Conceptual Origami

Andrew Notier unfolds the social construction of mathematics for us.

The beating heart of mathematics, its foundational axiom, is the idea that provides it with precision and efficacy: *start with symmetry*. Mathematics in its purest form is the manipulation of symmetries through conceptual origami. Through the clever and creative folding of symmetries, brilliant mathematicians create the sometimes beautiful and often critically important constructs through which we understand our universe. When we ask questions of the cosmos using mathematics, we map these symmetries onto the world.

A Handful of Digits

To better appreciate how this works, let us first look at the most ubiquitous symbols of mathematics, commonly known as the cardinal numbers.

Let's start with the basics. By 'basics', I mean that this may initially seem so trivially, stupidly obvious that you can't see why anyone would bother to point it out. Hang in there, and hopefully it will become apparent that this is going somewhere.

The notion we're going to start with is fingers. Yep, that basic. So basic that (as it's generally assumed) the entire reason we have a base ten number system is because humans have ten digits on their hands. Psychologists have shown that mathematical achievement in children is strongly correlated with using their fingers and thumbs to solve math problems; so fingers and thumbs are right there at the evolutionary inception of mathematical thinking.

When someone states that they have five digits on their right hand, there is an underlying assumption smuggled into that statement that's easy to miss: for the purpose of the counting, each of those digits is exactly the same. To be clear, obviously no one believes that each of their digits is identical to the others in every way. If you switch around your thumb and middle finger there are going to be problems in manipulating objects. The point is that the statement "I have five digits on my right hand" creates the category *digits*, and for the purpose of counting, each instantiation of a digit is *exactly* equal. It doesn't matter whose fingers and thumbs we count, the order they're counted in, or which ones are included in the tally. We add them all up exactly the same way, as if they were completely interchangeable. In other words, a *symmetry* has been imposed onto the category of *digits* in which they each individually count as one unit: in respect of counting, they are completely interchangeable. In mathematical jargon, each unit is translationally symmetrical with every other unit in the system, and the system will remain invariant regardless of how the units are swapped around. This symmetry is the foundational attribute that allows mathematics to function with exactness and elegance. To ask a question using mathematics is simply to manipulate this and other symmetries in specific ways.

The First Fruits of Mathematical Knowledge

To illustrate the point with objects perhaps a little less peculiar than fingers, let's move on to apples, since apples are often among the first objects used to instruct children in basic mathematical concepts.

Let's say that we have a bag of apples. Everything in the bag is apples, and the apples in the bag are a complete set for the system under discussion. There are a number of lines of inquiry one may pursue to find out information about these apples. Some of the questions posed might be descriptive. Descriptive questions can provide information about such things as the color of the apples, how they taste, if any of them are rotten, or if they are big or small by some subjective measure. However, if you want to use mathematics to ask a question about the apples in this bag, you need to impose a symmetry onto the system, which requires ignoring these types of descriptions.

The most obvious, and common, way to impose a symmetry onto this system is to assert that each of the objects in the bag – each apple – is equal to exactly one unit. Now that you've imposed a symmetry, you can ask all sorts of mathematical questions about the system, the bag of apples. A simple question such as "How many apples are in the bag?" is a starting point, but it can become as intricate as you like. For instance: "If you need to provide lunch for 20 people and there are 10 apples in the bag, what percentage of the people can have an apple?" Or, "If you have 10 apples and 20 people, what is the least number of slices you can cut the apples into in order to provide everyone with the same number of slices?" This undoubtedly all seems obvious, but it is crucial to understand that you can begin to use mathematics to ask questions about a system only when you have imposed symmetry onto that system.

Asking mathematical questions of your bag of apples will not give you any information about the system except insofar as it relates to the archetype you have imposed. Imagine that a little girl named Emma is hungry and asks you if she can have an apple. You have a bag of apples in the refrigerator, so you reply "Yes" and walk her to the kitchen to get one. You remove a red apple from the bag to offer her, and Emma recoils in horror. "That's what witches use to poison princesses!" she shrieks: "I only eat the green ones." It turns out that while you thought you had apples in the fridge, according to Emma there are zero because she had an entirely different model in mind when she asked for an apple. To her, the red ones do not count.

An astute reader may notice that the word *count* could be doing double duty in that last sentence: as both an indication of inclusion/exclusion and as a tallying of units. The *Online Etymology Dictionary* says that the word 'count' is derived from the Latin word *computare*, which means 'to sum up' or 'reckon together'… in Old French, *computare* became *conter*, meaning 'to add up' or 'to tell a story', and over time this developed into the word *count* and its modern usage. This etymological detour illustrates the idea that we must tell a story as to how things reckon together, or *categorize*, in order to add them up. In other words, we create a categorical prototype and then map that onto the system we want to count. So by Emma's reckoning, green apples were the only type in the category she was counting. That's okay, because it isn't just Emma who doesn't think all apples are the same. Grocery stores feel the same way.

Numbers Speed Away Indiscretely

Until this point we have been considering symmetries involving discrete objects – fingers or apples as separate units, equal but isolated from the other units in the system. The grocery store does not look at apples this way at all. The symmetry they map onto the system is a continuous measurement of weight, in which the number of discrete apples is irrelevant. When you bring your bag of ten apples to the counter to pay for them, you're charged for the weight of all of the apples combined: the total amount of *apple*, not the tally of discrete units.

There is a particular conceptual difference between these discrete and continuous measurements that benefits from a closer look. When counting apples, while it is not always obvious that the symmetry between units is a conceptual contrivance, it is obvious that each unit is discrete. With continuous measurements, the issue becomes somewhat inverted. Conceptually, each unit of a continuous measurement is perfectly symmetrical in a way that discrete objects are not. A gram of weight or a meter of length is exactly the same as every other gram or meter regardless of the system the measurement is mapped onto. The symmetry is perfect, but now it is the discreteness which is illusory. As Georg Cantor demonstrated at the end of the nineteenth century, the numbers between one and two (or between any specified numbers, in fact) comprise an uncountable infinity. For example, when trying to pick a place to begin counting, you can always add another zero right after the decimal in 1.01 to get the new number 1.001; you can similarly always add another nine at the end of 1.9 before you get to two, and so on *ad infinitum*. In this respect, a unit of measurement is only as discrete as the number of decimal places one choses to measure, in an infinite continuum of uncountable numbers.

Consider the speedometer on a car. A speedometer measures how fast the vehicle is moving at a particular instant in time. It will be read as either kilometers or miles per hours depending on where you are in the world, but the units of measurement are an arbitrary convenience: they could be cubits per turn of the sandglass. The point is that in terms of both the time and distance measured, the number derived is only as precise as the measurement taken. If your car shows how fast you are travelling to one decimal place, and displays a reading of 55.2 mph, that's not precisely the speed you are travelling. You are actually going 55.2…. *out to some unknown decimal place on an infinite continuum* mph. (Well, it is conceivable that you *are* going *exactly* 55.2 mph, but there is absolutely no way of knowing that without taking an infinitely precise measurement, which is not possible.) To be fair, it doesn't really matter if you're going 55.2 mph or 55.223067 mph. No one cares, not even a particularly pernickety traffic cop. Indeed, in order to catapult spacecraft into the farthest reaches of the Solar System with unbelievable precision, NASA only calculates pi out to fifteen decimal places. So for all practical purposes, the lack of discreetness in measurement is inconsequential because we can know things with enough precision to ask any questions of our universe we can presently conceive. But the vagueness is always there beneath the surface, lurking, haunting the minds of mathematicians and philosophers alike.

The Magic of Mathematics

The realm in which mathematics seems truly magical is when its conceptual symmetries are mapped onto systems that turn out to be symmetrical themselves. When physicists refer to the elegance and beauty of an equation, this is down to this seemingly supernatural marriage of the conceptual abstraction of perfect symmetry in mathematics with the observed reality onto which this abstraction is mapped. When this happens, as the great mathematician Emmy Noether famously proved, we have stumbled across one of the fundamental laws of nature. Noether's Theorem states that whenever there is a symmetry in a physical system, there is an associated conservation law – such as the law of conservation of energy, or the law of conservation of electric charge. These are the laws of physics which provide the predictability and order we need to navigate an otherwise impossibly complex and chaotic world.

Just as a perfect square of paper provides the initial framework for an origami artist to fold delightfully elegant and complex shapes, to *start with symmetry* is the inceptive essence of mathematics. With this realization comes the awareness that when a question begins with a perfect symmetry the mystifying precision of math becomes tautological: the answers gleaned from a mathematically posed question cannot help but reflect the underlying perfect construction upon which the question was formed. By continuing to ask questions with this unparalleled conceptual tool, brilliant thinkers will continue to push the frontiers of human knowledge to reach a more profound understanding of our universe.

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