one, since the elements $v \in U$ have open neighborhoods U_v where intφ holds. In other words, the topological semantics for KB logic defined by (3.11) is a generalization of the familiar topological semantics of S4.

For the proof of a topological completeness theorem for KB in section 5 we need an equivalent reformulation of the concept of nuclei in order to construct an adequate nucleus for the canonical topological model of KB. More precisely, we have to show that the nuclei of OX are equivalent to the sublocales of OX. For this purpose, we heavily rely on the more detailed presentations of Picado and Pultr (2012, Chapter III) and Johnstone (2002, Proposition 1.1.13, p. 481).

(3.16) Definition. For a topological space (X, OX) a subset $S \subseteq OX$ is a sublocale of OX iff

(i) S is closed under all meets;

(ii) For every $s \in S$ and every $x \in OX$, the Heyting implication $x \Rightarrow s \in S$.♦

(iii) S is a dense sublocale of OX iff $\emptyset \in OX$.♦

Probably the best-known example of a sublocale (not necessarily under this name) is the Boolean lattice of regular open sets $O^*X \subseteq OX$. Actually, $O^*X \subseteq OX$ is a very special sublocale. According to Isbell's theorem for every topological space O^*X is the smallest dense sublocale of OX.

Denote the set of sublocales of OX by SL(OX). Then SL(OX) is a complete lattice with respect to set-theoretical intersection $\cap$. Even more, SL(OX) is a complete co-Heyting algebra with the sublocale {X} as bottom element and the sublocale OX as top element (cf. Picado and Pultr (2012, 3.2.1. Theorem, p.28)). What we need for the proof of the completeness theorem in the next section, is that there is a bijection between nuclei and sublocales (cf. Johnstone (2002, Proposition 1.1. 13). This may be seen as follows: A nucleus $OX \xrightarrow{B} OX$ is uniquely determined by its image $B(OX) \xrightarrow{i} OX$. Thus, a nucleus uniquely determines a sublocale. On the other hand, the inclusion map of a sublocale $S \xrightarrow{i} OX$ has an adjoint frame map $OX \xrightarrow{j} S$ such that the concatenation $OX \xrightarrow{j} S \xrightarrow{i} OX$ is a nucleus. This correspondence is an order reversing bijection between nuclei and sublocales (cf. Johnstone (2002, Proposition 1.1.3., p. 486)), Picado and Pultr (2012, 5.3.2. Proposition. p. 32)). This bijection between nuclei and sublocales yields:

<u>(3.17) Proposition</u>. For all $A \subseteq OX$ there is a (unique) smallest sublocale $S_A \in SL(OX)$ such that $A \subseteq S_A$, namely, the intersection of all sublocales that contain A.

<u>Proof</u>. The class of sublocales that contain A is not empty, since $A \subseteq OX$. Since SL(OX) is complete with respect to arbitrary set-theoretical intersections there is a smallest sublocale S_A that contains A, namely, the intersection of all sublocales that contain A.♦

In the next section, we will construct a canonical topological model (H, OH) for weak KB such that in OH a subset $H_B \subseteq OH$ can be defined that uniquely determines a nucleus B_H of OH that can be used to prove that the weak version of KB defined in (2.2) is complete with respect to the canonical model (H, OH, B_H).

4. A Topological Completeness Theorem for weak KB. The 1-1-relation between nuclei and sublocales explicated in the previous section will be used in the following to construct a canonical topological model (H, OH, B_H) for Stalnaker's combined logic KB of knowledge and belief that can be used to prove a completeness theorem for KB. This proof follows closely the lines of the analogous proof for the standard topological completeness proof of the epistemic logic for K as carried out in Aiello et alii (2003). The only novelty is the construction of an appropriate dense nucleus B_H that takes care of the belief operators B that are compatible with K.

We start with the construction of a topological space (H, OH) for the canonical topological model of KB. Let φ be any well-formed formula of the bimodal extension L_{KB} of classical Boolean propositional logic. Call a set Γ of formulas L_{KB}-consistent if for no finite set $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ we have KB $\vdash \neg (\{\varphi_1 \& \ldots \& \varphi_n)$. A consistent set Γ is called maximally consistent if there is no consistent set of formulas properly containing Γ. Sufficiently many maximally consistent sets of formulas exist due to Lindenbaum's lemma (cf. Blackburn et al. (2001, Lemma 4.17, p. 197). It is well known that Γ is maximally consistent iff for any formula φ of L_{KB}, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, but not both. Now we can define a topological space of maximally consistent sets of formulas:

(4. 1) Proposition. The canonical topological space (H, OH) of L_{KB} is defined by the following items:

(i) H is the set of all maximally consistent sets Γ_{max} of formulas of L_{KB}.

(ii) OH is the set of subsets of H generated by arbitrary unions of the following basic sets S_K: = {[Kφ]; φ is any formula of L_{KB}}, where [φ] := {Γ_{max} ∈H; φ ∈ Γ_{max}}.

Proof. In other words, the basic sets of the topology of H are the families of the form U_ϕ = {{Γ_{max} ∈H; Kφ ∈ Γ_{max}}. (H, OH) is a topological space, called the topological space of the canonical model of L_{KB}. We have to show that S_K is a basis for a topology of H. This is carried out exactly in the same way as is done for the analogous assertion for L_K in Lemma (3.2) in Aiello et alii (2003) by replacing L_K by L_{KB}.♦

It follows that for the operator K of (H, OH) a truth lemma can be proved in the same way as is done in Aiello et al. (2003) for the interior operator of the canonical topological space of S4. The only missing ingredient for a completeness proof of KB is the construction of an appropriate belief operator for (H, OH). This will be carried out now. The key for this construction is the following observation:

(4.2) Lemma. For all formulas φ of L_{KB} one has [Bφ] = [BKφ] = [KBφ], i.e., the sets [Bφ] are basic open sets of the topological space (H, OH).

Proof. By the axiom (PI) of positive introspection and the fact that knowledge implies belief one obtains that Bφ is equivalent to KBφ and to BKφ. Hence, by definition [Bφ] = [KBφ] is an open set of (H, OH).♦

Using the 1-1-relation between nuclei and sublocales proved in section 5 for the set {[Bφ]} ⊆ OH there exists a minimal sublocale that contains the open subsets [Bφ] ∈ OH. This sublocale uniquely determines a nucleus that will also be denoted by B. The next step in proving the completeness theorem is to prove that not only K, but also B satisfies a truth lemma for (H, OH).

As explained in the last part of section 3 the set $\{[B\varphi];\ \varphi \in L_{KB}\} \subseteq OH$ uniquely defines a sublocale and therefore a nucleus of OH that also is denoted as B, i.e., B Clearly, $B \in NUC(OH)_d$, i.e., B is a dense nucleus, since $[B(\varphi \wedge \neg\varphi)] = [B(\emptyset)] = [\emptyset] = \emptyset$. We will show that B satisfies a truth lemma for the canonical space (H, OH). This is done by observing that due to the axioms (2.1) of KB one has $B\varphi = BK\varphi = KB\varphi$. Thus, in the class of basic open sets $\{[K\varphi];\ \varphi \in LK_B\}$ there is the subclass $\{[B\varphi];\ \varphi \in L_{KB}\}$ of basic open sets of the form $[B\varphi]$.

The lattice SL(OX) of sublocales of OX is closed with respect to arbitrary intersections. Hence the sublocale O_BX of all sublocales that contain all elements of $\{[B\varphi];\ \varphi \in LK_B\}$ exists. As a sublocale, O_BX uniquely determines a dense nucleus that may still be denoted by B. Finally, this nucleus B defines the belief operator $i \bullet B \bullet int$ that is also denoted by B.

After these preparations, we can define the canonical topological model of L_{KB} as follows:

<u>(4.3) Proposition</u>. The canonical topological model of KB is defined as $M = (H, OH, B, \mu)$

(i) The elements of H are the maximally consistent sets of formulas Γ_{max}.

(ii) The topology OH is generated by the basis of open sets $\{[K\varphi];\ \varphi \in L_{KB}\}$.

(iii) The belief operator B is defined by the nucleus generated by $\{[B\varphi];\ \varphi \in L_{KB}\}$.

(iv) $\mu(\varphi) := \{\{\Gamma_{max};\ \Gamma_{max}$ is a maximally consistent set of formulas of L_{KB} with $\varphi \in \Gamma_{max}\}$.

<u>Proof</u>. The proof that (H, OH) is a topological space is completely analogous to the proof of the analogous assertion for the canonical topological space for S4 proved in Aiello et al.

(2003).[11] Since $B \in NUC(OH)_d$ is clearly a dense nucleus, the assertion that (K, B) satisfies the rules and axioms of a weak Stalnaker system follows from (3.13).♦

(4.4) Theorem (Truth Lemma TL). Let (H, OH, B, μ) be the canonical topological model of KB. For all modal formulas φ of L_{KB} and $w \in H$ one has:

$$w \models_{LKB} \varphi \text{ iff } w \in [\varphi].$$

Proof. Induction of the complexity of φ. The base case follows from the definition from the first clause of (3.16). The case of the Booleans is also well known, see (Aiello et al. (2003, p. 895). The interesting cases are the modal operators K and B. The proof for K is just a rehearsal of the well-known proof of the truth lemma for the unimodal case for K given in Aiello et al. (ibid.). Thus, it only remains to prove TL for B. The proof is naturally divided into two parts: (1) "From truth to membership" (If $w \models_{LKB} \varphi$ then $w \in [\varphi]$) and (2) "From membership to truth" (If $w \in [\varphi]$ then $w \models_{LKB} \varphi$).

ad (1) From truth to membership: Assume $w \models B\mathrm{int}\varphi$. That means that there is a $U \in OX$ such that $w \in BU$ & $\forall v(v \in U \Rightarrow v \models \mathrm{int}\varphi))$. Since the TL holds for K one obtains that this is equivalent to that there is $U \in OX$ ($x \in BU$ & $\forall v(v \in U \Rightarrow v \in K\varphi)$. This means that $U \subseteq [K\varphi]$. Due to the soundness of KB this entails that in S4 the implication $U \rightarrow [\mathrm{int}\varphi]$ is valid. By (3.5)(i) a nucleus B is inflationary. Hence, *a fortiori* $B(U \rightarrow [\mathrm{int}\varphi])$. Since B is a normal operator (cf. (3.14)), this entails that $B(U) \rightarrow B([\mathrm{int}\varphi]))$ is valid. Since KB is sound with respect to topological spaces *cum* nucleus, this entails $[BU] \subseteq [B\mathrm{int}\varphi]$. This implies that $w \in [BU]$ implies $w \in [B\mathrm{int}\varphi]$. In other words, „Truth implies membership".

[11] This topology is more precisely described in Aiello et al. (2003, p. 896): The canonical topology on canonical space is the intersection of the Kripke and Stone topology. This entails that this space is compact and dense-in-itself.

ad (2) From Membership to Truth: Proof by induction on the complexity of formulas and reductio ad absurdum. Assume $w \in [\mathrm{Bint}\varphi]$, and assume that the first part of TL "From membership to truth" has been proved for B. Suppose $w \not\models \mathrm{Bint}\varphi$. Then by definition of $w \not\models \mathrm{Bint}\varphi$ one has

(i) $$\text{NOT } (\exists U \in OX\ (w \in BU\ \&\ \forall v(v \in U \Rightarrow v \models \mathrm{int}\varphi))$$

This is equivalent to

(ii) $$\forall U \in OX\ (w \notin B(U) \text{ OR } \exists v\ (v \in U\ \&\ v \not\models \mathrm{int}\varphi))$$

In order to carry the reductio one has to find a U for which (ii) is false. Obviously, this is the case for $U = \mathrm{int}\varphi$, since we have assumed that $w \in [\mathrm{Bint}\varphi]$. One has $[\mathrm{int}\varphi] \subseteq [\mathrm{Bint}\varphi]$ since nuclei are inflationary by definition (3.5) and we can assume that $v \models \mathrm{int}\varphi$ iff $v \in [\mathrm{int}\varphi]$. In other words, the reductio ad absurdum has been carried out and thereby the proof of the truth lemma for K and B is completed.

(4.5) Completeness Theorem for a weak version of Stalnaker's logic KB. For any consistent set of formulas Γ of L_{KB} one has

$$\text{If } \Gamma \models \varphi \text{ then } \Gamma \vdash_{KB} \varphi.$$

Proof. Suppose that NOT($\Gamma \vdash_{KB} \varphi$). For the proof of (4.6) we have to prove that this supposition entails NOT($\Gamma \models \varphi$). Then $\Gamma \cup \{\neg \varphi\}$is consistent, and by a Lindenbaum Lemma it can be extended to a maximally consistent set $\Gamma_{max} \in H$ with $\{\neg \varphi\} \in \Gamma_{max}$, i.e., $\Gamma_{max} \in [\neg \varphi]$. According to the truth lemma (4.5) for (H, OH), this is equivalent to $\Gamma_{max} \models \neg\varphi$, whence NOT($\Gamma_{max} \models \varphi$). and we have constructed the required counter-model.♦

5. Concluding Remarks. The interpretation of topological nuclei B $\in$ NUC(OX) as belief operators B not only gives us a natural proof of a topological completeness theorem for Stalnaker's combined logic of knowledge and belief, more generally it shows that the framework of topology offers many more possibilities for formal epistemology and epistemological logic than one might have thought. Not all of these formal possibilities may be epistemologically and logically meaningful.[12] To find out which ones are meaningful, is a matter of future empirical research, so to speak. In any case, taking into account nuclei leads to a pluralist conceptualization of the relation between knowledge and belief that is opposed to more traditional accounts that either characterize knowledge as a special kind of belief (knowledge as true justified belief) or belief as a kind of restricted knowledge (as is proposed, for instance, by Stalnaker who proposes to conceive belief as knowledge of the possibility of knowledge, i.e., B = intclint). The topological account sketched in this paper is more flexible: according to it, the knowledge operator K provides a framework for the definition of a numerous family of belief operators all of which fit K in the sense that all pairs (K, B) satisfy the rules and axioms of weak KB logic. This could be used to define multimodal systems $KB_1B_2 \ldots B_n$ with "competing" belief operators $B_i \in NUC(OX)_d$ that all share a common knowledge basis defined by K.

The topological epistemology sketched in this paper may be characterized as a hierarchical knowledge first epistemology: Its basic level is given by a topological structure (X, OX) defined by the knowledge operator K itself. The next level of topological epistemology is given by a "derivation" of the topological structure (X, OX), namely, the complete Heyting algebra

[12] A case in question is the following: Nuclei can be defined not only for topological Heyting algebras OX but for all complete Heyting algebras H whatsoever, for instance for NUC(OX). Hence, one can define a tower NUC(OX)), (NUC(NUC(OX))), … . Thereby, the higher floors of this tower can be interpreted as beliefs about beliefs etc. Such a tower of nuclei has, or does not have, an infinite height, depending on the complexity of the topological space (X, OX). An extreme case is the (not very plausible) epistemic logic of knowledge S5, for which one obtains OX = NUC(OX) = NUC(NUC(OX)) = … = NUC^n(OX) for all n.

NUC(OX) of nuclei. The belief operators B that are compatible with K live on this floor. The lattice NUC(OX) of belief operators has a very complex structure even for familiar apparently simple spaces (X, OX) such as the Euclidean spaces. A more detailed investigation is an issue of future research. Several types of belief operators can be distinguished. Among them are belief operators defined by dense subsets F of (X, OX) and, particularly important, iStalnaker's operator B_S distinguished as the riskiest, but still consistent belief operator that is compatible with K.

References.

Aiello, M., van Benthem, J.F.A.K., Bezhanishvili, G., 2003, Reasoning about Space: The Modal Way, Journal of Logic and Computation 13(6), 889 - 920.

Baltag, A., Bezhanishvili, Özgün, A., Smets, S., 2014, The topology of full and weak belief. In Proceedings of the 11th International Tbilisii Symposium on logic, language, and computation (TbiLLC 2015, revised selected papers (pp. 205 – 228).

Baltag, A., Bezhanishvili, Özgün, A., Smets, S., 2019, A Topological Approach to Full Belief, Journal of Philosophica Logic, 48, 205 – 244.

Blackburn, P., de Rijke, M., Venema, Y., 2010, Modal Logic, Cambridge, CUP.

Chagrov, A., Zakharyaschev, M., 1997, Modal Logic, Oxford, Clarendon Press.

Johnstone, P., 1982, Stone Spaces, Cambridge, Cambridge University Press.

Johnstone, P., 2002, Sketches of an Elephant. A Topos Theory Compendium, Volume 2, Oxford, Clarendon Press.

Macnab, D.S., 1981, Modal Operators on Heyting Algebras, Algebra Universalis 12, 5 – 29.

Picado, J., Pultr, A., 2012, Frames and Locales. Topology without Points, Birkhäuser.

Simmons, H., 1977, A Framework for Topology, in A. MacIntyre, L. Pacholski, J. Paris (eds.), Logic Colloquium '77, Amsterdam, North-Holland Publishing Company, 239 – 251.

Simmons, H., 1980, Spaces with Boolean Assemblies, Colloquium Mathematicum XLIII(1), 23 – 39.

Stalnaker, R., 2006, On Logics of Knowledge and Belief, Philosophical Studies 128, 166 – 199.

Willard, St., 2004, General Topology, Dover Publications, New York.