Synthese
https://doi.org/10.1007/s11229-020-02951-2

Prototypes, poles, and tessellations: towards a topological theory of conceptual spaces

Thomas Mormann[1]

Received: 25 April 2020 / Accepted: 3 November 2020

Abstract
The aim of this paper is to present a topological method for constructing discretizations (tessellations) of topological conceptual spaces. The method works for a class of topological spaces that the Russian mathematician Pavel Alexandroff defined more than 80 years ago. The aim of this paper is to show that Alexandroff spaces, as they are called today, have many interesting properties that can be used to explicate and clarify a variety of problems in philosophy, cognitive science, and related disciplines. For instance, recently, Ian Rumfitt used a special type of Alexandroff spaces to elucidate the logic of vague concepts in a new way. Moreover, Rumfitt's class of Alexandroff spaces can be shown to provide a natural topological semantics for Susanne Bobzien's "logic of clearness". Mainly due to the work of Peter Gärdenfors and his collaborators, conceptual spaces have become an increasingly popular tool of dealing with a variety of problems in the fields of cognitive psychology, artificial intelligence, linguistics and philosophy. For Gärdenfors's conceptual spaces, geometrically defined discretizations (so-called Voronoi tessellations) play an essential role. These tessellations can be shown to be extensionally equivalent to topological tessellations that can be constructed for Alexandroff spaces in general. Thereby, Rumfitt's and Gärdenfors's constructions turn out to be special cases of an approach that works for a more general class of spaces, namely, for weakly scattered Alexandroff spaces. The main aim of this paper is to show that this class of spaces provides a convenient framework for conceptual spaces as used in epistemology and related disciplines in general. Weakly scattered Alexandroff spaces are useful for elucidating problems related to the logic of vague concepts, in particular they offer a solution of the Sorites paradox (Rumfitt). Further, they provide a semantics for the logic of clearness (Bobzien) that overcomes certain problems of the concept of higher-order vagueness. Moreover, these spaces help find a natural place for classical syllogistics in the framework of conceptual spaces. The specialization order of Alexandroff spaces can be used to refine the all-or-nothing distinction between prototypical and non-prototypical stimuli in favor of a gradual distinction between more or less prototypical elements of conceptual spaces. The greater conceptual flexibility of the topological approach helps avoid some inherent inadequacies of

Extended author information available on the last page of the article

Published online: 03 January 2021

the geometrical approach, for instance, the so-called “thickness problem”. Finally, it is shown that the Alexandroff spaces offer an appropriate framework to deal with digital conceptual spaces that are gaining more and more importance in the areas of artificial intelligence, computer science and related disciplines.

Keywords Conceptual spaces · Polar spaces · Alexandroff spaces · Prototypes · Topological tessellations · Digital topology · Voronoi tessellations

1 Introduction

The aim of this paper is to present a topological method for constructing discretizations (tessellations) of conceptual spaces. This method works for a class of spaces that the Russian mathematician Pavel Alexandroff defined some 80 years ago [cf. Alexandroff (1937)]. Alexandroff spaces, as they are called today, exhibit a 1–1 correspondence between their specialization orders and their topological structures.

Recently, Ian Rumfitt used a special class of Alexandroff spaces to elucidate the logic of vague concepts in a new way [cf. Rumfitt (2015, chapter 8)]. According to his approach, the color spectrum and other similar conceptual spaces that characterized concepts by prototypes or paradigms give rise to classical systems of concepts that have the structure of complete atomic Boolean algebras.[1]

Rumfitt was not the first to study conceptual systems defined via prototypes. For some 20 years or so Peter Gärdenfors and his collaborators have shown that conceptual spaces serve as a useful modeling tool in the fields of cognitive psychology, artificial intelligence, linguistics, and philosophy.[2] The core idea of the conceptual space approach is that concepts can be represented geometrically as regions of (metrically structured) similarity spaces. Using prototypes and the metrical structure of similarity spaces, Gärdenfors constructed geometrical discretizations of conceptual spaces by so-called Voronoi tessellations. The topological account of conceptual spaces to be presented in this paper has several advantages over Gärdenfors’s geometrical account. For instance, the vexing question of choosing the “right” metric of a conceptual space (from infinitely many candidates) can be avoided.[3] Moreover, the so-called “thickness problem” of Douven et al. can be dissolved.

[1] Complete atomic Boolean algebras are isomorphic to 2^L, where 2^L is the power set of a set L.

[2] For some interesting recent work on the role of prototypes in the theory of conceptual spaces see Douven and Gärdenfors (2019). Recent applications of the theory of conceptual spaces in linguistics, cognitive science and philosophy may be found in Zenker and Gärdenfors (2015), and Kaipainen et al. (2019).

[3] This move does not solve, of course, the problem of determining the topology of the conceptual space in question. But at least, for the “conceptual engineer”, i.e., the scientist who is in charge to design appropriately structured conceptual spaces, the task of designing a topological structure is less demanding than to determine the full metrical structure of a conceptual space. After all, the topological structure is fully determined by the metrical structure. Moreover, it may be that a conceptual space has no metrical structure at all.

For a fuller account of a “design theory” of conceptual spaces, see Douven and Gärdenfors (2019) and some remarks in section 5 of this paper.

Gärdenfors's meanwhile "classical" geometrical approach of conceptual spaces endowed with a convexity structure and the topological approach are not unrelated to each other. As will be shown, Gärdenfors's geometrical construction of conceptual spaces gives rise to the construction of topologically defined Alexandroff spaces.[4] More precisely, Voronoi tessellations are extensionally equivalent to topologically defined discretizations that rely only on the topological features of Alexandroff spaces. Furthermore, Rumfitt's as well as Gärdenfors's constructions turn out to be special cases of an approach that works for a more general class of spaces, namely, for weakly scattered Alexandroff spaces. For these spaces, the corresponding Boolean algebras of regular open regions yield natural atomic tessellations. This suggests that the class of Alexandroff spaces provides a convenient framework for conceptual spaces in general. Formulated differently an important task of cognitive science is to understand how conceptual spaces can be endowed with spatial structures that can serve as a basis for the elaboration of interesting classifications of stimuli or experiences.[5]

For defining a spatial structure on a set of experiences, Gärdenfors proposed to employ so-called Voronoi tessellations based on a Euclidean structure and a finite set of prototypes of the underlying conceptual space:

> A Voronoi tessellation based on a set of prototypes is a simple way of classifying a continuous space of stimuli. The partitioning results in a discretization of the space. The prime cognitive effect is that the discretization speeds up learning. … [A] Voronoi tessellation is a cognitively economical way of representing information about concepts. Furthermore, having a space partitioned into a finite number of classes means that it is possible to give names to the classes. (Gärdenfors (2000, p. 89))

[4] Thus, the topological approach and the geometrical approach of conceptual spaces should not be considered as incompatible alternatives. Rather, both the geometrical approach (based on the concept of convexity) and the topological approach should be considered as having a common conceptual root, namely, the concept of a general closure operator. In a formally precise way this is explained in "Appendix A". This generalized perspective on the task of how to represent experience in appropriately structure conceptual spaces should not be considered as an idle attempt of generalizing for the sake of generalizing. Rather, it should be taken as a proposal to find an adequate general framework for the task of presenting possible experiences in a rich and flexible way.

[5] With some good will, the "attribute spaces", introduced by Carnap long ago, may be considered as forerunners of conceptual spaces in Gärdenfors's sense [cf. Carnap (1971)]. In contrast to attribute spaces, regions of conceptual spaces that correspond to concepts are non-homogeneous in the sense that some (generating) points are more prototypical than others. While Rumfitt and Gärdenfors assume a strict dichotomy between prototypical and non-prototypical elements, this paper shows how to define a gradual distinction between more or less prototypical (central) elements of a conceptual space. This is done by using the so-called "specialization order" that is characteristic for Alexandroff spaces. More precisely, in the framework of Alexandroff spaces, prototypical elements are characterized as maximal elements of the specialization (quasi-)order, while all other elements have a more or less high degree specialization (prototypicality).

For a more detailed comparison of the similarities and differences of Carnap's and Gärdenfors's approaches, see Sznajder (2016, section 6).

As will be shown in the following, the topological essence of Gärdenfors's and Rumfitt's discretizations of continuous conceptual spaces is based on a structural correspondence between the specialization order and the topological structure discovered by Alexandroff in the 1930s.

A geometrically defined Voronoi tessellation uniquely determines a topological tessellation that is extensionally equivalent to a regular open tessellation constructed by Alexandroff's method. The constructions of Rumfitt and Gärdenfors boil down to different, very special cases of Alexandroff's method of constructing topological spaces from partial orders. Thus, Alexandroff spaces may be considered a natural topological habitat of conceptual spaces. They provide a natural framework for conceptual spaces that deal with empirically meaningful concepts. More precisely, this claim can be explicated as follows:

(1) Empirically meaningful concepts must be stable in the sense that if such a concept applies to a situation x, it also applies to minor variations x' of x. This stability is accompanied by a certain degree of conceptual vagueness. Stable concepts do not single out empirical objects with absolute precision. This should be considered a virtue rather than a vice. Otherwise, concepts would no longer be empirically applicable.
(2) Arbitrary conjunctions of stable concepts should be stable. This requirement expresses a reasonable conceptual modesty. Otherwise, we could eliminate the inherent vagueness of empirical concepts by purely logical means by forming (infinite) conjunctions of more and more concepts that eventually result in an absolutely precise conceptualization of reality.

Topologically, (1) and (2) can be satisfied by requiring that a conceptual space S has the structure of an Alexandroff space (S, OS) such that concepts are characterized as elements of the Boolean algebra O*S of regular open subsets of S. In this paper, we rely on a topological account of concepts, i.e., concepts are characterized as topologically well-formed regions in contrast to Gärdenfors's geometrical approach to conceptual spaces, which represents natural concepts as convex regions with convexity defined geometrically with the aid of the Euclidean space of an underlying conceptual space. In comparison to the geometrical account, the topological one to be presented in this paper is more austere insofar as different geometrical structures may be considered realizations of one and the same topological structure.[6]

The topological concepts used in this paper require a conceptual apparatus that likely not all philosophers are acquainted with. Topology, as used in this paper, goes beyond the vague idea that "topology" is just a sort of "generalized geometry" as expressed in the well-known pun: "A topologist is someone for whom the shapes of a coffee mug and a donut do not essentially differ." Thus, for the reader's

[6] From an abstract point of view, topological structures and convex structures are not unrelated to each other: Both may be mathematically characterized as closure structures [cf. van de Vel (1993)].

convenience, the necessary rudiments of the mathematical theory of topology are briefly recalled in "Appendix A".[7]

The topology employed in this paper for elucidating the structure of conceptual spaces can be characterized as "nonclassical". This topology considerably differs from "classical" topology emerging from the study of Euclidean spaces and their relatives. While "classical" topological spaces may be succinctly characterized as spaces that satisfy the Hausdorff axiom[8] and often even stronger separation axioms, "nonclassical" topological spaces such as Alexandroff spaces do not satisfy the Hausdorff axiom. Formulated in a positive way, "nonclassical" topology is characterized by the fact that it is strongly related to a certain order structure (called the specialization order[9]) such that the topological structure is characterized completely by the order. For "classical" topological spaces, the specialization order is trivial [cf. Goubault-Larrecq 2013; Kuratowski and Mostowski 1976; Steen and Seebach Jr. 1978]. For Alexandroff spaces, it is, however, highly non-trivial and suffices to characterize the topological structure.

Now and then, topology has been mentioned in the literature about conceptual spaces as is, for instance, exemplified by Gärdenfors's books (2000, 2014). There, topology is understood in a vague sense as a kind of generalized Euclidean geometry. This attitude is not to be criticized per se. The only point we want to emphasize is that this is not the way in which topological concepts are used in the following. The topologies that we use in this paper are essentially different from the Euclidean topology. Nevertheless, the topological approach put forward in this paper may be considered as a contemporary attempt to respond positively to the admonition that Plato is said to have engraved above the door to his academy:[10]

$$\text{ΑΓΕΩΜΕΤΡΙΚΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ} \tag{1.1}$$

[7] All terms defined in the "Appendix" are underlined when they are used for the first time in the main text.

[8] The Hausdorff separation axiom for topological spaces requires that two distinct points x and y of the space have disjoint open neighborhoods U(x) and U(y), or, in other words, that x and y can be separated from each other topologically. Many of the familiar topological spaces such as Euclidean spaces, and, more generally, metrical spaces satisfy the Hausdorff axiom. For a precise definition of the axiom and other separation axioms, see A11.

[9] Since this structure is very important for the rest of this paper, it may be expedient to give a preliminary informal description just now. Given a topological space (X, OX), the elements $a \in OX$ may be interpreted as properties that the elements $x \in X$ may or may not possess ("x has the property a" iff $x \in a$). Then an element x may be defined as "more special", "more central" or "more typical" than an element y (denoted by $y \leq x$) iff x has at least as many properties as y. In many papers dealing with conceptual spaces such a (quasi-)order of specialization is implicit assumed when geometrical illustrations by Venn-like diagrams are used to distinguish between central and not-so-central cases of concepts. See, for instance, the "bird space" where "penguin" occupies a less central position than "robin" (Osta-Vélez and Gärdenfors (2020, p. 6)).

As will be explained later, this order structure is based on the topological structure of X, it is called the specialization quasi-order of the topological space (X, OX) and denoted by $(X, \leq)$. A precise topological definition of $(X, \leq)$ is given in A6.

[10] Ageometrikos maedeis eisito: Let no one untrained in geometry enter.

Those of us, who consider this maxim still relevant for contemporary philosophy, have no reason to restrict our attention to classical geometry of Euclid. Rather, we should attempt to make sense of it in terms of a modern understanding of geometry. To put it bluntly then, today, geometry understood as a theory of space in general, is *topology*.

Up to now, geometrical methods in this modern sense, i.e., in the sense of topological methods, have been an under-exploited resource in many areas of philosophy and related disciplines such as cognitive psychology, and cognitive science in general. This holds, in particular, for the issue of how (possible) experiences or stimuli are represented in appropriate formal structures aka conceptual spaces. Following a recent proposal of Douven and Gärdenfors, let us conceive a conceptual system as

> an agreement between the members of a community that a particular meaning domain be partitioned in a particular way. A concept is then a particular element of such a partitioning. (Douven and Gärdenfors (2019, p. 2))

In a very simplified way, then, a conceptual system—as a method of partitioning a domain of events, possible experiences—can be described as a partitioning of a domain of experiences, stimuli or something similar. Without giving criteria to distinguish between interesting and uninteresting partitions, however, a theory of conceptual systems as a theory of partitions does not get off the ground. Here, geometry and topology come to the rescue. Instead of being content to characterize a concept system abstractly as a partition, one may conceive it as a sort of map that partitions a space in various regions. Maps are tools for providing orientation for some parts or aspects of the world. Consider, for instance, a Voronoi tessellation of a conceptual space. Quite literally, a Voronoi tessellation indicates where something of interest for us, is located in a certain space and how it is spatially related to other, more or less similar entities. Locating an experience somewhere in a conceptual space helps categorize it: Moreover, provided that two cognitive agents use the same map (in a sense to be specified), they can compare their experiences, assess how similar they are, and deliberate what kinds of (common) actions are advisable to carry out in a situation. Thereby maps also enable us to communicate and, more specifically, even to reason about experiences (cf. Douven and Gärdenfors (2019)).

Modern maps do not presuppose a Euclidean structure of the domain that is mapped. This is already evident for the many kinds of "topological" mappings and diagrams we are using in our everyday epistemic activities. For instance, the various kinds of subway maps that people use for orientation, the countless types of diagrams and other graphical presentations up to the most indirect, theoretically and physically highly sophisticated ways of producing computer-aided images of medical phenomena by methods of positron emission tomography (PET) and similar methods (see for instance Zvolsky (2014)).

All these ways of representing and making sense of aspects of the world are based on maps that are not just catalogues of what there is, rather, they are conceptually grounded symbolizations that heavily depend on highly sophisticated geometrical, or better, topological theories. Thus, the issue of geometrical and topological representations of events, experiences and processes for all kinds of sciences is of the outmost importance for modern cognitive sciences and related disciplines. Non-trivial representations require a certain amount of formal, in particular mathematical tools that cannot be justified in advance. Thus, in order to persuade the reader that it is worth the effort to get acquainted with topology at least to a modest degree, it is argued that the concept of weakly scattered Alexandroff spaces (WSA) is useful in quite a few areas:

(i) WSA spaces help elucidate problems related to the logic of vague concepts, in particular, a novel solution of the Sorites paradox (proposed by Rumfitt).
(ii) WSA spaces provide a natural semantic for Bobzien's "logic of clearness" and help overcome certain problems of the concept of higher-order vagueness.
(iii) WSA are essential for finding a natural place for classical syllogistics in the framework of conceptual spaces.
(iv) The specialization order of WSA spaces refines the all-or-nothing distinction between the prototypical and the non-prototypical in favor of a gradual distinction between more and less central elements.
(v) Some intrinsic shortcomings of the geometrical account of conceptual spaces can be avoided by the topological approach, for instance, the so-called "thickness problem".
(vi) A further advantage of WSA spaces is their ability to deal with issues of digital conceptual spaces that have become ever more important in the areas of artificial intelligence, computer simulation, digital imaging and related disciplines.

This paper is structured as follows. In Sect. 2, as starting point of the project of the topological elucidation of conceptual spaces, we investigate the topological structure of polar spaces in detail. This type of conceptual spaces was recently introduced by Ian Rumfitt in his book *The Boundary Stones of Thought* (Rumfitt 2015) as a convenient framework for dealing with the logic of vague concepts. Polar spaces may be considered an elementary example of the general topological account elaborated in this paper. As is shown they also provide a natural semantics for Bobzien's logic of clearness that has been designed to cope with certain problems of higher-order vagueness.

In Sect. 3 the relation between the topologically defined tessellations of polar spaces and the better known geometrically defined Voronoi tessellations of Gärdenfors's conceptual spaces is explicated. Section 4 addresses the topology and order structure of a subclass of Alexandroff spaces that is especially useful for studying conceptual spaces, namely, weakly scattered Alexandroff spaces. It is shown that this class of spaces may be considered the most general class of spaces that gives rise to well-behaved classifications and categorizations of objects. Section 5 argues that the topological approach can contribute to the so-called design theory of (natural) concepts. We conclude with some general remarks on the intricate intertwinement of the aspects of order, algebra, and topology in the framework of Alexandroff spaces in Sect. 6.

Moreover, the paper has two appendixes. In "Appendix A", for the reader's convenience, basic definitions and facts of topology are collected that are used throughout the paper. "Appendix B" contains a list of various kinds of Alexandroff spaces that are interesting for the purposes of this paper. A detailed presentation of the theory of Alexandroff spaces, in particular for modal logic, can be found in Aiello et al. (2003, 2007, Chapter 5).

2 The topological structure of polar spaces

In this section, we begin the explication of the topological structure of conceptual spaces by investigating the topological structure of polar spaces. These spaces were recently introduced by Rumfitt to discuss the logic of appropriate vague concepts such as color concepts [cf. Rumfitt (2015, chapter (8.4)]. In this section, we recall the basic ideas of Rumfitt's approach and show in particular that these spaces are very simple Alexandroff spaces.

Let X be a set of colored objects that serves as the underlying set of a conceptual space for color experiences. We are looking for a discretization of X, i.e., a partition of X that allows us to classify color experiences into different categories. This can be done with the aid of certain paradigmatic or prototypical experiences of red, blue, yellow and so on [cf. Gärdenfors (2000)]. More precisely, Rumfitt argues that the classification of colors is best conceptualized as a procedure based on a comparison of certain color experiences to be considered as paradigmatic or prototypical:

> The spectrum enables us to attach senses to colour terms not because it shows boundaries, but because it displays colour paradigms or poles. Sainsbury likens colour paradigms to 'magnetic poles exerting various degrees of influence: some objects cluster firmly to one pole, some to another, and some, though sensitive to the forces, join no cluster'. … I prefer a simpler analogy, which likens paradigms to gravitational poles, that is, massive bodies. If a small body is sufficiently close to a gravitational pole, it will be drawn towards it, rather as we are drawn to classify as red those objects that are sufficiently close in colour to a paradigm, or pole, of red. Rumfitt (2015, p. 236)

The essential mathematical structure to be extracted from this example is as follows. Assume that there is a given set X of objects to be classified and a subset P of X to be considered as a set of distinguished elements that are "paradigmatic" or "prototypical" objects. Rumfitt (2015) calls them poles. See also Sainsbury (1996 (1990)). These poles are used to classify the ordinary objects. This procedure is rendered precise by the following definition:

Definition 2.1 Let X be a set and $P \subseteq X$ be a subset of distinguished elements to be interpreted as prototypes or poles. Assume that for every object $x \in X$, there is a nonempty set $m(x) \subseteq P$ of poles p. The poles $p \in m(x)$ are said to be maximally close to x. For each x, the set m(x) is assumed to satisfy the following two requirements:

(i) $\forall x \in X\ (\emptyset \neq m(x) \subseteq P)$.
(ii) $\forall x \in X\ (m(x) = \{x\} \Leftrightarrow x \in P)$.

The map $X—m \rightarrow 2^P$ defined by (i) and (ii) is called a pole distribution and denoted by (X, m, P).◆

Requirements (2.1) (i) and (ii) entail that poles do some classificatory work by classifying the elements of X. First, poles are distinguished from non-poles as those elements that are, so to speak, "self-classifying", i.e., $m(p)=\{p\}$.[11] Second, and this is an observation that is more interesting, a pole distribution (X, m, P) defines a topology on X. This is done with the help of an interior kernel operator 2^X—int→ 2^X:

Proposition 2.2 *Let (X, m, P) be a pole distribution and A⊆X. Define the operator int by*

$$x \in int(A) := (x \in A \;\&\; \forall_{p \in P}(p \in m(x) \Rightarrow p \in A)) \Leftrightarrow (\{x\} \cup m(x) \subseteq A). \blacklozenge$$

The operator int is a topological interior kernel operator that defines an Alexandroff topology. Informally formulated, x ∈ int(A) iff x ∈ A and all poles that are maximally close to x, i.e., that are elements of the set m(x), also belong to A. In other words, the interior of A comprises those elements of A whose maximally close poles also belong to A.

Equivalently, the topology corresponding to a pole distribution (X, m, P) is defined by a closure operator 2^X—cl→ 2^X:

Definition 2.3 *Let (X, m, P) be a pole distribution and A⊆X. Define the operator cl by*

$$x \in cl(A) := (x \in A \text{ or } \exists_{p \in P}(p \in A \text{ and } p \in m(x)) \Leftrightarrow (\{x\} \cup m(x)) \cap A \neq \emptyset). \blacklozenge$$

Informally speaking, the closure of a set, cl(A), comprises the members of A and all objects for which at least one of their maximally close poles is in A. In other words, cl(A) comprises all elements of A that are in A or that have at least a connection to elements of A.

The topological space (X, OX) defined by the operator int (or cl) is called the polar space of the distribution (X, m, P). The proof that int and cl are topological operators involves a routine check that these operators satisfy the Kuratowski axioms (A.2); see Rumfitt (2015, pp. 243–246). A closer inspection of definitions (2.2) or and (2.3) reveals that they even satisfy the Alexandroff condition (A.1) (iii), namely, arbitrary intersections (unions) of open (closed) sets are open (closed):

Proposition 2.4 *The topology (X, OX) defined by a pole distribution (X, m, P) is an Alexandroff topology.* ♦

Rumfitt defines the topology (X, OX) given by a pole distribution (X, m, P), but he does not describe the topology in any detail. In particular, he does not mention that (X, OX) is an Alexandroff space, nor does he explicitly show that O*X is atomic.

[11] The function m need not be defined with the aid of a full-fledged metric on X as Gärdenfors seems to assume.

Endowed with the topology defined by a pole distribution (X, m, P), the color spectrum (X, OX) is a very special Alexandroff space; namely, (X, OX) is a $T_{1/2}$-space such that the singletons {p} ⊆ P are open and all singletons {x} for x ∈ X − P are closed. More precisely, the following proposition holds:

Proposition 2.5 *Let (X, OX) be a topological space defined by a pole distribution (X, m, P). Then, X is a weakly scattered $T_{1/2}$–Alexandroff space, i.e., the elements p ∈ P are open and cl(P) = X. For all y ∈ X, let V(y) be the smallest open set that contains y. For elements p ∈ P and x ∈ X − P one calculates:*

$$\mathrm{int}(p) = \{p\}. \quad \mathrm{int}(x) = \emptyset. \quad V(x) = \{x\} \cup m(x). \quad V(p) = \{p\}.$$
$$\mathrm{cl}(x) = \{x\}. \quad \mathrm{cl}(p) = \{x; p \in m(x)\}. \quad \mathrm{int}(\mathrm{cl}(p)) = \{x; \{p\} = m(x)\}.$$

The specialization order of (X, OX) is given by $x < y := x \neq y$ and $y \in m(x)$. Thus, for polar spaces (X, OX) the specialization order (X, ≤) is of depth 1. It just amounts to the distinction between prototypical and non-prototypical elements of X. The Boolean algebra O*X of regular open sets of (X, OX) is isomorphic to the power set 2^P.

Proof To prove these claims, one must only check definitions. Be it sufficient to prove that V(x) = {x} ∪ m(x) and the claim that $O^*X = 2^P$. According to the definition of the interior kernel operator int one has y ∈ int({x} ∪ m(x)) ⇔ y ∈ {x} ∪ m(x) & ∀p (p ∈ m(y) ⇒ p ∈ {x} ∪ m(x)). Clearly, every element in {x} ∪ m(x) satisfies this condition. On the other hand, any smaller set properly contained in {x}∪ m(x) does not satisfy the condition. For two different elements x and y, V(x) and V(y) are different. Hence, X is a T_0-space. Thus, the set {{x} ∪ m(x), x ∈ X} forms a unique minimal basis for the topology OX of X.

An isomorphism 2^P—r→O*X can be inductively defined as follows: r(p) := intcl(p), and if r(A) and r(B) are already defined, r(A ∪ B) := j(i(r(A)) ∪ i(r(B))), the maps O*X—i→OX and i and OX—j→O*X as defined in A.5. A canonical minimal basis of OX is given by the set {V(x); x ∈ X} = {{x} ∪ m(x); x ∈ X}.◆

Proposition 2.5 characterizes polar spaces as a special class of Alexandroff spaces (X, OX) [cf. Bezhanishvili et al. (2003)].

The cardinalities of OX and O*X may be quite different. Already for the color circle S with a finite set P of prototypical colors ("red", "orange", "yellow", …) OS is of uncountable cardinality while O*S has finitely many elements $2^{|P|}$. This is a compelling argument that O*S rather than OS should be taken as the set that represents the Boolean algebra of concepts of a conceptual space (cf. Rumfitt 2015). One should note that O*X associated with a polar conceptual space (X, OX) come equipped with a binary similarity relation:

Definition 2.6 Let O*X be the Boolean lattice of regular open subsets of the topological space (X, OX). A reflexive and symmetric similarity relation on O*X is defined by

$$A \sim B := cl(A) \cap cl(B) \neq \emptyset \quad \text{for } A, B \in O * X. \blacklozenge$$

This relation is not necessarily transitive. It can be used to distinguish between dif-fe-rent conceptual spaces X and Y that have isomorphic Boolean lattices O*X and O*Y but that differ in their similarity relations $\sim_X$ and $\sim_Y$ defined on O*X and O*Y, respectively. An elementary example is given by the linear color spectrum Lin(S) and the circle spectrum Cir(S) (cf. Gärdenfors (2000) and Rumfitt (2015)). Both have the same number of prototypes, say, "red", "yellow", "green", "blue", and "purple", but their similarity relations may differ. For the linear spectrum one obtains

$$\text{Lin(S)} = <\text{red} \sim \text{yellow} \sim \text{green} \sim \text{blue} \sim \text{purple} > \tag{2.7}$$

In contrast, for the circular color spectrum one obtains

$$\text{Cir(S)} = <\text{red} \sim \text{yellow} \sim \text{green} \sim \text{blue} \sim \text{purple} \sim \text{red} > \tag{2.7'}$$

These two similarity structures are different since in Lin(S) "red" and "purple" are not similar to each other, while in Cir(S) they are similar. Nevertheless, the corresponding Boolean lattices of regular open concepts of the two conceptual spaces are isomorphic (as Boolean algebras) to the power set 2^P of the set P = {"red", "yellow", "green", "blue", "purple"}.

Rumfitt rightly emphasizes that his topological representation of the color spectrum is just an example—analogous results hold for all conceptual spaces that "involve predicates whose meaning is given by reference to paradigms or poles" (Rumfitt 2015, p. 255).

His main concern is not a topological reconstruction of the color spectrum as a (polar) topological space, rather, the topology of the color spectrum is only the basic ingredient for his solution of the Sorites paradox.[12] This solution relies on the fact that for regular open interpretations of classical Boolean propositional logic "the

[12] Cut down to its bones, the topological core of Rumfitt's solution of the Sorites paradox consists in exploiting the peculiar properties of topologically defined regular open interpretations of Boolean logic. The details are as follows. Let O*X be the Boolean algebra of regular open sets of (X, OX). A regular open interpretation of a propositional language L (with propositional variables a, b, …, and the Boolean connectives ∧, ∨, …) in O*X is a map L—r→O*X such that

$$\text{(i)} \quad r(a \wedge b) := r(a) \cap r(b), \quad \text{(ii)} \quad r(a \vee b) := int(cl(r(a) \cup r(b)), \quad \text{and} \quad \text{(iii)} \quad r(\neg a) := int\mathbf{C}r(a).$$

The crucial feature of a regular open interpretation is that it yields a semantic of classical Boolean logic that may render a disjunction a ∨ b true in X without implying that for all x ∈ X either a or b is true at x. The underlying topological fact is simply that for a regular open interpretation r a disjunction a ∨ b of a and b has the interpretation int(cl(r(a) ∪ r(b)). This set may be strictly larger than the union r(a) ∪ r(b) of the disjuncts r(a) and r(b). This fact may be used to cope with the Sorites paradox, see Rumfitt (2015, p. 253) or Mormann (2020, Section 5). As it seems, Rumfitt assumes that the topological product of polar spaces is again a polar space. This is, as can be easily shown, not correct. An elementary example is the Khalimsky plane. This space is the Cartesian product of two polar spaces but not a polar space itself, see (2.10), (2.11), and (A.12)(iv). Rumfitt's arguments are not affected by this slight inaccuracy, however. Be this as it may, the approach of the present paper has no difficulty of dealing with products of polar spaces, see (2.8)ff.

truth of a disjunction does not entail the truth of any of its disjuncts"(Rumfitt (215, 254)).

Before we go on, it may be expedient to explain in more detail the general significance of (2.5) for the theory of conceptual spaces. Succinctly stated, (2.5) ensures that a conceptual space X endowed with a pole distribution (X, m, P) has a Boolean lattice O*X of regular open subsets that is atomic and isomorphic to the power set 2^P of the set of its prototypes P.

As is pointed out in Rumfitt (2015), the linear and the circular color spectrum Lin(S) and Cir(S) belong to this class of spaces. Proposition (2.5) states that these spaces possess well-behaved conceptual systems O*X. More precisely, O*X is a complete atomic Boolean algebra generated by intcl(p), $p \in P$. Thus, the conceptual systems related to this kind of conceptual spaces have a very simple classical structure.

Already Rumfitt's solution of the Sorites paradox shows that the topological approach may be useful for attacking intricate philosophical problems. For further applications it is expedient, however, to consider a somewhat broader class of topological spaces than polar spaces, namely, weakly scattered Alexandroff spaces (X, OX), for which polar spaces provide only the simplest example. More generally, one obtains:

Proposition 2.8

(i) *The topological product ($X \times Y$, $O(X \times Y)$) of the polar spaces (X, OX) and (Y, OY) is a weakly scattered Alexandroff space whose specialization order ($X \times Y, \leq \times \leq$) has depth 2.*

(ii) *Finite products of weakly scattered Alexandroff spaces are weakly scattered Alexandroff spaces. If the specialization orders (X, ≤) and (Y, ≤) are of depth m and n, respectively, then the specializiation order ($X \times Y, \leq$) is of depth $m+n$.*

(iii) *Let ~ be an equivalence relation on weakly scattered Alexandroff space (X, OX) such that the quotient map X—q→X/~ is open. Then (X/~, OX/~) is a weakly scattered Alexandroff space.*

Proof (i) Let (X, m, P) and (Y, n, Q) be the pole distributions of (X, OX) and (Y, OY), respectively. Assume $(x, y) \in X \times Y$, $(p, q) \in P \times Q$, $x < p$, and $y < q$. In $X \times Y$ the chain $(x, y) < (p, y) < (p, q)$ is of length 2. Hence the specialization order of the topological product ($X \times X$, $OX \times Y$) has depth 2.

(ii) If MX and MY are the dense sets of maximal elements of the specialization orders (X, ≤) and (Y, ≤), respectively, then $MX \times MY$ is the dense set of isolated elements of the weakly scattered Alexandroff space ($X \times Y$, $OX \times Y$). The depth of the specialization of ($X \times Y, \leq$) is calculated as in (i).

(iii) Since the set ISO(X) of isolated points of X is mapped by q onto the open set {[p], p ∈ P} of isolated points of X/~, the space (X/~, OX/~) is weakly scattered and q(ISO(X)) is dense in X/~. Obviously, the quotient space q(X) is Alexandroff.◆

Proposition 2.8 ensures that there are plenty of weakly scattered Alexandroff spaces. The following neat result shows that this class of spaces has topological properties that possess interesting modal interpretations:

Proposition 2.9 *Let (X, OX) be a weakly scattered Alexandroff space. Then the following equivalent conditions hold:*

(i) *X satisfies the McKinsey axiom, i.e., int(cl(A)) ⊆ cl(int(A)) for all A ⊆ X.*
(ii) *The boundary operator bd of (X, OX) satisfies bd(bd(A)) = bd(A) for all A ⊆ X.*

Proof Bezhanishvili et al. (2003, Propositions 2.1, 2.4, and 2.8 prove (among other things) that (2.9)(i)–(iii) are all equivalent with the assumption that (X, OX) is weakly scattered.◆

In order to see that (2.9) has interesting modal interpretations it is expedient first of all to recall a trail-blazing result of McKinsey and Tarski (1944). According to these authors, the modal system S4 is the logic of topology in the sense that a proposition is valid in S4 if and only it is valid in all topological spaces. McKinsey and Tarski's result has been generalized in many ways, in particular by establishing a correspondence between certain classes of topological spaces on the one hand and certain extensions S4.X of S4 logic on the other. As is well known, the extension of S4.1 of S4 by the McKinsey axiom corresponds to the logic of weakly scattered spaces [cf. Bezhanishvili et al. 2003, 2004; Gabelaia 2001]. In other words, weakly scattered Alexandroff spaces are models for the modal logic S4.1.[13]

As a second example that weakly scattered Alexandroff spaces have some useful applications in modal logic, Bobzien's logic of clearness may be mentioned. Very succinctly, this can be explicated as follows. For general topological spaces (X, OX) one easily calculates that, instead of (2.9)(ii), only the weaker equation bd(bd(bd(A))) = bd(bd(A)) holds for all A. There may be A ⊆ X with bd(bd(A)) ≠ bd(A).[14] Proposition (2.9) (ii) ensures that weakly scattered Alexandroff spaces behave particularly well with respect to boundaries bd, since for them the stronger formula bd(bd(A)) = bd(A) holds. In several papers Susanne Bobzien has argued that the logic of vague concepts should be cast in the framework of a modal logic based on an operator C such that CA is to be read as "It is clear that A" (cf. Bobzien (2012, 2015)). More precisely, she argues that the operator C should satisfy at least the axioms of the modal system S4. The operator C can be used to define an operator U of "unclearness" such that UA is to be read as "It is not clear that A, and

[13] Not all weakly scattered spaces are Alexandroff, of course. See "Appendix B".

[14] An elementary example is given by the real line (**R**, O**R**): For the set of rational numbers **Q** one obtains bd(bd(**Q**)) = Ø and bd(**Q**) = **R**.

it is not clear that not-A". Obviously, the operators C and U are related to each other in the same way as the topological operators int and bd are related. Bobzien compellingly argues that U should satisfy the law $U^2A = UA$, and proves that $U^2A = UA$ is equivalent with the assumption that her "clearness" operator C satisfies the McKinsey axiom. In sum, by (2.9) weakly scattered Alexandroff spaces offer a natural topological semantic for Bobzien's modal "logic of clearness".

Rumfitt's and Bobzien's solutions of the Sorites paradox and related problems of the theory of vagueness show that weakly scattered Alexandroff spaces offer interesting insights into the subtleties of the theoretical logic of vagueness. These applications do not exhaust the usefulness of Alexandroff spaces, however. In the rest of this section we want to show at least in outline that the topology of Alexandroff spaces also has an immense practical importance as framework of the discipline of digital topology.

Digital topology deals with the geometrical and topological investigation of digitized objects or digitized images and provides both theoretical and computational frameworks for image computing. It plays an essential role in various fields related to digital images, such as image analysis, computer graphics, pattern recognition, shape modeling and computer vision. It emerged during the second half of the twentieth century with the birth of computer graphics and digital image processing (cf. Couprie et al. (2020), Chen (2015)).

As a starting point for digital topology one may consider the "digital line" or "Khalimsky line". The Khalimsky line is a polar space that can be defined as follows:

Definition 2.10 Let $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ be the set of entire numbers. Denote the set of odd numbers by $\mathbf{2Z+1} = \{\ldots -3, -1, 1, 3, \ldots\}$. Then a pole distribution ($\mathbf{Z}$, m, $\mathbf{2Z+1}$) is defined by the map

$$m(2n) = \{2n - 1,\ 2n + 1\}, \qquad m(2n + 1) := \{2n + 1\}.$$

The corresponding polar topological space ($\mathbf{Z}$, $\mathbf{OZ}$) is called the "digital line" or the "Khalimsky line".♦

Corollary 2.11 Let $\mathbf{Z_{2m}}$ denote the quotient space $\mathbf{Z/2mZ}$ for some integer m. Then the canonical quotient map $\mathbf{Z}$—q→$\mathbf{Z_{2m}}$ is open (with respect to the Khalimsky topology and the quotient topology) and defines a finite topological space ($\mathbf{Z_{2m}}$, $\mathbf{OZ_{2m}}$). It is called the Khalimsky circle. ($\mathbf{Z_{2m}}$, $\mathbf{OZ_{2m}}$) is a polar space and may be considered as a kind of digital model of the circular color spectrum (X, m, P) with $P = \{1, 3, \ldots 2m - 1\}$.♦

Applying (2.5) to (2.11) the pole distribution ($\mathbf{Z}$, m, $\mathbf{2Z+1}$) defines a topological space ($\mathbf{Z}$, $\mathbf{OZ}$) such that the singletons $\{2m\}$ of the "even points" $2m \in \mathbf{Z}$ are closed, and the singletons $\{2m+1\}$ of the odd points $2m + 1 \in \mathbf{Z}$ are open subsets. The smallest open set of $\mathbf{OZ}$ containing 2m is $\{2m - 1, 2m, 2m + 1\}$, and $O^*(\mathbf{Z})$) is the power set 2^L, $L = \mathbf{2Z+1}$.

Definition 2.12 The topological product (**Z × Z,** O**Z × Z**) of two copies of the digital line (**Z,** O**Z**) is called the "digital plane" or "Khalimsky plane". Higher-dimensional digital spaces ($\mathbf{Z}^n$, $O\mathbf{Z}^n$), $n \geq 3$, are defined analogously.◆

It is easily checked that the digital spaces ($\mathbf{Z}^n$, $O\mathbf{Z}^n$) for $n \geq 2$ are NOT polar spaces (as the digital line (**R**, O**R**)) but WSA spaces. More precisely, one calculates that the digital plane ($\mathbf{Z}^2$, $O\mathbf{Z}^2$) has three kinds of points: The singletons $\{(2m+1, 2n+1)\}$ are open, the singletons $\{(2m, 2n)\}$ are closed, and singletons $\{(2m+1, 2n)\}$ and $\{(2m, 2n+1)\}$ are neither open nor closed (cf. Adams and Franzosa (2008, Ch. 11.3, p. 367ff)). Analogously, the higher-dimensional digital spaces ($\mathbf{Z}^n$, $O\mathbf{Z}^n$) have many kinds of points that are neither open nor closed. Nevertheless, their open singletons are dense. Hence, they are weakly scattered Alexandroff spaces that are not polar spaces.

The most important quality of the digital spaces ($\mathbf{Z}^n$, $O\mathbf{Z}^n$) is that they can serve as "discrete" or "digital" models of the Euclidean spaces ($\mathbf{R}^n$, $O\mathbf{R}^n$). For the digital line this claim is rendered precise by the following proposition:

Proposition 2.13 *Let* (**R**, O**R**) *be the real line with the Euclidean topology. Define the map* $\mathbf{R}\text{—}q\rightarrow\mathbf{Z}$ *by*

$$q(x) = \begin{array}{l} x \text{ iff } x = 2m,\ m \in \mathbf{Z}, \\ 2m + 1 \text{ for } 2m < x < 2m + 2,\ m \in \mathbf{Z}. \end{array}$$

Then the digital line ***(Z, OZ)*** *is the quotient space of the real line (**R**, **OR**) by the map q. Further, the quotient map (**R**, **OR**)—q→(**Z**, **OZ**) is an open continuous map. Thereby, the polar topological space (**Z**, **OZ**) is shown to be a* <u>*connected*</u> *topological space*.♦

Corollary 2.14 The higher-dimensional digital spaces ($\mathbf{Z}^n$, $O\mathbf{Z}^n$) are weakly scattered connected Alexandroff spaces that are open quotient spaces of the Euclidean spaces ($\mathbf{R}^n$, $O\mathbf{R}^n$) by the product maps ($\mathbf{R}^n$, $O\mathbf{R}^n$)—q^n→($\mathbf{Z}^n$, $O\mathbf{Z}^n$).◆

Corollary (2.14) has been said to provide the foundation for the disciplines of digital topology and geometry (cf. Kopperman (1994)). In a nutshell, (2.14) establishes the fact that digital structures ($\mathbf{Z}^n$, $O\mathbf{Z}^n$) can serve as a (partially faithful) models of the continuous structures ($\mathbf{R}^n$, $O\mathbf{R}^n$).[15] Or, to emphasize more clearly the relevance of (2.14) for the issue of conceptual spaces: Continuous conceptual spaces (i.e., conceptual spaces based on Euclidean spaces $\mathbf{R}^n$ or appropriate Euclidean substructures) can be replaced (at least in principle) by digital conceptual spaces $\mathbf{Z}^n$ (and appropriate derivatives). If one wants to deal with digital data that (due to the

[15] The task of finding out what are the digital counterparts of various kinds of continuous phenomena (if there are any), may be highly non-trivial. An early classical result in this field is the digital Jordan curve theorem according to which a continuous curve of a digital circle defines a tessellation of the digital plane in two parts.

ubiquitous usage of computers) become ever more important in many areas in science and elsewhere, then such a replacement is unavoidable.[16]

The example of higher-dimensional digital spaces ($\mathbf{Z}^n$, $O\mathbf{Z}^n$) (and other digital manifolds such as the Khalimsky circle [cf. Melin (2009)] should be considered as a convincing argument for the thesis that polar spaces do not offer a convenient framework for a comprehensive topological theory of conceptual spaces. Rather, a more adequate framework for such a theory is provided by the more general class of WSA spaces. To put it bluntly, WSA spaces seem to be a convenient and even "necessary" generalization of polar spaces.[17]

3 Topological and geometrical tessellations

The aim of this section is to discuss several types of tessellations (discretizations) of conceptual spaces that define conceptual classifications based on geometrical or topological structures of conceptual spaces. Because the Boolean algebra O*X is atomic for polar spaces (X, OX), the resulting tessellation is particularly simple and essentially unique.

Definition 3.1 A regular open tessellation of a space (X, OX) is a set $T := \{A_\lambda \in O^*X\}$ of disjoint regular open subsets $A_\lambda \in O^*X$ with supremum $\vee A_\lambda = X$ in O*X. The A_λ are called the cells of T. Note that the supremum $\vee A_\lambda$ of the A_λ is to be taken in O*X (not in OX).[18] The set $bd(T) = X - \cup A_\lambda$ is called the boundary of T. If the A_λ are atoms of O*X, then T is called an atomic tessellation. Points of X that are not in any cell A_λ are said to be on the boundary bd(T) of the tessellation T.◆

3.1 Examples of topological tessellations

(i) Let (X, OX) be a topological space, $A \in O^*X$, $A \neq \emptyset$, X. Denote the Boolean complement of A in O*X by A* (A* = int(CA)). Then, T = {A, A*} is a regular open tessellation of X with two open cells A and A* and boundary bd(T) = bd(A) (= bd(A*)). More generally, let $A_1, \ldots, A_n$ be n regular open subsets of X with $\vee A_i = X$. The intersections of the A_i generate a regular open tessellation of X that has m cells, $m \leq 2^n - 1$.

[16] This paper is, of course, not the appropriate place to deal with digital topology and its many applications in any greater depth. The literature on digital topology is immense. Here, it must suffice to mention just some introductory texts, e.g., Rosenfeld (1979), Kovalevsky (2006), Kong et al. (1991), and Melin (2008, 2009).

[17] Already Gärdenfors in his geometrical account of conceptual spaces uses the possibility of constructing new conceptual spaces from products (or quotients) of already given conceptual spaces (cf. Gärdenfors (2000)). Likewise, Rumfitt uses finite products of the color spectrum to deal with the Sorites. These products are WSA spaces, not polar spaces [cf. Rumfitt (2015, Ch.8)].

[18] It should be observed that the supremum $\vee A_\lambda$ is to be taken in O*X. Thus, it may be strictly larger than the set-theoretical union $\cup A_\lambda$ of the A_λ.

(ii) A particularly important (geometrical and topological) tessellation is the tessellation of the Euclidean line **R** given by open intervals: T := {(2m, 2m + 2); m ∈ **Z**}.[19] This tessellation may be called the Khalimsky tessellation.
(iii) Tessellations of higher-dimensional Euclidean spaces $\mathbf{R}^n$ can be defined analogously.♦

Regular open tessellations for topological spaces (X, OX) exist in profusion. The point is to find tessellations that are interesting for some reason or other. For instance, one may ask whether or not a space has a atomic regular open tessellation. As is easily shown, the real line R and, more generally, Euclidean spaces R^n do not possess atomic tessellations. In contrast, polar spaces (X, OX) possess atomic regular tessellations:

Proposition 3.3 *Let (X, m, P) be a pole distribution for X. Then, the topological space (X, OX) has a canonical regular open atomic tessellation defined by T := {intcl(p); p ∈ P} by the atoms of the Boolean algebra O*X and* $X = V_{p \in P}$ *intcl(p).*

Proof Let (X, OX) be defined by (X, m, P). As proved in (2.6), the Boolean algebra O*X is isomorphic to the power set 2^P. Thus, the atoms of O*X generate a regular open tessellation T of X. The atoms of T are the regular open sets int(cl(p)) = {x; {p} = m(x)}, p ∈ P. Elements of X to which more than one pole is maximally close are located at the boundary of T, i.e., bd(T) = {x; {p, p'} ⊆ m(x)} for some p, p' ∈ P and p ≠ p'.♦

By (2.5) Rumfitt's polar spaces define topological tessellations in a natural way. In this section we show that also Gärdenfors's geometrically defined conceptual spaces may be used for this purpose. More generally, we are going to explain how Gärdenfors's conceptual spaces may be conceived as Alexandroff spaces endowed with a distinguished topological tessellation defined by their Voronoi tessellation.

According to Gärdenfors, conceptual spaces are similarity spaces. A similarity space is a metrical space whose metric is used to define a binary similarity relation on it. Distances in the space are meant to measure similarity: the shorter the distance between objects, the more similar they are.

Let us now recall briefly the basics of the most prominent class of tessellations of conceptual spaces, namely, the so-called Voronoi tessellations (cf. Gärdenfors (2000), Decock and Douven (2015), Okabe et al. (1992), Zenker and Gärdenfors (2015)). For the sake of simplicity, let us restrict our attention to Voronoi tessellations of the Euclidean plane E. Assume E is endowed with the familiar Euclidean metric d and P = {p_1, …, p_n} is a finite set of distinct points of E. The bisector B(p_i, p_j) between p_i and p_j is defined as the set of points x ∈ E such that d(x, p_i) = d(x, p_j). Since d is Euclidean, B(p_i, p_j) is a straight line that divides the plane E into two open half planes. Thereby one obtains a tessellation T_{ij} of the Euclidean plane E

[19] As will be shown in a moment, the tessellation (3.2)(ii) is closely related to the construction of the digital line (**Z,** 0**Z**).

defined by $T_{ij} := L(p_i, p_j) \cup B(p_i, p_j) \cup R(p_i, p_j)$ such that $p_i \in L(p_i, p_j)$ and $p_j \in R(p_i, p_j)$. The open convex sets $L(p_i, p_j)$ and $R(p_i, p_j)$ are called the cells of T_{ij} and $B(p_i, p_j)$ is called its boundary. Then, a general Voronoi tessellation may be conceived as the result of the intersection of $n!/(2!\,(n-2)!)$ pairs of half-planes each defined by the bisectors of the pairs (p_i, p_j) of different points p_i and p_j. Thereby the plane is divided into n convex open cells together with their boundaries.

Clearly, a geometrically defined Voronoi tessellation of Euclidean space defines a regular open topological tessellation in the sense of (3.1). By construction, all open Voronoi cells are convex and disjoint from each other [cf. Gärdenfors (2000, p. 88), Okabe et al. (1992)]. As is well known, they are not only open but even regular open. From the very definition of Voronoi cells, points not in any cell are the points positioned at an equal distance to two (or more) paradigmatic points p_i. Hence, they are located on the topological boundaries of the cells defined by the p_i. This fact can be used to show that a Voronoi tessellation based on the metrical structure of Euclidean space E also yields a pole distribution (E, m, P):

Proposition 3.4 *Let T be a Voronoi tessellation of the Euclidean plane E defined by a finite set P of prototypes p_1, …, p_n. Then a topological pole distribution (E, m, P) is defined by taking the Voronoi generators p_1, …, p_n as the set P of poles of a pole distribution $X—m \rightarrow 2^P$ defined as:*

$$m(x) = \{p_i;\ x \in cl(< p_i >);\ < p_i > \text{ the Voronoi cell generated by } p_i)\}.$$

Proof Let (E, OE) be the polar topological space defined by (E, m, P). Since the cells $<p_i>$ of the Voronoi tessellation are convex, they are regular open. Hence, (E, m, P) defines a regular open tessellation $T = \{<p_i>;\ p_i \in P\}$. This tessellation is, of course, not atomic with respect to Euclidean topology. It is, however, by definition, a regular atomic tessellation with respect to the polar topology defined (E, m, P). Its regular open atoms are just the Voronoi cells $<p_i>$. In terms of the pole distribution m, one has x ∈ E contained in a cell int(cl(p)) iff $m(x) = \{p\}$.♦

In other words, the cells of the topological tessellation of E defined by (E, m, P) coincide with the cells of the Voronoi tessellation of E. Moreover, the geometrically defined boundary of the Voronoi tessellation coincides with the topologically defined boundary. In sum, every geometrical Voronoi tessellation of the Euclidean space E defined by a finite set P of prototypes gives rise to a topological tessellation defined by a pole distribution (E, m, P). The two tessellations are extensionally equivalent in the sense that their cells and boundary areas coincide. Thus, they may be conceived as two different interpretations of the same set-theoretical data.

Moreover, tessellations of a space (X, OX) define an equivalence relation ~ on X in a natural way:

Definition 3.5 Let T be a (topological or geometrical) tessellation of (X, OX). The elements x, y ∈ X are equivalent with respect to T iff the following holds:

$$x \sim y := x = y \text{ or there is a cell A of T and } x, y \in A. \blacklozenge$$

Examples 3.6 (i) Let (**R**, O**R**) be the Euclidean line and $T = \{(2m, 2m+2), m \in \mathbf{Z}\}$ the tessellation (3.2). Choose the points $2m+1 \in (2m, 2m+2)$ as representatives of the resulting (non-trivial) equivalence classes of the relation ~ defined by the tessellation on **R** by $x \sim y := x = y$ or $2m < x, y < 2m+2$. Then (**R**/~, O**R**/~) is just the Khalimsky digital line (**Z**, O**Z**).

(ii) Let (**Z**, O**Z**) be the Khalimsky line and $m \geq 2$ a natural number. On **Z** consider the equivalence relation: $x \sim y := x - y \in 2m\mathbf{Z}$**.** Then the quotient space (**Z**/~, O**Z**/~) is called the Khalimsky digital circle ($\mathbf{Z}_{2m}$, $O\mathbf{Z}_{2m}$) (cf. Melin (2008, p. 14)). Let [x] $\in \mathbf{Z}_{2m}$ denote the class of elements represented by $x \in \mathbf{Z}_{2m}$. The space ($\mathbf{Z}_{2m}$, $O\mathbf{Z}_{2m}$) is a polar space with poles represented by the elements [1], [3], …, [2m – 1]. An element [2k] with $0 \leq k \leq m-1$ represents the class of elements that have the same distance from the poles 2k – 1 and 2k + 1. In other words, the Khalimsky circle defined for 2m can be considered as a digital model of the circular color spectrum with m prototypical colors.♦

Compared with the geometrical construction of a Voronoi tessellation of the Euclidean plane, a topological tessellation requires much fewer structural presuppositions in the sense that a Euclidean structure of space is much more specific than a topological one. This is a conceptual advantage insofar as certain problems caused by the presence of representational artifacts disappear. For example, for Euclidean spaces, there are many different metrical structures that define the same underlying topological structure.[20] With respect to these different metrics, one and the same set P of prototypical points may give rise to different Voronoi tessellations. Which should be chosen as the "right" one? This is a question that may have no unique answer. A topological approach does not have the burden to answer it.

Another problem that may be attributed to the peculiarities of the specific mathematical apparatus used for the definition of a Voronoi tessellation T of a conceptual space concerns the boundary area bd(T) of T. This issue has been dubbed the "thickness problem" [cf. Douven et al. (2013) and Douven (2019)].

The "thickness problem" can be explicated as follows. Consider a Voronoi tessellation of the Euclidean plane. By its very construction, the boundaries of the Voronoi cells are "thin" compared to their interior since they are composed of lines consisting of points that have equal distances to two (or more) prototypical points. Douven et al. rightly point out that this assumption for most conceptual spaces is

[20] A prominent case is provided by the family of Minkowski metrics $d_i(x, y)$ for $1 \leq i \leq \infty$. This problem is briefly discussed for d_1 (Manhattan metric) and d_2 (Euclidean metric) in Gärdenfors (2000, chapter 3.9). The Euclidean metric d_2 offers a structural advantage in that the cells of its Voronoi tessellations are always convex with respect to the standard convex structure of Euclidean space. This does not hold for d_1 (cf. Hernández Conde 2017).

not very plausible. For instance, for the conceptual space of the color spectrum, the boundary, say, between "red" and "orange," is defined by points positioned at exactly the same distance from the prototypical points of "red" and "orange." Empirically, this does not make much sense. What does it matter that a certain shade of color is positioned at the same distance from a prototypical "red" and a prototypical "orange"? Moreover, in a general case, there is no reason to assume that boundaries are "thin" compared to the regular open cells of Voronoi tessellation. Douven et al. (2013) proposes overcoming this shortcoming by the introduction of "collated Voronoi diagrams" that arise as a result of projecting similar ordinary Voronoi diagrams onto each other such that the set-theoretical union of their boundaries define a blurred and more or less "thick" area to take into account the vagueness of concepts and their boundaries.

For topological tessellations, no "thickness" problem arises, since they do not distinguish between "thick" and "thin" as geometrical tessellations do (in an artificial way). The following example shows that the topological approach easily deals with tessellations with cells whose boundaries are "thicker" than the cells themselves:

Example 3.7 Let X be the set {α, ω} ∪ **N,** with **N** the natural numbers and α and ω are two objects that are different from all elements of **N** and from each other. Take P = {α, ω} and define a pole distribution (X, m, P) by m(i) = {α, ω}, i ∈ **N**, m(α) = {α}, and m(ω) = {ω}. The corresponding topological structure (X, OX) is given by

$$\begin{aligned}
&\mathrm{cl}(\alpha) = \{\alpha\} \cup \mathbf{N}, \quad \mathrm{cl}(i) = \{i\}, \quad \mathrm{cl}(\omega) = \{\omega\} \cup \mathbf{N}\\
&\mathrm{int}(\alpha) = \{\alpha\}, \quad \mathrm{int}(\mathbf{N}) = \emptyset, \quad \mathrm{int}(\omega) = \{\omega\}\\
&\mathrm{intcl}(\alpha) = \{\alpha\}, \quad \mathrm{intcl}(\omega) = \{\omega\}, \quad \mathrm{bd}(\omega) = \mathrm{bd}(\alpha) = \mathbf{N}.\\
&\mathrm{bd}(i) = \{i\}, \quad \mathrm{V}(i) = \{i, \alpha, \omega\}, \quad \mathrm{cl}(N) = \mathbf{N}.
\end{aligned}$$

The specialization order of (X, OX) is given by i < α, ω for all i ∈ **N**, and there is no other nontrivial relation between the elements of X.◆

The cardinalities of the boundaries bd(ω) and bd(α) of the regular open cells {α} and {ω} are much greater than the cardinalities of the regular open cells {α} and {ω} themselves. A natural basis for OX is given by {{α}, {ω}, {i, α, ω}; i ∈ N}. In contrast, the cardinality of the algebra of regular open sets O*X is much smaller. O*X is isomorphic to the Boolean algebra with 4 elements generated by {α} and {ω}. Thus, moving from OX to O*X amounts to a considerable gain of conceptual parsimony (cf. Sect. 5).◆

The example (3.7) shows that the topological approach has no difficulty in dealing with the "thickness" of boundaries. The concept of topological tessellation is flexible enough to allow cells with boundaries that are intuitively much "thicker" than the cells they are boundaries of.

4 Weakly scattered Alexandroff spaces as a general framework for topological conceptual spaces

The aim of this section is to show that weakly scattered Alexandroff spaces may be considered as a convenient topological framework for conceptual spaces in general. Weakly scattered Alexandroff spaces provide a natural generalization of polar spaces. They possess all their nice features but are more flexible and have a larger domain of applications. To put it bluntly, they may be considered as the "right" generalization of polar spaces.

To set the stage, let us begin by recalling the essential features of the Alexandroff approach. An Alexandroff space X is defined as a topological space for which arbitrary intersections (unions) of open (closed) sets are open (closed) and not only finite ones). Clearly, every topological space (X, OX) with only a finite number of elements is an Alexandroff space. Finite spaces do not exhaust the class of Alexandroff spaces, however. Rather, Alexandroff topology becomes a particularly interesting field of topology exactly for spaces of infinite cardinality. Thus, the Alexandroff topology of a color circle and similar conceptual spaces defined by prototype distributions (X, m, P) qualifies as an interesting Alexandroff topology [cf. Rumfitt (2015)] as well as digital topological spaces such as the Khalimsky plane ($\mathbf{Z} \times \mathbf{Z}$, O$\mathbf{Z} \times \mathbf{Z}$).

All Alexandroff spaces (X, OX) are completely characterized by their specialization orders (X, $\leq$) defined by $x \leq y := x \in cl(y)$. A particularly well-behaved subclass of Alexandroff spaces is the class of spaces whose specialization orders (X, $\leq$) have maximal elements, i.e., for each x ∈ X there is a y such that $x \leq y$ and y is maximal in (X, $\leq$). Clearly, polar spaces and their products are weakly scattered. Given an Alexandroff space (X, OX) (or equivalently, a partial order (X, $\leq$) one may distinguish two kinds of "extreme" elements: x ∈ X is an extreme element with respect to $\leq$ if and only if x is a maximal element of the specialization order, and x ∈ X is extreme with respect to the topology OX iff {x} ∈ OX, i.e., iff x ∈ ISO(X). Fortunately, these two concepts of "extreme elements" coincide:

Proposition 4.1 *Let (X, OX) be a weakly scattered Alexandroff space with specialization order (X, ≤). The sets ISO(X) of isolated points and the set MX of maximal points of the specialization order (X, ≤) coincide. Conceiving (X, OX) as a conceptual space, the "extreme" elements of X can be conceived as the prototypical elements p of the regular open concepts int(cl(p)) ∈ O*X.*

Proof Let p be a maximal element of (X, $\leq$) and assume that {p} is not open. Then int(p) = Ø. By definition of int this is equivalent **C**cl**C**(p) = Ø, i.e., cl(**C**p) = X. Since X is Alexandroff one obtains X = cl(**C**p) = $\cup_{a \neq p}$ cl(a). Hence p ∈ cl(a) and $a \neq p$, i.e., $p < a$ for at least one a. This is a contradiction against the maximality of p. Hence {p} is open.

Now assume {p} is open and suppose p is not maximal and p ∈ cl(a) = **C**int**C**a. Clearly, p ∈ **C**a, since $a \neq p$. Since {p} is open one obtains int(p) ⊆ int**C**a. This is a contradiction. Hence p is maximal.◆

Now we can formulate the concluding theorem of this paper that characterizes weakly scattered Alexandroff spaces as a convenient class for dealing with order-theoretical, algebraic, and topological aspects of conceptual spaces:

Theorem 4.2 *Let (X, OX) be a weakly scattered T_0 Alexandroff space with specialization order (X, ≤), and let ISO(X) be the set of maximal elements. Then (X, OX) satisfies the McKinsey axiom, and the Boolean lattice O*X of regular open elements of OX is an atomic Boolean algebra with atoms intcl(p), p ∈ ISO(X) as generators. One obtains a regular open atomic tessellation of X by $X = V_{p \in ISO(X)}$ intcl(p), i.e., $O^*X = 2^L$, with L being the set of atoms of O*X, i.e., L = {int(cl(p)), p ∈ ISO(X)}.*

Proof Let ISO(X) be the set of maximal elements of the specialization order (X, ≤) of (X, OX). By definition, for each x ∈ X there is at least one p ∈ ISO(X) such that x ≤ p. Since the Alexandroff topology is the upper topology of the specialization order (X, ≤), singletons {p} are open, and closures cl(p) of {p} are the down sets ↓p := {x; x ≤ p}.

The sets int(cl(p)) := {x; ↑x ⊆ ↓p} for p ∈ ISO(X) are atoms of O*X. For different p, p*, sets intcl(p) and intcl(p*) are disjoint and regular open, as int(cl(p)) ∩ int(cl(p*)) = intcl({p} ∩ {p*}) = Ø because the operator intcl is a nucleus, i.e., it distributes over finite intersections of open sets [cf. Johnstone (1982 (ch. II, p. 48)].

To prove that intcl(p) is an atom in O*X is seen as follows: assume that x ∈ A = intcl(A) ⊂ intcl(p). Since intcl(p) is open, one has ↑x ⊆ cl(p)). This entails that x ≤ p and therefore that p ∈ ↑x ⊆ A. Hence, intcl(p) ⊆ intcl(A) = A and A = intcl(p).

We now prove that any regular open A ∈ O*X has the form $A = V_{n \in M'}$ intcl(n) for some M' ⊆ M. Let $M_A := \{p;\ p \in A \cap M\}$. Clearly, $V_{n \in AM}$ intcl(p) ⊆ A. If we can show that $A \subseteq V_{n \in AM}$ intcl(p), we are done. Assume that x ∈ A and define $M_x := \{p;\ x \leq p \text{ and } p \in ISO(X)\}$. Clearly, $M_x \subseteq M_A$. Hence, x ∈ cl(M_A). Assume that y ∈ ↑x. This is the case iff x ≤ y and this entails that $M_y \subseteq M_x$. Thus, we obtain ↑x ⊆ cl(M_A). This means that x ∈ intcl(M_A) = intcl($\cup_{n \in AM}\{p\}$) = $V_{n \in AM}$ intcl(p). Rather, $A \subseteq V_{p \in AM}$ intcl(p).◆[21]

Theorem 4.2 shows that the topological account of conceptual spaces has an important advantage over the geometrical account of conceptual spaces based on the concept of Euclidean convexity, insofar as it takes care of the genuinely "logical" aspects of concept systems. In Gärdenfors's account of conceptual spaces the logical, i.e., the syllogistic aspects of concept systems are virtually absent. These syllogistic aspects of concept systems are encapsulated in the traditional logical calculus dealing with connectives such as "AND", "OR", "NOT", etc.

The geometrical approach, interested mainly in constructing discretizations of a conceptual space (by Voronoi tessellations or otherwise), has no means to adequately

[21] According to theorem (4.2), for O*X to be an atomic Boolean lattice, it suffices that (X, OX) is Alexandroff and weakly scattered. These two requirements are, however, not necessary to ensure that O*X is atomic. There are non-weakly scattered Alexandroff spaces and non-Alexandroff weakly scattered spaces with regular open atomic tessellations O*X, see "Appendix B".

represent most of the classical logical operations on concepts defined for them from the time of Aristotelian syllogistics. This is due to the fact that these operations do not go well with convexity. Among the classical logical operations of concepts such as disjunction, conjunction, negation and others, only the conjunction of concepts has a well-behaved geometrical representation in conceptual spaces structured by convexity. That is, if A and B are concepts represented by convex regions of a conceptual space (X, co) the conjunction A & B of A and B is represented by the set-theoretical intersection of these regions. Other logical operations such as disjunction and negation of concepts cannot be represented in a plausible way by convex regions. The most obvious case is negation. Assume that a concept A is represented by a convex region of a conceptual space (X, co). For the sake of definiteness, take X to be the Euclidean plane endowed with standard Euclidean vector convexity [cf. Gärdenfors 2000; Douven 2019a, b]. If only convex subsets of X are recognized as (natural) concepts the set-theoretical complement **C**A of A is not a concept since **C**A is usually not convex. The convex hull co(**C**A) does not work much better, since it is usually much larger than **C**A and has a non-trivial intersection with A. Disjunctions do not score better. The set-theoretical union A ∪ B of convex sets A and B is usually not convex. Hence, it cannot represent a concept. On the other hand, the convex hull of co(A ∪ B) is usually too great to serve as a logically plausible supremum A ∨ B of A and B, since the law of distributivity:

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C) \tag{4.3}$$

does not hold in the lattice Co(X) of convex subsets of X. In sum, the familiar classical logical connectives of concepts have no place in the framework of conceptual spaces defined with the aid of Euclidean convexity operators. In contrast, topological operators account have no problems with the Boolean logic of concepts. Every conceptual space X endowed with a topology OX comes along with a naturally defined regular open interpretation of the Boolean lattice O*X. Moreover, if (X, OX) is weakly scattered Alexandroff, then O*X is even atomic with distinguished isolated elements a ∈ ISO(X) as generators of its atoms.[22]

Let us conclude this section with some brief remarks on the relation between geometrically and topologically defined conceptual spaces. Very succinctly then, the relation between geometrically and topologically defined conceptual spaces may be expressed as follows. Starting from a "classical" geometrically defined conceptual space (endowed with Voronoi tessellation defined via prototypes and Euclidean

[22] The fact that negations, disjunctions and other familiar combinations of concepts can be represented naturally in a topological framework should perhaps not be taken as the ultimate and definite argument that "not red", "red or blue or green" are natural concepts in exactly the same sense as "red", "blue", and "green". But for every even minimally useful calculus of concepts negations and disjunctions of concepts are indispensable. An example treated in this paper in some detail is Rumfitt's solution of the Sorites paradox dealing with "non-red" etc. Thus, a theory of concepts that does not deal with the issue of logical connectives has to give explicit reason why it does so. Otherwise, it is to be assessed as seriously incomplete. In sum, I'd tend to answer the question (asked by a reviewer of an earlier version of this paper) whether the fact that Gärdenfors's account of conceptual spaces does not deal with negative, disjunctive and other combinations of concepts is to be judged "as a bug or a feature" in favor of the first option.

convexity), one obtains a corresponding topologically defined polar space (X, OX) with the same set of prototypes and the same set of open cells. This set of cells topologically defines a regular atomic tessellation of (X, OX) whose elements are the atoms of the Boolean algebra O*X of the underlying topological space (X, OX). Thus, in contrast to a conceptual space defined by a geometrical convexity, a conceptual space endowed with a topological structure comes with a ready-made and well-behaved classical system of concepts, namely, the complete atomic Boolean algebra O*X. Moreover, a closer look at the construction of the conceptual space reveals that the full-fledged apparatus of Euclidean geometry is not necessary to construct a topological discretization of X. Rather, a more austere structure suffices, namely, a weakly scattered Alexandroff topology.

Weakly scattered Alexandroff spaces possess regular atomic tessellations that can be used to construct discretizations of conceptual spaces. As distinguished from polar spaces, for weakly scattered Alexandroff spaces, the dichotomy between prototypical and non-prototypical elements is replaced by a gradual distinction between "more" and "less prototypical" elements defined by the specialization order.[23]

Already Rumfitt's elucidation of the logic of vague concepts has shown that the very simple class of polar spaces offers a fruitful explication of many aspects of conceptual spaces. The more comprehensive class of weakly scattered Alexandroff spaces offers a sufficiently flexible framework for dealing with various aspects of conceptual discretization and categorization arising in cognitive science and related disciplines.

5 Towards a topological design theory of conceptual spaces

The basic assumption of the conceptual spaces approach is that concepts can be usefully represented as well-formed subsets of a conceptual space. To speak meaningfully about well-formedness requires that the space in question is structured in one way or another. Then, the basic task of this approach is to find appropriate structures that allow us to characterize empirically useful concepts as structurally well-formed subsets. Topological structures have shown to be basic for all kinds of "spaces" that are used and investigated in numerous and variegated realms of knowledge (for instance, physics (of course), game theory, biology, neural sciences, economics, logic) [see for example Adams and Franzosa 2008; Curto 2017; Rabadán and Blumberg 2020]. Thus, it appears reasonable to expect that topological structures may also play an important role in the theory of conceptual spaces. Topological concepts are flexible enough to be adapted to various empirical and theoretical necessities. It is a matter of "theory-guided" empirical research to find out which topological structures for which types of conceptual spaces are the most useful ones.

[23] As stated already in the introduction, such a gradual distinction has been assumed often more or less implicitly by many authors, nice examples can be found in the recent paper Osta-Vélez and Gärdenfors (2020).

Topological structures are flexible enough to take into account a variety of criteria (which sometimes pull in opposite directions) that a system of "good," i.e., "natural concepts," should satisfy. This means that topology may be a helpful device for setting up a general "design theory" that aims to determine how "good" discretizations of conceptual spaces by "natural concepts" should look like. An account of such a theory has recently been put forward by Douven (2019a, b) and Douven and Gärdenfors (2019). They present a list of desiderata for the design of conceptual spaces that "good" (or even "optimal") systems of concepts should satisfy. More precisely, according to these authors, a good conceptual system should satisfy the requirements of parsimony, informativeness, contrast, and learnability. These design principles may pull in different directions. Thus, the overall task is to find a kind of equilibrium or balance between the various principles. The following shows how Douven and Gärdenfors's list of design principles are accounted for by the topological approach (based on weakly scattered Alexandroff spaces):

5.1 Parsimony

The conceptual structure should not overload the system's memory.

Topological response Already elementary examples like the color spectrum show that the cardinality of the Boolean lattice O*X of regular open sets is much lower than the cardinality of the Heyting lattice OX of open sets. This is evidence that the choice of O*X (instead of OX) as the set of representatives of concepts is in line with the principle of parsimony.♦

5.2 Informativeness

Concepts should be informative, meaning that they should jointly offer good and roughly equal coverage of the domain of classification cases.

Topological Response According to the topological approach, the extensions intcl(p) of concepts jointly offer complete coverage of the domain of classification cases since O*X provides an atomic topological tessellation $V_{p \in p}$intcl(p) of X. In a somewhat different vein, informativeness is expressed by the fact that the set P of prototypes is dense in X, i.e., cl(P) = X.♦

5.3 Presentation

The conceptual structure should be such that it allows one to choose for each concept a prototype that is a good representative of all items falling under the concept.

Topological response According to the topological approach, the prototypes for all concepts are defined as isolated points of (X, OX) or equivalently as maximal elements of the specialization order (X, ≤).♦

5.4 Contrast

The conceptual structure should be such that prototypes of different concepts can be so chosen such that they are easy to tell apart.

Topological response Different prototypes p and p' are topologically separated in the sense that their extensions intcl(p) and intcl(p') are separated, i.e., intcl(p) ∩ intcl(p') = Ø. ◆

5.5 Learnability

The conceptual structure should be learnable, ideally from a small number of instances.

Topological response This can topologically be taken into account by requiring that for "good" conceptual spaces X, for each element x ∈ X there is a path $x < x_1 < \ldots < x_m$ from x to x_m, and x_m maximal with respect to the specialization order (X, ≤) of (X, OX). For polar spaces this requirement is satisfied by very short paths of length 2 that have prototypes as their maximal elements. For more general spaces, for instance, topological products of polar spaces, the paths are longer and pass through intermediate elements that may be characterized as "more or less prototypical".[24] This amounts to a more complex categorization than that which is determined by just one prototypical pole.◆

As Douven and Gärdenfors point out the task of designing a good (or even optimal) conceptual structure for a conceptual space may be understood as problem of conceptual engineering. Engineers have to take into consideration certain constraints that restrict the feasibility of their design. Not everything that appears "in principle" possible, is practically feasible. This holds, in particular, for the cognitive scientist who aspires to design conceptual spaces that are to be used for building programs, robots, or other artificial devices that are designed for accomplishing various cognitive tasks. To be specific, consider the growing importance of computer simulations in many areas of science. Many of the conceptual spaces involved in such simulations use to be digital spaces of one kind or another. This predicament requires digitized conceptual spaces. Thus, a comprehensive design theory of conceptual systems has to take into account this fact. This is a desideratum that goes far beyond the level of concretization that Douven and Gärdenfors's design theory has achieved up to now. For instance, in Douven (2019a, b) the author presents several examples of "conceptual structures for disc-shape similarity spaces" (Douven 2019a, b, pp. 125–127). These examples, intuitively appealing and plausible as they may be, are not much more than intuitive illustrations or geometrical metaphors for design problems. In order to render them more precise and less metaphorical, the conceptual engineers have to take seriously the constraints that determine the material architecture of conceptual spaces. One important constraint that gains more and more relevance in contemporary cognitive science and artificial intelligence is the recognition

[24] Detailed discussions about the essential role of "intermediate elements" in digital spaces may be found in Adams and Franzosa (2008, chapter 11.3), Melin (2008) and elsewhere.

of the digital ("discrete") character of many conceptual spaces. The other side of this coin is the necessity to admit the limited relevance of considerations based on plausible intuitive Euclidean spaces.

The task of finding appropriate design criteria for conceptual systems can perhaps be compared with the task of finding in physics fruitful "relative a priori principles" that characterize general features of theories concerning aspects such as continuity, probability, and the causality of laws [cf. Cassirer (1937), Friedman (2001)]. For instance, in physics states spaces of systems have often been assumed a priori to satisfy certain geometrical constraints, e.g., the constraint of being Riemannian manifolds. This Riemannian constraint is mathematically highly non-trivial and not very intuitive, at least not for someone who is accustomed only to elementary Euclidean geometry. It took a long time for scientists and mathematicians to formulate this constraint in a mature form (and for philosophers to understand it). Something analogous may be expected for the design principles of the cognitive sciences.

6 Concluding remarks

In this paper arguments have been put forward in favor of the thesis that conceptual spaces can be usefully endowed with the topological structures of weakly scattered Alexandroff spaces. Alexandroff topologies are an expedient conceptual device for addressing the important role that prototypes and paradigmatic elements play in human and non-human categorization. In the simplest case, this is evidenced already by polar spaces defined by pole distributions. Weakly scattered Alexandroff spaces are more general than polar spaces but inherit most of the useful properties exhibited by the former. In weakly scattered Alexandroff spaces the mathematical structures of order, algebra, and topology (Bourbaki's "mother structures") are interwoven in an intricate manner that can be used to tackle a variety of problems that arise for issues of conceptualization, categorization, and logic.

Topological structures are fundamental spatial structures and arguably even the most fundamental ones. Thus, if one subscribes to Gärdenfors's thesis that in order

> to understand the structure of our thoughts and to be able to build artificial systems with similar cognitive capacities, we should aim at unveiling our conceptual spaces [Gärdenfors (2000, 262)],

we should invest some effort in the task of understanding the topological structures of our conceptual spaces and other conceptual spaces that we are interested in.

Acknowledgements Many thanks to the referees for very helpful corrections, comments, and suggestions on previous versions of this paper. Further, I'd like to thank Javier Belastegui for some useful remarks and criticisms. Financial aid from the Research Project GIU/19051, funded by the University of the Basque Country UPV/EHU, is gratefully acknowledged.

Appendix A: Elements of topology

For the reader's convenience, this appendix lists some basic definitions and facts of topology that are used in this paper. An excellent textbook of elementary set-theoretical topology is Willard (2004).

Definition A.1 Let X be a set with power set 2^X. A topological space (X, OX) is a relational structure with $OX \subseteq 2^X$ satisfying the following:

(i) Ø, X ∈ OX;

(ii) Finite intersections and arbitrary unions of elements of OX are elements of OX.

The elements of OX are called open sets of (X, OX). The set-theoretical complements of open sets are called closed sets.[25] As usual, when there is no danger of confusion, a topological space (X, OX) is simply denoted by X.

(iii) A topological space (X, OX) is an Alexandroff space iff arbitrary intersections (unions) of open (closed) sets are open (closed).♦

If X has more than one element, then different topological structures xist on X. In particular, there are two extreme topological structures (X, O_0X) and (X, O_1X) defined by $O_0X := \{Ø, X\}$ and $O_1X := 2^X$. The topology (X, O_0X) is called the indiscrete topology on X, and the topology (X, O_1X) is called the discrete topology. With respect to set-theoretical inclusion all topological structures (X, OX) on X lie between these two (rather uninteresting) extremal topologies: $O_0X \subseteq OX \subseteq O_1X$. A topology O_aX is coarser than a topology O_bX iff $O_aX \subseteq O_bX$. Equivalently, O_bX is said to be finer than O_aX. Thus, O_0X is the coarsest topology on X, and O_1X is the finest topology on X.♦

The following definition collects some standard methods to construct new topologies from old ones:

Definition A.2 Let (X, OX) and (Y, OY) be two topological spaces. Recall that a (set-theoretical) map X—f→Y is continuous iff $f^{-1}(OY) \subseteq OX$. The map f is open if and only if $f(OX) \subseteq OY$.

(i) The product topology $O(X \times Y)$ on $X \times Y$ is the finest topology such that the projections $X \times Y$—p_X→X and $X \times Y$—p_Y→Y are continuous with respect to $O(X \times Y)$ and O(X) and $O(X \times Y)$ and O(Y), respectively.

(ii) Let Z—i—>X be an inclusion map of a subset $Z \subseteq X$. If (X, OX) is a topological space, the induced topological structure (Z, OZ) is the coarsest topology on Z such that the map i is continuous, i.e., $OZ = Z \cap OX$.

(iii) Let ~ be an equivalence relation on X and X—q→X/~ the canonical quotient map. The quotient topology OX/~ on X/~ is the finest topology such that q is continuous.♦

[25] A set may be open and closed. For instance, the sets Ø and X are open and closed for all topological structures (X, OX). Sets that are open and closed, are sometimes called clopen. A topological space is called connected iff Ø and X are the only clopen subsets of X.

Topological structures (X, OX) can be defined in many equivalent ways. For our purposes, particularly useful is a definition in terms of closure operators cl or interior kernel operators int. These operators must satisfy the so-called Kuratowski axioms:

Definition A.3 A topological closure operator is an operator 2^X—cl→2^X satisfying the four requirements (i) – (iv) below. Dually, a topological interior kernel operator is a map 2^X—int→2^X satisfying requirements (i)*–(iv)*:

(i) cl(A ∪ B) = cl(A) ∪ cl(B) (i)* int(A ∩ B) = int(A) ∩ int(B). (Distributivity)

(ii) cl(cl(A)) = cl(A). (ii)* int(int(A)) = int(A). (Idempotence)

(iii) A ⊆ cl(A). (iii)* int(A) ⊆ A. (Extension)

(iv) cl(Ø) = Ø. (iv)* int(X) = X. (Normality)

Closure operators and interior kernel operators are interdefinable: Denoting the set-theoretical complement of A by CA, one obtains cl(A) = Cint(CA)) and int(A) = Ccl(CA)). Every topological closure operator cl uniquely defines a topological structure (X, OX) and vice versa. Given cl, the class of open sets OX is defined by OX := {B; B = Ccl(A); A ⊆ X}. Dually, given a topological interior kernel operator int, the corresponding topological structure OX is defined by OX := {A; A = int(A), A ⊆ X}.

For A ⊆ X, the boundary bd(A) of A is defined as bd(A) := cl(A) ∩ cl(CA) = **C**(int(A) ∪ int(**C**A)). Moreover, it is possible to define cl (and int) in terms of bd.◆

Topological closure operators are only one of many different types of closure operators used in mathematics. As is well known, also the concept of convexity may be defined in terms of closure operators:

Definition A.4 A convex closure operator (or convexity) on a set X is defined as an operator 2^X—co→2^X that satisfies the following requirements:

(i) A ⊆ B ⇒ co(A) ⊆ co(B). (Monotony)

(ii) co(co(A)) = co(A). (Idempotence)

(iii) A ⊆ co(A). (Extension)

(iv) co(Ø) = Ø. (Normality)

(v) For all y ∈ co(A) there is a finite set F ⊆ A such that y ∈ co(F). (Algebraicity).

A set A is called convex with respect to the operator co iff co(A) = A. The convex operator co is of arity ≤ n provided its convex sets are precisely the sets A with the property that co(F) ⊆ A for each F ⊆ A with cardinality#F ≤ n.◆

The familiar Euclidean convexity is a convex operator of arity 2.[26] More interesting is the observation that the topological closure operator cl of an Alexandroff space (X, OX) is also a convex closure operator in the sense of (A.4). By definition (see A.9) a set A is closed in the Alexandroff topology iff A = ↓A. Then x ∈ ↓A iff there is an a ∈ A with x ≤ a. In other words, x ∈ ↓a = co(a). Thus, one may choose F(x) = {a} as the finite set F(x) with F(x) ⊆ A and x ∈ co(F(x)). Hence, the "lower convexity", defined by the Alexandroff topological operator, is of arity 1.

In other words, a conceptual space (X, OX) endowed with an Alexandroff topological structure OX (defined by the operator cl) may be conceived as a space endowed with a convex structure (defined as well by cl).[27] Admittedly, this "Alexandroff convexity" is rather different from the Euclidean convexity that most conceptual spaces are assumed to be endowed with. Nevertheless, this fact suggests that Alexandroff's and Gärdenfors's approaches to conceptual spaces are not totally alien to each other.

Quite often, a topological structure (X, OX) and a convex structure (X, co) coexist on the same set X. Prominent example are the Euclidean spaces $\mathbf{R}^n$. In this situation it is expedient to require appropriate compatibility conditions of the two structures [for details see van de Vel (1993, ch. III].

Proposition A.5 *Let (X, OX) be a topological space. An open subset A ∈ OX is regular open iff A = int(cl(A)). The set of all regular open subsets of X is denoted by O*X. O*X is well known to be a complete Boolean algebra. There is a map OX—j → O*X defined by j(A) := int(cl(A)) and an inclusion map O*X—i → OX such that* $j \bullet i = id_{O*X}$ *and* $id_{OX} \subseteq i \bullet j$.♦

Definition A.6 Let (X, OX) be a topological space.

(i) An element x ∈ X is isolated iff {x} ∈ OX. The set of isolated points of X is denoted by ISO(X).

(ii) A subset A ⊆ X is dense in X iff cl(A) = X.

(iii) The space (X, OX) is weakly scattered iff ISO(X) is dense in X, i.e., cl(ISO(X)) = X.

(iv) The space (X, OX) satisfies the McKinsey axiom iff int(cl(A)) ⊆ cl(int(A)) for all A ⊆ X.♦

[26] Indeed, all convexity operators that Gärdenfors considers in contributions to the approach of conceptual spaces are of arity 2. More precisely, his basic primitive concept is a ternary relation B(x, y, z) of elements x, y, and z of a conceptual space X. The relation B(x, y, z) is to be read as "y is between x and z". For any two points x and z the relation B defines a subset [x, z] of elements between x and z, i.e., [x, z] := {y; B(x, y, z)}. Then a set A ⊆ X is convex with respect to B iff x, z ∈ A entails [x, z] ⊆ A. This defines a convex closure operator co for which A.3(5) is just the requirement that y ∈ A iff there are x, z ∈ A and such that y ∈ B(x, y, z).

[27] The standard convex operators of Euclidean spaces and the topological operators of Alexandroff spaces are both convex closure operators. Thus, they can be treated as two cases of a general theory of convex structures [cf. van de Vel (1993)]. This suggests that topology and convexity, used as devices for structuring conceptual spaces should be considered as special cases of the more general approach of an approach that conceives conceptual spaces as closure structures.

Definition A.7 (*Specialization quasi-order of a topology*) Let X be a set. A quasi-order on X is a binary relation $\leq$ such that for all x, y, z ∈ X, the following conditions (i) and (ii) are satisfied:

(i) $x \leq x$. (Reflexivity)

(ii) $x \leq y$ and $y \leq z$ implies $x \leq z$. (Transitivity)

(iii) If also $x \leq y$ and $y \leq x$ implies $x = y$ is satisfied the quasi-order $\leq$ is said to be a partial order, and the structure $(X, \leq)$ is called a poset.

(iv) A subset C of a partial order $(X, \leq)$ is a chain iff all elements x, y ∈ C are comparable, i.e., $x \leq y$ or $y \leq x$. If C is a finite chain in X with $\#C = n + 1$, the length of C is n. The length of the longest chain is called the depth of partial order.

A topological space (X, OX) defines a quasi-order $(X, \leq)$ by $x \leq y := x \in cl(y)$. This quasi-order is called the specialization quasi-order of (X, OX). The set of maximal elements of $(X, \leq)$ with respect to this order is denoted by MX.◆

For many traditional topological spaces such as the Euclidean spaces (E, OE), the specialization order $(E, \leq)$ is trivial, i.e., $x \leq y$ iff $x = y$, or, equivalently, iff MX = X. In contrast, for non-trivial Alexandroff spaces (X, OX) the specialization quasi-order is non-trivial, i.e., MX ≠ X.

Proposition A.8 (Upper topology defined by a quasi-order $(X, \leq)$) *Let* $(X, \leq)$ *be quasi-order. For* $A \subseteq X$*, define the upper set of A by* $\uparrow A := \{x;\ a \leq x$ *for some* $a \in A\}$. *The upper topology (X, OX) corresponding to* $(X, \leq)$ *is defined by* $OX := \{\uparrow A;\ A \subseteq X\}$. *The closed sets of this topology are the lower sets of* $(X, \leq)$ *defined by* $A := \downarrow A := \{y;\ y \leq a$ *for some* $a \in A\}$. *X endowed with the upper topology of the quasi-order* $(X, \leq)$ *is an Alexandroff topological space (X, OX).*[28]◆

Proposition A.9 *Let (X, OX) be an Alexandroff space with the specialization quasi-order* $(X, \leq)$. *Then, the upper topology of X is isomorphic to (X, OX). In other words, the topology of an Alexandroff space (X, OX) is completely determined by its specialization quasi-order* $(X, \leq)$. *An element a ∈ X is maximal with respect to the specialization order if and only if a ∈ ISO(X), i.e., {a} ∈ OX.*◆

Separation axioms

Let (X, OX) be topological space.

(i) X is a T_0-space iff for every x ∈ X and every y ≠ x there exists an open set A ∈ OX such that either x ∈ A and y ∉ A or x ∉ A and y ∈ A.

(ii) X is a $T_{1/2}$-space iff every point x ∈ X is either open or closed.

(iii) X is a T_1-space iff every point x ∈ X is closed.

(iv) (X, OX) is a T_2-space (or Hausdorff space) iff for distinct points x and y there are open sets A ∈ OX and B ∈ OX containing x and y such that x ∉ A and y ∉ B.

[28] Analogously, the lower set $\downarrow A$ of $A \subseteq X$ is defined by $\downarrow A := \{y;\ y \leq a$ for some $a \in A\}$. Thereby, the so-called lower topology of $(X, \leq)$ is defined as the set of all lower sets $\{\downarrow A;\ A \subseteq X\}$. In this paper, however, there is no need to consider this topology of $(X, \leq)$.

(v) The separation axioms $T_0 - T_2$ satisfy a chain of proper implications: $T_2 \Rightarrow T_1 \Rightarrow T_{1/2} \Rightarrow T_0$.◆

The following examples show that Euclidean and Alexandroff spaces behave quite differently with respect to separation axioms:

Examples

(i) The standard Euclidean topology OR of the real line R is generated by open intervals (a, b) = {x; a < x < b}. Two distinct points x and y can be separated by open intervals U(x) and U(y), which are disjoint from each other. Hence, (R, OR) is a T_2–space. A fortiori, all points are closed, and no point is open.

(ii) Let (N, ≤) be the set of natural numbers endowed with their natural order ≤. A topological space (N, ON) is defined by stipulating that Ø and the sets ↑n := {m; n ≤ m} are open for each n ∈ N. Then (N, ON) is an Alexandroff space that satisfies T_0 but not $T_{1/2}$. No point of (N, ON) is open, and the only closed point of (N, ON) is 0.

(iii) The Khalimsky line (**Z**, O**Z**) (as a polar space) satisfies the axiom $T_{1/2}$ but not T_1.

(iv) The Khalimsky plane (**Z** × **Z**, O**Z** × **Z**) satisfies T_0 but not $T_{1/2}$. The even points (2m, 2n) ∈ **Z** × **Z** are closed, the odd points (2m + 1, 2n + 1) ∈ **Z** × **Z** are open, and the “mixed points” (2m, 2n + 1), (2m +1, 2n) ∈ **Z** × **Z** are neither open nor closed.◆

Proposition A.12 (i) *An Alexandroff space (X, OX) satisfies T_1 iff it is discrete, i.e., $OX = 2^X$.*

(ii) *A topological space (X, OX) is a T_0-Alexandroff space iff its specialization quasi-order (X, ≤) is a partial order.*[29]

(iii) *For an Alexandroff space with specialization quasi-order (X, ≤) define an equivalence relation on X by x ~ y := x ≤ y and y ≤ x. Then (X/~, OX/~) is a T_0-Alexandroff space.*◆

Appendix B: Examples of Alexandroff spaces

B.1

All finite topological spaces (X, OX) are Alexandroff spaces with O*X atomic. The Sierpinski space (X, OX) with X = {a, b}, OX = {Ø, {a}, {a, b}} is weakly scattered. In general, finite topological are not weakly scattered. The smallest example is the

[29] In classical topology, the separation axiom T_1 is considered a minimal requirement that must be satisfied for a topological space to be considered reasonable. (A.13)(i) shows that Alexandroff spaces fall outside the realm of classical theory: Only trivial (discrete) Alexandroff spaces are T_1, and some important Alexandroff spaces such as the digital plane (**Z** × **Z**, O(**Z** × **Z**)) fail to be $T_{1/2}$.

space $X=\{a, b, c\}$ with topology $OX=\{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}\}$. The only isolated point of (X, OX) is {c}, but {c} is not dense in (X, OX), since clearly $cl(c)=\{c\}$. Nevertheless, O*X is atomic, namely, O*X is the Boolean algebra with four elements $\{\emptyset, \{a, b\}, \{c\}, \{a, b, c\}\}$, generated by the atoms {a, b} and {c}.

B.2

Polar spaces X provide the simplest class of Alexandroff spaces that may have infinitely many elements. Examples treated in detail include the linear color spectrum and the circular color spectrum (color circle) with finitely many poles but infinitely many shades of colors.

Perhaps the most important example of a polar space is the Khalimsky line (or digital line) (**Z**, O**Z**) defined by the pole distribution **(Z,** m**, 2Z + 1)** (cf. 2.10). Indeed, the Khalimsky line may be considered as the "foundation of digital topology" (cf. Kopperman (1994)).

B.3

Finite products of polar spaces are weakly scattered Alexandroff spaces. More generally, finite products of weakly scattered Alexandroff spaces are weakly scattered Alexandroff spaces.

B.4

Not all Alexandroff spaces with infinitely many elements are weakly scattered. An example is the Alexandroff space (N, ON) defined by the standard linear order $(N, \leq)$ as the specialization order. As is easily observed, the set of isolated points of this space is empty. Hence, (N, ON) is not weakly scattered. Nevertheless, the Boolean algebra O*N is atomic, namely, the minimal Boolean algebra of two elements generated by the atom N.

B.5

More generally, there are infinite trees $(X, \leq)$ whose Alexandroff topologies (X, OX) have regular open lattices O*X that are the atomic Boolean algebras 2^n, $n=1, 2, \ldots$ Just take X to be the disjoint union of n copies of **N** endowed with the natural order. Identify the minimal elements ("0") of all copies of **N** with each other. The result is a tree with n infinite linear branches. Clearly, (X, OX) is not weakly scattered, because X has no isolated points at all. However, the Boolean algebra O*X of regular open sets of the Alexandroff space (X, OX) is the atomic Boolean algebra 2^n generated by the regular open upper sets $\uparrow x_1, \ldots, \uparrow x_n$, where each x_i generates a branch of the tree as its open hull $\uparrow x_i$.

B.6

Not all Alexandroff spaces are regular atomic. An example is given by the specialization order (X, ≤) of the infinite binary tree.[30] Let X be the set of finite 0–1-sequences $(\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_i = 0, 1$, endowed with the following partial order: For x, y ∈ X one has x ≤ y := x is an initial subsequence of y, i.e. y = (x, z) and z ∈ X. Then (X, ≤) is an infinite binary tree with root the empty sequence Ø. The two children of Ø are the sequences (0) and (1), the children of (0) and (1) are (0, 0) and (0,1), and (1, 0) and (1, 1), respectively, and so on.

For all x ∈ X, the subtrees ↑x are regular open subsets of (X, OX). They are not atomic since for any x, one may find x < x' < x'' < … such that ↑x ⊃ ↑x' ⊃ ↑x'' ⊃ …. One then obtains infinite strictly decreasing sequences of regular open elements of O*X.◆

B.7

There are weakly scattered spaces (X, OX) with atomic O*X that are not Alexandroff: Let (**R**, O**R**) be the set of real numbers **R** endowed with the topology engendered by the standard Euclidean topology and the elements of the set **Q** of rational numbers. Then, the rationals are isolated points of (**R**, O**R**) such that cl(**Q**) = **R** since every open neighborhood U(s) of an irrational number s contains a rational number q ∈ **Q**. The singletons {s} of irrational numbers s ∈ **R** – **Q** are closed but not open: if {s} were open, s ∈ intcl(s) ⊆ intcl(**R** – **Q**). However, clearly intcl(**R** – **Q**) = int(**R** – **Q**) = Ø, since **Q** is open in this topology. Hence, {s} is closed but not open. This shows that (**R**, O**R**) is not Alexandroff, since the intersection {s} of the open neighborhoods of s is not open. The Boolean lattice O***R** is atomic, since clearly O***R** = P**Q** due to the fact that the only atomic elements of O***R** are singletons {q} with q ∈ **Q**.◆

References

Adams, C., & Franzosa, R. (2008). *Introduction to topology, pure and applied*. Upper Saddle River: Prentice Hall.

Aiello, M., van Bentham, J., & Bezhanishvili, G. (2003). Reasoning about space: The modal way. *Journal of Logic and Computation, 13*(6), 799–919.

Aiello, M., Pratt, I. E., & van Benthem, J. F. A. K. (2007). *Handbook of spatial logics*. Springer.

Alexandroff, P. (1937). Diskrete Räume, Rec. Math. [Mat. Sbornik], N.S., 1937, Volume 2 (44), Number 3, 501–519.

Bezhanishvili, G., Mines, R., & Morandi, P. J. (2003). Scattered, Hausdorff-reducible, and hereditarily irresolvable Spaces. *Topology and its Applications, 132,* 291–306.

Bezhanishvili, G., Esakia, L., & Gabelaia, D. (2004). *Modal logics of submaximal and nodec spaces, Festschrift for Dick de Jongh on his 65th birthday*. University of Amsterdam.

[30] I owe this example to Imanol Mozo.

Bobzien, S. (2012). If it's clear, then it's clear that it's clear, or is it? Higher-order vagueness and the S4 axiom. In K. Ierodiakonou & B. Morison (Eds.), *Episteme, etc: Essays in honour of Jonathan Barnes* (pp. 189–212). Oxford: Oxford University Press.

Bobzien, S. (2015). Columnar higher-order vagueness, or vagueness is higher-order vagueness. *Proceedings of the Aristotelian Society, Supplementary, 89,* 61–89.

Carnap, R. (1971). A basic system of inductive logic, part I. In R. Carnap & R. C. Jeffrey (Eds.), *Studies in inductive logic and probability* (Vol. I, pp. 33–165). Berkeley: University of California Press.

Cassirer, E. (1937 (1956)). Determinism and indeterminism in modern physics. New Haven and London: Yale University Press.

Chen, L. M. (2015). *Digital and discrete geometry*. New York: Springer.

Couprie, M., Cousty, J., Kenmochi, Y., & Coeurjolly, D. (2020). Guest editorial: Special issue on discrete geometry for computer imagery. *Journal of Mathematical Imaging and Vision, 62,* 625–626.

Curto, C. (2017). What can topology tell us about the neural code? *Bulletin (New Series) of the American Mathematical Society, 54*(1), 63–78.

Decock, L., & Douven, I. (2015). Conceptual spaces as philosophers' tools. In F. Zenker & P. Gärdenfors (Eds.), *Applications of conceptual spaces, the case for geometric knowledge representation* (pp. 207–222). New York: Springer.

Douven, I. (2019a). The rationality of vagueness. In R. Dietz (Ed.), *Vagueness and rationality in language use and cognition* (pp. 115–134). New York: Springer.

Douven, I. (2019b). Putting prototypes in place. *Cognition, 193,* 1–13. https://doi.org/10.1016/j.cognition.2019.104007.

Douven, I., Decock, L., Dietz, R., & Égré, P. (2013). Vagueness: A conceptual spaces approach. *Journal of Philosophical Logic, 42,* 137–160.

Douven, I., & Gärdenfors, P. (2019). What are natural concepts? *A Design Perspective, Mind and Language*. https://doi.org/10.1111/mila.12240.

Friedman, M. (2001). Dynamics of reason. The 1999 Kant lectures at Stanford University, Stanford, California, CSLI Publications.

Gabelaia, D. (2001). Modal definability in topology. Master's Thesis, University of Amsterdam.

Gärdenfors, P. (2000). *Conceptual spaces. A Bradford Book*. Cambridge: MIT Press.

Goubault-Larrecq, J. (2013). *Non-Hausdorff topology and domain theory. Selected topics in point-set theory*. Cambridge: Cambridge University Press.

Hernández Conde, J. V. (2017). A case against convexity in conceptual spaces. *Synthese, 194*(10), 4011–4037.

Johnstone, P. (1982). *Stone spaces*. Cambridge: Cambridge University Press.

Kaipainen, M., Zenker, F., Hautamäki, A., & Gärdenfors, P. (2019). *Conceptual spaces: Elaborations and applications*. New York: Springer.

Kong, T. Y., Kopperman, R., & Meyer, P. R. (1991). A topological approach to digital topology. *The American Monthly, 98*(10), 901–917.

Kopperman, R. (1994). The Khalimsky line as a foundation of digital topology. In Y.-L. O. A. Toet, D. Foster, H. J. A. M. Heijmans, & P. Meer (Eds.), *Shape in picture* (pp. 3–20). Berlin: Springer.

Kovalevsky, V. (2006). Axiomatic digital topology. *Journal of Mathematical Imaging and Vision*, *26*, 41–58.

Kuratowski, K., & Mostowski, A. (1976). *Set theory. With an introduction to descriptive set theory*. Amsterdam: North-Holland.

McKinsey, J. C. C., & Tarski, A. (1944). The algebra of topology. *Annals of Mathematics*, *45*(2), 141–191.

Melin, E. (2008). *Digital geometry and Khalimsky spaces, Uppsala dissertations in mathematics 54*. Uppsala: Uppsala University.

Melin, E. (2009). Digital Khalimsky manifolds. *Journal of Mathematical imagining and vision, 33,* 267–280.

Mormann, T. (2020). Topological models of columnar vagueness. Erkenntnis online. https://doi.org/10.1007/s10670-019-00214-2.

Okabe, A., Boots, B., & Sugihara, K. (1992). *Spatial tessellations: Concepts and applications of Voronoi diagrams*. New York: Wiley.

Osta-Vélez, M., & Gärdenfors, P. (2020). Category-based induction in conceptual spaces. *Journal of Psychology, 96,* 1–14.

Rabadán, R., & Blumberg, A. J. (2020). *Topological data analysis for genomics and evolution. Topology in biology*. Cambridge: Cambridge University Press.

Rosenfeld, A. (1979). Digital topology. *The American Mathematical Monthly*, *86*(8), 621–630.
Rumfitt, I. (2015). *The boundary stones of thought. An essay in the philosophy of logic*. Oxford: Oxford University Press.
Sainsbury, R. M. (1996 (1990)). Concepts without boundaries. In: Keefe and Smith (Eds.), Vagueness. A Reader (pp. 251– 64). Cambridge MA: MIT Press.
Steen, L. A., & Seebach, J. A., Jr. (1978). *Counterexamples in topology* (2nd ed.). New York: Springer.
Sznajder, M. (2016). What conceptual spaces can do for Carnap's late inductive logic. *Studies in History and Philosophy of Science, Series A, 56,* 62–71.
Van de Vel, M. L. J. (1993). *Theory of convex structures*. Amsterdam: North-Holland.
Willard, S. (2004). *General topology*. Mineola, NY: Dover Publications.
Zenker, F., & Gärdenfors, P. (2015). *Applications of Conceptual Spaces. The Case for Geometric Knowledge Representation*. New York: Springer.
Zvolsky, M. (2014). Tomographic image reconstruction. An introduction. Lecture on medical physics by Erika Garutti and Florian Grüner, DESY, University of Hamburg.

Affiliations

Thomas Mormann[1]

Thomas Mormann
thomasarnold.mormann@ehu.eus

[1] Department of Logic and Philosophy of Science, University of the Basque Country UPV/EHU, Avenida de Tolosa 70, 20080 Donostia-San Sebastian, Spain