COUNTEREXAMPLES TO SOME CHARACTERIZATIONS OF DILATION

MICHAEL NIELSEN AND RUSH T. STEWART

1. Introduction

Pedersen and Wheeler (2014) and Pedersen and Wheeler (2015) offer a wide-ranging and in-depth exploration of the phenomenon of dilation. We find that these studies raise many interesting and important points. However, purportedly general characterizations of dilation are reported in them that, unfortunately, admit counterexamples. The purpose of this note is to show in some detail that these characterization results are false.

We start by rehearsing the relevant definitions (Section 2) and some known results about dilation (Section 3). Section 4 addresses the results in Pedersen and Wheeler (2014), and Section 5 addresses the results in Pedersen and Wheeler (2015). We conclude, in Section 6, by pointing out where Pedersen and Wheeler's proofs go wrong.

2. Definitions

We use $(\Omega, \mathcal{F})$ to denote a measurable space. A probability measure P on $(\Omega, \mathcal{F})$ is a non-negative, finitely additive set function with domain $\mathcal{F}$ and $P(\Omega) = 1$. $\mathbb{P}$ denotes a set of probability measures on $(\Omega, \mathcal{F})$, and $\mathbb{P}_*$ denotes the weak*-closure of the convex hull of $\mathbb{P}$. The *lower probability* of and event A in $\mathbb{P}$ is $\underline{P}(A) = \inf\{P(A) : P \in \mathbb{P}\}$, and the *upper probability* of and event A in $\mathbb{P}$ is $\overline{P}(A) = \sup\{P(A) : P \in \mathbb{P}\}$. We call $(\Omega, \mathcal{F}, \mathbb{P}, \underline{P})$ a *lower probability space*.

If $P(E) > 0$, then the conditional probability of A given E is defined, as usual, by

$$P(A \mid E) = P(A \cap E)/P(E). \tag{1}$$

Upper and lower conditional probabilities are defined, respectively, by $\underline{P}(A \mid E) = \inf\{P(A \mid E) : P \in \mathbb{P}\}$ and $\overline{P}(A \mid E) = \sup\{P(A \mid E) : P \in \mathbb{P}\}$.

Let $\mathcal{E} = \{E_i : i \in I\}$ be a positive, measurable partition of Ω. Measurability of the partition means that $E \in \mathcal{F}$ for all $E \in \mathcal{E}$, while by "positive" we mean that $P(E) > 0$ for all $E \in \mathcal{E}$. Say that $\mathcal{E}$ *dilates* an event $A \in \mathcal{F}$ just in case, for each $E \in \mathcal{E}$

$$\underline{P}(A \mid E) < \underline{P}(A) \leq \overline{P}(A) < \overline{P}(A \mid E). \tag{2}$$

The next definitions are used in the alleged characterizing conditions for dilation. If $A, E \in \mathcal{F}$, let

$$S_P(A, E) = \frac{P(A \cap E)}{P(E)P(A)} \tag{3}$$

provided $P(A)P(E) > 0$. If $P(A)P(E) = 0$, then we can let $S_P(A, E) = 1$, as in Seidenfeld and Wasserman (1993), but we won't encounter that situation in this note. Now let

$$S_{\mathbb{P}}^-(A, E) = \{P \in \mathbb{P} : S_P(A, E) < 1\} \text{ and } S_{\mathbb{P}}^+(A, E) = \{P \in \mathbb{P} : S_P(A, E) > 1\}. \tag{4}$$

If $\epsilon > 0$, then let

$$\underline{\mathbb{P}}(A \mid E, \epsilon) = \{P \in \mathbb{P} : |P(A \mid E) - \underline{P}(A \mid E)| < \epsilon\} \tag{5}$$

and

$$\overline{\mathbb{P}}(A \mid E, \epsilon) = \{P \in \mathbb{P} : |P(A \mid E) - \overline{P}(A \mid E)| < \epsilon\}. \tag{6}$$

Finally, let $\mathbb{R}_+$ be the set of positive real numbers, and, if I is a set, let $\mathbb{R}_+^I$ be the set of functions from I into $\mathbb{R}_+$. We also call members of $\mathbb{R}_+^I$ I-sequences.

3. Some Known Results

The following proposition summarizes some known necessary and sufficient conditions for dilation. See Pedersen and Wheeler (2014) and Seidenfeld and Wasserman (1993).

Proposition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \underline{P})$ be a lower probability space, let $\mathcal{E} = \{E_i : i \in I\}$ be a positive, measurable partition of Ω, and let $A \in \mathcal{F}$. The following condition is sufficient for $\mathcal{E}$ to dilate A:*

(a) $\underline{\mathbb{P}}(A) \cap S_{\mathbb{P}}^-(A, E_i) \neq \emptyset$ *and* $\overline{\mathbb{P}}(A) \cap S_{\mathbb{P}}^+(A, E_i) \neq \emptyset$ *for all $i \in I$.*

And the following conditions are necessary for $\mathcal{E}$ to dilate A:

(b) $\underline{\mathbb{P}}(A \mid E_i) \subseteq S_{\mathbb{P}}^-(A, E_i)$ *and* $\overline{\mathbb{P}}(A \mid E_i) \subseteq S_{\mathbb{P}}^+(A, E_i)$ *for all $i \in I$.*

(c) *There is $(\epsilon_i)_{i \in I} \in \mathbb{R}_+^I$ such that for every $i \in I$:*

$$\underline{\mathbb{P}}(A \mid E_i, \epsilon_i) \subseteq S_{\mathbb{P}}^-(A, E_i) \quad and \quad \overline{\mathbb{P}}(A \mid E_i, \epsilon_i) \subseteq S_{\mathbb{P}}^+(A, E_i). \tag{7}$$

Remark 1. The proof that dilation implies (c) in Pedersen and Wheeler (2014) uses several concepts from functional analysis. But an elementary proof is possible. Indeed, if dilation occurs, then $\underline{P}(A \mid E_i) < \underline{P}(A)$ for all $i \in I$. Let $0 < \underline{\epsilon_i} < \underline{P}(A) - \underline{P}(A \mid E_i)$. If $P \in \underline{\mathbb{P}}(A \mid E_i, \underline{\epsilon_i})$, then $P(A \mid E_i) - \underline{P}(A \mid E_i) < \underline{\epsilon_i}$. Hence, $P(A \mid E_i) < \underline{P}(A) \leq P(A)$, which means that $P \in S_{\mathbb{P}}^-(A, E_i)$. An analogous proof, from the fact that dilation implies $\overline{P}(A) < \overline{P}(A \mid E_i)$ for all $i \in I$, yields, for each i, an $\overline{\epsilon_i} > 0$ such that $\overline{\mathbb{P}}(A \mid E_i, \overline{\epsilon_i}) \subseteq S_{\mathbb{P}}^+(A, E_i)$. We then let $\epsilon_i = \min(\underline{\epsilon_i}, \overline{\epsilon_i})$, and (c) is proved.

Proposition 1 lists two necessary conditions and one sufficient condition for dilation, but no jointly necessary and sufficient conditions. In Section 6, we will refer to a result due to Wasserman and Seidenfeld that provides a condition that is both necessary and sufficient for dilation in simple models (1994, Result 1). Our leading example, Example 1, is an instance of such a model.

4. Counterexamples to the 2014 Results

We will now record the main results from Pedersen and Wheeler (2014) and show that they are false.

Claim 1 (Pedersen and Wheeler, 2014, Proposition 5.1). Let $(\Omega, \mathcal{F}, \mathbb{P}, \underline{P})$ be a lower probability space such that $\mathbb{P}$ is weak*-closed and convex, let $\mathcal{E} = \{E_i : i \in I\}$ be a positive, measurable partition of Ω, and let $A \in \mathcal{F}$. Then the following are equivalent:

(1i) $\mathcal{E}$ dilates A.

(1ii) There is $(\epsilon_i)_{i \in I} \in \mathbb{R}_+^I$ such that for every $i \in I$:

$$\underline{\mathbb{P}}(A \mid E_i, \epsilon_i) \subseteq S_{\mathbb{P}}^-(A, E_i) \quad and \quad \overline{\mathbb{P}}(A \mid E_i, \epsilon_i) \subseteq S_{\mathbb{P}}^+(A, E_i). \tag{8}$$

(1iii) There are $(\underline{\epsilon_i})_{i \in I} \in \mathbb{R}_+^I$ and $(\overline{\epsilon_i})_{i \in I} \in \mathbb{R}_+^I$ such that for every $i \in I$:

$$\underline{\mathbb{P}}(A \mid E_i, \underline{\epsilon_i}) \subseteq S_{\mathbb{P}}^-(A, E_i) \quad and \quad \overline{\mathbb{P}}(A \mid E_i, \overline{\epsilon_i}) \subseteq S_{\mathbb{P}}^+(A, E_i). \tag{9}$$

Furthermore, each radius $\underline{\epsilon}_i$ may be chosen to be the unique positive minimum of $|P(A \mid E_i) - \underline{P}(A \mid E_i)|$ attained on $C_i^+ := \{P \in \mathbb{P} : S_P(A, E_i) \geq 1\}$, and similarly each radius $\overline{\epsilon}_i$ may be chosen to be the unique positive minimum of $|P(A \mid E_i) - \overline{P}(A \mid E_i)|$ attained on $C_i^- := \{P \in \mathbb{P} : S_P(A, E_i) \leq 1\}$.

On the most natural reading of Claim 1, it is a conjunction: it asserts that the statements (i)-(iii) are equivalent and conjoins to that assertion an additional assertion about how $\underline{\epsilon}_i$ and $\overline{\epsilon}_i$ *may* be chosen. To falsify this claim, then, it is enough to show that the assertion about the equivalence of (i)-(iii) is false. In personal communication, Pedersen and Wheeler have indicated that "the statement of our theorem in the *Erkenntnis* paper [Pedersen and Wheeler (2014)] is not as explicit as our generalization of this result, Theorem 1, in our ISIPTA 2015 paper [Pedersen and Wheeler (2015)]" and that "'Furthermore, each radius $\underline{\epsilon}_i$ may be chosen' ought to read 'Furthermore, each radius $\underline{\epsilon}_i$ *should* be chosen' [emphasis added]." But the replacement of "may" with "should" changes the content of Claim 1 significantly. Since the result is stated with "may," we think that it is worth showing that Claim 1 is false as stated. Later, we will show that Pedersen and Wheeler's result remains false even when "may" is replaced by "should", as is done more explicitly in their 2015 paper.

Before giving the counterexample, note that (1ii) and (1iii) are clearly equivalent. That (1ii) implies (1iii) is obvious, and the implication from (1iii) to (1ii) is proved by setting $\epsilon_i = \min(\underline{\epsilon}_i, \overline{\epsilon}_i)$ for all $i \in I$. Moreover, by Proposition 1, (1i) implies (1ii) whether or not $\mathbb{P}$ is weak*-closed and convex. To show that (1i)-(1iii) are *not* equivalent, then, we must show that (1iii) does *not* imply (1i):

Proposition 2. *In Claim 1, (1iii) is **not** sufficient for $\mathcal{E}$ to dilate A.*

In the proof of this proposition and throughout the rest of the note, we will use the following simple example.

Example 1. *Let P_0 and P_1 be probabilities on the discrete measurable space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ defined by*

	ω_1	ω_2	ω_3	ω_4
P_0	1/8	1/2	1/3	1/24
P_1	5/24	1/24	1/24	17/24

Let $\mathbb{P} = \{P_0, P_1\}$. Let $A = \{\omega_1, \omega_2\}$, $E_1 = \{\omega_1, \omega_3\}$, $E_2 = \{\omega_2, \omega_4\}$ and $\mathcal{E} = \{E_1, E_2\}$. △

Proof of Proposition 2. We show that $(\Omega, \mathscr{P}(\Omega), \mathbb{P}_*, \underline{P})$, $\mathcal{E}$, and A, as defined in Example 1 satisfy (1iii) but not (1i). We begin by noting that $\mathbb{P}_*$ is identical to the convex hull of $\{P_0, P_1\}$. For each $\alpha \in [0, 1]$ let

$$P_\alpha = (1 - \alpha)P_0 + \alpha P_1. \tag{10}$$

Due to the one-to-one correspondence between $[0, 1]$ and $\mathbb{P}_*$, it is useful to think of $\mathbb{P}_*$ as a set of probabilities indexed by the unit interval. Some straightforward calculations show that

$$P_\alpha(A) = \frac{5 - 3\alpha}{8}, \quad P_\alpha(A \mid E_1) = \frac{3 + 2\alpha}{11 - 5\alpha}, \quad P_\alpha(A \mid E_2) = \frac{12 - 11\alpha}{13 + 5\alpha}. \tag{11}$$

Since, $\underline{P}(A) = 1/4 < 3/11 = \underline{P}(A \mid E_1)$, it follows that $\mathcal{E}$ does not dilate A. Thus, (1i) does not hold. To finish, we must show that (1iii) holds. An easy way to see that (1iii) does indeed hold is to graph the curves in (11).
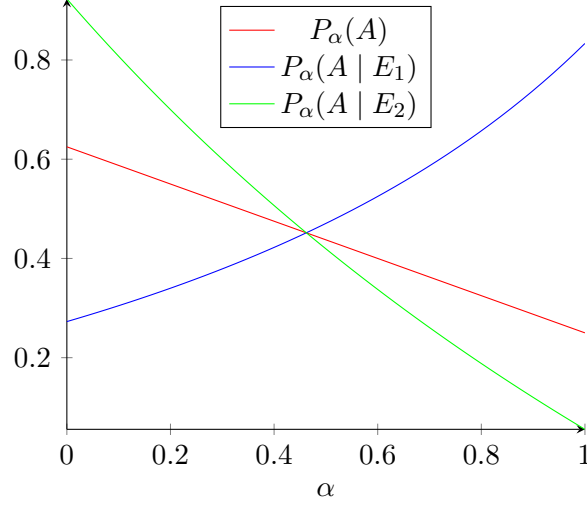
Observe that for each $E \in \{E_1, E_2\}$ all $P_\alpha \in \mathbb{P}_*$ that are sufficiently close to minimizing (resp. maximizing) $\{P'(A \mid E) : P' \in \mathbb{P}_*\}$ are members of $S_{\mathbb{P}_*}^-(A, E)$ (resp. $S_{\mathbb{P}_*}^+(A, E)$). In other words, observe that for some small $\underline{\epsilon}_1, \bar{\epsilon}_1 > 0$, the blue curve lies below the red line on $[0, \underline{\epsilon}_1)$ and above the red line on $(1 - \bar{\epsilon}_1, 1]$. Similarly, there are small $\underline{\epsilon}_2, \bar{\epsilon}_2 > 0$ such that the green curve lies below the red line on $(1 - \underline{\epsilon}_2, 1]$ and above the red line on $[0, \bar{\epsilon}_2)$.

In case the graph is not transparent, we now check all of the relevant calculations in detail.

We begin by showing that there exists $\underline{\epsilon}_1 > 0$ such that $\underline{\mathbb{P}}_*(A \mid E_1, \underline{\epsilon}_1) \subseteq S_{\mathbb{P}_*}^-(A, E_1)$. Let $\underline{\epsilon}_1 = 1/11$. Then, $P_\alpha \in \underline{\mathbb{P}}_*(A \mid E_1, \underline{\epsilon}_1)$ iff

$$\frac{3 + 2\alpha}{11 - 5\alpha} - \frac{3}{11} < \frac{1}{11},$$

which holds iff

$$0 \leq \alpha < \frac{11}{42}.$$

Now, $P_\alpha \in S_{\mathbb{P}_*}^-(A, E_1)$ iff

$$\frac{3 + 2\alpha}{11 - 5\alpha} < \frac{5 - 3\alpha}{8},$$

which holds iff

$$0 < 31 - 74\alpha + 15\alpha^2.$$

The function $f(\alpha) = 31 - 74\alpha + 15\alpha^2$ is decreasing in the interval $[0, 1]$ because $f'(\alpha) = 30\alpha - 74 < 0$ if $\alpha \in [0, 1]$. Since $f(11/42) \approx 12.6 > 0$, it follows that for all $\alpha \in [0, 11/42)$, $f(\alpha) > 0$. Hence, if $P_\alpha \in \underline{\mathbb{P}}_*(A \mid E_1, \underline{\epsilon}_1)$, then $P_\alpha \in S_{\mathbb{P}_*}^-(A, E_1)$.

Next, we show that there exists $\bar{\epsilon}_1 > 0$ such that $\overline{\mathbb{P}_*}(A \mid E_1, \bar{\epsilon}_1) \subseteq S_{\mathbb{P}_*}^+(A, E_1)$. Let $\bar{\epsilon}_1 = 1/6$. Note that $\overline{P}(A \mid E_1) = 5/6$. Then, $P_\alpha \in \overline{\mathbb{P}_*}(A \mid E_1, \bar{\epsilon}_1)$ iff

$$\frac{5}{6} - \frac{3 + 2\alpha}{11 - 5\alpha} < \frac{1}{6},$$

which holds iff

$$\frac{13}{16} < \alpha \leq 1.$$

Now, $P_\alpha \in S_{\mathbb{P}_*}^+(A, E_1)$ iff

$$\frac{3 + 2\alpha}{11 - 5\alpha} > \frac{5 - 3\alpha}{8},$$

which holds iff

$$0 > 31 - 74\alpha + 15\alpha^2.$$

As we noted in the first case, the function $f(\alpha) = 31 - 74\alpha + 15\alpha^2$ is decreasing in the interval $[0,1]$. Since $f(13/16) \approx -19.2 < 0$, it follows that for all $\alpha \in (13/16, 1]$, $f(\alpha) < 0$. Hence, if $P_\alpha \in \overline{\mathbb{P}_*}(A \mid E_1, \bar\epsilon_1)$, then $P_\alpha \in S_{\mathbb{P}_*}^+(A, E_1)$.

Next, we show that there exists $\underline{\epsilon}_2 > 0$ such that $\underline{\mathbb{P}_*}(A \mid E_2, \underline{\epsilon}_2) \subseteq S_{\mathbb{P}_*}^-(A, E_2)$. Let $\underline{\epsilon}_2 = 1/18$. Note that $\underline{P}(A \mid E_2) = 1/18$. Then, $P_\alpha \in \underline{\mathbb{P}_*}(A \mid E_2, \underline{\epsilon}_2)$ iff

$$\frac{12 - 11\alpha}{13 + 5\alpha} - \frac{1}{18} < \frac{1}{18},$$

which holds iff

$$\frac{95}{104} < \alpha \leq 1.$$

Now, $P_\alpha \in S_{\mathbb{P}_*}^-(A, E_2)$ iff

$$\frac{12 - 11\alpha}{13 + 5\alpha} < \frac{5 - 3\alpha}{8},$$

which holds iff

$$0 > 31 - 74\alpha + 15\alpha^2.$$

As usual, the function $f(\alpha) = 31 - 74\alpha + 15\alpha^2$ is decreasing in the interval $[0,1]$. Since $f(95/104) \approx -24.1 < 0$, it follows that for all $\alpha \in (95/104, 1]$, $f(\alpha) < 0$. Hence, if $P_\alpha \in \underline{\mathbb{P}_*}(A \mid E_2, \underline{\epsilon}_2)$, then $P_\alpha \in S_{\mathbb{P}_*}^-(A, E_2)$.

Finally, we show that there exists $\bar\epsilon_2 > 0$ such that $\overline{\mathbb{P}_*}(A \mid E_2, \bar\epsilon_2) \subseteq S_{\mathbb{P}_*}^+(A, E_2)$. Let $\bar\epsilon_2 = 1/13$. Note that $\overline{P}(A \mid E_2) = 12/13$. Then, $P_\alpha \in \overline{\mathbb{P}_*}(A \mid E_2, \bar\epsilon_2)$ iff

$$\frac{12}{13} - \frac{12 - 11\alpha}{13 + 5\alpha} < \frac{1}{13},$$

which holds iff

$$0 \leq \alpha < \frac{13}{198}.$$

Now, $P_\alpha \in S_{\mathbb{P}_*}^+(A, E_2)$ iff

$$\frac{12 - 11\alpha}{13 + 5\alpha} > \frac{5 - 3\alpha}{8},$$

which holds iff

$$0 < 31 - 74\alpha + 15\alpha^2.$$

By now the reader will anticipate our assertion that the function $f(\alpha) = 31 - 74\alpha + 15\alpha^2$ is decreasing in the interval $[0,1]$. Since $f(13/198) \approx 26.2 > 0$, it follows that for all $\alpha \in [0, 13/198)$, $f(\alpha) > 0$. Hence, if $P_\alpha \in \underline{\mathbb{P}_*}(A \mid E_2, \bar\epsilon_2)$, then $P_\alpha \in S_{\mathbb{P}_*}^+(A, E_2)$.

We have now shown that $(\underline{\epsilon}_i)_{i \in \{1,2\}}$ and $(\bar\epsilon_i)_{i \in \{1,2\}}$ satisfy (9) for each $i \in \{1, 2\}$, thus (1iii) holds. $\square$

Remark 2. Let $\epsilon = \min(\underline{\epsilon}_1, \bar\epsilon_1, \underline{\epsilon}_2, \bar\epsilon_2) > 0$ in the proof of Proposition 2. Then, by what we have just shown, the following condition—which is, in general, stronger than (1iii), but, in the case $|I| < \infty$, equivalent to (1iii)—is not sufficient for dilation:

(1iii') There exists $\epsilon > 0$ such that for all $i \in I$

$$\underline{\mathbb{P}}(A \mid E_i, \epsilon) \subseteq S_{\mathbb{P}}^-(A, E_i) \quad \text{and} \quad \overline{\mathbb{P}}(A \mid E_i, \epsilon) \subseteq S_{\mathbb{P}}^+(A, E_i).$$

The corollaries to Pedersen and Wheeler's Proposition 5.1 are also false. To show this, we need an additional definition. Let

$$S_*^-(A, E) = \{P \in \mathbb{P}_* : S_P(A, E) < 1\} \text{ and } S_*^+(A, E) = \{P \in \mathbb{P}_* : S_P(A, E) > 1\}. \quad (12)$$

(We note that in the proof of Proposition 2, we used the notation $S_{\mathbb{P}_*}^-(A, E)$ (resp. $S_{\mathbb{P}_*}^+(A, E)$) instead of $S_*^-(A, E)$ (resp. $S_*^+(A, E)$). The only reason we introduce the latter, redundant notation now is in order to follow Pedersen and Wheeler as closely as possible.)

Claim 2 (Pedersen and Wheeler, 2014, Corollary 5.2). Let $(\Omega, \mathcal{F}, \mathbb{P}, \underline{P})$ be a lower probability space, let $\mathcal{E} = \{E_i : i \in I\}$ be a positive, measurable partition of Ω, and let $A \in \mathcal{F}$. Then the following are equivalent:

(2i) $\mathcal{E}$ dilates A.

(2ii) There is $(\epsilon_i)_{i \in I} \in \mathbb{R}_+^I$ such that for every $i \in I$:

$$\underline{\mathbb{P}_*}(A \mid E_i, \epsilon_i) \subseteq S_*^-(A, E_i) \text{ and } \overline{\mathbb{P}_*}(A \mid E_i, \epsilon_i) \subseteq S_*^+(A, E_i). \quad (13)$$

(2iii) There are $(\underline{\epsilon}_i)_{i \in I} \in \mathbb{R}_+^I$ and $(\overline{\epsilon}_i)_{i \in I} \in \mathbb{R}_+^I$ such that for every $i \in I$:

$$\underline{\mathbb{P}_*}(A \mid E_i, \underline{\epsilon}_i) \subseteq S_*^-(A, E_i) \text{ and } \overline{\mathbb{P}_*}(A \mid E_i, \overline{\epsilon}_i) \subseteq S_*^+(A, E_i). \quad (14)$$

Furthermore, each radius $\underline{\epsilon}_i$ may be chosen to be the unique positive minimum of $|P(A \mid E_i) - \underline{P}(A \mid E_i)|$ attained on $C_i^+ := \{P \in \mathbb{P}_* : S_P(A, E_i) \geq 1\}$, and similarly each radius $\overline{\epsilon}_i$.

Claim 3 (Pedersen and Wheeler, 2014, Corollary 5.3). Let $(\Omega, \mathcal{F}, \mathbb{P}, \underline{P})$ be a lower probability space, let $\mathcal{E} = \{E_i : 1 \leq i \leq n\}$ be a finite positive, measurable partition of Ω, and let $A \in \mathcal{F}$. Then the following are equivalent:

(3i) $\mathcal{E}$ dilates A.

(3ii) There is $\epsilon > 0$ such that for every $i = 1, ..., n$:

$$\underline{\mathbb{P}_*}(A \mid E_i, \epsilon) \subseteq S_*^-(A, E_i) \text{ and } \overline{\mathbb{P}_*}(A \mid E_i, \epsilon) \subseteq S_*^+(A, E_i). \quad (15)$$

Proposition 3. *In Claim 2, (2iii) does **not** imply (2i). In Claim 3, (3ii) does **not** imply (3i).*

Proof. Let $(\Omega, \mathscr{P}(\Omega), \mathbb{P}, \underline{P})$, $\mathcal{E} = \{E_1, E_2\}$, and A be defined as in Example 1.

In the proof of Proposition 2, we have already shown that (2iii) holds. Now note that $\underline{P}(A) = P_1(A) = 1/4 < 3/11 = P_0(A \mid E_1) = \underline{P}(A \mid E_i)$, so (2i) doesn't hold. Thus, (2iii) does not imply (2i).

As $\mathcal{E}$ is finite, the conditions of Claim 3 are met, and we have just shown that (2i) does not hold, so (3i) does not hold. In Remark 2, we have already shown that (3ii) holds. Thus, (3ii) does not imply (3i). □

We now turn to the results in Pedersen and Wheeler (2015).

5. Counterexamples to the 2015 Results

As we noted above, the "Furthermore..." clauses of Claims 1 and 2 of the previous section are incorporated more explicitly into the characterizing conditions of dilation in Pedersen and Wheeler's later work. In this section, we show that the resulting claims are still false.

Claim 4 (Pedersen and Wheeler, 2015, Theorem 1). Let $(\Omega, \mathcal{F}, \mathbb{P}, \underline{P})$ be a lower probability space, let $\mathcal{E} = \{E_i : i \in I\}$ be a positive, measurable partition of Ω, and let $A \in \mathcal{F}$. Then the following are equivalent:

(4i) $\mathcal{E}$ dilates A.

(4ii) There is $(\epsilon_i)_{i \in I} \in \mathbb{R}_+^I$ such that for every $i \in I$:

$$\underline{\mathbb{P}}_*(A \mid E_i, \epsilon_i) \subseteq S_*^-(A, E_i) \quad \text{and} \quad \overline{\mathbb{P}}_*(A \mid E_i, \epsilon_i) \subseteq S_*^+(A, E_i). \tag{16}$$

(4iii) There is $(\epsilon_i)_{i \in I} \in \mathbb{R}_+^I$ such that for every $i \in I$:

$$\underline{\mathbb{P}}(A \mid E_i, \epsilon_i) \subseteq S_{\mathbb{P}}^-(A, E_i) \quad \text{and} \quad \overline{\mathbb{P}}(A \mid E_i, \epsilon_i) \subseteq S_{\mathbb{P}}^+(A, E_i), \tag{17}$$

where for each $i \in I$, $\epsilon_i \le \min(\underline{\epsilon}_i, \overline{\epsilon}_i)$ and $\underline{\epsilon}_i$ is the unique minimum of $|P(A \mid E_i) - \underline{P}(A \mid E_i)|$ attained on $C_i^+ := \{P \in \mathbb{P}^* : S_P(A, E_i) \ge 1\}$ and $\overline{\epsilon}_i$ is the unique minimum of $|P(A \mid E_i) - \overline{P}(A \mid E_i)|$ attained on $C_i^- := \{P \in \mathbb{P}^* : S_P(A, E_i) \le 1\}$.

We note that (4iii) is stronger than the mere existence claim in (2iii), since it requires that ϵ_i be no greater than the minimizers $\underline{\epsilon}_i$ and $\overline{\epsilon}_i$. More analysis of Example 1 shows, however, that (4iii) is still not strong enough to imply dilation.

Proposition 4. *In Claim 4, (4iii) does **not** imply (4i).*

Proof. Let $(\Omega, \mathscr{P}(\Omega), \mathbb{P}_*, \underline{P})$, $\mathcal{E}$, and A, be defined as in Example 1. In the proof of Proposition 2, we have already shown that (2i) does not hold, so (4i) does not hold.

At this point, there is a somewhat unfortunate overlap of notations. $\underline{\epsilon}_i$ and $\overline{\epsilon}_i$ have different meanings in the proof of Proposition 2 and (4iii). To resolve this difficulty, let us rewrite the quantities from the proof of Proposition 2 as

$$\delta_1 = \underline{\epsilon}_1, \delta_2 = \overline{\epsilon}_1, \delta_3 = \underline{\epsilon}_2, \delta_4 = \overline{\epsilon}_2.$$

To be clear, then, in the remainder of the current proof $\underline{\epsilon}_i$ and $\overline{\epsilon}_i$ refer to the minimizers defined in (4iii).

Now, to complete the proof, we must show that (4iii) holds. To that end, it suffices to show, for $i = 1, 2$, that $\min(\underline{\epsilon}_i, \overline{\epsilon}_i) > 0$, because we can then let $\epsilon_i = \min(\underline{\epsilon}_i, \overline{\epsilon}_i, \delta_{2i-1}, \delta_{2i}) > 0$, $i = 1, 2$. By what we have already shown in the proof of Proposition 2, the resulting $(\epsilon_i)_{i \in \{1,2\}}$, will satisfy condition (4iii).

First, consider $\underline{\epsilon}_1$. Observe that the functions $\underline{f}_1(\alpha) = |P_\alpha(A \mid E_1) - \underline{P}(A \mid E_1)|$ and $g_1(\alpha) = S_{P_\alpha}(A, E_1)$ are both strictly increasing and continuous in the interval $[0, 1]$. (We note that this, and all the inferences that follow, can be seen quite easily by graphing the relevant functions.) Let α_1 be the unique point in $[0, 1]$ such that $g_1(\alpha_1) = 1$. Then, $C_1^+ = [\alpha_1, 1]$. Thus, the unique minimum, $\underline{\epsilon}_1$, of $\underline{f}_1$ on C_1^+ is attained at α_1. Solving the relevant quadratic equation, we find that $\alpha_1 \approx 0.462$. So, $\underline{\epsilon}_1 = f(\alpha_1) > 0$ because $\underline{f}_1$ is strictly increasing on $[0, 1]$ and $\underline{f}_1(0) = 0$.

Second, consider $\overline{\epsilon}_1$. Observe that the function $\overline{f}_1(\alpha) = |P_\alpha(A \mid E_1) - \overline{P}(A \mid E_1)|$ is strictly decreasing and continuous in the interval $[0, 1]$. With α_1 defined as in the previous paragraph, we have $C_1^- = [0, \alpha_1]$. Thus, the unique minimum, $\overline{\epsilon}_1$, of $\overline{f}_1$ on C_1^- is attained at α_1. As above, we have $\alpha_1 \approx 0.462$, so $\overline{\epsilon}_1 = \overline{f}_1(\alpha_1) > 0$ because $\overline{f}_1(1) = 0$.

We now draw the intermediate conclusion that $\min(\underline{\epsilon}_1, \overline{\epsilon}_1) > 0$ and proceed to the cases involving E_2.

Consider $\underline{\epsilon}_2$. Observe that the functions $\underline{f}_2(\alpha) = |P_\alpha(A \mid E_2) - \underline{P}(A \mid E_2)|$ and $g_2(\alpha) = S_{P_\alpha}(A, E_2)$ are both strictly decreasing and continuous in the interval $[0, 1]$. Let α_2 be the unique point in $[0, 1]$ such that $g_2(\alpha_2) = 1$. Then, $C_2^+ = [0, \alpha_2]$. Thus, the unique minimum, $\underline{\epsilon}_2$, of $\underline{f}_2$ on C_2^+ is attained at α_2. Solving, we again find that $\alpha_2 \approx 0.462$, so $\underline{\epsilon}_2 = \underline{f}_2(\alpha_2) > 0$ because $\underline{f}_2(1) = 0$.

Finally, consider $\bar{\epsilon}_2$. Observe that the function $\bar{f}_2(\alpha) = |P_\alpha(A \mid E_2) - \overline{P}(A \mid E_2)|$ is strictly increasing and continuous in the interval $[0, 1]$. With α_2 defined as in the previous paragraph, we have $C_2^- = [\alpha_2, 1]$. Thus, the unique minimum, $\bar{\epsilon}_2$, of $\bar{f}_2$ on C_2^- is attained at α_2. As before, we find that $\bar{\epsilon}_2 = \bar{f}_2(\alpha_2) > 0$ because $\bar{f}_2(0) = 0$.

We conclude that $\min(\underline{\epsilon}_2, \bar{\epsilon}_2) > 0$, which completes the proof. $\square$

Finally, we show that the following claim is false.

Claim 5 (Pedersen and Wheeler, 2015, Corollary 1). Let $(\Omega, \mathcal{F}, \mathbb{P}, \underline{P})$ be a lower probability space, let $\mathcal{E} = \{E_i : 1 \leq i \leq n\}$ be a finite, positive, measurable partition of Ω, and let $A \in \mathcal{F}$. Then the following are equivalent:

(5i) $\mathcal{E}$ dilates A.

(5ii) There is $\epsilon > 0$ such that for each $i = 1, ..., n$:
$$\underline{\mathbb{P}}_*(A \mid E_i, \epsilon) \subseteq S_*^-(A, E_i) \quad \text{and} \quad \overline{\mathbb{P}_*}(A \mid E_i, \epsilon) \subseteq S_*^+(A, E_i). \tag{18}$$

(5iii) There is $\epsilon > 0$ such that for each $i = 1, ..., n$:
$$\underline{\mathbb{P}}(A \mid E_i, \epsilon) \subseteq S_{\mathbb{P}}^-(A, E_i) \quad \text{and} \quad \overline{\mathbb{P}}(A \mid E_i, \epsilon) \subseteq S_{\mathbb{P}}^+(A, E_i), \tag{19}$$

where $\epsilon \leq \min(\underline{\epsilon}_i, \bar{\epsilon}_i)$ and $\underline{\epsilon}_i$ is the unique minimum of $|P(A \mid E_i) - \underline{P}(A \mid E_i)|$ attained on $C_i^+ := \{P \in \mathbb{P}^* : S_P(A, E_i) \geq 1\}$ and $\bar{\epsilon}_i$ is the unique minimum of $|P(A \mid E_i) - \overline{P}(A \mid E_i)|$ attained on $C_i^- := \{P \in \mathbb{P}^* : S_P(A, E_i) \leq 1\}$.

Proposition 5. *In Claim 5, (5iii) does **not** imply (5i).*

Proof. Let $(\Omega, \mathscr{P}(\Omega), \mathbb{P}_*, \underline{P})$, $\mathcal{E}$, and A, be defined as in Example 1. Continuing from the proof of Proposition 4, we note that we have shown that (4i) does not hold, so (5i) does not hold. Moreover, we showed that $(\epsilon_i)_{i \in \{1,2\}}$ satisfies (4iii). It is easily seen, then, that $\epsilon = \min(\epsilon_1, \epsilon_2) > 0$ satisfies (5iii). $\square$

6. Remarks on Pedersen and Wheeler's Proof

We conclude our note with a few remarks about the mistakes in Pedersen and Wheeler's alleged proof of the Claims above. We focus on their proof that (1iii) implies (1i) (Pedersen and Wheeler, 2014, p. 1340).[1]

Here is the relevant portion of their proof with a few harmless changes to the notation:

> Suppose that $\mathcal{E}$ does not dilate A. Then there is $i \in I$ such that $\underline{P}(A \mid E_i) \geq \underline{P}(A)$ or $\overline{P}(A) \geq \overline{P}(A \mid E_i)$. We may assume without loss of generality that $\underline{P}(A \mid E_i) \geq \underline{P}(A)$ for some $i \in I$. First, if $\underline{P}(A) \leq \overline{P}(A) \leq \underline{P}(A \mid E_i)$, then choosing a minimizer $p \in \mathbb{P}$ of $\underline{P}(A \mid E_i)$, we see that $S_P(A, E_i) \geq 1$. Second, if $\underline{P}(A) < \underline{P}(A \mid E_i) < \overline{P}(A)$, then for every $\epsilon > 0$ we can find a convex combination $P \in \mathbb{P}$ of $P', P'' \in \mathbb{P}$ assigning a probability to A within ϵ-distance below $\underline{P}(A \mid E_i)$, where $\underline{P}(A) \leq P'(A) < \underline{P}(A \mid E_i) < P''(A) \leq \overline{P}(A)$, so $S_P(A, E_i) > 1$. Third, if $\underline{P}(A) = \underline{P}(A \mid E_i) < \overline{P}(A)$, then choosing a minimizer $P \in \mathbb{P}$ of $\underline{P}(A)$, we see that $S_P(A, E_i) \geq 1$. Evidently, the conditions of the main claim cannot be jointly satisfied.

---

[1]Note that there is a typo in their paper where they describe the direction of implication; "(iii) $\Leftarrow$ (i)" should be replaced by "(iii) $\Rightarrow$ (i)".

To prove the contrapositive of the desired implication, Pedersen and Wheeler must show that the negation of (1iii) follows from the assumption that $\mathcal{E}$ doesn't dilate A. In the passage above, it seems that they are attempting to argue, from the assumption that there is no dilation, that there is some $i \in I$ such that for all $\epsilon > 0$ there is some $P \in \underline{\mathbb{P}}(A \mid E_i, \epsilon)$ such that $P \notin S_{\mathbb{P}}^-(A, E_i)$. The argument errs in the second and third cases that it considers. What is needed in both cases is a P whose *conditional value for A given E_i* is ϵ-close to $\underline{P}(A \mid E_i)$. That is the meaning of $P \in \underline{\mathbb{P}}(A \mid E_i, \epsilon)$. What Pedersen and Wheeler provide, in both cases, is a P whose *unconditional value for A* is ϵ-close to $\underline{P}(A \mid E_i)$. As it is not true in general that a P with unconditional value for A that is ϵ-close to $\underline{P}(A \mid E_i)$ will also have a conditional value for A given E_i that is ϵ-close to $\underline{P}(A \mid E_i)$, the proof is fallacious. In particular, in the third case, it is not true in general that a P that attains $\underline{P}(A)$ also attains $\underline{P}(A \mid E_i)$. The reader can verify that our Example 1 illustrates this fact.

We are grateful to an anonymous referee for showing us that the mistake in Pedersen and Wheeler's proof can also be explained using Result 1 in Wasserman and Seidenfeld (1994). That result provides a condition that is both necessary and sufficient for dilation in a binary partition. Since our central example uses only a binary partition and does not exhibit dilation, it follows that it violates Seidenfeld and Wasserman's characterizing condition. We will not go into further details but simply note that the mistake in Pedersen and Wheeler's proof that we identified just above amounts to erroneously restricting attention to a special case of Wasserman and Seidenfeld's condition, namely, the case in which P attains both $\underline{P}(A)$ and $\underline{P}(A \mid E_i)$.

References

Pedersen, A. P. and G. Wheeler (2014). Demystifying dilation. *Erkenntnis 79*(6), 1305–1342.

Pedersen, A. P. and G. Wheeler (2015). Dilation, disintegrations, and delayed decisions. In *Proceedings of the 9th International Symposium on Imprecise Probability: Theories and Applications*, pp. 227–236.

Seidenfeld, T. and L. Wasserman (1993). Dilation for sets of probabilities. *The Annals of Statistics 21*(3), 1139–1154.

Wasserman, L. and T. Seidenfeld (1994). The dilation phenomenon in robust Bayesian inference. *Journal of Statistical Planning and Inference 40*(2), 345–356.