

# Making Theorem-Proving in Modal Logic Easy

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## 1 Introduction

In an appendix, Mendelson (1964, p. 259) gives an account of a formulation of a system in which Karl Schütte (1951) proves consistency of number theory. A formulation of classical sentential logic can be abstracted from this, identical with the system discussed later by Belnap and Anderson (1959) and Hunter (1971, pp. 125-34). The rules of inference of this system all share the feature that the premise(s) are shorter than the conclusion. Accordingly, the rule variously called cut or modus ponens is not among them. Backwards application of the rules to any formula generates a tree the branches of which contain successively shorter formulas. Eventually, at the tip of each branch there is a formula which is either an axiom or no rule can be applied to it to shorten it further, and the tree is complete. If the formula at the tip of every branch is an axiom, then the original formula is a theorem and the tree has generated a proof. Otherwise, the formula is not a theorem.

Since theorem-proving becomes a very simple mechanical process in this system, it has obvious pedagogic virtues. It is worthwhile investigating whether this feature can be extended to a system of normal modal logic based on the same underlying system of classical sentential logic. The results of such an investigation are presented here. In the next section, the system of classical sentential logic is outlined and some of its general features briefly discussed. This system is extended to a system **K** of normal modal logic in the following section, which is developed to illustrate the ease with which theorems can be demonstrated. The effect of adding the cut rule, yielding a system **K**<sup>+</sup>, is discussed in section 4.

## 2 Classical Sentential Logic

The system **S** of sentential logic has negation and disjunction as the primitive logical constants and the vocabulary consists of the signs:

$$\sim, \vee, (, ),$$

together with an infinite list of sentence letters,  $A_1, A_2, A_3, A_4, \dots$ . Well-formed formulas are defined as usual and other connectives are introduced metalogically in the usual way. Brackets are sometimes eliminated by the use of dots in standard fashion. A disjunction  $\phi_1 \vee \phi_2 \vee \dots \vee \phi_n$  is called *an expression of the law of excluded middle* if one of its disjuncts is the negation of another. The single axiom schema is

(Sax) Any expression of the law of excluded middle,

i.e. axioms are of the kind

$$(i) \quad \psi \vee \phi \vee \sim\phi,$$

where there may be no disjunct  $\psi$ , and the order of the disjuncts is immaterial. There are two rules:

$$(S1) \quad \frac{\psi \vee \phi}{\psi \vee \sim\sim\phi}$$

$$(S2) \quad \frac{\psi \vee \sim\phi \quad \psi \vee \sim\chi}{\psi \vee \sim(\phi \vee \chi)} .$$

Again, it is understood that there may be no disjunct (side formula)  $\psi$ , and the order of the disjuncts is immaterial.

To illustrate the application of the axioms and rules, consider the proof of

$$(\phi \supset \psi) \wedge (\psi \supset \chi) \cdot \supset \cdot \phi \supset \chi.$$

In primitive notation, this is  $\sim\sim(\sim(\sim\phi \vee \psi) \vee \sim(\sim\psi \vee \chi)) \vee \sim\phi \vee \chi$ , which can be demonstrated by the following proof tree:

$$\begin{array}{c} \text{Axiom} \\ \hline \sim\psi \vee \sim\phi \vee \chi \vee \psi \\ \hline \sim\psi \vee \sim\phi \vee \chi \vee \sim\sim\psi \quad S1 \\ | \end{array}$$

$$\begin{array}{c}
 \text{Axiom} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \text{Axiom} \\
 \frac{\varphi \vee \sim(\sim\psi \vee \chi) \vee \sim\varphi \vee \chi}{\sim\sim\varphi \vee \sim(\sim\psi \vee \chi) \vee \sim\varphi \vee \chi} \text{ S1} \qquad \qquad \qquad \frac{\quad | \quad \sim\psi \vee \sim\varphi \vee \chi \vee \sim\chi}{\sim\psi \vee \sim\varphi \vee \chi \vee \sim(\sim\psi \vee \chi)} \text{ S2} \\
 \hline
 \frac{\sim(\sim\varphi \vee \psi) \vee \sim(\sim\psi \vee \chi) \vee \sim\varphi \vee \chi}{\sim\sim(\sim(\sim\varphi \vee \psi) \vee \sim(\sim\psi \vee \chi)) \vee \sim\varphi \vee \chi} \text{ S1}
 \end{array}$$

The tree is easily established by starting at the bottom with the formula to be proved, and applying the rules backwards as appropriate until an axiom is reached on every branch. (For typographical reasons, the left-hand branch of the uppermost fork in the right-hand branch has been split into two parts, and the vertical line, ‘|’ indicates the join.) Because the premises of the rules are shorter than their conclusions, backwards application will eventually lead to a disjunction in which all the disjuncts are either atomic formulas or negations of atomic formulas if an axiom is not reached beforehand, when no further application of the rules is possible. So decidability is straightforward.

System **S** is easily shown to be sound and semantically complete (Hunter 1971, pp. 130-2). But to catch all valid consequences as deductions, **S** is extended to system **S**<sup>+</sup> by addition of the cut rule:

$$\text{(S3)} \quad \frac{\varphi \quad \sim\varphi \vee \psi}{\psi} .$$

No further theorems are introduced by the addition of this rule:

**Theorem 1** *The theses of **S**<sup>+</sup> are the same as those of **S**.*

*Proof.* Clearly, all theses of **S** are included in those of **S**<sup>+</sup>, since its axioms and rules of inference are part of **S**<sup>+</sup>. To show the converse, suppose  $\vdash_{\mathbf{S}^+} \varphi$ . Construction of the appropriate truth table shows that S3 preserves truth, so the conclusion of any application of S3 is always a valid consequence of its premises. Accordingly, if S3 is applied in the proof of  $\varphi$ , it applies only to axioms, which are clearly valid, or their valid consequences, and introduces only valid consequences of these, so  $\varphi$  is valid. But by the completeness of **S**,  $\vdash_{\mathbf{S}} \varphi$ , and  $\varphi$  is a thesis of **S**. Q.E.D.

A simple corollary of this is that although the cut rule does not hold as a general rule of inference for **S** (to be used in deductions), it does hold in the form of a rule of proof:

**Corollary 1.1** *Cut holds as a rule of proof in **S**; i.e.*

$$\text{If } \vdash_{\mathbf{S}} \varphi \text{ and } \vdash_{\mathbf{S}} \varphi \supset \psi, \text{ then } \vdash_{\mathbf{S}} \psi.$$

*Proof.* Assume  $\vdash_{\mathbf{S}} \varphi$  and  $\vdash_{\mathbf{S}} \varphi \supset \psi$ . Then  $\models \varphi$  and  $\models \varphi \supset \psi$  by soundness of **S**, from which it follows that  $\models \psi$ . Then  $\vdash_{\mathbf{S}} \psi$  by semantic completeness of **S**. Q.E.D.

From this corollary it is a straightforward matter to establish

**Corollary 1.2** *The substitutivity of provable equivalents holds in  $\mathbf{S}$ ; i.e.*

$$\vdash_{\mathbf{S}} \varphi \equiv \varphi' \Rightarrow \vdash_{\mathbf{S}} \psi \equiv \psi',$$

where  $\psi'$  is the result of replacing an occurrence of  $\varphi$  in  $\psi$  by  $\varphi'$ .

But corollary 1.1 is not sufficient for the extended completeness of  $\mathbf{S}^+$ , which requires rule S3:

**Theorem 2** *Extended completeness of  $\mathbf{S}^+$*

$$\text{If } \psi_1, \dots, \psi_n \models \varphi, \text{ then } \psi_1, \dots, \psi_n \vdash_{\mathbf{S}^+} \varphi$$

*Proof.*  $\psi_1, \dots, \psi_n \models \varphi$  holds iff  $\models \psi_1 \wedge \dots \wedge \psi_n \supset \varphi$  by application of truth tables. So, by the completeness of  $\mathbf{S}^+$ ,  $\vdash_{\mathbf{S}^+} \psi_1 \wedge \dots \wedge \psi_n \supset \varphi$ . The latter implies  $\psi_1, \dots, \psi_n \vdash_{\mathbf{S}^+} \varphi$  by  $n$  applications of S3 ensuring

$$\begin{aligned} \varphi_{j+1}, \dots, \varphi_n \vdash_{\mathbf{S}^+} \sim \varphi_1 \vee \dots \vee \sim \varphi_j \vee \psi &\Rightarrow \\ \varphi_j, \varphi_{j+1}, \dots, \varphi_n \vdash_{\mathbf{S}^+} \sim \varphi_1 \vee \dots \vee \sim \varphi_{j-1} \vee \psi. \end{aligned}$$

### 3 The Modal System $\mathbf{K}$

The language is now extended with the addition of the necessity operator,  $\Box$ . Well-formed formulas are defined as usual and the possibility operator,  $\Diamond$ , is defined as  $\sim \Box \sim$ . The modal system  $\mathbf{K}$  is defined by adding to the axioms and rules of  $\mathbf{S}$  all axioms of the kind

$$(ii) \quad \sim(\varphi \vee \sim\varphi) \vee \Box(\psi \vee \sim\psi)$$

together with the rule

$$(RKN) \quad \frac{\varphi \vee \sim\psi_1 \vee \dots \vee \sim\psi_n}{\Box\varphi \vee \sim\Box\psi_1 \vee \dots \vee \sim\Box\psi_n}, n \geq 0.$$

As before, the order of the disjuncts is not significant.

Clearly, the rule of necessitation is simply a special case of RKN with  $n = 0$ :

$$(RN) \quad \vdash \varphi \Rightarrow \vdash \Box\varphi$$

This makes it clear why an additional side formula cannot be added as a disjunct  $\chi$  to the premise and conclusion of RKN thus:

$$\frac{\chi \vee \varphi \vee \sim\psi_1 \vee \dots \vee \sim\psi_n}{\chi \vee \Box\varphi \vee \sim\Box\psi_1 \vee \dots \vee \sim\Box\psi_n}, n \geq 0.$$

By analogy with the rule of necessitation, this would sanction the rule

$$\vdash \chi \vee \phi \Rightarrow \vdash \chi \vee \Box\phi,$$

which in the special case that  $\chi$  is  $\sim\phi$  would lead to the trivialisation  $\vdash \phi \supset \Box\phi$ . Although it would certainly trivialise modal logic if this general schema were to hold, there are, however, special cases that are not objectionable. Theorems such as expressions of the law of excluded middle are certainly necessary if anything is, and special cases of the form  $\phi \vee \sim\phi \supset \Box(\phi \vee \sim\phi)$  should be reckoned among the theorems. Because they cannot be accommodated by extending the rule RKN with side formulas, they have been added for completeness as additional axioms.

In order to illustrate the ease with which theorems can be proved with this formulation, the system is developed to show how standard results are established. We begin with the derived rule

$$(RKP) \quad \frac{\sim\phi \vee \psi_1 \vee \dots \vee \psi_n}{\sim\Diamond\phi \vee \Diamond\psi_1 \vee \dots \vee \Diamond\psi_n} \quad n \geq 0.$$

*Proof*

$$\frac{\frac{\frac{\sim\phi \vee \psi_1 \vee \dots \vee \psi_n}{n \times S1}}{\sim\phi \vee \sim\sim\psi_1 \vee \dots \vee \sim\sim\psi_n} \text{ RKN}}{\Box\sim\phi \vee \sim\Box\sim\psi_1 \vee \dots \vee \sim\Box\sim\psi_n} \text{ S1}}{\sim\sim\Box\sim\phi \vee \sim\Box\sim\psi_1 \vee \dots \vee \sim\Box\sim\psi_n} \text{ S1}}{\sim\Diamond\phi \vee \Diamond\psi_1 \vee \dots \vee \Diamond\psi_n} \text{ Df } \Diamond$$

The standard theorems of basic normal modal logic are easily proved. Note that the proofs don't call for specific strategies relying on previously proved theorems. Each theorem is proved from scratch, directly from the axioms.

$$Kt1 \quad \Box(\phi \wedge \psi) \equiv. \Box\phi \wedge \Box\psi$$

*Proof* First the left-to-right implication, remembering that the disjuncts in RKN can be taken in any order.

$$\frac{\frac{\text{Axiom}}{\sim\phi \vee \sim\psi \vee \phi} \text{ RKN, S1}}{\sim\Box\sim(\sim\phi \vee \sim\psi) \vee \Box\phi} \quad \frac{\frac{\text{Axiom}}{\sim\phi \vee \sim\psi \vee \psi} \text{ RKN, S1}}{\sim\Box\sim(\sim\phi \vee \sim\psi) \vee \Box\psi} \text{ S2, S1}}{\sim\Box\sim(\sim\phi \vee \sim\psi) \vee \sim(\sim\Box\phi \vee \sim\Box\psi)}$$

Then the right-to-left implication:

$$\begin{array}{c}
 \begin{array}{c} \textit{Axiom} \\ \sim\phi \vee \sim\psi \vee \sim\sim\phi \end{array} \qquad \begin{array}{c} \textit{Axiom} \\ \sim\phi \vee \sim\psi \vee \sim\sim\psi \end{array} \\
 \hline
 \sim\phi \vee \sim\psi \vee \sim(\sim\phi \vee \sim\psi) \qquad \textit{S2, S1} \\
 \hline
 \sim\Box\phi \vee \sim\Box\psi \vee \Box\sim(\sim\phi \vee \sim\psi) \qquad \textit{RKN} \\
 \hline
 \sim\sim(\sim\Box\phi \vee \sim\Box\psi) \vee \Box\sim(\sim\phi \vee \sim\psi) \qquad \textit{S1}
 \end{array}$$

The corresponding principle of distribution of possibility over disjunction,

$$\text{Kt2} \quad \Diamond(\phi \vee \psi) \equiv. \Diamond\phi \vee \Diamond\psi,$$

can be similarly proved in two parts:

*Proof* The left-to-right implication:

$$\begin{array}{c}
 \begin{array}{c} \textit{Axiom} \\ \sim\phi \vee \phi \vee \psi \end{array} \qquad \begin{array}{c} \textit{Axiom} \\ \sim\psi \vee \phi \vee \psi \end{array} \\
 \hline
 \sim(\phi \vee \psi) \vee \phi \vee \psi \qquad \textit{S2} \\
 \hline
 \sim\Diamond(\phi \vee \psi) \vee \Diamond\phi \vee \Diamond\psi \qquad \textit{RKP}
 \end{array}$$

The right-to-left implication:

$$\begin{array}{c}
 \begin{array}{c} \textit{Axiom} \\ \sim\phi \vee \phi \vee \psi \end{array} \qquad \begin{array}{c} \textit{Axiom} \\ \sim\psi \vee \phi \vee \psi \end{array} \\
 \hline
 \sim\Diamond\phi \vee \Diamond(\phi \vee \psi) \qquad \textit{RKP} \qquad \sim\Diamond\psi \vee \Diamond(\phi \vee \psi) \qquad \textit{RKP} \\
 \hline
 \sim(\Diamond\phi \vee \Diamond\psi) \vee \Diamond(\phi \vee \psi) \qquad \textit{S2}
 \end{array}$$

Necessity distributes over implication as an implication which can't be strengthened to an equivalence:

$$\text{Kt3} \quad \Box(\phi \supset \psi) \supset. \Box\phi \supset \Box\psi$$

*Proof*

$$\begin{array}{c}
 \begin{array}{c} \textit{Axiom} \\ \phi \vee \sim\phi \vee \psi \end{array} \qquad \begin{array}{c} \textit{Axiom} \\ \sim\psi \vee \sim\phi \vee \psi \end{array} \\
 \hline
 \sim(\sim\phi \vee \psi) \vee \sim\phi \vee \psi \qquad \textit{S2, S1} \\
 \hline
 \sim\Box(\sim\phi \vee \psi) \vee \sim\Box\phi \vee \Box\psi \qquad \textit{RKN}
 \end{array}$$

Attempting to prove the converse of Kt3 results in a tree with a branch tipped by a formula which is not an axiom and cannot be shortened by further

application of any of the rules:

$$\frac{\frac{\frac{\text{Axiom}}{\sim A_2 \vee \sim A_1 \vee A_2}}{\sim \Box A_2 \vee \Box(\sim A_1 \vee A_2)} \text{RKN}}{\Box A_1 \vee \Box(\sim A_1 \vee A_2) \quad \sim \Box A_2 \vee \Box(\sim A_1 \vee A_2)} \text{S2, S1}}{\sim(\sim \Box A_1 \vee \Box A_2) \vee \Box(\sim A_1 \vee A_2)}$$

Necessity doesn't distribute over disjunction with an equivalence, but just in accordance with the implication

$$\text{Kt4} \quad \Box \varphi \vee \Box \psi \supset \Box(\varphi \vee \psi)$$

*Proof*

$$\frac{\frac{\frac{\text{Axiom}}{\sim \varphi \vee \varphi \vee \psi}}{\sim \Box \varphi \vee \Box(\varphi \vee \psi)} \text{RKN}}{\frac{\frac{\frac{\text{Axiom}}{\sim \psi \vee \varphi \vee \psi}}{\sim \Box \psi \vee \Box(\varphi \vee \psi)} \text{RKN}}{\sim(\Box \varphi \vee \Box \psi) \vee \Box(\varphi \vee \psi)} \text{S2}}}$$

Although the converse of Kt4 doesn't hold, what we do have is

$$\text{Kt5} \quad \Box(\varphi \vee \psi) \supset \Diamond \varphi \vee \Box \psi$$

*Proof*

$$\frac{\frac{\frac{\text{Axiom}}{\sim \varphi \vee \varphi \vee \psi}}{\sim(\varphi \vee \psi) \vee \varphi \vee \psi} \text{S2}}{\sim \Box(\varphi \vee \psi) \vee \sim \Box \sim \varphi \vee \Box \psi} \text{RKN, S1}}$$

Correspondingly, possibility distributes over conjunction in the form of the implication

$$\text{Kt6} \quad \Diamond(\varphi \wedge \psi) \supset \Diamond \varphi \wedge \Diamond \psi$$

*Proof*

$$\frac{\frac{\frac{\text{Axiom}}{\sim \varphi \vee \sim \psi \vee \varphi}}{\sim \Diamond \sim(\sim \varphi \vee \sim \psi) \vee \Diamond \varphi} \text{RKP, S1}}{\frac{\frac{\frac{\text{Axiom}}{\sim \varphi \vee \sim \psi \vee \psi}}{\sim \Diamond \sim(\sim \varphi \vee \sim \psi) \vee \Diamond \psi} \text{RKP, S1}}{\sim \Diamond \sim(\sim \varphi \vee \sim \psi) \vee \sim(\sim \Diamond \varphi \vee \sim \Diamond \psi)} \text{S2, S1}}}$$

The converse of Kt6 doesn't hold, but what we do have is  $\Box \varphi \wedge \Diamond \psi \supset \Diamond(\varphi \wedge \psi)$ , or

$$\text{Kt7} \quad \sim\Diamond\varphi \wedge \Diamond\psi \supset \Diamond(\sim\varphi \wedge \psi)$$

*Proof*

$$\frac{\frac{\text{Axiom}}{\varphi \vee \sim\psi \vee \sim\varphi} \quad \frac{\text{Axiom}}{\varphi \vee \sim\psi \vee \sim\sim\psi}}{\varphi \vee \sim\psi \vee \sim(\sim\sim\varphi \vee \sim\psi)} \text{S2}}{\sim\sim(\sim\sim\Diamond\varphi \vee \sim\Diamond\psi) \vee \Diamond\sim(\sim\sim\varphi \vee \sim\psi)} \text{S1, RKP}$$

Together with negation, possibility and necessity are related to one another much as disjunction and conjunction are related by DeMorgan's laws. A few theorems will put us in a position to introduce a general relationship of duality.

$$\text{Kt8} \quad \Box\varphi \equiv \sim\Diamond\sim\varphi$$

*Proof*

$$\frac{\frac{\text{Axiom}}{\sim\varphi \vee \sim\sim\varphi}}{\sim\sim\Box\sim\varphi \vee \sim\Box\sim\varphi} \text{S1, RKN} \quad \frac{\frac{\text{Axiom}}{\sim\varphi \vee \varphi}}{\sim\sim\Box\sim\varphi \vee \sim\Box\sim\varphi} \text{S1, RKN, S1}$$

We now define  $\Box^n$  as a string of n necessity operators:

**Df**  $\Box^n$   $\Box^0\varphi$  is  $\varphi$

$$\Box^n\varphi \text{ is } \Box\Box^{n-1}\varphi.$$

A string of n possibility operators,  $\Diamond^n$ , is defined analogously.

$$\text{Kt9} \quad \sim\Diamond^n\varphi \equiv \Box^n\sim\varphi$$

*Proof* Basis: where n = 0, the theorem reduces to  $\sim\varphi \equiv \sim\varphi$ .

Induction step. The left-to-right and right-to-left implications are proved in turn.

$$\frac{\frac{\text{Axiom}}{\varphi \vee \sim\varphi}}{\Diamond\varphi \vee \Box\sim\varphi} \text{Df.}\Diamond, \text{RKN, S1}}{\vdots} \frac{\frac{\Diamond^{n-2}\varphi \vee \Box^{n-2}\sim\varphi}{\sim\Box\sim\Diamond^{n-2}\varphi \vee \Box\Box^{n-2}\sim\varphi} \text{RKN, S1}}{\Diamond^{n-1}\varphi \vee \Box^{n-1}\sim\varphi} \text{Df.}\Diamond} \frac{\sim\sim\sim\Box\sim\Diamond^{n-1}\varphi \vee \Box\Box^{n-1}\sim\varphi} \text{S1, RKN, S1}$$



$$\begin{array}{c}
 \textit{Induction hypothesis} \\
 \frac{\Box^{n-1}\sim\varphi \supset \sim\Diamond^{n-1}\varphi}{\sim\Box^{n-1}\sim\varphi \vee \sim\Diamond^{n-1}\varphi} \textit{Df.}\supset \\
 \frac{\sim\Box^{n-1}\sim\varphi \vee \sim\Diamond^{n-1}\varphi}{\sim\Box\Box^{n-1}\sim\varphi \vee \Box\sim\Diamond^{n-1}\varphi} \textit{RKN} \\
 \frac{\sim\Box\Box^{n-1}\sim\varphi \vee \Box\sim\Diamond^{n-1}\varphi}{\sim\Box\Box^{n-1}\sim\varphi \vee \sim\sim\Box\sim\Diamond^{n-1}\varphi} \textit{S1}
 \end{array}$$

And analogously,

$$\text{Kt10} \quad \sim\Box^n\varphi \equiv \Diamond^n\sim\varphi.$$

*Proof Basis:* where  $n = 0$ , the theorem reduces to  $\sim\varphi \equiv \sim\varphi$ .

*Induction step.* The left-to-right and right-to-left implications are proved in turn.

$$\begin{array}{c}
 \textit{Axiom} \\
 \frac{\varphi \vee \sim\varphi}{\Box\varphi \vee \sim\Box\sim\varphi} \textit{RKN, S1} \\
 \vdots \\
 \frac{\Box\Box^{n-2}\varphi \vee \sim\Box\sim\Diamond^{n-2}\sim\varphi}{\Box^{n-1}\varphi \vee \Diamond^{n-1}\sim\varphi} \textit{Df.}\Diamond \\
 \frac{\Box^{n-1}\varphi \vee \Diamond^{n-1}\sim\varphi}{\Box\Box^{n-1}\varphi \vee \sim\Box\sim\Diamond^{n-1}\sim\varphi} \textit{RKN, S1} \\
 \frac{\Box\Box^{n-1}\varphi \vee \sim\Box\sim\Diamond^{n-1}\sim\varphi}{\sim\sim\Box\Box^{n-1}\varphi \vee \sim\Box\sim\Diamond^{n-1}\sim\varphi} \textit{S1}
 \end{array}$$

$$\begin{array}{c}
 \textit{Induction hypothesis} \\
 \frac{\Diamond^{n-1}\sim\varphi \supset \sim\Box^{n-1}\varphi}{\sim\Diamond^{n-1}\sim\varphi \vee \sim\Box^{n-1}\varphi} \textit{Df.}\supset \\
 \frac{\sim\Diamond^{n-1}\sim\varphi \vee \sim\Box^{n-1}\varphi}{\Box\sim\Diamond^{n-1}\sim\varphi \vee \sim\Box\Box^{n-1}\varphi} \textit{RKN} \\
 \frac{\Box\sim\Diamond^{n-1}\sim\varphi \vee \sim\Box\Box^{n-1}\varphi}{\sim\sim\Box\sim\Diamond^{n-1}\sim\varphi \vee \sim\Box\Box^{n-1}\varphi} \textit{S1}
 \end{array}$$

### Def Modality

A modality is simply a sequence of monadic operators  $\sim$ ,  $\Diamond$ , and  $\Box$ , including the empty sequence. An affirmative modality is a modality containing no negation.

**Def** Dual of an Affirmative Modality

The dual,  $\mathcal{M}^d$ , of an affirmative modality  $\mathcal{M}$  is the affirmative modality obtained by uniformly substituting each  $\diamond$  in  $\mathcal{M}$  by  $\square$  and each  $\square$  in  $\mathcal{M}$  by  $\diamond$

Examples of modalities are  $\sim$ ;  $\square \sim$ ;  $\diamond \diamond \square$ ;  $\diamond$ ;  $\square \square \square \diamond \sim \square$ ; – (the empty sequence); etc. Now, a negation sign attached to one end of a modality can be moved to the other end, provided all the  $\square \tilde{\bullet}$ s and  $\diamond \tilde{\bullet}$ s are interchanged for one another, to produce a new formula logically equivalent with the original one. Suppose, for example, that the negation is on the left of the modality. By Kt9 or Kt10, depending on whether the negation sign is followed by  $\diamond^n$  or  $\square^n$ , the negation can be moved n places to the right and the modality replaced by its corresponding dual,  $\square^n$  or  $\diamond^n$ , respectively. The negation will then either (i) be to the left of an atomic formula, or (ii) precede another affirmative modality or (iii) precede another negation. In case (ii), either Kt9 or Kt10 is applicable as before, and the negation can be moved further to the right. In case (iii) the double negation can be eliminated. Either then negation now precedes an atomic formula, or the process can be repeated, and since formulas are of finite length, the negation will eventually precede an atomic formula. We thus have the

**Duality theorem** *If  $\mathcal{M}$  is an affirmative modality and  $\mathcal{M}^d$  is its dual, then*

$$\sim \mathcal{M} \varphi \equiv \mathcal{M}^d \sim \varphi.$$

Note, furthermore, that  $\mathcal{M}^{dd} \varphi \equiv \mathcal{M} \varphi$ .

## 4 Cut and Completeness

Consider the system  $\mathbf{K}^+$  obtained by adding the cut rule S3,

$$(S3) \quad \frac{\varphi \quad \sim \varphi \vee \psi}{\psi},$$

to  $\mathbf{K}$  as defined in the last section. The additional axiom schema (ii) of system  $\mathbf{K}$ , over and above the excluded middle axiom schema, is redundant in  $\mathbf{K}^+$ , being provable from the ‘paradox of implication’,  $\varphi \supset \psi \supset \varphi$  (which is just an instance of excluded middle) and S3, thus:

$$\frac{\begin{array}{l} \text{Axiom} \\ \psi \vee \sim \psi \\ \hline \square(\psi \vee \sim \psi) \end{array} \quad \text{RN} \quad \begin{array}{l} \text{Axiom} \\ \sim \square(\psi \vee \sim \psi) \vee \sim(\varphi \vee \sim \varphi) \vee \square(\psi \vee \sim \psi) \\ \hline \end{array}}{\sim(\varphi \vee \sim \varphi) \vee \square(\psi \vee \sim \psi)} \quad S3$$

Since  $\mathbf{S}$  is complete, the system  $\mathbf{K}^+$  includes the standard system of basic normal modal logic  $\mathbf{K}$  because it includes  $\text{Kt3}$  as a theorem as well as having necessitation and the cut rule as rules of inference. It includes, for example, system  $\mathbf{K}$  as defined by Hughes and Cresswell (1996, pp. 24-5). But it will transpire that the theorems of  $\mathbf{K}$  are exactly the same as those of  $\mathbf{K}^+$ , so that  $\mathbf{K}$  does capture all of basic normal modal logic. Using the standard algebraic technique, it will be shown that a normal modal algebra can be defined which is characteristic of system  $\mathbf{K}$  in the sense that it *verifies* exactly the theorems of  $\mathbf{K}$ . Since all the formulas verified by the algebra are provable in  $\mathbf{K}$ , and  $\mathbf{K}^+$  differs from  $\mathbf{K}$  only in the additional rule  $\text{S3}$ , which can be shown to preserve verifiability and so does not introduce non-verified formulas as theorems of  $\mathbf{K}^+$ , it follows that  $\mathbf{K}^+$  has the same theorems as  $\mathbf{K}$ .

The correspondence between the properties of the sentential operators  $\sim$  and  $\vee$  and the operators  $-$  and  $\cup$  of Boolean algebra is well known. Boolean algebras can be extended with an additional operator corresponding the necessity operator as follows:

**Def.** Normal Modal Algebra

Where  $\langle B, -, \cup \rangle$  is a Boolean algebra, a modal algebra is a quadruple  $\langle B, -, \cup, \ell \rangle$  which is closed under the operation  $\ell$  (i.e. if  $a \in B$ , then  $\ell a \in B$ ) and satisfies the following conditions:

- M1      $\ell 1 = 1$
- M2      $\ell(a \cap b) = \ell a \cap \ell b$

In order to consider how this corresponds to features of modal logic, a definition of a Boolean algebra is given and some standard results stated without proof for future reference.

**Def.** Boolean Algebra

A Boolean algebra is a triple  $\langle B, -, \cup \rangle$ , where  $B$  is a set of at least two elements closed under the operations  $-$  and  $\cup$  (i.e. if  $a, b \in B$ , then  $-a \in B$  and  $a \cup b \in B$ ), complying with the following definitions and axioms:

- Def 1     $1 = a \cup -a$
- Def 0     $0 = -1$
- Def  $\cap$     $a \cap b = -(-a \cup -b)$
- Def  $\subseteq$     $a \subseteq b$  iff  $a \cup b = b$
- B1      $a \cup b = b \cup a$
- B2      $a \cup (b \cup c) = (a \cup b) \cup c$
- B3     For all  $a$  and  $b$ , if for some  $c$ ,  $a \cup -b = c \cup -c$ , then  $a \cup b = b$
- B4     For all  $a$  and  $b$ , if  $a \cup b = b$ , then for all  $c$ ,  $a \cup -b = c \cup -c$

Some straightforward properties of Boolean algebras that will be used in the sequel are simply stated in the following theorem:

**Theorem 3** *If  $\langle B, -, \cup \rangle$  is a Boolean algebra, then for any  $a, b, c \in B$ ,*

- |     |                                     |     |  |
|-----|-------------------------------------|-----|--|
| (a) | $a \cup 1 = 1$                      | (f) | $-(a \cup b) = -a \cap -b$                       |
| (b) | $a \cup 0 = a$                      | (g) | $-(a \cap b) = -a \cup -b$                       |
| (c) | $a \cap 1 = a$                      | (h) | $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ |
| (d) | $--a = a$                           | (i) | $a \cup b = b$ iff $a \cap b = a$                |
| (e) | $-a \cup b = 1$ iff $b \subseteq a$ |     | (equivalent definition of $\subseteq$ )          |

An important property of modal algebras is established in the following theorem:

**Theorem 4** *If  $\langle B, -, \cup, \ell \rangle$  is a modal algebra, then for any  $a, b \in B$ ,*

$$a \subseteq b \Rightarrow \ell a \subseteq \ell b$$

*Proof* Suppose  $a \subseteq b$ , i.e.  $a \cap b = a$  [equivalent definition of  $\subseteq$ ]

$$\Rightarrow \ell a \cap \ell b = \ell(a \cap b) = \ell a \quad [\text{M2, supposition}]$$

$$\Leftrightarrow \ell a \subseteq \ell b \quad [\text{equivalent definition of } \subseteq]$$

Q.E.D.

Modal algebra can be related to modal logic by an assignment,  $V$ , which associates each sentence letter,  $A_i$ , with an element,  $a_i$  of the set  $B$ , and assigns an element of  $B$  to each compound formula in accordance with the general rules:

1. If  $V(\varphi) = a$ , then  $V(\sim\varphi) = -a$  and  $V(\Box\varphi) = \ell a$ .
2. If  $V(\varphi) = a$  and  $V(\psi) = b$ , then  $V(\varphi \vee \psi) = a \cup b$ .

A normal modal algebra is then said to *verify* a formula  $\varphi$  if, for every assignment  $V$ ,  $V(\varphi) = 1$ . A formula  $\varphi$  is *falsified* by a normal modal algebra if for some assignment  $V$ ,  $V(\varphi) \neq 1$ . We now show that every normal modal algebra verifies every theorem of system **K**.

**Theorem 5** *Every normal modal algebra verifies every theorem of system **K**.*

*Proof* We show that the axioms are verified and the rules in inference preserve the property of being verified.

Suppose that a normal modal algebra falsifies some expression of the law of excluded middle,  $\varphi \vee \sim\varphi \vee \psi$ . Then there is an assignment  $V$  for which  $V(\varphi \vee \sim\varphi \vee \psi) \neq 1$ . Thus, where  $V(\varphi) = a$  and  $V(\psi) = b$ ,  $(a \cup -a) \cup b \neq 1$ . But by Def. 1, this implies  $1 \cup b \neq 1$ , which contradicts theorem 3 (a).

Suppose that a normal modal algebra falsifies some instance of axiom (ii), so that for some assignment  $V$ ,  $V(\sim(\varphi \vee \sim\varphi) \vee \Box(\psi \vee \sim\psi)) \neq 1$ . Then, if  $V(\varphi) = a$  and  $V(\psi) = b$ ,

$$\begin{aligned}
 1 &\neq \neg(a \cup \neg a) \cup \ell(b \cup \neg b) \\
 &= \neg 1 \cup \ell 1 && \text{[Def. 1]} \\
 &= 0 \cup 1 && \text{[Def. 0, M1]} \\
 &= 1 && \text{[Theorem 3 (a)]}
 \end{aligned}$$

which is a contradiction.

Suppose that the rule of inference S1 does not preserve the property of being verified. Then for some formulas  $\varphi$  and  $\psi$ , and assignment  $V$ ,  $V(\varphi \vee \psi) = 1$  and  $V(\sim\sim\varphi \vee \psi) \neq 1$ . Thus, where  $V(\varphi) = a$  and  $V(\psi) = b$ ,  $a \cup b = 1$  and  $\neg\neg a \cup b \neq 1$ . But by theorem 3 (d), the latter reduces to  $a \cup b \neq 1$ , contradicting the former.

Suppose that the rule of inference S2 does not preserve the property of being verified. Then for some formulas  $\varphi$ ,  $\psi$  and  $\chi$ , and assignment  $V$ ,  $V(\psi \vee \sim\varphi) = 1$ ,  $V(\psi \vee \sim\chi) = 1$  and  $V((\psi \vee \sim(\varphi \vee \chi)) \neq 1$ . Thus, where  $V(\varphi) = a$ ,  $V(\psi) = b$  and  $V(\chi) = c$ ,  $b \cup \neg a = 1$ ,  $b \cup \neg c = 1$  and

$$\begin{aligned}
 1 &\neq b \cup \neg(a \cup c) \\
 &= b \cup (\neg a \cap \neg c) && \text{[Theorem 3 (f)]} \\
 &= (b \cup \neg a) \cap (b \cup \neg c) && \text{[Theorem 3 (h)]} \\
 &= 1 \cap 1 && \text{[Given } b \cup \neg a = 1 \text{ and } b \cup \neg c = 1\text{]} \\
 &= 1 && \text{[Theorem 3 (c)]}
 \end{aligned}$$

which is a contradiction.

To show that the rule of inference RKN preserves the property of being verified, suppose, for some formulas  $\varphi$ ,  $\psi_1, \dots, \psi_n$  and assignment  $V$ , that  $V(\varphi \vee \sim\psi_1 \vee \dots \vee \sim\psi_n) = 1$ . Thus, where  $V(\varphi) = a$ ,  $V(\psi_1) = b_1$  and  $\dots$  and  $V(\psi_n) = b_n$ , then  $a \cup \neg b_1 \cup \dots \cup \neg b_n = 1$ . We want to show that  $(\ell a \cup \neg \ell b_1 \cup \dots \cup \neg \ell b_n) = 1$ , so that  $V(\Box\varphi \vee \sim\Box\psi_1 \vee \dots \vee \sim\Box\psi_n) = 1$ . Now

$$\begin{aligned}
 1 &= a \cup \neg b_1 \cup \dots \cup \neg b_n = a \cup \neg(b_1 \cap \dots \cap b_n) && \text{[3 (f)]} \\
 \Leftrightarrow &a \subseteq (b_1 \cap \dots \cap b_n) && \text{[B1, 3 (e)]} \\
 \Rightarrow &\ell a \subseteq \ell(b_1 \cap \dots \cap b_n) && \text{[Th 4]} \\
 \Leftrightarrow &\ell a \cup \ell(b_1 \cap \dots \cap b_n) = \ell(b_1 \cap \dots \cap b_n) && \text{[Def } \subseteq\text{]} \\
 \Rightarrow &\ell a \cup \neg \ell(b_1 \cap \dots \cap b_n) = 1 && \text{[B4]} \\
 \Leftrightarrow &\ell a \cup \neg(\ell b_1 \cap \dots \cap \ell b_n) = 1 && \text{[M2]} \\
 \Leftrightarrow &\ell a \cup \neg \ell b_1 \cup \dots \cup \neg \ell b_n = 1 && \text{[3 (g)]}
 \end{aligned}$$

Accordingly,  $V(\Box\varphi \vee \sim\Box\psi_1 \vee \dots \vee \sim\Box\psi_n) = 1$ , and RKN preserves the property of being verified. Q.E.D.

The next step in the argument is to define a specific normal modal algebra which is *characteristic* of system  $\mathbf{K}$ , i.e. which verifies *exactly* the theorems of system  $\mathbf{K}$  and no others. For this purpose, a normal modal algebra  $\langle B, -, \cup, \ell \rangle$  is constructed in which the elements of  $B$  are equivalence classes of formulas provably equivalent in  $\mathbf{K}$ . The class of formulas equivalent with

$\varphi, \{\psi : \vdash_{\mathbf{K}} \psi \equiv \varphi\}$ , is denoted by  $|\varphi|$ . The operations of the modal algebra are then defined by laying it down that  $-|\varphi|$  is  $|\sim\varphi|$ ,  $\ell|\varphi|$  is  $|\Box\varphi|$  and  $|\varphi| \cup |\psi|$  is  $|\varphi \vee \psi|$ .

Any two theorems  $\varphi, \psi$  are provably equivalent. For  $\vdash_{\mathbf{K}} \varphi$ , which implies  $\vdash_{\mathbf{K}} \sim\psi \vee \varphi$ , and  $\vdash_{\mathbf{K}} \psi$ , which implies  $\vdash_{\mathbf{K}} \sim\varphi \vee \psi$ , therefore jointly imply  $\vdash_{\mathbf{K}} \sim\psi \vee \varphi \wedge \sim\varphi \vee \psi$ , and thus  $\vdash_{\mathbf{K}} \varphi \equiv \psi$ . All theorems are therefore members of  $|A_1 \vee \sim A_1|$ , which will be the unit element, 1, since this is defined as  $a \cup -a$ . The null element, 0, which is defined as  $-1$ , will be  $|\sim(A_1 \vee \sim A_1)|$ .

**Theorem 6** *The structure just defined is a normal modal algebra characteristic of  $\mathbf{K}$ .*

*Proof* To show that this is, indeed, a normal modal algebra, it must satisfy the conditions B1-4 and M1-2. For B1, note that  $\vdash_{\mathbf{K}} (\varphi \vee \psi) \equiv (\psi \vee \varphi)$ . Accordingly,  $\psi \vee \varphi \in |\varphi \vee \psi|$  and  $|\varphi \vee \psi| \in |\psi \vee \varphi|$ , and so  $|\varphi \vee \psi| = |\psi \vee \varphi|$ . Thus, where  $a$  is  $|\varphi|$  and  $b$  is  $|\psi|$ ,  $a \cup b = b \cup a$ .

For B2, where  $a$  is  $|\varphi|$ ,  $b$  is  $|\psi|$  and  $c$  is  $|\chi|$ , we have  $\vdash_{\mathbf{K}} \varphi \vee (\psi \vee \chi) \equiv (\varphi \vee \psi) \vee \chi$ . Thus,  $|\varphi \vee (\psi \vee \chi)| = |(\varphi \vee \psi) \vee \chi|$ , so that  $a \cup (b \cup c) = (a \cup b) \cup c$ .

For B3, where  $a$  is  $|\varphi|$ ,  $b$  is  $|\psi|$  and  $c$  is  $|\chi|$ ,

$$\vdash_{\mathbf{K}} (\varphi \vee \sim\psi \equiv \chi \vee \sim\chi) \supset (\varphi \vee \psi \equiv \psi) \quad [\text{Theorem of } \mathbf{S}]$$

$$\vdash_{\mathbf{K}} (\varphi \vee \sim\psi \equiv \chi \vee \sim\chi) \Rightarrow \vdash_{\mathbf{K}} (\varphi \vee \psi \equiv \psi) \quad [\text{Corollary 1.1}]$$

$$\therefore a \cup -b = c \cup -c \Rightarrow a \cup b = b.$$

For B4, suppose  $a \cup b = b$ , which means that, where  $a$  is  $|\varphi|$  and  $b$  is  $|\psi|$ , that  $|\varphi \vee \psi| = |\psi|$ . Thus,  $\vdash_{\mathbf{K}} \varphi \vee \psi \equiv \psi$ . Now consider  $\varphi \vee \sim\psi$ . From  $\vdash_{\mathbf{K}} \varphi \vee \psi \equiv \psi$  it follows by the substitution of provable equivalents (Corollary 1.2) that

$$\begin{aligned} & \vdash_{\mathbf{K}} \varphi \vee \sim\psi \equiv \varphi \vee \sim(\varphi \vee \psi) \\ \Rightarrow & \vdash_{\mathbf{K}} \varphi \vee \sim\psi \equiv \varphi \vee (\sim\varphi \wedge \sim\psi) \\ \Rightarrow & \vdash_{\mathbf{K}} \varphi \vee \sim\psi \equiv (\varphi \vee \sim\varphi) \wedge (\varphi \vee \sim\psi) \\ \Rightarrow & \vdash_{\mathbf{K}} \varphi \vee \sim\psi \equiv \varphi \vee \sim\psi \\ \Rightarrow & |\varphi \vee \sim\psi| = |\varphi \vee \sim\varphi| \\ \Rightarrow & |\varphi \vee \sim\psi| = |\chi \vee \sim\chi|, \quad \text{for any formula } \chi \\ \Rightarrow & a \cup -b = c \cup -c, \quad \text{where } c \text{ is } |\chi|. \end{aligned}$$

For M1, since 1 is  $|A_1 \vee \sim A_1|$ , then  $\ell 1$  is  $|\Box(A_1 \vee \sim A_1)|$ . Now since  $\vdash_{\mathbf{K}} \varphi \supset A_1 \vee \sim A_1$ , for any formulas  $\varphi$ , then in particular,  $\vdash_{\mathbf{K}} \Box(A_1 \vee \sim A_1) \supset A_1 \vee \sim A_1$ . The converse is an instance of the axioms schema (ii). Accordingly,  $\vdash_{\mathbf{K}} A_1 \vee \sim A_1 \equiv \Box(A_1 \vee \sim A_1)$ , and so  $|A_1 \vee \sim A_1| = |\Box(A_1 \vee \sim A_1)|$ . Thus,  $1 = \ell 1$ .

For M2,  $\vdash_{\mathbf{K}} \Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$ , so that  $|\Box(\varphi \wedge \psi)| = |\Box\varphi \wedge \Box\psi|$ , and  $\ell(a \cap b) = \ell a \cap \ell b$ .

The structure is, therefore, a normal modal algebra, and so verifies all the theorems of  $\mathbf{K}$ . To show that it falsifies all the non-theorems, suppose  $\varphi$  is such a non-theorem, and let  $V(\varphi) = |\varphi|$ . Now if  $|\varphi|$  were 1, so that  $|\varphi| = |A_1 \vee \sim A_1|$ , then it would hold that  $\vdash_{\mathbf{K}} \varphi \equiv A_1 \vee \sim A_1$ . But then in particular a proof tree of the right-to-left implication,  $\sim(A_1 \vee \sim A_1) \vee \varphi$ , could not be a proof of  $\sim(A_1 \vee \sim A_1)$  together with an arbitrary disjunct, and so must be a proof of  $\varphi$  with an arbitrary disjunct, contradicting the assumption that  $\varphi$  is not a theorem. Accordingly, any verified formula is a theorem of  $\mathbf{K}$ . Q.E.D.

**Theorem 7** *The theses of  $\mathbf{K}^+$  are the same as those of  $\mathbf{K}$ .*

*Proof.* Clearly, all theses of  $\mathbf{K}$  are included in those of  $\mathbf{K}^+$  since its axioms and rules of inference are part of  $\mathbf{K}^+$ . To show the converse, consider the rule S3, which is the only additional feature distinguishing  $\mathbf{K}^+$  from  $\mathbf{K}$ . Rule S3 preserves the property of being verified (by a normal modal algebra). For suppose, where  $V(\varphi) = a$  and  $V(\psi) = b$ , that  $V(\varphi) = 1$  and  $V(\sim\varphi \vee \psi) = 1$ . Then  $\sim a \cup b = 1$ , and since  $V(\varphi) = a = 1$ , this becomes  $0 \cup b = 1$ . But  $0 \cup b = b$  by theorem 3 (b), so that  $1 = b = V(\psi)$ , and S3 preserves the property of being verified. Accordingly, any application of S3 in proving a thesis of  $\mathbf{S}^+$  will be applied to verified axioms or their verified consequences, and so will yield only verified consequences. But all and only the verified formulas are theses of  $\mathbf{K}$ . Therefore, the theses of  $\mathbf{K}^+$  are theses of  $\mathbf{K}$ . Q.E.D.

## 5 Conclusion

The metalogical results of the last section establish the claim made in the introduction, that  $\mathbf{K}$  includes all the theses of  $\mathbf{K}^+$ . But the main point has been to present a formulation of the basic system of normal modal logic generally known as  $\mathbf{K}$  which allows the theorems to be very easily proved. I hope this will be found pedagogically more enlightening than treatments relying entirely on argument by reductio ad absurdum or which obscure the distinction between the model theory and the proof theory in an attempt to avoid the perceived difficulties of manipulating traditional Hilbert-style axiomatisations of modal logic.

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