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Research article

JTB Epistemology and the Gettier Problem in the Framework of Topological Epistemic Logic

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Abstract:
Traditional epistemology of knowledge and belief can be succinctly characterized as justified true belief (JTB) epistemology, namely by the thesis that knowledge is justified true belief, i.e., K = JTB. Since Gettier's (1963) classical paper, JTB-epistemology has come under heavy attack. The aim of this paper is to study JTB-epistemology and Gettier's criticism of it in the framework of topological epistemic logic. In this topological framework, Gettier situations, for which knowledge does not coincide with true justified belief, occur for formal reasons, i.e., there are models for which K ≠ JTB. On the other hand, topological logic offers natural models of JTB, i.e., models for which knowledge coincides with true justified belief.Moreover, for every model of Stalnaker's "combined logic KB of knowledge and belief" a canonical JTB model (its JTB doppelganger) can be constructed that is free of Gettier situations. In brief, the traditional JTB-epistemology can be shown to be a simplification of a more complex epistemological account of knowledge and justified true belief that assumes that these two concepts may differ. Further, for all models of Stalnaker's KB-logic, Gettier situations turn out to be topologically exceptional events in a precise sense, i.e., they are nowhere dense situations. This entails that Gettier situations are doxastically and epistemologically invisible in the sense that they can be neither known nor believed with respect to the knowledge operator and the belief operator of the models involved. In sum, the version of topological epistemic logic presented in this paper leads to a partial rehabilitation of the traditional JTB-account: Gettier situations, where knowledge does not coincide with justified true belief, are characterized topologically as anomalies or exceptional situations. On the other hand, Gettier situations necessarily occur

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for most universes of possible worlds. Only for a special subclass of universes (epistemically characterized by a rather strong concept of knowledge and topologically characterized as the class of nodec spaces) can Gettier situations be avoided completely. This description amounts to the thesis that, in general, JTB-epistemology is false. JTB remains correct, however, for a special class of universes of possible worlds, namely, nodec spaces. Moreover, in a precise topological sense, any topological space whatsoever can be shown to be "almost" a nodec space. This fact renders the assertion plausible that the classical JTB account is "almost correct."

Keywords:
Topological epistemic logic, JTB-epistemology, Gettier problem, Justified belief, Epistemic and doxastic invisibility

1. Introduction

The use of formal, often mathematical, models is ubiquitous in the natural and social sciences. There is no reason why in philosophy, understood as a science in a broad sense, this should not be the case. At least, it should be the case for analytic philosophy, conceived in a broad sense. According to Williamson:

> The aim of using models is to gain insight into phenomena by studying how they work under simplified, rigorously described conditions that enable us to apply mathematical or quasi-mathematical reasoning that we cannot apply directly to the phenomena in the wild. (Williamson (2013), p. 131)

Using mathematical models or other kinds of formal models in philosophy is not, of course, a foolproof method for obtaining philosophically interesting results. Rather, a philosophy that employs those formal methods is always in danger of indulging in mere mathematical window-dressing in order to appear "scientific" without substantial philosophical content. This is a classical problem of any mathematical (and more generally formal) philosophy. One of the founding fathers of this philosophical current was already aware of this:

> The acceptance or rejection of abstract linguistic forms, just as the acceptance or rejection of any other linguistic forms in any branch of science, will finally be decided by their efficiency as instruments, the ratio of the results achieved to the amount and complexity of the efforts required. (Carnap (1950), p. 40)

The aim of this paper is to investigate the fundamental epistemological problem of "What is knowledge?," put again on the agenda of modern epistemology by Gettier's (1963) classical paper, by using the conceptual tools of topological epistemic logic.[1]

The topological interpretation of modal operators is one of the oldest semantics for modal languages (cf. Aiello *et al*. (2007)), going back to the trailblazing paper of McKinsey and Tarski (1944). The perhaps more widely used Kripke semantics of S4 is just a special case of topological semantics. Or, reversely, for S4 the topological semantics is an extension of Kripke semantics in that a Kripke frame (X, R), with a reflexive and transitive relation $R \subseteq X \times X$, can be seen as an Alexandroff topological space (X, OX), the topology of which is defined by the partial order R.

The topological semantics of modal logics may be considered as more intuitive than "abstract" relational (Kripke) semantics and providing a richer source of geometric/spatial interpretations. This holds in particular for an epistemic reading. The interior semantics of topology is naturally epistemic and extends the relational semantics. Elementary topological operators such as the interior operator produce the epistemic logic S4 with no need for additional constraints. In other words, in its most general form, topologically modeled knowledge is factive and positively introspective; however, it does not necessarily possess stronger properties than S4. This in no way limits the usage of interior semantics for stronger epistemic systems that include other epistemic modalities such as beliefs or ignorance. The interior semantics provides a deeper insight into the evidence-based interpretation of knowledge than relational semantics. The topological approach shows that Gettier counterexamples are "exceptional" events. Topology has a toolkit of conceptual devices to deal with this kind of event, whereas Kripke models do not. Of course, one could amend the Kripke approach in an appropriate way. After all, topological structures are just one special type of relational structure. But then, topologically amended Kripke models become some kind of topological structure.

Using topological models does not need a special justification compared with using Kripke models. Topology is not a somehow dubious tool compared with the theory of relational structures on which Kripke frames are based. Topology and the theory of relational structures have the same mathematical dignity, so to speak. A central claim of my paper is that the *topological* notion of nowhere density (ND) and related *topological* concepts elucidate problems related to Gettier counterexamples in a new way. ND and its relatives are genuine topological concepts that do not occur in Kripke models.

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1 The topological concepts and terminology used in this paper are standard. Nevertheless, for the sake of definiteness, they will be explained in full detail in Section 2. The introduction of this paper only requires a superficial acquaintance with the basic ideas of topology.

As already mentioned, topological epistemic logic may be said to have begun with McKinsey and Tarski's (1944) paper which showed that the epistemic modality of knowledge (as it appears in expressions such as "it is known that A," "one knows that A," and others) can be formalized with the help of the topological kernel operator Int as it occurs in set-theoretical topological formulas $Int(A)$ (to be read as "the interior of the set A"). Here, a proposition A is to be understood as a set of possible worlds where this proposition holds. In recent decades, the approach of McKinsey and Tarski's seminal paper has been extended and further elaborated by many authors (see, for instance, the recent works of Baltag and others, and the works mentioned therein (Baltag *et al*. (2017, 2019, 2022)). Today, topological logic may be considered as a well-established and thriving approach of the field of epistemic logic and formal epistemology.

In the 60 years since its publication in 1963, Gettier's short paper has generated a huge literature engaged in the invention of ever more sophisticated thought experiments aiming to refute the classical justified true belief (JTB)-account of knowledge as true justified belief (cf. Turri (2012), Borges *et al*. (2017)). The present paper tackles the issue from a different angle. Following Williamson (2013, 2015) I propose to address the Gettier problem and related issues from the perspective of formal epistemic logic. More precisely, the issue is approached from the perspective of topological epistemic logic. This allows access to a rich reservoir of formal models that can be used to study these problems. This does not mean that the usage of formal models definitively decides matters epistemological. They may well be natural formal models that provide robust evidence against JTB, and, at the same time, that there are natural formal models that provide evidence in favor of JTB. The mere existence of formal models of one kind or another does not suffice to decide the question of whether JTB is a correct (or, at least a reasonable) account of knowledge and belief. Rather, to deal with this matter in an appropriate manner, it is necessary to delve more deeply into the realm of formal models of knowledge and belief that one intends to use for the elucidation of these concepts.[2]

How can topological epistemology be related to JTB in general and to the Gettier problem in particular? The general answer is that topological epistemic logic investigates (idealizing) topological models of knowledge, belief, and other epistemic concepts by modeling them as appropriately chosen topological operators. Then the basic issue for dealing with JTB in the

2 For a general survey of different research programs in epistemic logic and their use of idealizations see Yap (2014). Yap does not explicitly mention topological epistemic logic, but explicitly deals with Williamson's proposal to deal with Gettier situations in the framework of Kripke models of possible worlds. Mutatis mutandis her remarks on the issue of formal models and idealizations in the realm of epistemic logic apply also to topological epistemic logic (cf. Yap (2014, Section 3.3)).

framework of topological epistemology is whether there are compelling topological models for which knowledge is justified true belief or not.

Today, one of the most prominent formal accounts of knowledge is Stalnaker's "combined logic of knowledge and belief" KB. In the framework of KB, knowledge is represented as the interior kernel operator Int of a topological space (X, OX), with $OX = \{Int(A); A \in PX\}$, PX being the power set of subsets $A \in PX$ are to be conceived as propositions of classical propositional logic to be interpreted as sets of possible worlds as usual. While knowledge is topologically modeled rather unanimously by the interior operator Int, it is less clear how to define other epistemic operators such as belief. In Stalnaker's KB logic, belief is represented as $ClInt$[3]. Baltag *et al*. (2019) have shown that the operator $ClInt$ works quite well as a belief operator of models based on extremally disconnected spaces (ED-spaces). For general topological models, however, $ClInt$ is not a good belief operator. As early as in Stalnaker (2006) it was observed that for general topological spaces $ClInt$ is not even a normal operator in the sense of modal logic.

As will be argued in this paper, for general topological models the operator $ClInt$ should be replaced by $IntClInt$. On ED-spaces, $ClInt$ and $IntClInt$ coincide, and on general topological spaces $IntClInt$ preserves almost all qualities of a plausible belief operator that $ClInt$ exhibits on ED-spaces. In particular, as will be explained in detail in Section 3, $IntClInt(A)$ can be interpreted as *justified* belief B. This makes it possible to use topological logic to deal with the Gettier problem and related issues. By interpreting the operators Int and $IntClInt$ as knowledge and justified belief, respectively, the basic thesis of JTB-epistemology can succinctly be expressed by the identity

(1.1) $$Int(A) = A \cap IntClInt(A) \qquad \text{(JTB)}$$

for all propositions $A \in PX$. Informally expressed, (1.1) asserts that knowing that A coincides with the conjunction that A is obtained (true) and that A is believed with justification. The following reformulation of (1.1) will be useful later. For any proposition $A \subseteq X$ define

(1.1)' $$G(A) := A \cap IntClInt(A) \cap Int(A)^C.$$

3 Here, Cl is the topological closure operator, defined by $Cl(A) := Int(A^C)^C$, A^C the set-theoretical complement of A with respect to X. $ClInt$ and $IntClInt$ are concatenations of the operators Int and Cl.

$G(A)$ is called the Gettier proposition defined by A. If $G(A) = \emptyset$ for all A, this is to be interpreted as the classical JTB-epistemology holding for this universe.

The worlds $w \in G(A)$ are called Gettier worlds for A: A world w is a Gettier world for A, i.e., $w \in G(A)$, iff in w knowledge of A does not coincide with true justified belief of A. A topological model (X, OX) is free of Gettier worlds for all propositions A iff $G(A) = \emptyset$ for all A. Hence, a topological model (X, OX) that is free of Gettier worlds for all propositions $A \in PX$ is a model of JTB epistemology.

For the time being, this terminology may be not fully convincing as long as no argument has been given that $IntClInt$ can be interpreted as *justified belief*. This gap will be filled in Section 3 by explaining in more detail that $IntClInt$ can be interpreted as *justified belief* in a strong sense. This requires us to dwell more closely on the justificatory qualities of the topological operators Int and Cl that are inherited by the composition $IntClInt$ of these components.

Now the following natural question arises: Which topological models (X, OX) of knowledge and belief are models of JTB and which are not? The answer to this question will be given in several stages. Let us begin with the trivial extreme cases of total ignorance and omniscience. For them, one obtains:

(1.2) Proposition. The trivial topological spaces $(X, \{\emptyset, X\})$ and (X, PX) satisfy JTB, i.e., for them, (1.1) is valid.♦

JTB epistemology is not restricted, or course, to these trivial cases. Consider the following example: Recall that a topological space (X, OX) is almost discrete iff every open set is closed. As is well known, a topological space (X, OX) is almost discrete iff OX is a family of pairwise disjoint subsets of X such that $X = \bigcup OX$. In other words, OX defines an equivalence relation. For almost discrete topological spaces one easily calculates:

(1.3) Proposition. Almost discrete topological spaces (X, OX) satisfy JTB, i.e., $Int(A) = A \cap IntClInt(A)$ for all $A \in PX$.♦

By conceiving Int — in the spirit of McKinsey and Tarski's approach — as a modal operator, the topological theory of almost discrete spaces corresponds to the modal logic S5. Only a minority of logicians consider this logic to be a satisfying epistemic logic.

It is rather almost unanimously agreed that the logic of knowledge and belief is located somewhere in the interval of the modal logics between S4 and S5 (cf. Lenzen (1979)). S4 and

S5 are to be considered only as boundary stones for the rough determination of where the logic of knowledge and belief is located in the landscape of modal logics. It will be shown that being almost discrete is in no way a necessary requirement for a topological space to serve as a model for JTB.

(1.4) Proposition. Let $(\mathbb{N}, O\mathbb{N})$ be the natural numbers endowed with the finite/cofinite topology, i.e., the open sets of $O\mathbb{N}$ are $\emptyset$, $\mathbb{N}$, and all infinite subsets $A \subseteq \mathbb{N}$ with finite complements A^C. Then $(\mathbb{N}, O\mathbb{N})$ is a model of JTB, i.e., $Int(A) = A \cap IntClInt(A)$.♦[4]

For the moment, these examples may suffice to convince the reader that the JTB account of knowledge is not totally without the support of topology. On the other hand, there are many topological models of knowledge and belief for which JTB is not valid. Perhaps the best-known topological model is based on the familiar metrical real line $(\mathbb{R}, O\mathbb{R})$. One easily calculates that for this model JTB does not hold:

(1.5) Proposition. For the topological model of knowledge and belief based on the universe of possible worlds of the Euclidean line $(\mathbb{R}, O\mathbb{R})$ and $A := \mathbb{R} - \{1/n, n \geq 1\}$ one obtains

$$Int(A) = A - \{0\} \text{ and } A \cap IntClInt(A) = A.$$

That is, the universe of possible worlds that represents worlds and their relations by elements of the Euclidean line $(\mathbb{R}, O\mathbb{R})$ does not support JTB-epistemology.♦

Proof. One has $0 \notin Int(A)$ since every open neighborhood $U(0)$ of 0 contains an element $1/n$ for some n that is not contained in A. On the other hand, $IntClInt(A) = \mathbb{R}$, since all open neighborhoods of 0 and $1/n$ contain elements of $Int(A)$. Hence, the sets $Int(A)$ and $A \cap IntClInt(A)$ differ. In other words, $(\mathbb{R}, O\mathbb{R})$ does not satisfy the defining condition (1.1) of JTB.♦

Topological epistemology is, so to speak, "undecided" with respect to JTB: some topological models satisfy JTB, others do not.

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4 For later use, it may be observed that $(\mathbb{N}, O\mathbb{N})$ is also an extremally disconnected space (cf. Baltag *et al.* (2019), p. 215, and Section 4, Corollary 4.5). This implies that $(\mathbb{N}, O\mathbb{N})$ is a model of Stalnaker's combined logic KB.

Admittedly, models (1.2)–(1.4) of JTB are rather trivial and hardly provide a convincing explanation of why JTB played such a prominent role in traditional epistemology. More is needed to argue that JTB is topologically plausible in some sense. A more complete presentation of JTB-models in topological epistemology is called for. The present paper aims to provide exactly this.

The organization of this paper is as follows. The next two sections provide the prerequisites for the above purposes. In Section 2 we introduce the necessary formal apparatus to deal with concepts of knowledge and belief in a topological framework. In Section 3 we recall the basics of a topological version of Stalnaker's combined logic KB of knowledge and belief. This has been elaborated in detail by Baltag *et al.* in various publications (cf. Baltag *et al.* (2017, 2019, 2022) and elsewhere). Moreover, it is shown that Stalnaker's concept of belief is justified belief in quite a strong sense. Thus, Stalnaker's KB logic is directly relevant to the issue of JTB.

The central part of this paper (Section 4) deals with three issues. First, we show that there exist many "natural" topological JTB-models, i.e., models that satisfy the axiom (1.1) characteristic for JTB. In the standard language of topology, JTB holds for topological models based on topological spaces (X, OX) that are nodec[5] spaces. These spaces are defined by the special topological feature that all their nowhere dense subsets are closed. This has the result that the class of nodec spaces to provide a niche for the survival of the JTB-account. Second, we show that for every topological space (X, OX) whatsoever there exists a canonical nodec space $(X, O_{nod}X)$. This nodec space $(X, O_{nod}X)$ is a kind of doppelganger of (X, OX) in a precise topological sense, namely $(X, O_{nod}X)$ is very similar to the original topological space (X, OX) from which it is derived. Most topological models, however, are not JTB-models. The epistemic logic of nodec spaces is characterized as an extension of the standard modal logic S4, namely by the extension of S4 by the Zeman axiom $((\Box\neg\Box\neg\Box p)\rightarrow(p\rightarrow\Box p))$, where $\Box p$ is to be interpreted as "p is known" (cf. Zeman (1969), Bezhanishvili *et al.* (2004)).

The last topic treated in Section 4 is the discussion of a topological analogue of a double luck construction that has been used in many thought experiments proposed by the "Gettier industry" as a device for providing ever more sophisticated examples of Gettier situations refuting the classical JTB-account. This construction shows that the topological account is able to emulate important aspects of the standard informal thought experiments leading to Gettier situations.

5 A space (X, OX) is nodec iff all nowhere dense subsets of X are closed (nodec = no(where) de(nse) c(losed)) (cf. van Douwen (1993), Definition and Fact 1.14, p. 129). For some equivalent definitions of nodec spaces see Bezhanishvili *et al.* (2004, Theorem 2.5).

In Section 5 we show that for general topological models (X, OX) for which Gettier propositions exist, these propositions can be neither known nor believed with respect to the epistemic operators Int and $IntClInt$ that characterize the topological model (X, OX): Gettier cases are *epistemically* and *doxastically invisible*, i.e., if w is a world for which a Gettier situation holds with respect to a proposition A, an agent who relies on the operators Int and $IntClInt$ neither knows nor consistently believes that w is an A-world. This doxastic invisibility of Gettier cases may have contributed to the impression that the traditional JTB account (for which no Gettier cases exist) appears to be correct without it being so. We conclude with some general remarks on further possible directions of research on topological epistemology in Section 6.

2. The Topology of Knowledge and Belief

To set the stage, in this section we recall the absolutely necessary basics of elementary set-theoretical topology needed for the formulation of the interior semantics for epistemic logic of knowledge and belief as presented by Baltag *et al*. (cf. Baltag *et al*. (2017, 2019. 2022)). This semantics will be used throughout the rest of this paper. First, recall the definition of a topological space:

(2.1) Definition. Let X be a set with power set PX. A topological space is an ordered pair (X, OX) with $OX \subseteq PX$ that satisfies the following conditions:

(i) $\emptyset, X \in OX$.
(ii) OX is closed under *finite* set-theoretical intersections $\cap$ and arbitrary set-theoretical unions $\cup$.♦

The elements of OX are called the open sets of the topological space (X, OX). The set-theoretical complement A^C of an open set $A \subseteq X$ is called a closed set. The set of closed subsets of (X, OX) is denoted by CX. The interior kernel operator Int and the closure operator Cl of (X, OX) are defined as usual: the interior kernel $Int(A)$ of a set $A \in PX$ is the largest open set that is contained in A; the closure $Cl(A)$ of A is the smallest closed set containing A.[6] Topologies (X, OX) on a set X can be partially ordered set-theoretically.

6 For details, see Willard (2004), Steen and Seebach Jr. (1978), or any other textbook on set-theoretical topology.

(2.2) Definition. Let (X, OX) and $(X, O'X)$ be two topologies on X. OX is said to be coarser than $O'X$ iff OX is a subset of $O'X$, i.e., $OX \subseteq O'X$. If OX is coarser than $O'X$ this is also expressed by saying that $O'X$ is finer than OX.♦

Clearly, the coarsest topology on X is $O_0X = \{\emptyset, X\}$ and the finest topology is $O_1X = PX$. For all topologies OX one has

$$O_0X \subseteq OX \subseteq O_1X.$$

The set $TOP(X) := \{OX;\ OX \text{ is a topology on } X\}$ endowed with the partial order $\subseteq$ is well known to be a complete lattice $(TOP(X), \subseteq)$. The infimum in $(TOP(X), \subseteq)$ is just the set-theoretical intersection of topologies, the bottom element is $\{X, \emptyset\}$, and the top element is (X, PX).

The epistemological interpretation of $TOP(X)$ works as follows: $OX \in TOP(X)$ is to be interpreted as a cognitive agent who uses the interior kernel operator Int of as a knowledge operator for their epistemic activity. More precisely, $A \in PX$ is interpreted as a proposition A. A is true in a world $w \in X$ iff $w \in A$; otherwise, A is false in w. A proposition A entails a proposition D iff A is a subset of D, $A \subseteq D$. The other Boolean operators on PX are to be interpreted as usual. A proposition A is known at a world w iff $w \in Int(A)$. The fact $w \in Cl(A)$ is to be interpreted as the fact that w is considered conceptually possible to be an A-world. Or, in a more agent-centered language, an epistemiagent knows that x is in A iff w belongs to $Int(A)$. The assertion that w is an A-world is to be considered as equivalent to the assertion that the proposition A is true in the world w.

The partial order $\subseteq$ on the lattice $(TOP(X), \subseteq)$ has an obvious epistemological interpretation: if $OX \subseteq O'X \in TOP(X)$ a cognitive agent who uses OX has less knowledge than an agent who uses $O'X$. Moving from (X, OX) to $(X, O'X)$ may be conceived as a learning process in which the epistemic agent enhances their cognitive powers by extending their knowledge from OX to $O'X$. The maximal (discrete) topology O_1X may be interpreted as (trivial) omniscience with respect to the universe of possible worlds X.

The partial order $\subseteq$ on $TOP(X)$ will be important in later sections to assess the relation between traditional JTB-epistemology (for which no Gettier situations exist) and modern "post-Gettier" epistemology which recognizes the existence of Gettier cases.

The topological operators Int and Cl are well-known to satisfy the Kuratowski axioms (cf. Kuratowski and Mostowski (1976)):

(2.3) PROPOSITION (KURATOWSKI AXIOMS). Let (X, OX) be a topological space, A, $D \in PX$. Define the interior kernel operator Int of (X, OX) by $Int(A) := \cup\{U ; U \in OX \text{ and } U \subseteq A\}$. Dually, the closure operator Cl is defined by $Cl(A) := \cap\{K; K \in CX \text{ and } A \subseteq K\}$. The operators Int and Cl satisfy the following axioms:

(i) $Int(A \cap D) = Int(A) \cap Int(D)$. $Cl(A \cup D) = Cl(A) \cup Cl(D)$.
(ii) $Int(Int(A)) = Int(A)$. $Cl(Cl(A)) = Cl(A)$.
(iii) $Int(A) \subseteq A$. $A \subseteq Cl(A)$.
(iv) $Int(X) = X$. $\emptyset = Cl(\emptyset)$.♦

In the following, the Kuratowski axioms are used without explicit mention. Moreover, we will freely use the fact that the operators Int and Cl are interdefinable: $Int(A) = Cl(A^C)^C$ and $Cl(A) = Int(A^C)^C$.

Further, it is often expedient to conceive the operators Int and Cl as operators $Int{:}PX \to PX$ and $Cl{:}PX \to PX$ defined on PX in the obvious way. Hence, the concatenation of these operators makes perfect sense. In the following, concatenations such as $IntCl$ and $IntClInt$ will play an important role. For later use, we note the following:

(2.4) PROPOSITION. Let (X, OX) be a topological space with interior kernel operator Int, closure operator Cl, and A, $D \in PX$.

(i) $IntClIntCl(A) = IntCl(A)$ and $ClIntClInt(A) = ClInt(A)$.
(ii) $IntCl(Int(A) \cap D) = IntClInt(A) \cap IntCl(D)$.

PROOF. Identities (i) are well known and identity (ii) is also well known for A, $D \in OX$. For the following, however, we need also the stronger but less known fact that the identity (ii) even holds if D is not open. The proof of (2.4)(ii) can be found in Kuratowski and Mostowski (1976, Ch. I, §8).♦

(2.5) DEFINITION. A subset Z of a topological space (X, OX) is *nowhere dense* iff $IntCl(Z) = \emptyset$.♦

Informally expressed, nowhere dense subsets of (X, OX) are topologically "small" or "negligible." Looking at familiar topological spaces such as the Euclidean time, this informal expression is quite plausible. Epistemologically, nowhere dense sets may be interpreted as propositions for which the conceptual possibility cannot be known. Examples of nowhere dense subsets of the real line ($\mathbb{R}$, $O\mathbb{R}$) are the natural numbers $\mathbb{N} \subset \mathbb{R}$ and the Cantor dust D.

Before we address this issue, however, it is expedient to dwell a little more upon the general problem of how the epistemological concept of belief is to be explicated topologically. This issue is less clear than the corresponding problem for knowledge. Kuratowski (1922) proved that there are exactly seven different combinations[7] of the topological operators Int and Cl:

(2.6) $$Id^{8}, Int, Cl, IntCl, ClInt, IntClInt, ClIntCl.$$

It is not directly obvious whether any of the combinations of these seven operators can be meaningfully interpreted as a formal topological model of belief. For instance, the closure operator Cl is certainly not a plausible candidate for a belief operator since the inclusion $A \subseteq Cl(A)$ (required by (2.3)(iii)) had to be interpreted as the assertion that if w is an A-world, i.e., $w \in A$, then it would be believed that w is an A-world. This is certainly not true for a realistic concept of belief: there are many facts that are not believed to be facts. Further, it may be necessary for the following four intuitively plausible conditions to be satisfied by a "good" belief operator:

(2.7) Definition (Adequacy conditions for belief operators). Let (X, OX) be a topological space of possible worlds and A, $D \in PX$ be propositions. An operator $B{:}PX \to PX$ can be interpreted as a good belief operator only if it satisfies the following (in)equalities for all A and D:

(i) $NOT(A \subseteq B(A))$: There is a world w that is an A-world but is not believed to be an A-world.
(ii) $NOT(B(A) \subseteq A)$: There is a world w that is believed to be an A-world but is not an A-world.
(iii) $B(B(A)) = B(A)$: The proposition A is believed iff it is believed that A is believed.
(iv) $B(A \cap D) = B(A) \cap B(D)$: The conjunction of propositions A and B is believed iff A is believed and D is believed.♦

7 This means that there are topological spaces (X, OX) and propositions $A \in PX$ such that the operators of (2.6) yield seven different results: A, $Int(A)$, $Cl(A)$, $IntCl(A)$, $ClInt(A)$, $IntClInt(A)$, and $ClIntCl(A)$.

8 Id is the identity map $Id(A) = A$, considered as the empty concatenation of Int and Cl.

A closer look at (2.6) reveals that there is indeed one operator in this list that scores quite well as a plausible candidate for the office of a good belief operator, namely, the operator $IntClInt$. As is easily checked by elementary examples and calculations, $IntClInt$ satisfies (2.7)(i)–(iv). Even better, for all topological spaces (X, OX) the pair of operators $(Int, IntClInt)$ satisfies all axioms of Stalnaker's combined logic KB of knowledge and belief except the axiom of negative introspection (hereafter axiom (NI); cf. Stalnaker (2006), Baltag *et al*. (2019)).

3. Stalnaker's Combined Logic KB of Knowledge and Justified Belief

First, for the sake of definiteness, let us recall the basics of the syntax and semantics of the modal language to be employed in the following. We consider a bimodal extension L_{KB} of standard propositional logic defined by two modal operators K and B. The formulas of the language L_{KB} are defined on a countable set of propositional letters $PROP$, Boolean operator $\neg$, $\wedge$, and the modal operators K and B by the following grammar:

$$\varphi := p \mid \neg p \mid \varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi \mid, p \in PROP.$$

The abbreviations for the connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are standard. The unimodal fragments of L_{KB} defined by K and B are denoted by L_K and L_B , respectively. Now, the axioms and the inference rules of Stalnaker's system KB of a combined logic of knowledge and belief can be formulated as follows (cf. Stalnaker (2006), Baltag *et al*. (2017, 2019)):

(3.1) Definition (Axioms and inference rules of Stalnaker's logic of knowledge and belief).

(CL)	All tautologies of classical propositional logic.	
(K)	$K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$.	(Knowledge is additive)
(T)	$K\varphi \rightarrow \varphi$.	(Knowledge implies truth)
(KK)	$K\varphi \rightarrow KK\varphi$.	(Positive introspection of K)
(CB)	$B\varphi \rightarrow \neg B\neg\varphi$.	(Consistency of belief)
(PI)	$B\varphi \rightarrow KB\varphi$.	(Positive introspection of B)
(NI)	$\neg B\varphi \rightarrow K\neg B\varphi$.	(Negative introspection of B)

(KB) $K\varphi \rightarrow B\varphi$. (Knowledge implies belief)

(FB) $B\varphi \rightarrow BK\varphi$. (Full belief)

Inference Rules:

(MP) From φ and $\varphi \rightarrow \psi$, infer ψ. (Modus ponens)

(NEC) From φ, infer $K\varphi$. (Necessitation)♦

For the topological approach to knowledge and belief, axiom (NI) plays a special role. It has been shown that axiom (NI) holds only for topological models of a very special kind, namely, models based on extremally disconnected spaces (cf. Baltag *et al*. (2019), Stalnaker (2006)). All other axioms and rules of KB are satisfied by *all* topological spaces. Thus, by giving up (NI) considerable generality is gained. There is a cost, however. The validity of the axiom (NI) guarantees unique definability of the belief operator, i.e., for extremally disconnected[9] spaces, the belief operator is uniquely determined by the knowledge operator as $IntClInt$ (cf. Baltag *et al*. (2019), Stalnaker (2006), Mormann (2023)). This no longer holds for topological models that are not extremally disconnected. For the systems of knowledge and belief considered in this paper, it is only required that they are weak Stalnaker systems in the following sense:

(3.2) Definition (Weak KB logic). A bimodal logic with modal operators K and B based on the bimodal language L_{KB} is a *weak* KB- logic iff it satisfies the conditions

(i) $B(\varphi\rightarrow \psi) \rightarrow (B\varphi \rightarrow B\psi)$. (Kripke axiom K for B)

(ii) $BB\varphi \leftrightarrow B\varphi$. (Idempotence (4)* of B)

(iii) For the tandem (K, B), all of Stalnaker's axioms and rules given in (3.1) are satisfied *except axiom* (NI).♦

Note that the B-fragment of weak KB logic in the sense of (3.2) is a normal modal logic since the necessitation rule NEC for B is satisfied: from φ one may infer $B\varphi$.[10] Further, by (3.2) one has:

9 Recall that a space (X, OX) is extremally disconnected iff the closure of every open set is open: $ClInt(A) = IntClInt(A)$ (cf. Willard (2004, 15.G, p. 106)).

10 For various equivalent definitions of a normal modal logic, see Chellas (1980, Theorem 4.3, p. 115).

(3.3) Corollary. The B-fragment of weak KB-logic is a KD4*-logic. More precisely, the following axioms hold in weak KB:

(K) $B(\varphi\rightarrow\psi) \rightarrow (B(\varphi)\rightarrow B(\psi))$.
(D) $B\varphi \rightarrow \neg B\neg\varphi$.
(4)* $B\varphi \leftrightarrow BB\varphi$.♦

This result may be compared with the corresponding result for full KB logic that the B-fragment of full KB logic is a KD45 system (cf. Baltag *et al*. (2019, Proposition 4), Stalnaker (2006)).[11]

The B-fragment of a weak KB-system is slightly stronger than just a KD4-system, since the idempotence of B ((3.2)(ii)) requires not only $B\varphi\rightarrow BB\varphi$, i.e., axiom (4), but also its converse $BB\varphi\rightarrow B\varphi$. One observes that for (full) KB systems the operator B is idempotent. Hence, the B-fragment of full KB logic can be characterized more precisely by KD4*5.

The following proposition shows that definitions (3.1) and (3.2) fit well together:

(3.4) Proposition. Weak KB logic is strictly weaker than KB logic.

Proof. First, we show that the modal operator B of KB logic satisfies the Kripke axiom (K) of (3.2)(i). According to Stalnaker (2006) and Baltag *et al*. (2019) in KB-logic, one has

(3.5) $$B \leftrightarrow \neg K\neg K \leftrightarrow K\neg K\neg K.$$

As is easily checked, $K\neg K\neg K$ is a normal operator, i.e., it satisfies (K). Moreover, it is easily verified that (3.2)(ii) is satisfied, i.e., B is idempotent ((4)*). Hence, as it should be, KB logic is a weak KB logic. To show that weak KB is strictly weaker than KB, one has to find a formula that is valid for KB but not for weak KB. The formula $\neg K\neg K(\varphi \wedge \psi) \leftrightarrow \neg K\neg K\varphi \wedge \neg K\neg K\psi$ is such a formula.♦

(3.6) Definition. Given a topological space (X, OX), we define a topo(logical) model for L_K as $M = (X, OX, \mu)$, where $\mu{:}PROP\rightarrow PX$ is a valuation function from the set $PROP$ of propositional letters to PX.♦

11 Elementary examples based on the Euclidean line $(\mathbb{R}, O\mathbb{R})$ show that there are models of the weak KB logic whose B-fragments are not KD45 models.

The function $\mu{:}PROP{\to}PX$ can be inductively extended to a function $\mu{:}FORM(L_K){\to}PX$ (also denoted by μ) of the set of well-formed formulas $FORM(L_K)$ of L_K to PX in the usual way by defining:

(3.7) DEFINITION. Let $M = (X, OX, \mu)$ be a topological model of L_K. The interior semantics of L_{KB} with values in M is given by

(i) $\mu(p) \in PX$.
(ii) $\mu(\neg p) := \mu(p)^C$.
(iii) $\mu(\varphi \wedge \psi) := \mu(\varphi) \cap \mu(\psi)$.
(iv) $\mu(K\varphi) := Int\mu(\varphi)$.
(v) $\mu(B\varphi) := IntClInt\mu(\varphi)$.♦

Actually, the semantics of (3.7) is the semantics of the unimodal language L_K, since $B\varphi$ is defined in terms of $K\varphi$, namely $v(B\varphi) \leftrightarrow v(K(v(\neg(K\neg(v(K\varphi))))))$. For topological models (X, OX, μ) the truth of a formula φ at a world $w \in X$ is inductively defined as usual:

(i) $M, w \vDash p$ iff $w \in \mu(p)$.
(ii) $M, w \vDash \neg\varphi$ iff $NOT(M, w \vDash \varphi)$.
(iii) $M, w \vDash \varphi \wedge \psi$ iff $(M, w \vDash \varphi)$ and $(M, w \vDash \psi)$.
(iv) $M, w \vDash \varphi \vee \psi$ iff $(M, w \vDash \varphi)$ or $(M, w \vDash \psi)$.
(v) $M, w \vDash \varphi \to \psi$ iff $NOT(M, w \vDash \varphi)$ or $(M, w \vDash \psi))$.
(vi) $M, w \vDash K\varphi$ iff $\forall U(U{\in}OX$ $(w \in U$ and $\forall v \in U(M, v \vDash \varphi))$.
(vii) $M, w \vDash B\varphi$ iff $\exists U(U{\in}$ $OX(w \in IntClInt(U)$ and $\forall v \in U(M, v \vDash K\varphi))$.

We call a formula φ true in a topological model $M = (X, OX, \mu)$, denoted by $M \mid= \varphi$, if M, x $\mid= \varphi$ for all $x \in X$, and valid in a topological space $X = (X, OX)$, denoted by $X \mid= \varphi$, if $M \mid=$ φ for every topological model M based on X. Moreover, we say that φ is valid in a class K of topological spaces, denoted by $K \mid= \varphi$, if $X \mid= \varphi$ for every member of this class, and that it is valid, denoted by $\mid= \varphi$, if it is valid in the class of all topological spaces. Soundness and completeness with respect to this interior semantics are defined as usual. ♦

With this familiar formal apparatus in place, it has been proved in Mormann (2023) that weak KB logic is sound and complete. In this paper the knowledge operator of a topological

M is always interpreted as the topological interior kernel operator Int of (X, OX), and the belief operator is always interpreted as $IntClInt$.[12] Checking the pertinent definitions of a topology (2.1) and (2.3) and the axioms of weak KB-systems (3.2) one easily obtains the following:

(3.8) Proposition. Every topological model (X, OX, μ) defines a model of a weak Stalnaker system KB in the sense of (3.2). ♦

As already mentioned, with more effort the following stronger theorem can be proved:

(3.9) Theorem. A topological model (X, OX, μ) satisfies all rules and axioms (3.1) of a Stalnaker model ((NI) included) iff (X, OX) is an extremally disconnected space (ED-space) (cf. Baltag *et al*. (2019), Stalnaker (2006).♦

To deal with epistemological issues concerning the Gettier problem and related questions, the belief operator $IntClInt$ should not just be any kind of belief, but rather *justified* belief. For this claim we may argue as follows. First, we notice that the interior semantics of knowledge offers, so to speak, a built-in evidential justification of knowledge. Eventually this also gives rise to a strong justificatory component of the belief operator $IntClInt$. This may be explicated by interpreting the steps that go from knowledge Int over $ClInt$ to belief $IntClInt$ as follows.

For $A \in PX$ the proposition $Int(A)$ is true at a world $w \in X$ iff $w \in Int(A)$. By definition (2.3) of Int, this means that there is an open neighborhood $U(w)$ of w such that $w \in U(w) \subseteq Int(A)$. By definition one has $U(w) \in OX$. Epistemically, $U(w)$ may be interpreted as a piece of observable evidence that the cognitive agent possesses.

To render plausible the interpretation of the belief operator $IntClInt$ as *justified* belief, it is expedient to dwell more on the definition of Int and its justificatory aspects. For this purpose, the following definition is useful:

(3.10) Definition. A subbase of a topological space (X, OX) is a subset $SX \subseteq OX$ such that every element of OX is the set-theoretical union of a finite set of intersections of elements of SX. A base of BX of (X, OX) is a subbase SX of (X, OX) such that every element of OX is a set-theoretical union of elements of SX.♦

12 For all topological spaces, other definitions for belief operators are possible such that all topological spaces (X, OX) define models of weak KB logic, but they will not be considered in this paper. See Mormann (2023).

Informally expressed, a subbase of (X, OX) "generates" the topology OX. Clearly, every subbase SX defines a base BX by taking all finite intersections of elements of SX as elements of BX. Every topology (X, OX) has OX as its largest base. Often, it is convenient to look for smaller (sub)bases of OX, however. For instance, a useful subbase for the topological space of the Euclidean line $(\mathbb{R}, O\mathbb{R})$ is the set of all open rational intervals $\{(a, b); a < b, a, b \in \mathbb{Q}\}$. Epistemologically interpreted, a subbase SX of OX may be considered as the class of (propositions of) directly observable evidences that are available to a cognitive agent whose epistemic activity is characterized by OX (cf. Baltag *et al*. (2019)). Correspondingly, a base generated by subbase SX of OX may be considered as the class of finitely constructed (indirect) evidences that the cognitive agent carries out in the ongoing process of their research.

Defining knowledge topologically by the operator Int comes with the conceptual advantage that knowledge thus defined is naturally correlated with appropriate evidence. That w is an A-world is known is true iff there is an open neighborhood $U(w)$ of w such that $w \in U(w) \subseteq A$. Bringing into play the concepts of bases and subbases, we may say that a cognitive agent engaged in the task of claiming with justification that they *know* that w is an A-world has to find a finitely constructed piece of evidence $U(w)$ from a subbase SX of OX such that $w \in U(w) \in OX$ and $w \in U(w) \subseteq A$. This account of knowledge based on evidential justification can be expanded to a justificatory account of belief. According to (3.8)(v) the belief of a proposition A is defined as the proposition $IntClInt(A)$. By (3.7) the operator $IntClInt$ is a good belief operator in the sense that the pair $(Int, IntClInt)$ satisfies the rules and axioms of a weak Stalnaker system KB. Moreover, $Int(A)$ and $IntClInt(A)$ are extensionally very close to each other:

(3.11) LEMMA. The extensional difference between $Int(A)$ (knowing that A) and $IntClInt(A)$ (believing that A) is topologically small, i.e., nowhere dense:

$$IntCl(IntClInt(A) \cap Int(A)^C) = \emptyset \text{ for all } A \in PX.$$

PROOF. From (2.4) we calculate $IntCl(IntClInt(A) \cap Int(A)^C) = IntClInt(A) \cap IntCl(Int(A)^C) = IntClInt(A) \cap IntCl(Int(A)^C) \subseteq ClInt(A) \cap (ClIntInt(A))C = \emptyset$.♦

Now, $Int(A)$ as knowledge of A is certainly justified, since knowledge as such "by definition" is evidentially justified. Hence, a proposition $IntClInt(A)$ that differs from $Int(A)$ by a topologically negligible difference may also be considered as justified, although not as knowledge but at least as justified belief. Elementary examples for which justified belief may differ from (true) knowledge are easily constructed:

(3.12) Example. For the Euclidean line $(\mathbb{R}, O\mathbb{R})$ consider the set $A := [-1, 1] - \{0\}$. One calculates $Int(A) = (-1, 1) - \{0\}$ and $IntClInt(A) = (-1, 1)$, i.e., $Int(A) \neq A \cap IntClInt(A)$. Hence, it is believed that 0 is an A-world, but it is actually false that 0 is an A-world. Thus, it is not known that 0 is an A-world.♦

Example (3.12) suggests that in general $Int(A)$ and $IntClInt(A)$ extensionally differ only in rare exceptional cases. This is indeed the case: From (2.4)(ii) one calculates that the set-theoretical difference between these sets is topologically small, i.e., nowhere dense:

$$IntCl(IntClInt(A) \cap Int(A)^C) = \emptyset.$$♦

For the special case of topological models based on extremally disconnected spaces treated in detail by Baltag *et al*. (2019), the operators $IntClInt$ and $ClInt$ coincide. Recalling that in modal logic the closure operator Cl is naturally interpreted as (conceptual) possibility, on the class of extremally disconnected spaces "belief" may be elegantly characterized as "possibility of knowledge" (cf. Stalnaker (2006)).

For general topological models, however, $ClInt$ is no longer a good belief operator. Among other deficiencies, for spaces that are not extremally disconnected, $ClInt$ is not a normal operator, as already observed by Stalnaker (2006). If $ClInt$ is replaced by $IntClInt$, most of the qualities of $ClInt$ as a good belief operator are preserved. Only axiom (NI) is no longer valid. Instead of interpreting belief as "possibility of knowledge," for the general case belief is be taken as $IntClInt$ and to be interpreted as "knowledge of possibility of knowledge."

In sum, we may consider the operator $IntClInt$ as an example of a justified belief operator. Reading the operator $IntClInt$ as justified belief allows us to use the apparatus of topological epistemology for elucidating the Gettier problem that may be considered as the problem of how knowledge and justified belief are related.

4. Topological Models of Weak KB Logic and Their JTB Doppelgangers

In the previous sections we have established the elementary fact that all topological spaces (X, OX) can serve as the carriers of topological models of Stalnaker's weak logic of knowledge and belief, knowledge being represented by the topological interior operator Int and belief being represented by $IntClInt$. The tandem $(Int, IntClInt)$ of operators validates many plausible features of the concepts of knowledge and belief and their relations. In particular, all axioms of Stalnaker's KB logic hold except axiom (NI). On the other hand, we already know that not all topological spaces (X, OX) support the JTB-account of knowledge. For the topological model of the real line $(\mathbb{R}, O\mathbb{R})$, JTB (1.1) is already known to be invalid. Clearly, the counterexample for JTB given in proposition (1.5) is paradigmatic and could be multiplied ad libitum. This fact has been considered as sufficient to lay to rest the issue of JTB (cf. Williamson (2013)): JTB turns out to be falsified not only by countless informal counterexamples (cf. Turri (2012), Borges *et al*. (2017), Machery (2017)) but also for general formal reasons. For some, the existence of formal models of Gettier situations, where knowledge does not coincide with justified true belief, is a sufficient reason to lay to rest JTB.

But a closer look reveals that things are more complex. First, it should be taken into account that JTB is valid for at least some topological models. Actually, we can do much better than to rely on the rather trivial and contrived models of JTB mentioned in the introduction. In this section we will show that for any topological space (X, OX) we can canonically construct a topological space $(X, O_{nod}X)$ "in the neighborhood of (X, OX)" that *is* a JTB system, i.e., for which knowledge coincides with true belief. This JTB-system $(X, O_{nod}X)$ will be called the JTB doppelganger of (X, OX).

(4.1) Proposition. Let (X, OX) be a topological space with interior operator Int. Define a topological space $(X, O_{nod}X)$ with interior operator Int_{nod} as follows:

$$Int_{nod}(A) := A \cap IntClInt(A).$$

Then Int_{nod} is the interior operator of a topology on X that is at least as fine as OX, i.e., $OX \subseteq O_{nod}X$. Moreover, $O_{nod}X = O_{nodnod}X$. The topological closure operator Cl_{nod} of $(X, O_{nod}X)$ corresponding to Int_{nod} is given by

$$Cl_{nod}(A) = A \cup ClIntCl(A).$$

PROOF. See Njåstad (1965, Proposition 2, p. 962), or verify that the Kuratowski axioms are satisfied for Int_{nod} and Cl_{nod}.♦[13]

Proposition (4.1) opens a rich reservoir of topological JTB-models:

(4.2) THEOREM. Let $(X, O_{nod}X)$ be the topological space defined in (4.1) by Int_{nod}, $A \in PX$. Denote the belief operator defined on $(X, O_{nod}X)$ by $Int_{nod}Cl_{nod}Int_{nod}$. Then $(X, O_{nod}X)$ is a JTB-system with respect to Int_{nod} and $Int_{nod}Cl_{nod}Int_{nod}$:

$$A \cap Int_{nod}Cl_{nod}Int_{nod}(A) = Int_{nod}(A).$$

PROOF. The proof consists of an elementary calculation using well-known results of the concatenations of the topological operators Int and Cl and the less well known technical result of Kuratowski and Mostowski (2.3). To simplify the notation the abbreviations $I := Int$, $C := Cl$, $I' := Int_{nod}$, and $C' := Cl_{nod}$ are used. Then by definition we have $C'(A) = A \cup CIC(A)$ and we can prove for all $A \in PX$:

$$\begin{aligned}
& A \cap Int_{nod}Cl_{nod}Int_{nod}(A) \\
&= A \cap I'C'(A \cap ICI(A)) \\
&= A \cap I'((A \cap ICI(A)) \cup CIC(A \cap ICI(A))) \\
&= A \cap I'((A \cap ICI(A)) \cup C(IC(A) \cap ICICI(A))) \\
&= A \cap I'((A \cap ICI(A)) \cup CICI(A))) \\
&= A \cap I'((A \cap ICI(A)) \cup CI(A))) \\
&= A \cap I'((A \cup ICI(A)) \cap CI(A)) \\
&= A \cap I'(CI(A)) \\
&= A \cap CI(A) \cap ICICI(A) \\
&= A \cap ICI(A) \\
&= Int_{nod}(A).♦
\end{aligned}$$

Theorem (4.2) shows that there are many JTB-systems: Every ordinary topological system (X, OX) gives rise to a topological JTB-system $(X, O_{nod}X)$ defined on the same underlying set X. $(X, O_{nod}X)$ is obtained by slightly changing the original topological operator Int to the finer topological operator Int_{nod}. Theorem (4.2) can be further strengthened:

13 Njåstad and others call the topology $(X, O_{nod}X)$ the "α-topology" of (X, OX). Therefore, nodec spaces are just α spaces.

(4.3) THEOREM. Let $(X, O_{nod}X)$ be the nodec doppelganger of (X, OX) defined in (4.1). Then $Int_{nodnod}(A) = Int_{nod}(A)$ and $Int_{nod}Cl_{nod}Int_{nod} = IntClInt$.

PROOF. The proof for the first assertion is just a restatement of (4.2):

$$Int_{nod}nod(A) = A \cap Int_{nod}Cl_{nod}\, Int_{nod}(A) = A \cap IntClInt(A) = Int_{nod}(A).$$

The proof of the second condition consists of a direct calculation analogous to that in the proof of (4.2):

$$\begin{aligned}
& I'C'I'(A) \\
&= I'C'(A \cap ICI(A)) \\
&= I'((A \cap ICI(A)) \cup CIC(A \cap ICI(A))) \\
&= I'((A \cap ICI(A)) \cup C(IC(A) \cap ICIICI(A))) \\
&= I'((A \cap ICI(A)) \cup CICI(A))) \\
&= I'((A \cap ICI(A) \cup CI(A)) \\
&= I'CI(A) \\
&= CI(A) \cap ICICI(A) \\
&= ICI(A) \\
&= A \cap IntClInt(A). \blacklozenge
\end{aligned}$$

Recall that for a topological space (X, OX) a set $A \in PX$ is called regular open iff $IntClA = A$. The set of regular open sets is denoted by O^*X. Clearly, $O^*X \subseteq OX$. It is well known that O^*X is a Boolean algebra: Let A, $B \in O^*X$ and define the operations $\wedge^*$ and $\vee^*$ on O^*X by

$$A \wedge^* B := A \cap B \text{ and } A \vee^* B := IntCl(A \cup B).$$

Theorems (4.2) and (4.3) entail that the topological spaces (X, OX) and $(X, O_{nod}X)$ have very similar regular open structures:

(4.4) COROLLARY.

(i) The Boolean algebras O^*X and $O_{nod}{}^*X$ of regular open subsets of the spaces (X, OX) and $(X, O_{nod}X)$ coincide, i.e., $O^*X = O_{nod}{}^*X$.

(ii) The extensional difference between $Int(A)$ and $Int_{nod}(A)$ is nowhere dense for all $A \in PX$.

PROOF.

(i) By (4.3) and the definition of the Boolean algebras of regular open subsets one has

$$O^*X = \{IntClInt(A); A \in PX\} = \{Int_{nod}Cl_{nod}Int_{nod}(A); A \in PX\} = O_{nod}X^*.$$

(ii) By (2.3) and (2.4) one calculates for all propositions A:

$$
\begin{aligned}
&IntCl(Int_{nod}(A) \cap Int(A)^C) \\
&= IntCl(IntClInt(A) \cap A \cap Int(A)^C) \\
&= IC(ICI(A) \cap A \cap I(A)^C) \\
&= ICI(A) \cap IC(A \cap I(A)^C) \\
&= IC(I(A) \cap I(A)^C) \\
&= IC(\emptyset) \\
&= \emptyset.\blacklozenge
\end{aligned}
$$

Succinctly expressed, one may say that (X, OX) and its nodec doppelganger $(X, O_{nod}X)$ are very similar. The following, however, shows that they differ in one essential epistemological feature:

(4.5) COROLLARY. The topological doppelganger $(X, O_{nod}X)$ of (X, OX) is free of Gettier situations, i.e., for all propositions $A \in PX$ one has $G(A) = \emptyset$. In other words, for all worlds $w \in X$, knowledge $Int(A)$ and justified true belief $A \cap IntClInt(A)$ coincide.

PROOF. Using the same abbreviations as in (4.2) one has

$$
\begin{aligned}
&G(A) \\
&= A \cap ICI(A) \cap (A \cap ICI(A))^C \\
&= (A \cap ICI(A)) \cap A^C) \cup (A \cap ICI(A)) \cap ICI(A)^C) \\
&= \emptyset \cup \emptyset \\
&= \emptyset.\blacklozenge
\end{aligned}
$$

Corollary (4.5) can be used to achieve a kind of topological dissolution of the problem that the result of Gettier poses for the epistemology of knowledge and belief.

Our starting point is a critical assessment of a well-known argument of Baltag *et al*. (2019), according to which the semantics of belief based on the closure-interior belief operator $ClInt$ is preferable to Steinvold's semantics (cf. Steinsvold (2006)) for belief based on the co-derived set operator, since the latter falls prey to the well-known Gettier counterexamples:

> Where knowledge as the interior and belief as the co-derived set operator are studied together, we obtain the equality
>
> $$KA = A \cap BA,$$
>
> stating that *knowledge is true belief*. Therefore, this semantics yields a formalization of knowledge and belief that is subject to well-known Gettier counterexamples. … The closure-interior belief semantics improves on the co-derived set semantics for the following reasons: (1) belief as the closure of the interior operator does not face the Gettier problem, at least not in the easy way in which the co-derived set semantic does, when considered with the conception of knowledge as interior. More precisely, knowledge as interior cannot be defined as (justified) true full belief, since, in general, $Int(A) \neq ClInt(A) \cap A$, i.e., $KA \neq BA \wedge A$; (2) the class of DSO-spaces with respect to which KD45 is sound and complete under the co-derived set semantics is a proper subclass of the class of extremally disconnected spaces, which shows that the closure-interior semantics for KD45 is defined on a larger class of spaces. (Baltag *et. al*. (2019), pp. 219, 224–225).

Note that the semantics $ClInt$ may also fall prey to Gettier counterexamples: it is easily shown that the nodec doppelganger $(X, O_{nod}X)$ of an ED space (X, OX) is still an ED space. Hence, $(X, O_{nod}X)$ is a model of JTB. There are, fortunately, also ED spaces that are not nodec. For instance, it is well known that topological spaces that are Stone-dual to complete Boolean algebras are extremally disconnected, such as the Stone–Čech compactification $\beta(\mathbb{N})$ of the set of natural numbers with a discrete topology (cf. Baltag *et al*. (2019), p. 215). It can be shown that these extremally disconnected spaces are not nodec. In other words, in contrast to co-derivational semantics, the $ClInt$ semantics does not always fall prey to Gettier counterexamples. This is to be considered as a real advantage of $ClInt$ semantics.

More generally, the simultaneous discussion of different topologies (X, OX) and $(X, O_{nod}X)$ on the same underlying universe X is an argument in favor of an epistemological strategy that takes topology as a variable that can have different values depending on the specifics of the epistemic situation considered. That is to say, the "transformations" from (X, OX) to $(X, O_{nod}X)$

and vice versa (or similar ones) should not only be shown to be possible. Rather, it is to be discussed when and why it is expedient to carry them out out for some reason or other. This brings into play pragmatic considerations.

It is also interesting to consider what happens when we go beyond the class of extremally disconnected spaces. First, note that for spaces that are not extremally disconnected the operator $IntCl$ is not an acceptable belief operator, since it is not a normal operator. This was pointed out by Stalnaker (2006, p. 195). Therefore, this operator has to be replaced by $IntClInt$ (Mormann (2023)). This is an innocent change, since $IntClInt$ and $ClInt$ coincide on extremally disconnected spaces. Moreover, $IntClInt$ is a normal operator, and $(Int, IntClInt)$ satisfies all rules and axioms of KB except, of course, axiom (NI). Finally, it can be proved that for ($\mathbb{R}$, $O\mathbb{R}$) and similar spaces, $Int(A) \neq IntClInt(A) \cap A$, i.e., $KA \neq BA \wedge A$. In sum, the interior-closure-interior belief operator $IntClInt$ is also not subject to Gettier counterexamples in general. Only for nodec spaces does (Int, $IntClInt$) semantics falls prey to the Gettier paradox, since for this class of spaces one has $Int(A) = A \cap IntClInt(A)$ for all propositions A, i.e., $K(A) = A \cap B(A)$ in the terminology of Baltag *et al.* (2019). For the belief-fragment defined by $IntClInt$ one finds that it is a KD4*-logic.

The topological approach not only shows that $(Int, IntClInt)$ does not in general succumb to the Gettier counterexamples, it also explains the exceptional character of Gettier situations $G(A)$. This may be succinctly formulated by saying that the topological approach of this paper "explains away the Gettier problem" by offering an explanation for the exceptional character of Gettier situations:

(1) Gettier situations $G(A)$ are topologically characterized as nowhere dense. In informal language this means that for most worlds $x \in X$ the traditional JTB account holds.

(2) Gettier situations are dependent on the agent's knowledge situation: if the agent knew a little more, i.e., if they were located in $(X, O_{nod}X)$ and not in (X, OX), they would not experience a Gettier situation with respect to a proposition A.

(3) Being in a Gettier situation $x \in G(A)$ with respect to a proposition A remains cognitively opaque or doxastically intransparent to the agent, i.e., they neither know nor believe x to be in such a situation, since $Int(G(A)) = \emptyset$ and $IntClInt(G(A)) = \emptyset$.

For any topological space (X, OX) there exists a nodec doppelganger $(X, O_{nod}X)$ with very similar features (possessing the same sets PX of propositions and even the same Boolean lattice $O^*(X)$ of regular open sets). Nevertheless, in $O_{nod}X$ a cognitive agent is not confronted with Gettier situations. That is to say, being a Gettier proposition is a rather volatile feature of a proposition A. This is in line with the general feeling that many people have if they are confronted with a Gettier situation; they assess it as a somehow strange and weird situation. Even if this assessment is rather vague and imprecise, an adequate formal reconstruction of the situation should reflect this widespread assessment. This desideratum suggests a different attitude toward the problem of Gettier situations that contributes to its dissolution. The existence of Gettier situations as exceptional situations should be accepted.

Usually, for a topological model (X, OX) there are some propositions A that give rise to Gettier situations, $G(A) \neq \emptyset$. Extensionally, the set $G(A)$ is negligible or exceptional in a precise topological sense, namely $G(A)$ is nowhere dense, i.e., $IntCl(G(A)) = \emptyset$.

The elusive nature of Gettier situations $G(A)$ is further confirmed by the fact that a small enlargement of the cognitive capacity (which amounts to a refinement of the topological structure from OX to $O_{nod}X$) suffices to remove the Gettier situation caused by A. Seen from the perspective of possibly improving our intellectual capacity (or reducing our ignorance), a Gettier situation is little more than a temporary disturbance that can in principle be overcome. A cognitive agent α may be said to "feel cognitively at home" in a universe of possible worlds (X, OX) if for all propositions $A \in PX$ one has $G(A) = \emptyset$, i.e., if (X, OX) is nodec. To "feel cognitively at home" in X it is not necessary that α is an omniscient deity, it suffices for α's knowledge that the equation $OX = O_{nod}X$ holds.

The traditional JTB account of knowledge is "almost correct" in so far as for all propositions A the sets of Gettier situations $G(A)$, where JTB does not hold, are exceptional situations in the sense that the sets are nowhere dense, i.e., topologically "thin" or "small."[14] Moreover, even the exceptional character of these situations could be eliminated by improving α's knowledge replacing OX by $O_{nod}X$.

This possibility may be considered as a topological dissolution of the Gettier problem of epistemology: the Gettier phenomenon is recognized as such, namely that there may be propositions $A \in PX$ for which JTB does not hold, but at the same time the possibilities for overcoming these situations are presented.

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14 In the technical jargon of topology, the family of nowhere dense subsets of a topological space (X, OX) is characterized as an ideal (cf. Kuratowski (1966), Jankovic and Hamlett (1990)). Ideals are families of "small" or "negligible" sets. By the definition of an ideal, the finite unions of "small" sets are "small" and the subsets of "small" subsets are "small." This is a rather innocent, almost trivial "theory" that should raise no controversies even in "philosophical discussions." Nevertheless, it has become quite useful in many applications of topology. For the problem of Gettier counterexamples, this theory asserts that the extension of Gettier situations, i.e., situations in which knowledge does not coincide with justified true belief, is topologically small.

Thereby, topological epistemology "explains away" the Gettier problem. Here, "explaining away" does not mean that the Gettier counterexamples "disappear" in some miraculous way. Rather, there is a different way to live with them. Namely if one is content to have available a rather simple epistemological theory that works in most cases (Gettier cases are rare!), one may stick to JTB. Topologically this means that one makes the strong idealizing assumption that one's topological models of knowledge and belief are nodec spaces. If one is not prepared to make such a strong assumption concerning the epistemic structure of the universe, for this more realistic stance the cost is the acceptance of (rare) Gettier situations where the simple JTB theory fails.

For this way of "explaining away," we need not find a waterproof "fourth condition" of knowledge. After 60 years of obtaining less than fully convincing results, many people consider this endeavor as hopeless. The 60 years of post-Gettier epistemology has shown that ever more complicated definitions of knowledge (taking into account a "fourth condition") have only evoked ever more complicated counterexamples designed to defeat any allegedly definitive characterization of knowledge as JTB+X. To "explain away" the Gettier problem it is not necessary to find such a, perhaps unattainable, fourth condition X.

I propose to interpret the move from (X, OX) to $(X, O_{nod}X)$ as a "learning process." This learning process is to be distinguished from the one elaborated in Baltag *et al.* (2017, 2019). In these papers, a different kind of "learning process" is conceived of as an addition of *modal operators* B^{φ}. In contrast, in the present paper a "learning process" is understood as an addition of *knowledge*: a cognitive agent who uses $O_{nod}X$ and not OX has more knowledge available for their cognitive actions than one for whom only OX is available as knowledge.

This extra knowledge can be described as the result of a learning process that the agent has completed by replacing OX by $O_{nod}X$. It can be precisely described as follows. After completing this learning process, the cognitive agent knows all propositions (or facts) A of $O_{nod}X \subseteq PX$. As is well known, this the case iff

(4.6) $$A \subseteq ClInt(A) \text{ and } A \subseteq IntCl(A)$$

(cf. Reilly and Vamanamurthy (1985)). For the Euclidean line $(\mathbb{R}, O\mathbb{R})$, an elementary example of a set A that is open relative to $O_{nod}X$ but not relative to OX is given:

(4.7) Example. Let $(\mathbb{R}, O\mathbb{R})$ be the real line with the standard Euclidean topology $O\mathbb{R}$. For

$$A := \mathbb{R} - \{1/n; n = 1, 2, ...\}$$

we have $ClInt(\mathbb{R}) = \mathbb{R}$ and $IntCl(A) = \mathbb{R}$. This means that $A \in O_{nod}\mathbb{R}$, and therefore $Int(A) = A \cap IntClInt(A)$, i.e., $A \in O_{nod}\mathbb{R}$, but A is not open with respect to $O\mathbb{R}$, since we have $Int(A) = \mathbb{R} - (\{1/n\} \cup \{0\}) \neq A$. This example is typical for nodec spaces in so far as it is well known that these spaces can be characterized as those for which all nowhere dense subsets are closed. The space $(\mathbb{R}, O\mathbb{R})$ is not nodec, since there are nowhere dense sets such as $\{1/n; n = 1, 2, ...\}$ that are not closed in $(\mathbb{R}, O\mathbb{R})$. In $(\mathbb{R}, O_{nod}\mathbb{R})$, however, the set $\{1/n; n = 1, 2, ...\}$ is closed.

Stipulating that all nowhere dense sets are closed amounts to a considerable simplification of the topological structure. Epistemologically interpreted, this simplification is a global assumption about the knowability of the facts of the universe (X, OX).♦

Thus, nodec spaces are not just a niche where the traditional JTB epistemology can survive when belief is based on the operator $IntClInt$ (interpreting "belief" as "knowledge of not knowing that one does not know"). Rather, nodec spaces are also a means of explaining the exceptional character of Gettier situations. Thereby, they simultaneously take into account the unavoidable limitations and shortcomings of the JTB account and, at the same time, explain why JTB is "almost correct." In this manner the topological account of this paper avoids the alternative proposed by Ichikawa and Steup:

> Epistemologists who think that the JTB approach is basically on the right track must choose between two different strategies for solving the Gettier problem. The first is to strengthen the justification condition to rule out Gettier cases as cases of justified belief. This was attempted by Roderick Chisholm … . The other is to amend the JTB analysis with a suitable fourth condition, a condition that succeeds in preventing justified true belief from being "gettiered." Thus amended, the JTB analysis becomes a JTB+X account of knowledge, where the "X" stands for the needed fourth condition. (Ichikawa and Steup (2017, section 4))

The topological approach of the present paper offers an argument in favor of not participating in either of these two strategies: it does not intend to strengthen the justification condition nor does it intend to find the missing "fourth condition" for knowledge. Rather, it attempts to circumvent the Gettier problem by acknowledging the traditional JTB epistemology as "almost correct." More precisely, JTB is "almost correct" in the sense that "almost correct" is "correct except in exceptional circumstances." This paper renders precise what is to be understood by "exceptional circumstances" which are, usually and correctly, ascribed to Gettier situations $G(A)$.

Corollaries (4.4) and (4.5) offer — in principle — a way to remove the Gettier cases that in general beset topological models (X, OX) of weak KB. If the cognitive agent who employs Int and $IntClInt$ for their doxastic and epistemic actions improves their knowledge operator from Int to Int_{nod} and maintains their method of justified belief as $IntClInt$, then they can avoid being confronted with Gettier cases. In other words, Gettier situations could be avoided iff the agent improves their epistemic powers.

The spaces (X, OX) and $(X, O_{nod}X)$ may be conceived as the two stages of a learning process. The initial stage is topologically characterized by (X, OX). This stage is beset with epistemic anomalies, namely Gettier situations $G(A) \neq \emptyset$. It finds its ideal end in the universe $(X, O_{nod}X)$ that is free of Gettier situations. This means that the cognitive agent who uses Int_{nod} knows enough to avoid any Gettier situation for which knowledge differs from true justified belief.

Epistemologically, nodec spaces are characterized as the appropriate class of topological models for traditional JTB-epistemology which ignores the existence of Gettier situations. This may be considered as a *partial* rehabilitation of the JTB-account by topological epistemology.

The relation between the class of all topological spaces $TOP(X)$ on X and the class of nodec spaces $TOP_{nod}(X)$ defined on X may be described as follows:

(4.8) Proposition. The class $TOP_{nod}(X)$ of nodec spaces on X is a subclass of $TOP(X)$ such that the natural embedding $i{:}TOP_{nod}(X) \rightarrow TOP(X)$ has a left inverse $j{:}TOP(X) \rightarrow TOP_{nod}(X)$ ("nodecification") that maps every topological space (X, OX) onto its nodec doppelganger $(X, O_{nod}X)$ such that $TOP_{nod}(X) \rightarrow TOP(X) \rightarrow TOP_{nod}(X)$ is the identity map id on $TOP_{nod}(X)$.♦

In more general terms, the logical relation between the two classes $TOP(X)$ and $TOP_{nod}(X)$ of spaces can be further explicated as follows. Let $K \subseteq TOP$ be any class of topological spaces (for instance, the class of all topological spaces, the class of extremally disconnected spaces, the class of nodec spaces, etc.). Recall that S4 is the least set of formulas of the basic unimodal language L with basic modal operator o satisfying the axioms

(4.9) (i) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow o\psi)$, (ii) $\Box\varphi \rightarrow \varphi$, (iii) $\Box\varphi \rightarrow \Box\Box\varphi$

and closed under modus ponens, substitution, and necessitation $(\varphi/\Box\varphi)$. For a class of K of topological spaces, let $L(K)$ denote the set of formulas of L that are valid in K, interpreting the formulas of L in the familiar way (see 3.4). $L(K)$ is called the modal logic of K. Since the classical result of McKinsey and Tarski (1944) it has been well known that the modal logic of the class of all topological spaces TOP is S4.

In the past two decades the modal logics of many subclasses of TOP have been determined (see Bezhanishvili *et al*. (2004)). For topological epistemology the extension S4.2 of S4 is particularly interesting. As shown by Stalnaker (2006) and Baltag *et al*. (2019), S4.2 is the logic of extremally disconnected topological spaces (X, OX). Topologically, these spaces can be characterized as spaces whose closure operators satisfy $Cl(A \cap D) = Cl(A) \cap Cl(D)$ for open sets A and B (cf. Footnote 7). Expressed in epistemological terms, (X, OX) is extremally disconnected essentially iff axiom (NI) is satisfied and some other less contentious axioms are satisfied for $IntClInt$ and Int (Baltag *et al*. (2019)).

The interesting point is that the class of nodec spaces — as the class of topological spaces that satisfy JTB — also corresponds to a well-known extension of S4, namely to S4.Zem (cf. Bezhanishvili *et al*. (2004) and Zeman (1969)).

Thus, the status of JTB in topological epistemology may be understood as analogous to that of the extension S4.2 of S4, namely as a modal logic in which a special axiom holds: axiom S4.Zem (cf. Bezhanishvili *et al*. (2004), Theorem 3.4). Hence, from the point of view of formal topological epistemology, one should not ask simpliciter whether JTB is true or not. Rather, a more appropriate question is to ask for which class of topological models JTB is true. The neat answer to this question is that JTB holds for nodec models. This assertion is analogous to the statement that the topological knowledge operator Int satisfies the axioms of Stalnaker's combined logic KB (axiom (NI) included) iff the topological universe of possible worlds (X, OX) has the topological structure of an extremally disconnected space. Analogously, a topological model (X, OX) satisfies the classical JTB-account of knowledge as justified true belief iff it satisfies the Zeman axiom $((\Box\neg\Box\neg\Box p)\rightarrow(p\rightarrow\Box p))$.

(4.10) Proposition. Interpreting $\Box$ as the topological interior operator Int and the belief operator as $IntClInt$, for topological models (X, OX, v) the Zeman axiom $((\Box\neg\Box\neg\Box p)\rightarrow(p\rightarrow\Box p))$ holds iff the JTB-axiom (1.1) is valid: $Int(A) = A \cap IntClInt(A)$.

Proof. By definition of the topological interpretation of S4, we have that the characteristic axiom $((\Box\neg\Box\neg\Box p)\rightarrow(p\rightarrow\Box p))$ for S4.Zem holds for a topological model (X, OX, v) iff

$$IntClInt(A) \cap A \subseteq Int(A) \text{ for all } A \in PX.$$

On the other hand, the inequality $Int(A) \subseteq A \cap IntClInt(A)$ holds from the definition of Int and well-known properties of $IntClInt$ for all topological models (see 2.3). Hence $Int(A) = A \cap IntClInt(A)$. In other words, (X, OX) is a nodec space.♦

The logical characterization of nodec spaces as the class of spaces that satisfy S4.Zem establishes an analogy between JTB and other epistemologically interpretable modal logics that are defined as normal extensions of S4. The following list of examples of such modal logics is far from complete:

(4.11) Example. Some epistemically interpretable modal logics and their classes of topological spaces.

(i) The "logic of clearness" (Bobzien (2012)) is the logic S4.1 = S4 + $(\Box\neg\Box\neg p) \rightarrow (\neg\Box\neg\Box p)$. The class of topological models of S4.1 is the class of McKinsey spaces.
(ii) Stalnaker's combined logic of knowledge and belief KB (satisfying the rules and axioms (3.1) with (NI) included)) is the logic S4.2. The class of topological models of S4.2 is the class of extremally disconnected spaces.
(iii) The traditional epistemic logic JTB of knowledge as justified true belief is the logic S4. Zem = S4 + $((\Box\neg\Box\neg\Box p \rightarrow (p \rightarrow \Box p))$. The class of topological models of S4 .Zem is the class of nodec spaces.♦

In sum, topological epistemology suggests a relativization of the question of whether the traditional thesis that knowledge is justified true belief is correct or not. The answer to this question depends on the class of topological models chosen. Thus, traditionalists such as Sellars (see Section 6) who insist that traditional JTB is essentially correct should subscribe to S4.Zem as the appropriate logic for the epistemology of knowledge and belief, and those who acknowledge Gettier counterexamples will choose another extension of S4 as the appropriate logic of knowledge and belief. For instance, if they insist on (NI) as a necessary condition for a good belief operator they will choose S4.2.

5. Topology of Gettier Cases: A Topological "Double Luck" Construction and the Doxastic Invisibility of Gettier Situations

The previous section dealt with the topological and logical problems of a very special class of topological epistemological systems, namely systems for which the traditional account of knowledge as justified true belief is valid. As mentioned, since Gettier's paper this account has come under heavy attack. Nevertheless, some philosophers still doubt that Gettier cases have definitively refuted the traditional JTB-account. Nodec spaces may be considered as a niche where the traditional JTB-account can survive. Topological epistemology provides a *relative* justification of JTB.

This section intends to show that topological epistemology is also useful for bringing to the fore some interesting formal aspects of Gettier situations that have seldom or never been noticed in the decades during which the production of ever more sophisticated Gettier examples has flourished. First, let us consider the obvious but nevertheless somewhat enigmatic aspect of Gettier cases that they are exceptional or rare. Topology is an expedient device for rendering precise this impression.

Let $A \in PX$ be a proposition that describes a situation as a set of possible worlds. Then the Gettier proposition $G(A) := A \cap IntClInt(A) \cap Int(A)^C$ is a rare event in a precise topological sense:

(5.1) THEOREM.

(i) For all topological spaces (X, OX) and all $A \in PX$, the set $G(A)$ of Gettier worlds for A is *nowhere dense*, i.e., $IntCl(G(A)) = \emptyset$:

$$IntCl(G(A)) = IntCl(A \cap IntClInt(A) \cap Int(A)^C) = \emptyset.$$

(ii) For all nodec spaces $(X, O_{nod}X)$, one obtains the stronger result $G(A) = \emptyset$ for all $A \in PX$.

PROOF.

(i) Using once again the abbreviations of (4.2) and the technical lemma (2.3) one calculates:

$$IC(G(A))$$
$$= IC(A \cap ICI(A) \cap I(A)^C)$$

$$
\begin{aligned}
&= ICICI(A) \cap IC(A \cap I(A)^C) \\
&= ICICI(A) \cap IC(A \cap I(A)^C) \\
&= ICI(A) \cap IC(A \cap I(A)^C) \\
&= IC(A \cap IC(A) \cap I(A)^C) \\
&= IC(\emptyset) \\
&= \emptyset.
\end{aligned}
$$

(ii) This has already been proved in (4.5).♦

(5.1) (i) and (ii) confirm the intuitive impression that Gettier situations are rare or exceptional events. In the special case of nodec spaces, Gettier situations are extremally rare events — they never occur.

The fact that Gettier situations are nowhere dense events affects their epistemic status. They can be neither known nor believed by cognitive agents who use Int or $IntClInt$.

The proof that Gettier cases cannot be known by Int is elementary and amounts to a simple calculation using some axioms of KB:

(5.2) Proposition. Let (X, OX) be a topological model of KB-logic, $A \in PX$, and

$$G(A) := IntClInt(A) \cap A \cap Int(A)^C$$

be a Gettier situation of A. Then $G(A)$ cannot be known by a cognitive agent who uses the operator Int, i.e., the cognitive agent is ignorant of $G(A)$.

Proof. By definition the Gettier proposition $G(A)$ is known at a world $w \in X$ iff $w \in Int(G(A))$. An elementary calculation using the Kuratowski axiom (2.3) shows that this is impossible since $Int(G(A))$ is empty:

$$
\begin{aligned}
&Int(G(A)) \\
&= I(ICI(A) \cap A \cap I(A)^C) \\
&= IICIA) \cap IA) \cap I(I(A)^C) \\
&= I(A) \cap I(I(A)^C) \\
&= I(I(A)) \cap I(I(A)^C) \\
&= I(I(A) \cap I(A)^C)
\end{aligned}
$$

$$= I(\emptyset)$$
$$= \emptyset.\blacklozenge$$

Informally stated, Gettier sentences cannot be known.[15]

Theorem (5.2) can be strengthened by replacing Int by $IntClInt$. This is noteworthy in so far as $IntClInt$ is not factive.

(5.3) THEOREM. Let (X, OX) be a topological model of the weak KB-logic of knowledge and belief for the operators Int and $IntClInt$ as defined in (3.2). Let $G(A) := IntClInt(A) \cap A \cap Int(A)^C$. Then Gettier proposition $G(A)$ cannot be believed consistently, i.e., there is no world where $G(A)$ can be believed by a cognitive agent who relies on $IntClInt$ as justified belief, since $IntClInt(G(A)) = \emptyset$.

PROOF. Suppose that $w \in X$ is a world in which the Gettier case $G(A) \neq \emptyset$ is believed with respect to the belief operator $IntClInt$, i.e., $w \in IntClInt(IntClInt(A) \cap A \cap Int(A)^C) \neq \emptyset$. Then, using the axioms of KB-logic and (2.3) one calculates:

$$\begin{aligned}
& IntClInt(G(A)) \\
&= ICI(ICI(A) \cap A \cap I(A)^C) \\
&= ICIICI)(A) \cap ICI(A) \cap ICI(I(A)^C) \\
&= ICI\ A \cap ICI(I(A)^C) \\
&= IC(I(A) \cap I(A)^C) \\
&= I(\emptyset) \\
&= \emptyset.
\end{aligned}$$

Thus, $IntClInt(G(A)) = \emptyset$ for any $A \in PX$, i.e., there is no world w in which one can believe with justification that w is an A-world.♦

For all models, all Gettier situations are topologically rare situations. They may be eliminated by improving the cognitive agent's epistemic capacities, namely, by replacing the knowledge operator Int by the finer operator Int_{nod}. This replacement dissolves the cognitive anomalies exemplified by $G(A)$. Moreover, the move from Int to Int_{nod} is topologically small, since the extensional difference between $Int_{nod}(A)$ and $Int(A)$ is nowhere dense. Of course, an

15 For a recent discussion of the logic of ignorance see Fine (2018) and Fano and Graciani (2021).

omniscient cognitive agent is plagued with Gettier propositions. But omniscience is not a realistic aim for epistemic progress. In contrast, the move from (X, OX) to $(X, O_{nod}X)$ is a rather modest cognitive improvement that is already sufficient to avoid Gettier cases, since the JTB-doppelganger $(X, O_{nod}X)$ of (X, OX) is free of Gettier situations for all propositions A. Both models share the same operator of justified true belief $IntClInt$; only their knowledge operators Int and Int_{nod} differ slightly.

As has been observed by the "Gettier industry" of the past decades, Gettier examples may be constructed according to certain general recipes. As Turri put it:

> Gettier cases are constructed by a recipe. Start with a belief sufficiently justified to meet the justification requirement for knowledge. Then add an element of bad luck that would normally prevent the justified belief from being true. Lastly add a dose of good luck that "cancels out the bad," so the belief ends up true anyhow. (Turri (2012), p. 248)

Turri discusses the following well-known example of double luck adapted from Zagzebski (1996) as a classical Gettier case:

> (HUSBAND) Mary enters the house and looks into the living room. A familiar appearance greets her from her husband's chair. She thinks, "My husband is home," and then walks into the den. But Mary misidentified the man in the chair. It is not her husband, but his brother, whom she had no reason to think was even in the country. However, her husband was seated along the opposite wall of the living room, out of Mary's sight, dozing in a different chair. (Adapted from Zagzebski (1996), pp. 285–286)

This recipe of "double luck" has a topological analogue. This evidences that the topological model offers a useful, at least partially faithful representation of epistemological phenomena. To facilitate a better understanding of the role of topological models in formal epistemology, it may be expedient to recall briefly the role of mathematical models in physics. In many models of Newtonian mechanics, force (acceleration) f is considered as the derivative of velocity v, i.e., $f = v'$. This does not mean that the mathematical concept of derivative has a (causal) role in physical processes. It only means that there is a (partial) structural correspondence between the "logic of derivative and the "logic of physical processes." This structural correspondence is exemplified, for instance, by the fact that the law of addition of forces has a structural analogue to the mathematical law of vector addition.

To keep matters as simple and intuitive as possible, consider the following example based on the two-dimensional Euclidean plane endowed with its familiar Euclidean topology. The Euclidean plane may be identified in a natural way with a two-dimensional vector space. Let the one-dimensional line $\mathbb{R}$ diagonally embedded in $\mathbb{R}^2$ be $\mathbb{R}:=\{(x, x); x \in \mathbb{R}\}$. Let $A .= \mathbb{R}^2$ - $\mathbb{R} \cup \{(0, 0)\}$ and $E := \{(0, 0)\}$ the origin of the plane. Intuitively, the construction of A may be described as follows: One begins with a class of "ordinary" situations $\mathbb{R}^2$, removes a subclass of "exceptional" situations $\mathbb{R}$ ("bad luck"), and finally adds a class E of "exceptional exceptions" $E \subseteq \mathbb{R}$ with $Int(E) = \emptyset$ ("good luck"). The resulting set $A := \mathbb{R}^2 - \mathbb{R} \cup E$ may be considered as a topological version of the "double luck construction" that Turri and others describe as a general recipe for constructing Gettier situations. Indeed, the set A turns out to be a Gettier situation since one calculates for $G(A)$:

(5.4) $$G(A) = A \cap IntClInt(A) \cap Int(A)^C = \{(0,0)\} \neq \emptyset. \blacklozenge$$ [16]

Hence, the E-world $(0, 0)$ is a "Gettier world" with respect to A, i.e., A is a justified true belief at E-worlds but is not known there, since $Int(G(A)) = \emptyset$.

For formal epistemology, the relevant point of the topological model is the topological construction of a proposition that has an analogical structure to the construction process described informally and implicitly in the fantastic stories of Smith and Jones, fake barns, husbands and their brothers, etc., that lead to Gettier situations of various kinds. These situations are structurally described by propositions A for which knowledge of does not coincide with justified true belief:

$$Int(A) \neq A \cap IntClInt(A).$$

Following Zagzebski's recipe, the example of fake barns may be reconstructed in a topological model by the following correspondence:

(5.5) (Fake barn example topologized).
All barns (fake barns or real barns): $\mathbb{R}^2 := \{(x, y); (x, y) \text{ points of the plane}\}$

---

16 A moment's reflection reveals that this construction has little to do with the specific structure of $(\mathbb{R}^2, O\mathbb{R}^2)$ but can be considerably generalized to arbitrary Hausdorff spaces (X, OX), i.e., if (X, OX) is a Hausdorff space, $wG \in X$, then the subset $A := X \times X - D(X) \cup D(E)$ is a Gettier situation, i.e., $G(A) = E$ is a set of Gettier worlds where A is true, believed with justification, but not known. Here, of course, D is the diagonal function defined as $D: X \to X \times X$ by $D(x) := (x, x)$ for $x \in X$. Other topological constructions of Gettier situations are easily found.

Barns in "fake barn country" (fake barns or real barns): $\mathbb{R} \subseteq \mathbb{R}^2$
(0,0) (The only real barn in "fake barn country"): $\{(0, 0)\} \subseteq \mathbb{R} \subseteq \mathbb{R}^2$
Now define $A := \mathbb{R}^2 - \mathbb{R} + \{(0, 0)\}$. Then we easily calculate: $Int(A) \neq A \cap IntClInt(A)$.♦

After admitting that JTB is false in general, it is natural to look for an explanation of why JTB had and still has so much attraction in epistemology and common sense. Somehow, JTB seems close to being true. Topology offers a formal explanation for this fact. As I want to explicate in the following, a reason why JTB has the appeal of being essentially true is that counterexamples to JTB, namely, Gettier cases $x \in G(A)$ are — as "anomalies" for JTB — epistemically invisible in a precise sense. Thus, it is unsurprising that many people consider JTB as "essentially" correct even though they cannot deny the existence of Gettier situations. Moving from (X, OX) to $(X, O_{nod}X)$, thereby eliminating the Gettier situations existing in (X, OX), amounts to an extensionally small cognitive improvement that renders traditional JTB epistemology valid.

The topological account of knowledge and belief confirms Williamson's thesis, according to which JTB is refuted not only by an abundance of informal counterexamples but also by the fact that its failure can be predicted on general theoretical grounds. For appropriate topological models based on general topological spaces (X, OX), one can construct Gettier situations $G(A)$ by double luck constructions (or otherwise) that exhibit situations where knowledge does not coincide with true justified belief. On the other hand, topological epistemology also offers arguments for the assessment that JTB is almost true. It shows that Gettier situations do not occur for models based on nodec spaces. Moreover, topological epistemology shows that all topological models possess nodec doppelgangers free of Gettier situations.

6. Concluding Remarks

This paper has dealt with two complementary problems:

(1) How can Gettier situations $G(A)$ be constructed systematically in topological epistemology? Answer: Topological double luck constructions (and other devices) show the existence of Gettier situations for many topological models (X, OX).

(2) How can Gettier situations be avoided systematically for appropriate topological universes of possible worlds? Answer: If (X, OX) is any topological model of knowledge and belief whatsoever, its nodec doppelganger $(X, O_{nod}X)$ can be shown to be free of Gettier situations.

Thus, topological models of knowledge and justified belief provide robust evidence that traditional JTB-epistemology is not fully correct, independently of contrived thought experiments. On the other hand, the existence of JTB-doppelgangers for all topological models suggests that the classical JTB-account should not simply be dismissed as an obsolete erroneous theory. Rather, JTB offers a simplified account of knowledge and belief that works quite well in most cases but fails in exceptional cases. The conceptual surgery that is necessary to eliminate Gettier situations from a topological universe (X, OX) of possible worlds is extensionally small in the sense that for all propositions A the difference between $Int(A)$ and its JTB-doppelganger $Int_{nod}(A)$ is topologically negligible, i.e., nowhere dense. This fact may be interpreted as a *partial* rehabilitation of the classical JTB-account.

This rehabilitation is only partial in so far as this topological approach recognizes the unavoidability of Gettier situations that are to be considered as exceptional situations or anomalies that cannot be handled adequately by the JTB approach.

The unknowability and unbelievability of Gettier propositions confirm the impression that Gettier cases are somehow exceptional. Topological epistemology has, so to speak, a Janus face with respect to JTB: on the one hand it offers a justification for JTB by showing that models based on nodec spaces are free of Gettier situations; on the other hand, it provides a strict general refutation of the JTB that knowledge is justified true belief by formal mathematical arguments. Thereby, topological epistemology may be considered as a useful addition to the many informal arguments that often only rely on rather contrived thought experiments. In his classical paper "Epistemic Principles," Sellars (1975) asserted:

> The explication of knowledge as "justified true belief" though it involves many pitfalls [,] … is, I believe, essentially sound (Sellars (1975), p. 99)).

Sellars did not give arguments for his traditionalist assessment of this issue. He simply assumed it:

> In the present lecture I shall assume that it can be formulated in such a way as to be immune from the type of counterexamples with which it has been bombarded since Gettier's pioneering paper in Analysis. (Sellars (1975), ibid.)

Almost 50 years have passed since Sellars put forward his optimistic assessment that eventually a formulation of JTB would be found that is immune to Gettier's criticism. Today, Sellars' hope seems to be less realistic than ever. Since then, the production of ever more sophisticated counterexamples has continued (cf. Turri (2012), Borges *et al*. (2017)). Moreover, many philosophers have even lost interest in this issue. Topological epistemology offers a way out of this deadlock. The topological account of this paper proposes the concept of JTB as one among many possible topological versions of epistemological logic, each of which is characterized by one or more specific axioms. More precisely, JTB is characterized by the axiom characteristic for S4.Zem that can be topologically formulated as

(1.1) $$Int(A) = A \cap IntClInt(A).$$

This parallelism between JTB and other modal systems such as S4.1, S4.2, and S4.3 suggests that one should no longer ask the simple question: "Is knowledge justified true belief?" but rather "What type of topological models validates JTB?" With this modification, at once the more modest and more sophisticated question has a neat and satisfying answer:

(6.1) Theorem. (Restricted validity of the traditional JTB-account of knowledge as justified true belief): The JTB-epistemology is valid for topological models based on nodec spaces $(X, O_{nod}X)$. It is not valid for models that are not nodec.♦

Epistemically, the move from (X, OX) to $(X, O_{nod}X)$ that eliminates all Gettier situations $G(A)$ for all propositions $A \in PX$ can be characterized as a learning process that enlarges the cognitive powers of the cognitive agent from an initial state of knowledge defined by the operator Int to a more comprehensive knowledge defined by the finer operator Int_{nod}.

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