The Space of Mathematics

Philosophical, Epistemological, and Historical Explorations

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lems, involving calculation of the many examples, which need to be elaborated in order to clarify the usefulness of these particular concrete interpretations of the dialectical method of investigation. We very much need the assistance of interested philosophers and mathematicians.

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Structural Analogies Between Mathematical and Empirical Theories

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1. Introduction

Time and again philosophy of science has drawn analogies between mathematics and the empirical sciences, in particular physics. The orientation of these analogies, however, has been rather different. In the heyday of Logical Positivism philosophy of mathematics was considered the model for philosophy of the empirical sciences.[1]

For some time one can witness the opposite approach: the import of ideas from the realm of philosophy of empirical science to the realm of philosophy of mathematics. We would like to point out the following approaches of such a transfer:

Methodological Analogy
Lakatos proposed to transfer his "Methodology of scientific research programs" ("MSRP") from the sphere of empirical science to the realm of mathematics. He claimed that there exists a *methodological* parallelism between the empirical and ("quasiempirical") mathematical theories: in both realms one uses the method of "daring speculations and dramatic refutations".[2]

Functional Analogy
Quine proposed a functional analogy between mathematical and empirical knowledge. His approach is based on the holistic thesis that mathematical concepts like "numbers", "functions" or "groups" play in the global context of

1 This leads to deductively oriented conceptions of empirical science, in particular to the so called "received view", cf. (Nagel 1961), (Suppe 1974).
2 Cf. (Lakatos 1978, p. 41). There are not too many authors who have continued Lakatos' approach despite its very high esteem in all quarters of philosophy (and history) of mathematics. We mention (Howson 1979), (Hallett 1979), and (Yuxin 1990).

scientific knowledge essentially the same role as physical concepts like "electron", "potential" or "preservation of symmetry".[3]

We do not want to critizise these two approaches in detail here, we simply claim that neither Lakatos' "methodology of scientific research programmes" nor Quine's holism has to be the last word in this philosophical issue. We would like to sketch another analogy between empirical and mathematical knowledge, namely, a *structural analogy between empirical and mathematical theories*. It might not be incompatible with those mentioned above. In particular, it may be considered as a specification of Quine's holistic approach. The structural analogy between mathematical and empirical theories, which we want to explain, is based on the general thesis that cognition, be it empirical, mathematical or of any other kind, e.g. perceptual, has a *representational structure: Cognition is representation.*

This thesis can be traced back at least to Kant who maintained that cognition is governed by representational structures, e.g. by the forms of intuition (space and time), and certain categories of understanding like causality. Among its more recent adherents we may mention C. S. Peirce and E. Cassirer. But we do not want to deal here with the problem of general representational character of cognition from a transcendental philosophical viewpoint as Kant did. We would rather like to make plausible that cognition is representation by presenting an inductive argument.

First of all, let us recall that the representational character of cognition is not restricted to scientific knowledge but pervades all kinds of cognition, e.g. *perception* and *measurement*. For one reason or another, the similarity between these kinds of cognition and scientific theories is often underestimated or even denied: perception seems to belong to the merely subjective sphere of the individual, and measurement, though it may be objective, seems to lack a theoretical component. Hence, neither perception nor measurement seem to have much in common with scientific cognition. This impression is wrong. They all share a common representational character and this feature is of crucial importance for their epistemological structure.

Now, by ascribing a representational structure to perception, measurement, and scientific knowledge we do not want to claim that the representational structure in all of them is the same. This does not seem plausible. It may well be that the representations in these different areas are determined by quite different constraints, cf. (Goldman 1986, part II). This, however, does not exclude the possibility that they all should be dealt with a comprehensive epistemology which investigates cognition in all its different realizations and

3 Cf. (Quine 1970), also (Putnam 1979), and (Resnik 1988).

treats them from a unified perspective.[4] Let us mention some trends towards that epistemology.

Measurement as Representation

Measurement is representation in a quite direct and intuitive sense, namely, *measurement is representation of empirical facts and relations by numerical entities and relations*. The explication of this elementary idea in the framework of the so called *representational theory of measurement* has been developed, among others, by Stevens and Suppes, cf. (Suppes 1989), (Mundy 1986). This has led to a comprehensive classification of the various kinds of measurement scales and their transformation groups and invariants. It can be considered as an empirical analogue to Klein's *Erlangen Programme* which aimed to classify the different geometries. This analogy often has been recognized but only recently have there been serious attempts to consider the representational theory of numerical measurement, Klein's classification of geometries, and the various attempts of a general "geometrization" of physics as special cases of a single coherent "general theory of meaningful representation", cf. (Mundy 1986).

Perception as Representation

A perceived object is the result of a representational, constructive process. This is shown very clearly by the various phenomena of perceptual constancy, for example color or gestalt constancy. Up to now, there is no unanimity among the different approaches of cognitive psychology and cognitive science of how the representational character of perception is to be understood precisely. The so called "bottom-up" and the "top-down" approaches conceptualize it in a quite different way, cf. (Goldman 1986). Nevertheless, practically all sciences dealing with perception agree (or at least are compatible) with the assertion that some kind of representation and symbolic construction is involved. Following Gödel an alleged analogy of "mathematical perception" and "visual perception" often has been taken as an argument for a robust platonism or realism which claims that the mathematicians "perceive" mathematical objects just as ordinary people perceive the more mundane things of the ordinary world.

This analogy of mathematical and ordinary perception may subjectivily be

4 For general considerations for such a comprehensive epistemology, cf. (Goodman / Elgin 1988, p. 16). It should be noticed, however, that already more than sixty years ago Cassirer set about the project of embedding philosophy of science in a general representational theory of symbolization in his *Philosophie der symbolischen Formen*, cf. (Cassirer 1985).

quite justified, i.e. it may be the case that mathematicians in the course of their research believe that they "perceive" mathematical objects as they "really are" just as the layman believes that he perceives the table in front of him just as it "really is". But the cognitive sciences teach us that – *pace* Gibson – this is a somewhat simplistic account of perception. Hence, it would be interesting to investigate the problem whether the analogy of mathematical and visual perception could survive as an argument for mathematical realism (platonism) if it is based on an updated account of perception.

Cognition as Representation

Quite generally, Cassirer claims that the search for representational *invariants* is not restricted to perception but common to all kinds of cognition, be it perception, measurement, thinking or whatever else. He considers the tendency towards "objectification" in perception only as the rudimentary form of a general tendency in conceptual, in particular mathematical, thought, where it is developed far beyond its primitive stage (cf. also (Cassirer 1944, p. 20)):

> "A critical analysis of knowledge reveals that the "possibility of object" depends upon the formation of certain invariants in the flux of sense impressions, no matter whether these be invariants of perceptions or of geometrical thought, or of physical theory" (Cassirer 1944, p. 21).

In the case of empirical and mathematical cognition the thesis is that empirical theories are representations, and correspondingly that mathematical theories are representations.

Thus, in order to clarify the representational character of empirical and mathematical cognition and to make plausible the structural analogy between empirical and mathematical theories we are led to the elucidation of the concept of *theory* in both realms of knowledge. This is one of the central conceptual problems of philosophy of science: to provide an adequate explication of this term.[5]

The outline of the paper is as follows: in the next section we sketch the representational character of empirical theories.[6] Then we want to show through the example of *group theory* in what sense a "typical" mathematical theory can

5 A large part of the criticism against Logical Empiricism can be formulated as criticism against the inadequate theory concept of this approach: an empirical theory simply is not a partially interpreted calculus as the so called "received view" holds it, and similarly a "real life" mathematical theory like algebraic topology or commutative algebra cannot be explicated adequately in terms of a calculus of meaningless formal signs.

6 We base our approach largely on Henry Margenau's "Methodology of modern physics", (Margenau 1935).

be considered as a representation in quite an analogous way as empirical theories are representations. Finally, we will point out how *category theory* can be considered as a useful formal tool for the representational reconstruction of mathematical and empirical theories. We will close with some remarks on the relation of our representational approach with some more traditional currents in the philosophy of mathematics.

2. *Empirical Theories as Representations: Data, Symbolic Constructs, and Swing*

To explain the thesis of the representational structure of empirical theories we start by distinguishing two levels of physical conceptualization. We thereby follow the approach of the philosopher and scientist Henry Margenau who among others distinguished between the *level of data* and the *level of symbolic constructs*.[7] He considers the following paradigmatic example:

> "...–we observe a falling body, or many different falling bodies; we then take the typical body into mental custody and endow it with the abstract properties expressed in the law of gravitation. It is no longer the body we originally perceived, for we have added properties which are neither immediately evident nor empirically necessary. If it be doubted that these properties are in a sense arbitrary we need merely recall the fact that there is an alternate, equally or even more successful physical theory – that of general relativity – which ascribes to the typical bodies the power of influencing the metric of space, i.e. entirely different properties from those expressed in Newton's law of gravitation" (Margenau 1935, p. 57).

It should be evident that this two-level-structure is not restricted to mechanics, it pervades all parts of physics.

Even if the realm of *symbolic constructs* in physics is *not* determined in the same rigid way as the realm of *data*, it is not completely arbitrary. There are general requirements concerning *symbolic constructs:*

> "Physical explanation would be a useless game if there were no severe restrictions governing the association of constructs with perceptible situations.

7 We rely on Margenau because his account is intuitive and avoids any unnecessary technical fuss. However, Margenau is not the only one, and not the first, who makes such a distinction: some more or less implicite remarks on the representational character of empirical theories can be found in Duhem's account of "The aim and structure of physical theory" ; see especially (Duhem 1954, Ch. 8). In a formally very sophisticated manner the distinction between *data* ("Intended Applications") and *symbolic constructs* ("Models") is elaborated in the so called structuralist approach of philosophy of science, cf. (Balzer / Moulines / Sneed 1987).

> For a long time it had been supposed that all permissible constructs must be of the kind often described as mechanical models or their properties, but this view is now recognized as inadequate. ... While no restriction can be made as to their choice, their use is subject to very strong limitations. It is easy to find a set of constructs to go with a given set of data, but *we require that there be a permanent and extensive correspondence between constructs and data* " (Margenau 1935, p. 64).

Putting together the ingredients of *data*, *symbolic constructs*, and their *correspondence* we propose the following general format of an empirical theory. According to the representational approach an empirical theory is *a representation* of the following kind:[8]

$$f: \mathcal{D} \longrightarrow \mathcal{C}$$

The realm $\mathcal{D}$ of *data* is *represented* via a mapping f by the realm $\mathcal{C}$ of *symbolic constructs*. The requirement that there must be *a permanent and extensive correspondence between constructs and data* is expressed by the requirement that the representing mapping f from $\mathcal{D}$ to $\mathcal{C}$ cannot be just any mapping but has to respect the structure of $\mathcal{D}$ and $\mathcal{C}$. Therefore some constraints have to be put upon it which may not always be satisfiable. Thus, the thesis that a theory has a representational structure implicitely makes the claim that the $\mathcal{D}$ can be represented by $\mathcal{C}$ in an appropiate way.[9] How this is to be understood precisely depends on how we conceptualize the realms of *data* and *symbolic constructs*.[10]

In philosophy of science, the specific nature and relation of these two levels of empirical theories have been a topic of much discussion. A rather popular account took $\mathcal{D}$ as the observable and $\mathcal{C}$ as the non-observable. But this has not been the only approach. Others have considered $\mathcal{D}$ as the empirical, and $\mathcal{C}$ as the theoretical. It cannot be said that unanimity has been achieved how these levels of conceptualization have to be understood precisely. Probably *The One and Only Right Explication* does not exist. In any case, the

8 This is in some respect an oversimplification: as it turns out, a (mathematical or empirical) theory can be reconstructed as a whole bunch of representations of the above mentioned kind. Thus, more precisely we should consider a representation $f: \mathcal{D} \longrightarrow \mathcal{C}$ as the smallest meaningful element of a theory.

9 This claim corresponds to the "empirical claim of a theory" of the structuralist approach, cf. (Balzer / Moulines / Sneed 1987) and the "theoretical hypothesis" of the state space approach of van Fraassen and Giere, cf. (van Fraassen 1989) or (Giere 1988).

10 In the case of mathematical theories f will be a *functor* between the *Category $\mathcal{D}$ of Data* and the *Category $\mathcal{C}$ of Symbolic Constructs*.

emphasis should not be laid upon what the constituents $\mathcal{D}$ and $\mathcal{C}$ "are in themselves"; what is important is the functional aspect of representation. Thus, the representational approach puts the essence of a theory neither in the things the theory talks about nor in the concepts used by the theory but, so to speak, in the interstices, in the ontologically ambigous space of the representational relation between *data* and *symbolic constructs*.

We do not want to offer any argument for our specific position in this issue, but simply characterize our stance by the following remarks:

(i) The distinction between *data* and *symbolic constructs* is no absolute distinction, i.e. in one context entities can work as *data* and in another context as *symbolic constructs*. In particular, *data* must not be considered as the "inmediately given" of some Logical positivists. Hence, Margenau made the proposal to replace the term *"data"* by the less misleading term *"habita"*, i.e. that what one has at his disposal or takes as the starting point of a research undertaking, cf. (Margenau 1935). In a similar vein Goodman points out, that "the given" or even "the immediatly given" does not exist. Epistemology should consider "the given" as "the taken", i.e. as something which has been taken as the relative starting point or relative basis, (Goodman 1978, p. 10).

(ii) It is an important task for the philosophical reconstruction of empirical and mathematical theories to explicate in a precise manner the *structure* and the *functions* of the *correspondence between data and symbolic constructs*.[11]

What is the representation of *data* by *symbolic constructs* good for? This is a very deep problem whose surface we can only scratch in the context:

(a) *Symbolic constructs* generate a "conceptual surplus" which can be used for determining and predicting previously unacessible aspects of *data*. For example, partially known kinetic data are embedded into the framework of *symbolic constructs* like *forces*, *Hamiltonians*, or *Lagrangians* in order to obtain new information not available without them.

11 In the framework of the structuralist approach this correspondence is explicated in the following way: a theory T has a (more or less determined) domain I of "Intended Application" – Margenau's *data* – and a domain K of conceptual structures – Margenau's *symbolic constructs*. Thus T is an ordered couple T = <K, I>. The global claim of T is that I can be embedded in K in such a way that a connected class of *data* corresponds to a connected class of *symbolic constructs*. The most detailed account of this approach presently available is (Balzer / Moulines / Sneed 1987).

(b) *Symbolic constructs* have an explanatory function and serve to embed the *data* into a coherent explanatory theoretical framework. That is, the correspondence between *data* and *symbolic constructs* is the basis of physical explanation. To use once again the just mentioned example: a kinetic system may be explained causally by referring to theoretical constructs like *forces*.

Hence, physical explanation can be described as a movement of the following kind: it starts in the range of *data*, swings over into the field of *symbolic construction*, and returns to *data* again:

$$\mathcal{D} \longrightarrow \mathcal{C} \longrightarrow \mathcal{D}$$

More generally, we can characterize the activity of the scientists, be it explanation, prediction, or conceptual exploration, as an oscillating movement between the area of *data* and the area of *symbolic construction*. Following Margenau we want to call it *swing* – more philosophically oriented minds may even call it a *hermeneutic circle*. Somewhat more explicitly, this last expression may be justified as follows: we start with a limited and partial "Vorverständnis" of the *data*. Then the *data* are embedded and represented in an interpretatory framework of *symbolic construction* which may be used to yield a fuller understanding of the *data*.

The purpose of the *swing* is manifold: it may be used to obtain new information about the *data*, or to provide an explanation for them, or even to excite new conceptual research concerning the *symbolic constructs*.[12]

For the purpose of this paper we want to emphasize the following features of *data*, *symbolic constructs*, and *swing* :

Relativity: whether an entity *e* belongs to the realm of *data* or to the realm of *symbolic constructs* is context-depending: in one context *e* may be considered as a *datum*, in another context as a *symbolic construct*.

Plurality: the realm of *symbolic constructs* is not uniquely determined by the realm of *data*: there may be several rival (incompatible) *symbolic constructs* for one and the same *data*.

Usefulness, Economy and Explicitness: the *symbolic constructs* are constructed with certain purposes, they have to be useful. This usually implies economy and explicitness. To invoke an example of N. Cartwright taken from the empirical sciences:

> " A good theory aims to cover a wide variety of phenomena with as few principles as possible. ... It is a poor theory that requires a new Hamiltonian for each new physical circumstance. The great explanatory power of quantum mechanics comes from its ability to deploy a small number of well-understood Hamiltonians to cover a broad range of cases, and not from its ability to match each situation one-to-one with a new mathematical representation. That way of proceeding would be crazy" (Cartwright 1983, pp. 144/145).

symbolic constructs in mathematical theories have to have the same or at least similar virtues. The merits of Poincaré's theory of fundamental groups reside in the fact it is possible to cope with a wide range of seemingly disparate phenomena through the concept of fundamental groups. The fact that each manifold has a fundamental group is in itself only of limited interest. In order to be useful, one must be able to effectively calculate this construct. And, equally important, it must turn out that the fundamental group reflects important traits of the spaces.

Quite generally, in mathematics *symbolic constructs* represent or express *the possible* contexts of *data*. According to Peirce's *Pragmatic Maxim* the possible contexts of *data* may be identified with their *meaning:*

> "Consider what effects, that might conceivably have practical bearings, we conceive the objects of our conception to have. Then our conception of these effects is the whole of our conception of the object" (Peirce 1931/35 [5.402]).

This can be spelt as follows: the "practical bearings" a mathematical object might conceivably have are its (functional) relations to other mathematical objects. For example, the meaning of a particular manifold reveals itself in its possible relations with other manifolds or, more generally, with other mathematical entities and, as we will sketch in the following section for manifolds, an important part of these possible relations can be described in the framework of group theory. On the other hand, *category theory* can be considered as the realization of a kind of functional *Pragmatic Maxim* according to which the meaning of a mathematical object is to be seen in its relations to other mathematical objects. *category theory* does not care much about the objects of mathematical theories so that it is formally possible to eliminate them in favor of relations. Thus, from a category-theoretical point of view, an entity gets its meaning *not* by its underlying substance, i.e. its underlying sets of members or its internal properties, but through its external relations to other

12 It might be interesting to note that for perception some authors have proposed an analogous cycle between the "objects" perceived *(data)* and the "schemata" *(constructs)* structuring perception, cf. (Neisser 1976, p. 20f). In general, in cognitive science there is more or less agreement on the assertion that perception uses a mixture of "bottom-up" *(data ⇒ constructs)* and "top-down" *(constructs ⇒ data)* processes to realize the perceived object or situation, cf. (Goldman 1986, p. 187).

objects of the category. Impressive examples of this fact are provided by the so called "arrow style" definitions. A simple case is the definition of the kernel of a group homomorphism. A more spectacular one is Lawvere's category theoretical reconstruction of the concepts of set and element through the concept of a subobject classifier which is a pure "arrow-style" concept and makes no use of any set theoretical concept, cf. (Goldblatt 1979).

3. *The Case of Mathematics: Data, Symbolic Constructs, and Swing in Group Theory*

The structural analogy between empirical and mathematical knowledge which we want to exhibit consists in the fact that the structure of mathematical theories and research can be explained in terms of *data, symbolic constructs,* and *swing* as well. Our example is a tiny part of group theory.

3.1. *Groups as Symbolic Constructs*

We would like to explain the idea of groups as *symbolic constructs* through the example of the *fundamental group of manifolds* introduced by Poincaré at the turn of the century.

The *fundamental group* of a manifold is the prototype of a *symbolic construct* which plays a central role for the solution of the following general topological problem:

Given two manifolds M_1 and M_2 the problem is to prove that they are unrelated in the sense that there does not exist a continous map $f: M_1 \longrightarrow M_2$. A famous case in this context is *Brouwer's fixpoint theorem* that can be considered as a paradigmatic example.

In order to solve such a problem it often turns out to be convenient, even necessary, to replace the manifolds themselves by appropiate *symbolic constructs,* in our simple case by their fundamental groups $\pi_1(M_1)$ and $\pi_1(M_2)$, and to convert the geometrical problem into an algebraic one. That is to say, one shows that there does not exist an algebraic map, i.e. a group homomorphism, between the fundamental groups $\pi_1(M_1)$ and $\pi_1(M_2)$. Then, due to the correspondence between manifolds and their fundamental groups, one can conclude that there does not exist a continous map between the manifolds themselves.[13]

13 Compared with the *Data* manifold the definition of the *Symbolic Construct* "fundamental group" is somewhat complicated. One may know the *Data* quite well, i.e. one may be able to identify or to distinguish them quite easily without being able to calculate their fundamental groups. In the case of *Brouwer's Fixpoint Theorem* it is easy to calculate the fundamental groups of the manifolds involved.

Thus, in the same way as for empirical theories, in the case of Poincaré's theory of the fundamental group we have the two constituents: the *level of data* – the manifolds – and the *level of symbolic constructs* – the fundamental groups of the manifolds. Further, we have the *swing* from the manifolds and their geometric relations to the groups and their algebraic relations back to the manifolds. To put it bluntly: the proof of *Brouwer's fixpoint theorem* is essentially nothing but the *swing*.

Let us finally consider briefly the topic of plurality. In the later development of topology it turned out that the fundamental groups are by no means the only *symbolic constructs* for manifolds. A huge generalization is provided by the so called higher homotopy groups π_i (M) ($i \geq 2$) for manifolds. The fundamental group is only the first of an infinite series of grouplike *symbolic constructs.*[14]

3.2. *Symbolic Constructs of Groups*

Usually new mathematical entities first appear as *symbolic constructs*[15] However, once such a construct has been established in the mathematical discourse, it rather quickly takes the role of a datum *for* which further *symbolic constructs* are built. In the case of groups we want to consider the *symbolic construct* of characters related to a group G.

A character of a group G is a representation of G into the complex numbers $\mathbb{C}$, i.e. a function from G into $\mathbb{C}$ with certain special properties we need not consider in any detail. The *symbolic construct* of the set of characters $\mathbb{C}$ (G) forms a vectorspace and serves as a model of the group G, and can be used as a powerful tool to investigate its structure. A well known example is the *theorem of Burnside* which deals with the solvability of certain finite groups. For quite a long time the only available proof of this theorem made crucial use of the *symbolic construct* of characters, although the statement of *Burnside's theorem* can be formulated quite independently from this concept.

Again we are confronted with the typical *swing* : Starting from the level of *data,* in our case they are the set (or category) of groups, we move into the

14 It is remarkable that these higher homotopy groups are not known completely even for quite "elementary" manifolds like the 2-dimensional sphere.

15 Groups as *Symbolic Constructs* are introduced for the first time by Lagrange and Euler in the course of their investigations on quadratic forms and potential rests, cf. (Dieudonné 1976), (Wussing 1969).

field of *symbolic construction*, the set (or category) of vectorspaces, and return to the realm of groups again.

Here too we can witness a pluralism of *symbolic constructs*. Problems of groups in general, and problems of solvability in particular, *need not* be treated by characters. Much later, a proof of *Burnside's theorem* was found which does not depend on the *symbolic construct* of characters.

4. Formal Tools: Category Theory

The representational reconstruction of mathematical theories sketched here for a small part of group theory has the advantage that a lot of work for its formal elaboration has already been done. To nobody's surprise it turns out that one can use the tools of *category theory* for the representational reconstruction of mathematical theories.

As is well known, the *fundamental group of Poincaré* can be considered as a *functor* $\mathcal{P}$ from the *category of manifolds* to the *category of groups:*

$$\mathcal{P}: \mathcal{M} \longrightarrow \mathcal{G}$$

In a similar vein one can interprete the correlation of groups and characters as a functor $\mathcal{C}$ from the *category of groups* $\mathcal{G}$ to the *category of vectorspaces* $\mathcal{V}$:

$$\mathcal{C}: \mathcal{G} \longrightarrow \mathcal{V}$$

The fact that $\mathcal{P}$ and $\mathcal{C}$ are functors can be considered as a precise formulation of the requirement that there exists a permanent and extensive correspondence between *data* and *constructs*, i.e. relations between *data* have to correspond (at least partially) to relations between *constructs* and vice versa.[16]

According to category theory, *group theory* is a whole bunch of grouplike representations, or better, a net of grouplike representations. This net is to be conceptualized as an open net, i.e. as a net which is extended in different ways and directions: new knots are added, and new connections between already existing knots are constructed, and so on...

5. Conclusion.

The representational approach might prove to be especially useful in coping with some weaknesses which the philosophy of mathematics traditionally suffered from, i.e. the exaggerated inclination to stick to attitudes like *elementarism, fundamentalism*, and *ontologism*.

Elementarism claims that it is sufficient to understand elementary mathematical theories as the arithmetic of natural numbers, thus gaining complete philosophical insight into the whole enterprise of mathematics. The concepts of category theory used in the representational approach are *technical* concepts which are rather immune to an utterly elementarist approach. Thus the representational approach is closer to the conception of the working scientist.

Fundamentalism maintains that the most important task of the philosophy of mathematics is to provide an absolutely secure foundation of mathematics. *Fundamentalism* localizes the philosophical problematic of mathematics in its foundations, be it logic, set theory, or any other foundational discipline. This leads to a strictly *hierarchical* and *global* organization of mathematical knowledge. Contrary to this (inadequate) conception of mathematics the representational approach favors a local, more flexible organization of mathematics as a *non-hierarchical* net of interrelated units.

Finally, *ontologism* concentrates rather exclusively on global questions like the following: What "mode of being" pertains to mathematical entities? To this question there exists a whole spectrum of answers. On the one end, we find a solid platonism which assigns mathematical objects to an exclusive area whereas, on the other end, we find eliminative conceptions which try to reinterpret the domain of mathematics nominalistically or physicalistically. They hope to get rid of the ontological problems of mathematics once and for all. Between those extreme positions we find constructivist approaches which put various constraints on mathematical entities. In principle we do not think these approaches to be wrong but we would like to remark that these global ontological claims of philosophy towards mathematics appear, from a naturalistic perspective, to be rather strong. They do not correspond to anything in philosophy of empirical sciences. The question "What is an electron?" sounds strange while philosophers of mathematics frequently ask "What is a number?" and similar questions.[17]

The *relative*, *context-dependent* characterization of mathematical entities as *data* and *symbolic constructs*, however, leads the representational approach to a *distributed* and *variegated* ontology of the objects of mathematical discourse. One cannot assume that the mode of being of entities belonging to different levels of conceptualization is the same and remains fixed once and for all.

16 What this mean exactly depends on the specifics of the case in question, but it can be spelt out in the framework of a *general theory of meaningful representation*, cf. (Mundy 1986).

17 Often, this approach is related to *fundamentalism* in localizing the ontological question in a fundamental domain, e.g. in arithmetics of natural numbers or set theory.

Rather, the ontological status of mathematical entities is a *variable* and depends on the development of mathematics.

As Margenau pointed out already 50 years ago a simplistic *Yes/No* attitude concerning the existence of scientific entities is inappropriate:

> "Do masses, electrons, atoms, magnetic field strengths, etc., *exist* ? Nothing is more surprising indeed than the fact that ... most of us still expect an answer to this question in terms of yes and no. ... Almost every term that has come under scientific scruting has lost its initially absolute significance and acquired a range of meaning of which even the boundaries are often variable. Apparently the word *to be* has escaped this process" (Margenau 1935, p. 164).

Even if we assume for the sake of the argument that the isolated claim "π exists" makes sense, one must be a very hardheaded platonist to maintain that π exists *in the same way* as, say, an entity like an "extraordinary cohomology theory" – a rather complex entity which nevertheless can be made the object of study.

It should be noted that the problem of ontological diversification is not a special problem of the empirical sciences. It concerns the social sciences and common sense knowledge as well: Does it really make sense to maintain that objects like "San Sebastian", "the European Community", or "the development of capitalism in the 20th century" exist in the same way as the notorious apple tree in the philosopher's garden?

Thus, taking into account the structural analogy between empirical and mathematical theories as representations, we maintain that mathematics *shares* with empirical science this feature of a variegated ontological status of its objects. That is, we maintain that Margenau's remarks concerning the *blurred ontology* status of scientific entities also apply to mathematics. Regrettably this *common ground* of mathematics and empirical science is rarely recognized by philosophers of mathematics. In the realm of physics, for example, philosophers of mathematics often take a robust realism for granted, thereby accepting an artificial wall between mathematics and empirical science. The representational structure of empirical *and* mathematical theories, however, renders it dubious that there is a sharp and clear ontological distinction between the physical and the mathematical. At this point the representational approach meets Quine's holistic account of science, cf. (Resnik 1988).

Hence, taking into consideration this common feature of mathematics and empirical science it is evident that the philosophy of mathematics cannot restrict itself to the "foundations" or the "elements" of mathematics. It has to

pay attention to the ongoing process of mathematical progress.[18]

Stressing the *common ground* of empirical and mathematical cognition as it is exhibited in the representational structure of mathematical and empirical theories, the philosophy of mathematics could tap the sources of present day philosophy of empirical science. There, problems of ontology are dealt with in a far more liberal and sophisticated way than in the Logical Empiricism of the thirties. Nowadays, the ontological diversification of scientific entities is widely recognized, as is witnessed by the talk of causality, possibilities, or counterfactuals, even in quarters which consider themselves to belong to the analytic tradition. In particular, one may consider the ongoing debate on realism as an effort to overcome the far too simple dualism of "does exist" versus "does not exist".[19]

Up to now, however, philosophy of mathematics seems to have ignored this debate. In order to gain contact again with the rest of philosophy of science, philosophy of mathematics has to give up the pernicious concentration on such idiosyncratic "–isms" as *elementarism, fundamentalism* and *ontologism*.

We hope the representational approach can be considered to be a small step towards this goal.

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18 The analogue claim for the philosophy of empirical sciences has been widely recognized. We plea that it should be recognized in the case of mathematics too.

19 Cf. (Churchland / Hooker 1985).

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Reduction and Explanation: Science vs. Mathematics

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1. Introduction

The aim of this essay is to compare the explanatory roles which the notion of reduction has played in the philosophy science, on the one hand, and in the philosophy of mathematics, on the other, and to argue that in that respect there is a crucial difference between the two fields of study. Thus, for instance, in the philosophy of science the notions of explanation and reduction have been extensively discussed, even in formal frameworks, but there exist few successful and exact applications of the notions to actual theories, and, furthermore, any two philosophers of science seem to think differently about the question of how the notions should be reconstructed. On the other hand, philosophers of mathematics and mathematicians have been successful in defining and applying various exact notions of reduction (or interpretation), but they have not seriously studied the questions of explanation and understanding.

There are several reasons why reduction has been extensively discussed in the philosophy of science and in science itself. For example, it is often assumed that behind an observed, or otherwise given, phenomenon there exists a more fundamental reality to which the phenomenon can be reduced and which can be employed to explain and understand it. Secondly, it is usually thought that scientific research is not feasible if it cannot be reduced to methods which in some sense are objective and reliable. Philosophy and science abound in historical examples and consequences of these ontological and methodological forms of reductionism; such are radical empiricism and rationalism, the idea that the axiomatic method is reliable (these examples represent methodological reductionism deriving from the struggle for epistemic certainty), reductive materialism and idealism, the discussion concerning the reduction of biology to physics (which, in turn, represents ontological reductionism), discoveries of elementary particles (which are a consequence of a kind of ontological reductionism), etc.