Description, Construction, and Representation.
From Russell and Carnap to Stone

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Abstract . The first aim of this paper is to elucidate Russell’s construction of spatial points, which is to be considered as a paradigmatic case of the "logical constructions" that played a central role in his epistemology and theory of science. Comparing it with parallel endeavours carried out by Carnap and Stone it is argued that Russell’s construction is best understood as a structural representation. It is shown that Russell’s and Carnap’s representational constructions may be considered as incomplete and sketchy harbingers of Stone’s representation theorems. The representational program inaugurated by Stone’s theorems was one of the success stories of 20th century’s mathematics. This suggests that representational constructions à la Stone could also be important for epistemology and philosophy of science. More specifically it is argued that the issues proposed by Russellian definite descriptions, logical constructions, and structural representations still have a place on the agenda of contemporary epistemology and philosophy of science. Finally, the representational interpretation of Russell’s logical constructivism is used to shed some new light on the recently vigorously discussed topic of his structural realism.

1. Introduction . According to Russell "[t]he subject of denoting is of very great importance, not only in logic and mathematics, but also in the theory of knowledge" (*On Denoting,* 41). As one of the great achievements of Russell's essay one may consider the fact that he showed how "definite descriptions" help clarify the meaning of otherwise obscure denoting phrases such as "the present king of France", or "the centre of mass of the solar system at the first instant of the 21st century". Definite descriptions are a special kind of logical constructions. In this paper I'd like to concentrate on a type of logical construction that was especially important for Russell's epistemology and philosophy of science, namely the constructions of what may be called in non-Russellian terms theoretical entities such as spatial points, temporal instants, or particles. More precisely, in this paper I'll deal with Russell's construction of space points as sketched in *Our Knowledge of the External World* (Russell 1914) and carried out more fully in *Analysis of Matter* (1927).[1]

For Russell, the role of logical constructions in all areas of philosophy and science was guided by two principles, the "fundamental epistemological principle" and the "supreme maxim of scientific philosophizing". According to the first the following holds:

> "In every proposition that we can apprehend (i.e. not only in those whose truth or falsehood we can judge of, but in all that we can think about), all the constituents are really entities with which we have immediate acquaintance." (*On Denoting,* 56).

Not all we know we know by acquaintance. Particularly in science, a large part of our knowledge is indirect knowledge, or, as Russell put it, knowledge by description. The principal means to reach things for which we cannot obtain direct knowledge of, i.e., knowledge by acquaintance, are denoting phrases which, through logical analysis, can be reconstructed as a special kind of logical constructions, to wit, as definite descriptions. Although science has often been hailed to be the domain of knowledge par exellence, most scientific knowledge is (indirect) knowledge by description, whose role Russell described as follows:

> "The chief importance of knowledge by description is that it enables us to pass beyond the limits of our private experience. In spite of the fact that we can only know truths which are wholly composed of terms which we have experienced in acquaintance, we can yet have knowledge by description of things which we have never experienced. In view of the very narrow range of our immediate experience, this result is vital, and until it is understood, much of our knowledge must remain mysterious therefore doubtful." (*Problems of Philosophy* , 32).

Similarly as for denoting phrases the meaning of apparently irreducibly theoretical terms in physics such as "point", "instant", or "electron" are claimed to be logically reconstructible in terms of components we

[1] For a detailed discussion of Russell's construction of temporal points, i.e., instants, see Anderson (1989).

are acquainted with. This was expressed by the second principle of his epistemology, the "supreme maxim of scientific philosophizing", first stated explicitly in *The Relation of Sense-Data to Physics*: "Wherever possible, logical constructions are to be substituted for inferred entities." (Russell 1914, 149). The "supreme maxim" served as a kind of Occam's razor that cut the class of inferred entities to a minimum ensuring thereby that our claims for indirect knowledge by description were more than mere metaphysical speculations.

Russell's theory of scientific knowledge may be characterized in general terms as aiming to secure the move from perception to the objects of physical theory. As such, this aim was hardly original. Many a philosopher in those days subscribed to it in one way or other. For instance, the logical empiricists were struggling for decades to elucidate the intricate relation between the observational and the theoretical, but already the Neokantian philosopher of science Cassirer had claimed that the essential problem of philosophy of science was to understand how to move from "percepts" to "concepts" (cf. Cassirer 1910). The originality of Russell's theory of knowledge resided in the emphasis he laid upon the role of logic and mathematics for this endeavour.

Following Grayling one may distinguish two ways of how Russell thought this aim could be achieved: "[Either this move must] be inferential, in which it takes us from the incorrigible data of sense to something else, or it is analytic, that is, it consists in a process of constructing physical entities out of percepts." (Grayling 2003, 468). In this paper I'll concentrate on the analytic approach as Grayling calls it, which dealt with the construction of physical entities out of entities we are acquainted with. Recently, Demopoulos made the interesting proposal to conceive Russell's method of logical construction in terms of representation (Demopoulos 2003, 412f). For spacetime theories this requires that every abstract model of such a theory have an isomorphic representation constructed in terms of compresent (more precisely, copunctual) events. Thus reformulated, "the program of logical construction is a now familar part of the nature and methodology of representation theorems, a part which Russell understood very well." (ibidem). I do not fully share Demopoulos's positive assessment with respect to Russell's understanding of the representational paradigm. As I'll argue in this paper, Russell's representational construction of space-time points was mathematically flawed and sometimes conceptually obscure. Hence, it is well worth the effort to attempt a clarification of the concept of structural representation Russell and other philosophers such as Carnap were after in the first decades of the last century. I contend that the concept of structural representation and its conceptual ramifications were not too well understood in the early 20th century, not even in mathematics, to say nothing about philosophy. More specifically, I contend that the first profound theorem on matters representational ever proved in mathematics was Stone's Topological Representation Theorem of Boolean Algebras in his epoch-making papers *The Theory of Representation of Boolean Algebras* (Stone 1936) and *Applications of the Theory* *of Boolean Rings to General Topology* (Stone 1937).[2] The point I want to make is that Russell's construction of spatial points as well as Carnap's construction of qualities out of elementary experiences are best understood as sketchy and incomplete forerunners of what was carried out sucessfully by Stone a few years later.[3] To put it bluntly, from that time onwards Stone's work had set the benchmark of what was to be understood by structural representation. Russell's construction of points out of "events" and Carnap's constitution of quasiproperties out of "elementary experiences" aimed at something similar, but achieved it only partially.

While Stone's representational program was extremely sucessful in mathematics (cf. Johnstone (1982), Mac Lane (1970), the fledgling representational current in philosophy fared less well. Today, the program of logical or mathematical constructions as inaugurated by Russell in *Our Knowledge of the External World* and and pushed further by Carnap's *The Logical Structure of the World* is generally thought to be a proven failure. Maybe this dismissal was a bit too hasty, due to a too superficial assessment of the conceptual possibilities of the representational approach. Since Stone representational reconstruction has become a complex and intricate concept that can hardly be dismissed in the chivalresque manner many philosophers seem to believe, or so I want to argue.

The outline of this paper is as follows. To set the stage we succinctly present Stone's trail-blazing construction of a set-theoretical representation of Boolean algebras in section 2. Stone's representational account provides a precise conceptual framework in which the achievements and deficiencies of the representational enterprises of Carnap and Russell can be discussed in detail. That is to say, I propose to assess the fragmentary and sometimes obscure representational endevours of Russell and Carnap from the perspective of Stone's mathematically and conceptually mature account. In section 3 it is shown that Carnap's early construction of (quasi)qualities of 1923 can be interpreted as a fragment of Stone's representation of 1936. The topic of section 4 is Russell's sketch of the logical construction of points out of a system of "events" as carried out in *Analysis of Matter*. It can be shown that Russell's construction bears a striking similarity with that Stone carried out a decade later, in particular with his topological representation of 1937. Moreover, some of the shortcomings that plagued Russell's account can be attributed directly to his too close observance to the principle that logical constructions have to be based

[2] Of course, this contention has to be taken with a grain of salt: beside Stone there were other mathematicians who can claim to have made substantial contributions to matters representational, e.g. Birkhoff. In this paper, I am not interested in historical accuracy in the first line. Rather I propose to consider Stone as a paradigmatic example.

[3] As far as I know, no philosopher ever took notice of Stone's work during his lifetime. The closest link between Stone on the one hand, and logic and analytic philosophy on the other, seems to have been Tarski who knew Stone's work on Boolean algebras well. Indeed, Tarski's investigations on relational algebras (Tarski 1941) may be understood as a far-reaching generalization of Stone's work. Tarski mentioned the representation problem for relational algebras with direct reference to Stone's representation (ibidem, 546). One might expect a closer relation between the work of Stone and Whitehead, but I have not been able to find evidence for this. Without doubt, Whitehead had a profound influence on Russell in matters of logical constructions. For reasons of space, I cannot deal with it in this paper.

upon entities we know by acquaintance. In contrast to Russell's, Carnap's approach was less hampered by philosophical prejudices, but rather by its excessive generality which rendered it impossible to fully exploit the representational resources available. In section 5 the representational perspective on logical constructions is used to shed new light on some of the epistemological and metaphysical questions concerning Russell's constructional program, in particular on the recently vigorously debated problem of how to assess Newman's anti-structuralist argument against the feasibility of Russell's structural realism. We close with some general remarks on the feasibility of a representational approach in section 6.

2. Stone's Representation of Boolean Algebras.[4] In this section I'd like to present as succinctly as possible one of the most important theorems of 20th century's mathematics, to wit, Stone's representation theorem of Boolean algebras.[5] According to it every abstract Boolean algebra can be represented as subalgebra of a set-theoretical Boolean algebra such that the Boolean operations of meet, join and complement are represented by the set-theoretical operations of intersection, union, and set-theoretical complement. The reason for outlining a modernized version[6] of Stone's representational account is to show what Russell and Carnap could have achieved if the modern conceptual apparatus had been available to them. Stone's representational account offers a perspective to better understand what Russell and Carnap were after in their representational constructions. To put it in a nutshell, in matters representational Carnap and Russell were Stone's precursors without knowing it.

There are essentially two (equivalent) ways of defining Boolean algebras: either as special posets $(B, \leq)$ or as a relational structures of type $(B, \wedge, \vee)$, $\wedge$ and $\vee$ being the binary operations of meet and join satisfying some familiar axioms.[7] Both characterizations are related by the equivalence $a \leq b \Leftrightarrow a = a \wedge b$.

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[4] For this section I am very much indebted to Hannah Mormann. She helped me to get acquainted with Stone's *The Theory of Representation of Boolean Algebras* (1936) and *Applications of the Theory of Boolean Rings to General Topology* (1937).

[5] For some compelling arguments for this assessment, see Johnstone (1982).

[6] As is well known, the representation theory of Boolean algebras can be carried out either in terms of ideals or filters. Both accounts are strictly equivalent, since for Boolean algebras a maximal filter is just the set-theoretical complement of a maximal ideal. In his papers Stone used throughout the language of ideals. In particular he proved the crucial result that a Boolean algebra has enough prime ideals to ensure the existence of a faithful set-theoretical representation. In his (2003) Demopoulos asserts that Stone proved what today is often called the Ultrafilter Theorem for Boolean algebras (ibidem, Footnote 26). Strictly speaking, this is not the case: Stone proved the corresponding theorem for maximal ideals (cf. Stone 1936, Theorem 64, 105). Demopoulos rightly asserts that Russell's theorem on the existence of points as maximal classes of compresent events has a striking similarity with the Ultrafilter theorem. Since Russell's "points" and Carnap's "similarity circles" closely resemble maximal filters, it is expedient to present Stone's account in the garb of filters and not in its original version based on ideals. A slick presentation of Stone's results in terms of filters can be found in Koppelberg (1985), a more leisurely pace in terms of ideals is offered to the reader in Davey and Priestley (1990). Today, the topological applications of Stone's theorem, which established deep but quite unexpected relations between algebra and topology are considered as the most spectacular ones (cf. Stone 1937). Surprisingly, Russell's construction of points snugly fits in this topological version of Stone's theorem.

[7] For precise definitions one may consult any textbook, e.g. Davey and Priestley (1990), Halmos (1963), or Koppelberg (1989).

A Boolean algebra defined in one of these ways may be called an abstract[8] Boolean algebra because its order relation and its operations are defined abstractly in purely relational terms. Concrete Boolean algebras arise in the following way: If X is any set denote the set of all subsets of X by PX. The set PX becomes a Boolean algebra (PX, ≤) by defining the set-theoretical inclusion $\subseteq$ as the order relation ≤. A Boolean algebra B isomorphic to a subalgebra of the powerset PX of some set X is called a concrete or set-theoretical Boolean algebra. All finite Boolean algebras are concrete Boolean algebras. More precisely, if B is a Boolean algebra with 2^n elements it is isomorphic to the power set P({1, 2, …, n}) of the set {1, 2, …, n}. Hence, it is natural to ask if every Boolean algebra is isomorphic to a subalgebra of a Boolean algebra of type PX. Stone's representation theorem for Boolean algebras answers this question positively. More precisely it constructs for any abstract Boolean algebra B a set X and a structure-preserving representation B---r--->PX such that B is isomorphic to the subalgebra r(B) of PX. Before we discuss the proof of Stone's theorem in some detail it should be noted that the representation theorem is a non-elementary theorem in the sense that for its proof the axiom of choice (AC) or some similar principle such as Zorn's Lemma has to be used. The first step for the proof of the theorem is the introduction of the concept of a filter:

(2.1) Definition . Let (B, $\leq$) be a Boolean algebra. A subset F $\subseteq$ B is a filter if and only if the following conditions are satisfied:

(1) Ø ≠ F.

(2) F is upward closed, i.e., $x \in F$ and $x \leq y$ imply $y \in F$.

(3) F satisfies the finite intersection property, i.e., $x, y \in F$ entails $x \wedge y \in F$.

A maximal filter (ultrafilter) is a filter that is not properly contained in any other filter.

Every $b \in B$ defines a filter called the principal filter F(b) defined by F(b) := $\{x; b \leq x \in B\}$. In general, F(b) is not a maximal filter. Without further assumptions it is not at all clear that a given Boolean algebra has any maximal filter. To ensure the existence of maximal filters one has to rely on the axiom of choice or a similar principle (cf. Davey and Priestley 1990, chapter 9). With the help of such a principle it is possible to prove the following proposition that is crucial for the construction of Stone's representation:

(2.2) Proposition . Any filter of a Boolean algebra B is contained in a maximal filter.

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[8] This terminology can be traced back to Stone.

Now we are able to describe the basic ingredient of Stone's representation theorem, to wit, the set of points some of whose distinguished subsets will serve as set theoretical representatives of the elements of B. By definition (2.1) a maximal filter of B is an element of the power set PB of B. Hence one may define a set-theoretical representation B-------r----->PPB by defining

(2.3) $\quad r(b) := \{F; b \in F, F \text{ is a maximal filter of } B\}.$

The map (2.3) is often called the Stone map. According to (2.2) r(b)≠ Ø for all b ∈ B: by definition b ∈ F(b) ⊆ F, F some maximal fillter. Before we go on, the following remark may be in order. At first look, this representation, based on the elusive concept of maximal filter, looks to be a rather unwieldy construction. It is therefore remarkable that Carnap and Russell, apparently independent of each other and almost ten years before Stone, used exactly the same construction.

Informally, an element b of B is represented by the set of all maximal filters that contain it. This representation is order-preserving in the sense that if b ≤ c then F(b) ⊆ F(c) by (2.1)(ii). Order-preservation alone, however, is no guarantee that the representation r in (2.3) is interesting. As long as we do not know that B has enough maximal filters, it may happen that r is pretty trivial. Fortunately, proposition (2.2) can be strengthened to ensure that there are sufficiently many maximal filters in the sense that for two distinct elements of there is a maximal filter F that contains one of them but not the other. It should be noted that already Russell was aware that is was necessary to prove the existence of certain maximal classes of compresent events that could serve as logical constructions of "points" (cf. AMA, 299/300). Indeed, he proved his existence theorem with the aid of Zermelo's well-ordering theorem (which is well-known to be equivalent with the axiom of transcendental induction used by Stone). On the other hand, Carnap took the existence of maximal similarity classes (his analogues of maximal filters) for granted.

Having established the fact that there are enough maximal filters one proves that the representation r is a monomorphism in the sense that r(b) = r(c) entails b = c. Next we have to show that r not only preserves the order relation ≤, but also the Boolean operators ∧, ∨, and *. This is done by exploiting the fact that for Boolean algebras maximal filters are prime (cf. Davey and Priestley 1990, Theorem 9.8, 186). Thereby we eventually can prove

<u>(2.4) Theorem (Stone's Representation Theorem (Set-theoretical Version) .</u> Let B be a Boolean algebra. Denote the set of maximal filters of B by MAX(B) and the Stone map as defined in (2.3) by r. Then B has a faithful set-theoretical representation B-----r----->P(MAX(B)) with the following properties:

(1) r is an order-preserving 1-1 map,i.e., a ≤ b if and only if r(a) ⊆ r(b).

(2) r is a Boolean homomorphism in that the Boolean operations $\wedge$, $\vee$, and * of B are represented by the corresponding set-theoretical operations.

Actually these assertions only form a part of Stone's results in his (1936). Moreover, in his (1937) he showed that the representing sets r(b), b $\in$ B, could be characterized topologically. More precisely, he pointed out that the r(b) figured as the open and closed ("clopen") sets of a certain topology. In other words, the representation r was not only a set-theoretical representation but actually a topological representation. This result has no counter-part in Carnap's representational construction of qualities, but, as we shall see, traces of a topological interpretation of event structures may be found in Russell's representational construction of points.

3. Carnap's Representational Construction of Qualities . Russell's influence on early Carnap, and particularly on the *Aufbau* is a matter of dispute. While Quine did not hesitate to interpret the *Aufbau* as an attempt to execute the epistemological program Russell had layed out in *Our Knowledge of the External World* a more careful reading of the *Aufbau* shows that it is positively misleading to "Russell" Carnap's *Aufbau* in the straight-forward way advertised by Quine and other empiricists (cf. Richardson 1990). More recently, Pincock argued that Quine as well as the revisionist exegetes such as Richardson misconstrued the complex relation between the *Aufbau* and Russell's reconstructivist program (cf. Pincock 2002). In this paper I don't want to take side in this issue. Rather, I'd like to concentrate on an early (unpublished) manuscript of Carnap's (Carnap 1923) that may be considered as precursor of the *Aufbau* and that does square quite well with Russell's program. It may be considered as a direct application of his "supreme maxim of scientific philosophizing". This assessment is not intended to contend anything concerning the more complex problem of how to characaterize the intricate relation between Carnap's and Russell's philosophical stances in general.

After these disclaimers let us begin with the real thing, to wit, a representational interpretation of Carnap's earliest attempt of a logical construction of qualities (cf. Mormann 1994). This endeavour was set out in his unpublished manuscript *Die Quasizerlegung - Ein Verfahren zur Ordnung nichthomogener Mengen mit den Mitteln der Beziehungslehre (Quasi-Division - A Method for Ordering Non-Homogeneous Sets by Means of the Theory of Relations)* (Carnap 1923). The task of the method proposed by Carnap is best explained by invoking the "principle of abstraction" as explained in *Our Knowledge of the External World*:

> "When a group of objects have that kind of similarity which we are inclined to attribute to possession of a common quality, the principle in question (i.e. the principle of abstraction, T. M.) shows that membership of the group will serve all the purposes of the supposed common quality, and that therefore, ... the group or class of similar objects

> may be used to replace the common quality, which need not be assumed to exist." (OKEW, p. 51).

In line with Russell's program Carnap described the aim of *Quasizerlegung* in more detail as follows:

> "Suppose there is given a set of elements, and for each element the specification to which it is similar. We aim at a description of the set which only uses this information but ascribes to these elements quasicomponents or quasiqualities in such a way that it is possible to deal with each element separately using only the quasiqualities, without referene to other elements." (Carnap 1923)

More formally the task of *Quasizerlegung* can be described as follows: Given a similarity structure (S, ~), i.e., a set of objects endowed with a reflexive and symmetric relation ~. The relation ~ is to be interpreted as a similarity relation: a ~ b this is to be interpreted as the fact that a and b are similar. Then the task is to construe qualities q such that any two elements a and b of S that are similar to each other share a common quality q; if, however, a and b are not similar, they do not share a quality q. In line with the supreme maxim of scientifique philosophizing these qualities should not simply be inferred, rather they have to be constructed. This meant they had to be characterized extensionally, i.e., as sets of elements that have this quality. Denoting the power set of S by PS the correspondence between elements of S and their qualities may be conceived as a function S----r--->PPS by which every element $a \in S$ is represented by the set r(a) := {q; q quality of r}. Up to now we have only said that a quality q is to be characterized extensionally as a subset of S. This is quite vague. Not any subset of S will be acknowledged as a quality. Rather, a quality q is to be characterized as a subset $q \subseteq S$ that is compatible (in a sense to be specified in a moment) with the similarity relation ~ defined on S. Carnap proposed to take as qualities so called *similarity circles*:

<u>(3.1) Definition</u>. Let (S, ~) be a similarity structure. A subset $q \subseteq S$ is a <u>similarity circle</u> of (S, ~) iff it is a maximal set of similar elements, i.e., iff it satisfies the two conditions:

(1) All elements of q are similar.

(2) No element outside q is similar to every element of q.

Carnap took the existence of similarity circles for granted. This is justified only for finite similarity structures. To prove the existence of similarity circles for similarity structures with infinitely many elements, one has to rely on the axiom of choice, or, equivalently and conceptually simpler, on Zorn's Lemma. The proof is elementary and we need go into it here. Rather, under the assumption that the existence of similarity circles has been ensured in one way or other, the main result of *Quasizerlegung* can be stated as follows:

(3.2) Theorem . Let (S, ~) be a similarity structure, and a, b $\in$ S. Define the map S ----r----> PPS by r(a) := {q| q is a similarity circle and a $\in$ q}. Then a and b are similar if and only r(a) $\cap$ r(b) ≠ Ø. Moreover, if a is similar to all elements to which b is similar, then r(a) $\subseteq$ r(b).

The first thing to note is that this theorem is to be read as a representation theorem. It asserts that every abstract similarity structure (S, ~), as Stone might have said, has a faithful set-theoretical representation S----r----->PPS in the sense that the abstract similarity relation a ~ b is represented by the concrete set-theoretical relation of non-empty intersection r(a) $\cap$ r(b) ≠ Ø of elements as defined in PPS. Already this observation suggests that Carnap's project bears a certain similarity with that of Stone. Indeed, it can be shown that Carnap's theorem (3.2) is a fragment of Stone's representation theorem (2.4). In order to understand why this is the case the following remarks may be in order: First observe that Boolean algebras (B, $\leq$) may be considered as a special class of Carnapian similarity structures: Every Boolean algebra (B, $\leq$) gives rise to a similarity structure (B, ~) by defining a ~ b := a $\wedge$ b ≠ 0 or a = b = 0.[9] Then the Boolean partial order $\leq$ can be reconstructed from the similarity relation ~ by defining a $\leq$ b := {x; x ~ a} $\subseteq$ {x; x ~ b}. Hence, Boolean algebras may be conceived as special similarity structures. Moreover, the relation between maximal filters (defined on Boolean algebras (B, $\leq$) and similarity circles (defined for similarity structures (B, ~)) can be explicated as follows:

(3.3) Proposition . Let (B, $\leq$) be a Boolean algebra and (B, ~) be the corresponding similarity structure. Then a maximal filter F of (B, $\leq$) is a similarity circle of (B, ~) which satisfies the finite intersection property, i.e., that a, b $\in$ F entails a $\wedge$ b $\in$ F.

(3.4) Corollary . Conceiving a Boolean algebra (B, $\leq$) as a similarity structure (B, ~) Stone's representation B---r--->PMAX(B) is a special case of Carnap's representation B-----> PPB mapping the elements b of B to the sets r(b) of maximal similarity circles containing them and satisfying the finite intersection property.

Thus, conceiving Boolean algebras as special similarity structures reveals that Carnap and Stone were engaged in essentially the same project, to wit, the construction of set-theoretical representations of similarity structures. Their projects differed in that Stone's was much more specific: he was interested in a special class of similarity structures, to wit, Boolean algebras.[10] Moreover his representations were more

[9] As we shall see in the next section, this similarity relation corresponds exactly to Russell's relation of "overlapping".
[10] As a direct effect of this concentration on "Boolean similarity structures" one may consider the introduction of the finite intersection requirement as an explicit condition on "admissible" similarity circles.

sophisticated since he went on to show that they were not just a set-theoretical representations but topological ones that described the represented elements in topological terms (cf. Stone 1937).

To rule out any misunderstandings: Corollary (3.4) does not contend that Carnap was Stone's forerunner in giving a full-fledged proof of the latter's representation theorem for Boolean algebras. Carnap never considered Boolean algebras as a special case of similarity structures and he never undertook the slightest efforts to couch his representation in topological terms. Moreover, Stone was well aware of the fact that he was engaged in "building a general abstract theory [of representation] and must accordingly be occupied to a considerable extent with the elaboration of technical devices" (Stone 1937, 376). In contrast, Carnap in his representational entreprises employed only a rather modest formal apparatus. Nevertheless it seems justified to ascribe to him the achievement of having proved a fragment of Stone's representation theorem. By hindsight, one may say that the main drawback of Carnap's quasi-analytical constructional system put forward in *Quasizerlegung* was its excessive generality. He should considered more specific kinds of similarity structures for which more specific results could have been obtained.

4. Russell's Representational Construction of Points . It is not easy to fairly treat Russell's "construction of points" as set out in chapter xviii of *The Analysis of Matter*. His mathematical contentions are often vague and sometimes strictly interpreted outright mistaken. On the other hand, his programmatic statements concerning the tasks his logical constructions were to achieve are lucidly formulated and sometimes far ahead of his times. In order to come to grips mathematically with Russell sketch of a representational constitution of space points in terms of events I propose to cast his approach in terms of Stone's representational theory. This needs some preparation.

Russell's starting point for the representational constitution is a class of "events" characterized vaguely as a class of well-formed spacetime regions. In the terminology of the *Aufbau* "events" correspond to Carnap's "Elementarerlebnisse". They may be considered as the "Grundelemente" of Russell's system. On the set E of events there is defined the relation of overlapping ("compresence"). This relation enjoys the relational properties that are to be expected intuitively from such a relation, i.e. compresence is a reflexive and symmetric relation. Hence, ignoring its implicit geometrical connotations, Russell's system E of events, endowed with the relation of overlapping, is just a similarity structure (E, ~) in Carnap's sense. Recalling that also Boolean algebras can be conceived as special similarity structures one may say that Stone, Carnap, and Russell grounded their representational constructions on the very same formal base. Not only this, Russell started his constructional endevour by proposing to define a "point" as a maximal group of compresent events in the same way as Carnap had defined a quality as a maximal similarity circle:

(4.1) Definition . A point is a group G of events having the following properties:

(1) Any two members of the G are compresent.

(2) No event outside G is compresent with every member of G.

According to Russell, this definition works for the 1-dimensional case in the sense that for a group of compresent 1-dimensional events (conceived as intervals of the real line) "there will be some place ... which is occupied by all of them." (AMA, 294). The following example shows that this is not the case:

(4.2) Example . Let **R** be the real line endowed with the standard topology, and assume the class E of events to be the class of open intervals of **R**. Denote by G to the class of open intervals $\{x; -a < x < +b\}$ and $\{x; 0 < x < +c)$, a, b, c positive real numbers. Then any finite intersection of members of G has a non-empty intersection, but there is no point ("place") occupied by all of them, since there is no y with $0 < y$ contained in the intersection of all $\{x; -a < x < +b\}$.

Russell used a 2-dimensional example to show that (4.1) does not suffice to ensure a non-empty intersection when we pass beyond 1-dimensional manifolds (cf. (AMA, 295). Hence, in order to ensure non-empty intersection for the 4-dimensional space-time manifold he proposed to replace definition (4.1) by the following apparently stronger requirement:

(4.3) Definition . A point is (or is represented by) a group G of events having the following properties:

(1) Any five members of G are compresent.

(2) No event outside G is compresent with any four members of G.

As far as I can see he gave no compelling reason why "five" should work better than "two". Somehow he seemed to have believed that the number n necessary to ensure that a group of compresent events , conceived as regions of a manifold M of dimension k, had a point in common, was $k + 1$. Since he was interested in the logical construction of 4-dimensional spacetime he came to adopt $n = 4 + 1 = 5$, since for the 1-dimensional manifold **R** the number $n = 1 + 1 = 2$ allegedly worked. In any case, the example (4.2) for the 1-dimensional case satisfies (4.3) and even the stronger requirement

(4.4) Any n members of the group are compresent, n any finite number.

By the light of Stone's proof of the set-theoretical representation of Boolean algebras (4.4) is not yet the "correct" constraint. Rather, what is suggested by Stone's proof is that systems of set if they are to serve as points for a set-theoretical representation of Boolean algebras, should be maximal filters, i.e., they should satisfy the finite intersection property (2.1)(3) ensuring that a, b $\in$ F entails a $\wedge$ b $\in$ F. Evidently, (2.1)(3) implies (4.4) but not vice versa. The finite intersection property (2.1)(3) has become the unanimously accepted one in mathematics. For philosophical reasons, however, Russell could not swallow this condition. According to him, all events involved in the logical construction of points had to be of a certain minimal size, since otherwise they could not be known by acquaintance. If the overlapping of any two events was again an event, then any finite intersection of events counts as an event whereby an event might eventually lose the quality of being known by acquaintance. At least, this was Russell's argument against Whitehead's sketch of constructing points by his method of "enclosure-series" (cf. AMA, 291). Russell's argument is problematic for several reasons, in particular since it seems to smuggle in the concept of size of events as a new primitive. In any case, Russell left the relational structure of the domain of events utterly underdetermined and never defined what relational properties the basic relation of overlapping was assumed to have on which his construction of points was based. This distinguished his constructional sketch from that of Carnap and Stone who both defined their constructional bases quite carefully.

On the other hand, his construction went beyond that of Carnap's insofar as his attempted construction of points anticipated in a certain sense the <u>topological</u> version of Stone's representation theorem (cf. Stone 1937). While in his (1936) Stone had constructed only a set-theoretical representation of Boolean algebras, in (1937) he went on and proved that this set-theoretical construction had a natural topological interpretation. In some more detail, this may be explained as follows. Given the set-theoretical representation B----r---->PMAX(B) as defined in (2.4) in the line of Stone[11] it could be shown that MAX(B) of maximal filters was not just a set but carried a natural topological structure so that the images r(b) of the elements of the Boolean algebra B could be neatly characterized in topological terms. More precisely, the sets r(b) were not just contrived subsets of MAX(B) but turned out to be a so called "clopen", i.e., closed and open sets with respect to the topological structure defined on MAX(B), later to be called the Stone topology. Thus, Stone's construction had shown that every Boolean algebra not only had a set-theoretical but actually a topological representation.

On the one hand, Stone spaces are topologically well-behaved being compact Hausdorff spaces, on the other hand, they have some rather bizarre topological features, e.g. they were totally disconnected. Their discovery profoundly changed the mathematicians's ideas of what was to be understood by topology. By hindsight one may even say that Stone's topological representation of Boolean algebras paved the way for a new conceptualization of topology, later to be called "pointless topology" (cf. Johnstone 1982). In the traditional perspective point set topology, a topological space X was conceptualized as a set endowed

[11] Recall that Stone actually used ideals.

with some extra "topological" structure. This structure gave rise to certain lattices such as the Heyting lattice of open subsets or the Boolean lattice of regular open subsets of X. Stone's representation suggested a quite different conceptualization. Starting with a Boolean algebra B, one could construct a topological space X(B) that fitted B in the sense that B was isomorphic to the Boolean algebra of regular open subsets of X(B). Thereby the "essence" of a topological space was shown to lay in the algebraic structure B points turned out to be derivated entities. This opened up a completely novel perspective on topology.

The surprising fact is that in (AMA) we find Russell engaged in formulating just this program of a "pointless topology" or "pointless geometry" that came into being only several decades later.[12] Contrasting it with the more familiar point-set topology ("analysis situs") he explicitly stated the ambitious agenda of such an invisaged discipline as follows:

> "In *analysis situs*, both points and neighborhoods are given. We, on the other hand, wish to define our points in terms of "events", where "events" will have a one-one correspondence with certain neighborhoods. We want our "events" to correspond with neighborhoods which are above a certain minimum and below a certain maximum when, at a later stage, the empirical metric is introduced. We have to assign to our events such properties as will enable us to define the points of a topological space as classes of events. But we have to remember that we do not want to construct merely a topological space: what we want to construct is the four-dimensional space-time of the general theory of relativity." (AMA, 298).

The following remarks on his program may be in order. Russell's constraint that we are allowed to use only neighborhoods of limited size is due to his "principal epistemological principle" according to which we can only use components we are acquainted with. Too large or too small regions are rejected as entities we cannot be acquainted with. Implicitly, this distinction introduces a further undefined primitive term, namely, the size of a region. As has been shown by Roeper (1997), at least the exclusion of "too large" regions can be formulated in a precise way. As far as I know, nobody has ever followed Russell in excluding too small regions as well. In the following reconstructional sketch his size complication will be ignored. Also Russell's introduction of an empirical metric is too sketchy to deserve detailed reconsideration.

The most charitable way of interpreting Russell's proposal is to ignore his flawed "construction by quintets" reading his attempted construction of points instead in terms of Stone's topological representation of Boolean algebras (cf. Stone (1937)). In order to keep things as simple as possible, let us consider the specific example of the 2-dimensional Euclidean plane E. Assume E to be endowed with

[12] Russell credits Whitehead for the basic ideas of this approach. Indeed, Whitehead's sketch of constructing points in his *Process and Reality* may be considered as a forerunner of "pointless topology". As it seems, however, his work was not very influential in mathematical quarters.

its standard topological structure. Recall that the topological structure of E gives rise in a canonical way to the complete Boolean algebra O*E of regular open regions. These regions are to be considered as Russelian events - ignoring Russell's size restrictions. Two regions are defined to overlap if and only if there is a non-empty region contained in both. Then Boolean structure of O*E can be defined in terms of the overlapping relation by imposing appropriate axioms on it (cf. Lewis 1991, Chapter 3.4) In terms of Carnapian similarity structures, this amounts to define a similarity structure (O*E, ~) as the basic level of Russell's construction system. Following Stone's construction, the set O*E of events is assumed a complete Boolean algebra. The task, then, is to define "points" for O*E in the sense that events are represented as sets of points in such a way that the overlapping relation for events corresponds to the set-theoretial intersection of the point sets. In line with Stone, Russell, reinterpreted in the light of Stone, may be said to represent an event $e \in$ O*E by the set of maximal filters F of O*E containing it:

(4.5) $e \equiv \{F; e \in F$, F is a maximal filter on O*E$\}$.

This corresponds to Carnap's quasi-analytical representation of an element a of the similarity structure (O*E, ~) by the set of similarity circles containing it:

(4.6) $a \equiv \{F; a \in F$, F is a similarity circle on O*E$\}$.

Russell claimed to have reconstructed thereby the points of the Euclidean plane E as maximal filters of O*E and the topological structure of E in the usual sense of point set topology. For several reasons, this claim is untenable. One can object that in Russell constructed at most a point set, namely, the set of maximal filters of O*E. He did not prove how this set is to be endowed with a topology. He simply assumed that this set should carry the "same topology" as E. Unfortunately this is mistaken. Looking at Stone's proof it is evident that the topological structure of the point set constructed by Stone's method is not that of the Euclidean plane E. Rather, the topological space constructed from O*E is not E but the Stone space SE of E that is quite different from E. For instance, SE is compact and totally disconnected, while E is not compact and connected (cf. Stone 1937, Theorem 1, 378). In short, what Russell could have achieved by his method, was the construction of the Stone space S(E) but not that of E itself.[13]

Nevertheless, Russell was not too far off the mark: By more refined methods that cannot be discussed here it <u>is</u> indeed possible to reconstruct E from O*E (endowed with a more sophisticated overlapping relation) more or less in the sense Russell adumbrated. Hence one may say that in the light of Stone's achievements and the advent of modern "pointless" topology in the seventies and eighties of the last

[13] E and SE are not unrelated to each other: SE may be conceived as a kind of algebraic version of the space E. It is sometimes called "the absolute of E".

century Russell's program eventually has become feasible, although his own sketchy execution of it was less than perfect.

5. Representations, Ramsey-Sentences, and Structural Realism . As stated already in the introduction, for Russell there were two paths to move from the level of perceptions to the level of the objects of physics, either the inferential path or the representational one, as we may call it now. In general, the inferential option has attracted more attention than the representational one. In particular it has been pointed out that Russell's account may be seen in line with subsequent developments inaugurated by Ramsey, and later pursued further Carnap, Lewis, and others (cf. Carnap 1956, Lewis 1970). One may consider as one of the main achievements of Russell's theory of definite descriptions that it avoids reference to anything not known by acquaintance. If one provisionally equates knowledge by acquaintance with empirical knowledge Russell's and Ramsey's reconstructional aims become close neighbors. Both rely on the assumption that what really matters is the empirical content of a theory, and not its theoretical garb. Although scientists typically use theoretical terms and predicates in presenting a theory, these need not be treated as names. One can consider the theoretical terms and predicates as variables bound by existential quantifiers. Thereby one obtains the Ramsey sentence of a theory T as a sentence of the form

$$R(T) := \exists x_1 \ldots \exists x_n \; T(x_1, \ldots, x_n, o_1, \ldots, o_m) \tag{5.1}$$

Here $T(t_1, \ldots, t_n, o_1, \ldots, o_m)$ is a complex conjunction stating T's axioms in which empirical terms $o_1, \ldots, o_m$ and theoretical terms $t_1, \ldots, t_n$ are entangled in some way or other. Thereby, the Ramsey sentence can be interpreted as a kind of global and simultaneous definite description of all theoretical terms as sketched by Russell already in *On Denoting*.

The Ramsey sentence R(T) of a theory T provides a neat and elegant description of the theory's layer structure. Nevertheless it did not help much to solve the many controversies concerning the problem of how the relation between the theoretical and the empirical is to be conceptualized. This is evidenced by the recent debates on the issue of "structural realism" in which the discussion on the correct interpretation of the theory's Ramsey-sentence occupies a prominent place. Discussion about epistemological and ontological consequences of the Ramsey sentence of an empirical theory suffer from an extreme degree of abstractness. Usually not much more is told about the Ramsey sentence of a theory than that it is a quantified sentence of the form given above. Often, the "indissoluble entanglement" of the observational and the theoretical terms is alluded to (e.g. Zahar 2004, 10), but a detailed presentation of it is usually missing. This would be OK, if no serious difference concerning the interpretation of R(T) threatened. But it does. Quite incompatible interpretations of the Ramsey-sentence have put forward. In some sense they rehearse the classical debate Newman and Russell had on the (non)triviality of structural realism some seventy years ago.

Russell had put forward the radical claim that all we can validly infer from perceptions is the mathematical structure of the world (cf. (AMA, 254) which had for him the consequence that "[t]he only legitimate attitude about the physical world seems to be one of complete agnosticism as regards all but its mathematical properties." (ibidem, 270/271). To this extravagant claim that only the structure of the world can be known, Newman had rightly objected that ignoring some mild constraints of cardinality just any structure can be imposed on the world (cf. Newman 1928, 144). Hence, Russell's radical structuralist claim that only structure can be known, boils down to an extremely implausible agnosticism concerning the nature and scope of scientific knowledge about the world. Russell readily admitted that Newman had made a point:

> "It was quite clear to me, as I read your article, that I had not really intended to say what in fact I did say, that *nothing* is known about the physical world except its structure. I had always assumed spacio-temporal continuity with the world of percepts, that is to say, I had assumed that there might be co-punctuality between percepts and non-percepts, … . And co-punctuality I regarded as a relation which might exist among percepts and is itself perceptible." (Russell 1928 (1991), 413)

This argument is dismissed by Demopoulos and Friedman as not compatible with Russell's theory of knowledge by acquaintance, since one cannot assume acquaintance with "a cross category notion such as spatiotemporal contiguity or causality" (cf. Demopoulos and Friedman 1985, 192) as Russell did in his answer to Newman. It may be that Demopoulos and Friedman are right. To assess their verdict would require to delve deeper into the subtleties of Russell's theory of acquaintance. It should be noted, however, that Russell offers a second, more elementary argument to ensure a more plausible interpretation of his structuralism. Demopoulos and Friedman ignore this argument. Put forward in the last sentence quoted above it contends that copunctuality is a perceptible relation among concepts. This claim is compatible with his theory of acquaintance according to which we can be acquainted with relations. Cashed out in terms of structural representation it asserts we can assume to be acquainted with the similarity structure (E, ~) consisting of events *cum* compresence (or, more precisely, co-punctuality). In the light of this quite natural representational interpretation of Russell's claim it is then a non-trivial task to construct a point-set or a topological representation of (E,~) as explained in the previous section. Summarizing one may say that Russell's structural realism, correctly interpreted, still seems a viable option, pace Newman's criticism.

This conclusion is not unanimously accepted by those who consider the Ramsey sentence as a useful tool for describing a theory's structure. Demopoulos and Friedman maintain that Newman's argument can be couched in terms of the theory's Ramsey-sentence rendering it essentially trivial:

> "…[I]f our theory is consistent and if all its purely observational consequences are true, then the truth of the Ramsey-sentence *follows* as a theorem of set-theory or second order logic, provided our initial domain has the right cardinality - if it does not, then the consistency condition of our theory again implies the existence of a domain that does." (Demopoulos and Friedman1985, 635)

Zahar vigorously disagrees (cf. Zahar 2004). According to him Newman's criticism, as well as that of Demopoulos and Friedman miss the point of Russell's structural realism (properly understood as a <u>partial</u> structuralism) since it ignored the crucial distinction between the observational predicates and the theoretical ones:

> "No serious version of structuralist realism can get going without some distinction between the theoretical and the empirical. If all predicates of a scientific theory are taken to be interpreted only within the context of the claims made by the theory, it, that is, none is taken to be firmly anchored in experience independently of our attempted descriptions of the universe, then the constraints imposed by the Ramsey sentence would be hopelessly weak." (Zahar 2004, 10).

I think Zahar is right, and I take the divergent interpretations of the Ramsey-sentence as evidence that the Ramsey-sentence is rather a description of the problem than its solution. In this respect, the representational reconstruction of Russell's theory has an advantage. It clearly brings out that the existence of a structure-preserving representation is not a matter of cardinality, but depends on the structural kind of empirical system represented. To explain why, it is expedient to go back once again to Stone's representation and consider one of its important later ramifications as exhibited in Tarski's work on relational algebras (Tarski 1941). Relational algebras may be conceived as enriched Boolean algebras in the sense that they are Boolean algebras endowed with some further structure. With direct reference to Stone's representation, Tarski posed the problem to find out if relational algebras had a representation in the sense of Stone. He himself was not able to solve this problem, and only much later it could be shown that relational algebras in general do <u>not</u> have set-theoretical representations. The proof of this important theorem has, of course, nothing to do with trivial cardinality considerations. This may be taken as evidence that the traditional Ramsey account does not do full justice to representational constructions. Hence it may be expedient to modify it in such a way that the importance of structure-preserving representations becomes more visible. A not too far-fetched way would be this: Denote the empirical base system by E ("events") and the theoretically completed system by T(E) ("events cum points"). The task is to find a structure-preserving representation E---r--->T(E). To have specific examples at hand, consider Stone’s representation B---r--->PMAX(B) or Russell’s E----r---->PPE. Asserting that this task can be achieved is to claim that such an r exists. For Stone’s representation theorem this claim can be expressed in terms of the following "representational Ramsey-sentence":

(5.2) $\exists r[(x)(y)\ (x,y \in E \Rightarrow (x = y \Leftrightarrow r(x) = r(y)\ \&\ \ x \leq y \Leftrightarrow r(x) \leq r(y)\ \&\ ....]$

The bounded variable of this "Ramsey-sentence" is running over all all representations r and it is immune against trivializations à la Newman, since it amounts to the claim that there is a structure-preserving representation r, and this claim is, as has been explained above, definitively non-trivial.

The representational constructions, carried out explicitly by Stone and propagated by Russell as the core of his theory of physics exhibit a feature that distinguishes them from "arbitrary" constructions:

> "[S]tarting from hypotheses concerning undefined objects, … we have reason to believe that there are objects fulfilling these hypotheses, although, initially, we are unable to point out any such objects with certainty. Usually, in such cases, although many different sets are abstractly available as fulfilling the hypotheses, there is one such set which is much more important than the others … . The substitution of such a set for the undefined objects is "interpretation". This process is essential in discovering the philosophical import of physics." (AMA, 4 - 5)

For instance, the importance of Stone’s representational construction of points as maximal filters is not exhausted by the fact that it yields for each Boolean algebra B an isomorphic point-set algebra r(B). Rather, beyond the fact that it offered an isomorphic set-theoretical model of B, it opened the path to conceptual reinterpretations of the represented and the representing domains that had been literally unthinkable without it. This is amply confirmed by the tremendous impact of Stone’s representation that virtually reshaped the theories of Boolean algebras and topology. In order that this is possible even on a modest scale there has to exist a certain intensional affinity between the representing and the represented level that goes beyond a mere isomorphism. Carnap was after something similar when he characterized the range of the variables of the Ramsey sentence as follows:

> "The entities to which the variables in the Ramsey-sentence refer are characterized not purely logically, but in a descriptive way; and this is the essential point. These entities are identical with mathematical entities only in the customary extensional way of speaking. … In an intensional language there is an important difference between the intension 9 and the intension n(p) (number of planets). The former is L-determinate, the latter is not. Thus, if by "logical" or "mathematical" we mean "L-determinate" then the entities to which the variables in the Ramsey-sentence refer are not logical." (Carnap 1958, RC 102-07-05, quoted after Psillos 1999, p. 55)

The sketchy remarks of this section on the relations between the Ramsey-sentence, the representational constructions à la Stone, and structural realism certainly do not exhaust these topics, but at it should have been made clear that the case of Russell's "structuralism" is more intricate than one might have thought.

6. Concluding Remarks . The philosophical prospects of Russell's structuralist program of representational (re)constructions use to be assessed as rather bleak. For instance, Deomopoulos asserts that even the sucessful execution of this program in terms of representation theory "lends no support to the central epistemological contention of structuralism: from the fact that the representation is purely structure-preserving, it by no means follows that the knowledge expressed by the original theory - in this case, the theory of space-time - is purely structural." (Demopoulos 2003, 413). He is certainly right in contending that the thesis of radical structuralism according to which knowledge is purely structural cannot be maintained. But this is not the issue as has been admitted already by Russell himself in his answer to Newman. It is quite another thing, however, to maintain that the concept of structural representation in general is philosophically unimportant. A closer look at the theories of representations developed in the sciences reveals that representation is a "difficult" concept that up to now as been only partially understood by epistemologists and philosophers of science. As I wanted to show in this paper, the issue of definite descriptions, logical constructions, and structural representations may still have a place on the agenda of contemporary epistemology and philosophy of science. These disciplines could learn a lot by looking more closely to mathematics and the other sciences for their conceptual resources and technical tools (cf. Richardson 2003, 165). More specifically, the concept of structural representation, which unfolded in the evolution of 20th century's logic and mathematics in a spectacular manner, has been digested only partially by philosophy up to now. Perhaps the best example is the emergence of category theory[14], for which one of the decisive factors was Stone's representation theorem[15] (cf. Johnstone 1982, Mac Lane 1970). Studying in a Russellian spirit the concept of representational constructions may help overcome this less than optimal state of affairs.

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[14] For an account of category theory especially adapted for the needs of philosophers and with special emphasis on its relevance to Russell's philosophy, see Marquis (1993). It is not too far-fetched to characterize category theory as the theory of structure-preserving representations (morphisms). Although it came into being more than sixty years ago, philosophers have been rather reluctant to welcome it as relevant for matters philosophical, in stark contrast to the enthusiastic reception the "new logic" received one hundred years ago by Russell and other scientifically minded philosophers.

[15] The topological version of Stone's representation provided one of the earliest non-trivial examples of a functor, which is a fundamental concept of category theory. Almost explicitly, this is already contained in Stone's Theorem 4, 387 (1937) which asserts that the "algebraic theory of Boolean algebras and the topological theory of Boolean spaces are mathematically equivalent." (More precisely, Stone's functor is a duality.)

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