# The Dream of Recapture

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As a response to the semantic and logical paradoxes, theorists often reject some principles of classical logic. However, classical logic is entangled with mathematics, and giving up mathematics is too high a price to pay, even for nonclassical theorists. The so-called *recapture theorems* come to the rescue. When reasoning with concepts such as truth/class membership/property instantiation,[1] if ones is interested in consequences of the theory that only contain mathematical vocabulary, *nothing is lost by reasoning in the nonclassical framework*. It is shown below that this claim is highly misleading, if not simply false. Under natural assumptions, recapture claims are incorrect.

## 1. Recapture

We restrict our attention to recapture claims involving the so-called Strong Kleene logic (**K3**) originating in Kleene (1952). Minor modifications to the claims and proofs below yield failures of recapture for the closely related logics of First-Degree Entailment **FDE** and Logic of Paradox **LP**.

Informal versions of the recapture idea take various forms in the literature – see for instance (Field, 2008, p.101), or (Hjortland, 2017, 652). A nice and formally precise presentation of recapture claims is contained in Beall (2013). He proves in particular that

(RECAPTURE) $\qquad$ If $\Gamma \vDash_{\mathbf{CL}} \Delta$, then $\mathrm{e}(\Delta) \cup \Gamma \vDash_{\mathbf{K3}} \Delta$,

where:

- $\Gamma$ and $\Delta$ are sets of formulae of a first-order language, so that $\vDash_{\mathbf{CL}}$ and $\vDash_{\mathbf{K3}}$ are, respectively, the standard first-order consequence relations for classical logic and Strong Kleene;
- $\mathrm{e}(\Delta)$ is the "completeness set" for $\Delta$, i.e. the set of formulae of form

$$\forall \vec{v}(P\vec{v} \vee \neg P\vec{v})$$

[1]These are examples of concepts that are taken to satisfy naive rules such as the naive truth schema and naive comprehension, and that therefore are compatible with a solution to paradox cast in the logics considered below. Other notions of similar kind can be added to the list.

for all atomic formulae $P\vec{t}$ in $\Delta$ (with $\vec{u} := u_1, \dots, u_n$).

Recapture seems to tell us that, after all, there is no deep difference between classical logic and **K3** when it comes to specific consequences of an accepted set of assumptions $\Gamma$, and in particular those consequences whose primitive concepts satisfy excluded middle. Mathematical concepts are taken to belong to this category. Therefore, it would appear that no mathematics is lost when moving from classical logic to **K3**.

## 2. Failures of Recapture

Beall shows that recapture obtains when $\Gamma$ is taken to be a set of *formulae* closed under purely logical rules. We now show that a natural reformulation recapture, arguably more faithful to its application to the paradoxes, is bound to fail. For simplicity, we reason about a unary truth predicate $\mathrm{Tr}$, but other notions such as property application or class membership would work as well.

Let's start with the language of arithmetic $\mathscr{L}_{\mathbb{N}}$,[2] playing the role of our background language featuring basic operations needed for syntax and self-reference. Let also $\mathscr{L}_{\mathrm{Tr}} := \mathscr{L}_{\mathbb{N}} \cup \{\mathrm{Tr}\}$. Since nonclassical theories of truth in **K3** can be properly formulated only by means of rules of inference, we formulate our theories in sequent calculi. $\mathbf{PA}^{-}$ denotes the finite set containing sequents of form $\Rightarrow A$, for $A$ a recursive equation for the primitive symbols of $\mathscr{L}_{\mathbb{N}}$, and the initial sequents for identity:

$$\Rightarrow t = t, \qquad\qquad s = t, A(s) \Rightarrow A(t),$$

for $s, t$ terms of $\mathscr{L}_{\mathbb{N}}$ and $A(v) \in \mathscr{L}_{\mathrm{Tr}}$. These initial sequents guarantee that formulae of $\mathscr{L}_{\mathbb{N}}$ behave classically, i.e. they satisfy excluded middle. **PA** is obtained by adding to $\mathbf{PA}^{-}$ the induction rule

$$\frac{\Gamma, A(x) \Rightarrow A(x+1), \Delta}{\Gamma, A(\overline{0}) \Rightarrow A(t), \Delta}$$

for $A(v) \in \mathscr{L}_{\mathrm{Tr}}$ and $\Gamma, \Delta$ finite sets of formulae of $\mathscr{L}_{\mathrm{Tr}}$. This completes the description of the non-logical component of our background theory of syntax.

The purely logical systems we consider are standard sequent calculi for classical logic **CL** and **K3** – a sequent calculus for **K3** can be found for instance in (Nicolai and Stern, 2021, Appendix A). The principles characterizing truth will be the initial sequents

(pos-t) $\qquad A(x) \Rightarrow \mathrm{Tr}\ulcorner A(\dot{x})\urcorner, \qquad\qquad \mathrm{Tr}\ulcorner A(\dot{x})\urcorner \Rightarrow A(x),$

[2] For definiteness, we can take $\mathscr{L}_{\mathbb{N}}$ to be specified by the signature $\{\overline{0}, \overline{1}, \times, +\}$.

for $A(v)$ *positive*, that is where the truth predicate does not occur in the scope of an even number of negation signs. For the truth rules, a canonical formalization of syntax is assumed: the expression $\ulcorner A(\dot{x}) \urcorner$ stands for the result of substituting, within the code $\ulcorner A(v) \urcorner$ of $A(v)$, the variable $\ulcorner v \urcorner$ for the numeral of $x$. The rules are obviously sound (and therefore consistent) with respect to the Kripkean semantics from Kripke (1975). This is the case independently of whether one choose a classical or **K3**-based axiomatization of fixed-point semantics. The syntactic restriction on negation is motivated by the need to find uncontroversial principles that are sound both in classical and nonclassical logic.

The classical system resulting from the combination of **PA** and positive disquotation sequents is Halbach's theory of uniform positive disquotation **PUTB** (Halbach, 2014, Ch. 19.3). To properly state our results, it's convenient to employ the label **PUTB** for the arithmetical and truth sequents, and change only the underlying logical derivability relation. When writing **PUTB** $\vdash_S A$, we mean that the sequent $\Rightarrow A$ is derivable from the arithmetical and truth rules of **PUTB** together with the logical rules of $S$. We will also write $\mathbf{PUTB}_S$ to refer to the combination of the arithmetical and truth rules of **PUTB** with the logical system $S$. The expression $\mathrm{Con}(\mathbf{PUTB}_S)$ stands for the canonical consistency statement for the appropriate system.

Proposition 1. $\mathbf{PUTB} \vdash_{\mathbf{CL}} \mathrm{Con}(\mathbf{PUTB}_{\mathbf{K3}})$, *but*

$$\mathrm{e}(\mathrm{Con}(\mathbf{PUTB}_{\mathbf{K3}})) \cup \mathbf{PUTB} \nvdash_{\mathbf{K3}} \mathrm{Con}(\mathbf{PUTB}_{\mathbf{K3}}).$$

*Proof.* The claim after 'but' follows from Gödel's Second Incompleteness Theorem, since $\mathrm{Con}(\mathbf{PUTB}_{\mathbf{K3}})$ is a purely arithmetical sentence, and therefore

$$\mathbf{PUTB} \vdash_{\mathbf{K3}} \mathrm{e}(\mathrm{Con}(\mathbf{PUTB}_{\mathbf{K3}})).$$

Therefore, if the claim after 'but' held, $\mathbf{PUTB}_{\mathbf{K3}}$ would prove its own consistency. Since $\mathbf{PUTB}_{\mathbf{K3}}$ clearly satisfies the conditions for Gödel's second incompleteness theorem, this is impossible.

For the former claim, we appeal to well-known results in the study of Kripke-Feferman truth (Halbach and Horsten, 2006; Cantini, 1989; Feferman, 1991). The system $\mathbf{PUTB}_{\mathbf{K3}}$ is a sub-system of the system **PKF** from Halbach and Horsten (2006) formulated over the logic **K3** (in fact, **PKF** features *unrestricted* truth rules, not only for positive formulae). The proof theoretic analysis of **PKF** tells us that $\mathbf{PUTB}_{\mathbf{K3}}$ cannot prove more $\mathscr{L}_{\mathbb{N}}$-sentences than ramified analysis iterated up to any ordinal $\alpha < \omega^{\omega}$ or, equivalently, iterations of the Turing Jump up to any $\alpha < \omega^{\omega}$ (a.k.a. $\Pi^0_1$-$\mathbf{CA}^{<\omega^{\omega}}$).

By contrast, $\mathbf{PUTB}_{\mathbf{CL}}$ is much stronger. It proves the same arithmetical sentences as ramified analysis iterated up to any ordinal $\alpha < \varepsilon_0$ or, equivalenty, as $\Pi^0_1$-$\mathbf{CA}^{<\varepsilon_0}$

(or the system **KF** from Feferman (1991)). This means that there is a countable infinity of arithmetical statements, including a transfinite hierarchy of consistency statements, that are derivable in $\mathbf{PUTB}_{\mathbf{CL}}$ but not in $\mathbf{PUTB}_{\mathbf{K3}}$. In partcular, $\mathbf{PUTB} \vdash_{\mathbf{CL}} \mathrm{Con}(\mathbf{PUTB}_{\mathbf{K3}})$. *qed*

Let us now briefly relate Proposition 1 to RECAPTURE. First, Proposition 1 is a claim about provability. However, the adequacy of the logics **CL** and **K3** guarantees its equivalence with a claim about logical consequence. Secondly, and crucially, as stated Proposition 1 is not a direct counterexample to RECAPTURE, but a counterexample to a natural extension of it. In particular, **PUTB** is not only a set of formulae, but a collection of formulae, sequents, and rules. However, one should note that if the **K3**-theorist did not allow sequents and rules in the collection of their accepted assumptions, they would not even be able to *formulate* sound principles for truth and syntax: identity principles, induction, and truth rules, if formulated with a material **K3** conditional, will be unsound if not directly inconsistent.

## 3. Discussion

Proposition 1 tells us that the nonclassical theorist cannot rely on a general formulation of RECAPTURE: RECAPTURE fails once one fixes specific principles for syntax and truth. We discuss a few strategies that the **K3**-theorist may employ to react to these failures.

One may try to reject the counterexamples by rejecting T-POS. Perhaps the weakness of **K3** derives from the artificiality of the syntactic restriction on negation in the definition of **PUTB**. One would then need to reformulate RECAPTURE by specifying that the set $\Gamma$ of assumptions (inferences or formulae) needs to be not only sound, but also (philosophically) *complete* with respect to the **K3**-theorist's desiderata. For instance, one may want to require fully disquotational sequents of form $\mathrm{Tr}\ulcorner A(\dot{x})\urcorner \Leftrightarrow A(x)$ for all $A(v) \in \mathscr{L}_{\mathrm{Tr}}$. This would not work. If one replaces T-POS in **PUTB** with their unrestricted version, we would still obtain a failure of recapture. The closure of the unrestricted truth sequents under **CL** yields an inconsistent system, which proves, among other things, $0 = 1$. By contrast, the closure of unrestricted truth sequents under **K3** is again a sub-theory of **PKF**, and therefore consistent: $0 = 1$ cannot be a theorem.

A more plausible reaction is to embrace such failures, but notice that the recapture claim would hold – even for extensions of pure logic with truth and syntax – if all vocabulary involved in a proof behaved classically. This is equivalent to requiring, in the setting above, that only syntactic/arithmetical principles are employed in a proof. Perhaps what we should learn from recapture results is simply that logics such a **K3** require the addition of excluded middle across the board to collapse into classical logic? But in what sense this is "recapture"? And what is this telling us about our theories of paradox-breeding concepts? The advocate of classical logic is interested precisely in the entanglement of such concepts and purely mathematical concepts. Williamson (2018) cast doubt on the possibility of systematically distinguishing between mathematical and non-mathematical vocabulary, especially if one wants to preserve the role of applied mathematics in science. The truth predicate, for instance, is commonly used for *mathematical* purposes: to formulate more succinctly some theories in predicative analysis (Feferman, 1991), or to formulate natural representation systems for constructive ordinals (Beklemishev and Pakhomov, 2019). Other authors focus on the role of non-logical schemata such as induction, which is commonly applied outside pure mathematics (McGee, 2006; Halbach and Nicolai, 2018): inductive reasoning is crippled when one moves from **CL** to **K3**. The examples above suggest in fact that the task of distinguishing classical and nonclassical concepts is an all-or-nothing enterprise: as soon as *some* interaction is allowed, theorems of standard mathematics that are available classical theories quickly disappear in nonclassical ones. And there's no way of recapturing them in logics such as **K3**.[3]

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[3] There are ways to overcome these failures. Some specific logics avoids the kinds of counterexamples given as they can recover *all* classical theorems (Fischer et al., 2020), although they still fall short of some classical logical theorems. The addition of some additional nonlogical vocabulary in the nonclassical side may have the same effect, although this may not be a direct way of addressing the issue (Field, 2020).

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