Fermat's Least Time Principle Violates Ptolemy's theorem

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Abstract

Fermat's Least Time Principle has a long history. World's foremost academies of the day championed by their most prestigious philosophers competed for the glory and prestige that went with the solution of the refraction problem of light. The controversy, known as Descartes - Fermat controversy was due to the contradictory views held by Descartes and Fermat regarding the relative speeds of light in different media. Descartes with his mechanical philosophy insisted that every natural phenomenon must be explained by mechanical principles. Fermat on the other hand insisted an end purpose for every motion. For example, least time of travel and not the least distance of travel is the end purpose for motion of light. This implied a thinking nature, which was rejected by Descartes. Surprisingly, with contradictory assumptions regarding the relative speeds of light in different media, both Descartes and Fermat came to the same result that the ratio of sines of angles of incidence and refraction is a constant. Fermat's result came to be known as the 'Fermat's least time principle'. We show in this article that Fermat's least time principle violates a fundamental theorem in geometry – the Ptolemy's theorem. That leads to the invalidity of Fermat's principle.

Introduction

'Fermat's Least Time Principle[1-3] (FLTP) has a long history. Its development was described by Lamborn[1] as a battle! The phenomenon of refraction of light is a philosophical, mathematical and theoretical puzzle for the early moderns during the late 17th century and early 18th century. The three foremost, state run science academies of the world at the time, The Royal Society of London, The Royal academy of Science in Paris and The Royal Prussian Academy in Berlin competed for the glory and prestige that went with cultural advancement, in mathematics, science and philosophy. The priority in solving this puzzle was a prestigious enough. The Royal academy of Science in Paris was championed by Descartes, while The Royal Society of London was championed by Newton and The Royal Prussian Academy in Berlin championed by Gottfried Leibniz. The puzzle of FLTP arose in the context of refraction of light where it has different speeds in different media through which it passed. Descartes and Cartisians, Newtonians and Leibnizinians claimed that light travelled faster in denser media and slower in rarer media, while Fermat and Fermaticians and Huygens claimed that light travels slower in denser media and faster in rarer media; diametrically opposite claims indeed. Descartes with his mechanical philosophy and the motion of racket ball experiment, derived the law of sines for the refraction of light[1-7]. It was rejected by Fermat. For Fermat and Leibniz final causes decide the occurrence of natural phenomena[1-8]. This formed the basis of his arguments for the light choosing the least time path in preference to least distance path. The property or ability of light to choose the path of least time over the path of least distance showed a quality of intelligent decision making. This implies 'thinking nature'. Descartes rejected outright such attributes to light and natural phenomena; for him they occur because they have to occur as decided by God. He rejected Fermat's 'Method' stating that the method had probably been found by pure luck and not as a result of an industrious search for the solution of a general problem[9]. No one in 18th century was however, willing to admit that the nature of light contained a decision making intelligence. At the same time they were unwilling to admit that God was doing the thinking for light, directing its movements according to divine design. Surprisingly, Fermat too, derived the sine law with the assumption opposite to the assumption of Descartes regarding the relative speeds of light in different media.

Descartes with his mechanical philosophy described the motion of tennis racket-ball experiment analogous to motion of light, in his 1633 work on light – 'The World or Treatise on Light'[7]. Fermat presented his solution to the problem of refraction, in 'Method of Finding Maxima and Minima Synthesis for refraction'[7], and in his correspondence with de La Chambre in 1638[1]. Fermat says 'Our demonstration rests on a single postulate: that nature operates by the easiest and most expedient ways and means. This is a metaphysical statement. His principle is said to be a moral principle and not a physical one. Fermat calls his principle, 'The principle of natural economy' Later it became to be popularly known as Fermat's least time principle.

I never thought it would be so easy and so simple to prove the invalidity of such a powerful principle, as the 'Fermat's Least Time Principle' (FLTP). After my works on Fermat Point[10] and the inconsistency between Fermat Point and Fermat's Least Time Principle[11] a couple of years back, I turned to Alhazen problem, Leibniz's MDP Principle and other topics in optics. I also worked on a fundamental theorem in geometry viz., Ptolemy's theorem[12,13] (PT). Suddenly I saw the connection between PT and FLTP and found that FLTP violates PT. FLTP asserts that a ray of light follows the path between any two given points that takes the least time, that is, a path for which the sum of the two time intervals of travel - before and after reflection or refraction (essentially. the sum of two line segments from a given point) - is a minimum. A special case[13] of PT involving an inscribed equilateral triangle that brings in symmetry makes PT simpler and very useful for our analysis here. It also provides the necessary and sufficient condition for the minimality of the sum of two distances from a given point. Using this condition we find that FLTP and PT are mutually contradictory. In this article we give a simple proof to show that FLTP violates PT, a fundamental theorem of geometry, and is therefore invalid.

Key words

Ptolemy's theorem, Special case of Ptolemy's theorem, Fermat's Least Time Principle, Optics, Reflection, Refraction, Descartes, Snell's laws, Descartes-Fermat controversy, Minimum time path, Philosophy

Statement of Fermat's Least Time Principle[14]

Fermat's Least Time Principle states that out of all possible paths that it might take to get from one point to another light takes the path which requires the shortest time.

Statement of Ptolemy's theorem[15]

Ptolemy's theorem states that: The sum of the products of the opposite pairs of sides of a cyclic quadrilateral is equal to the product of the diagonals. Let ABCD be a cyclic quadrilateral (Fig. 1). Then according to Ptolemy's theorem, we get Eq. (1).

$$AB \times CD + BC \times AD = AC \times BD \qquad (1)$$

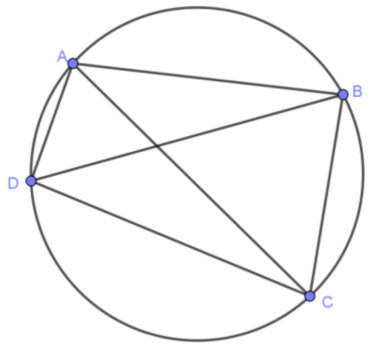

Fig. 1 The figure shows a cyclic quadrilateral ABCD used to describe Ptolemy's theorem.

Special case of Ptolemy's theorem[12,13,15]

A special case of Ptolemy's theorem arises when three sides of the quadrilateral are equal, that is, when there is an equilateral triangle in the quadrilateral (Fig. 2). The symmetry of the triangle leads to a simplification of Eq. (1) and gives Eq. (2) below.

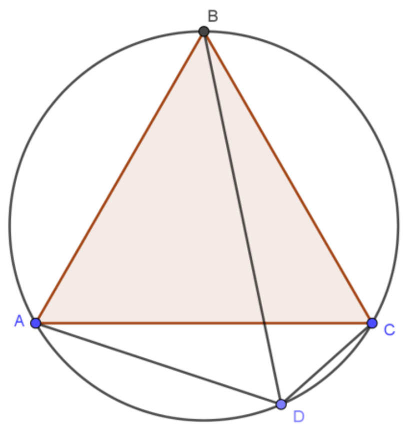

Fig. 2 The figure shows a cyclic quadrilateral ABCD inscribing the equilateral triangle ABC (shaded) used to describe the special case of Ptolemy's theorem.

Let ABC be an equilateral triangle, and P be any point on the circumcircle of the triangle. Then, the largest of the distances PA, PB, PC is equal to the sum of the other two distances.

$$AC.DB = BC.AD + AB.DC \qquad (2)$$

$$DB = AD + DC \qquad since\ AC = BC = AB \qquad (3)$$

Proof to show the invalidity of FLTP (Reflection)

Let A, B be the end points of the path of a ray of light undergoing reflection at a point C on a reflecting surface of plane or spherical convex or concave. Let the angle ACB be bisected by CD (Fig. 3). The ray of light from A incident at C is reflected to go along CB making equal angles i, r with CD.

$$A\hat{C}D = B\hat{C}D = i = r \qquad (4)$$

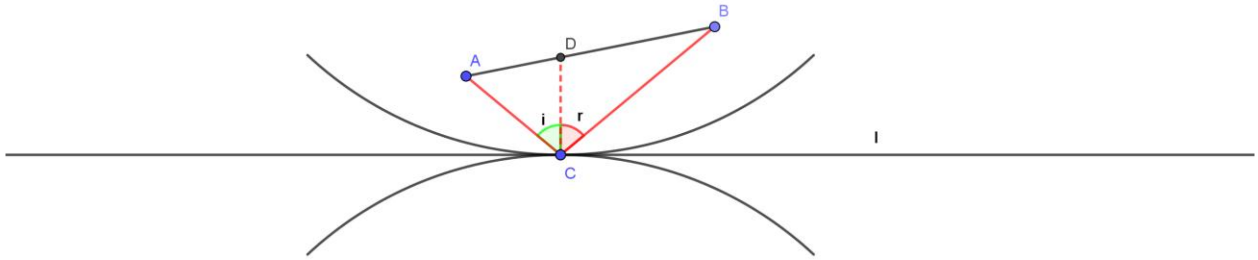

Fig. 3 The figure shows a point C on a reflecting surface, plane, l, or spherical convex or concave, at which a ray of light from a Point A is incident and is reflected along CB making equal angles with CD perpendicular to l at C.

According to Snell's law of reflection, the indent ray, AC and the reflected ray, CB make equal angles with the surface of reflection, l (or the normal, CD to the surface at the point of incidence). This is known as the equal angles law[4,5]. According to FLTP, the sum of the distances AC and BC is a minimum. Since distance of travel is proportional to the time of travel in reflection, it follows that the time of travel is also minimum.

Construction of the equilateral triangle and its circum circle for the travel distance[13,15]

Let us construct an equilateral triangle ABE on the line segment AB (Fig. 4) of the end points A, B of the path of the reflected ray couple, AC, CB. Let us also construct the circumcircle of the equilateral triangle ABE. Now we have the quadrilateral AEBC.

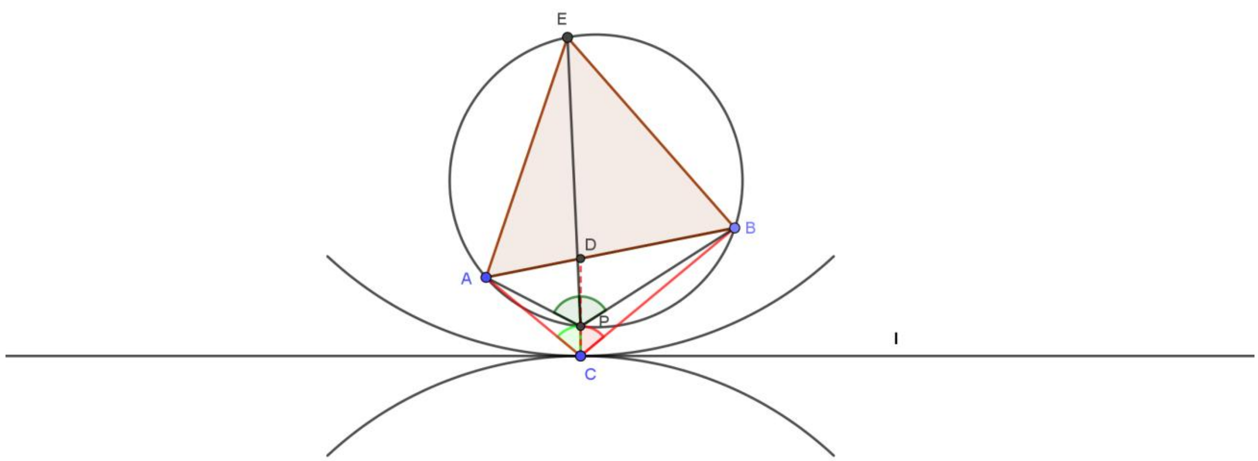

Fig. 4 The figure shows the equilateral triangle ABE (shaded) on the side AB and the circumcircle of the equilateral triangle ABE. It intersects CD at P.

The circumcircle does not pass through C. The quadrilateral AEBC is not cyclic. This is to be expected since the angle ACB is an arbitrary angle (since the incident ray AC is arbitrary) and the sum of the opposite angles at E and C of the quadrilateral is arbitrary.

Let the circumcircle intersect CD at P. Join P to A, E and B. Since all the four points A, E, B and P are concyclic by construction, the quadrilateral AEBP is cyclic.

FLTP violates Ptolemy's theorem

According to the above special case of the Ptolemy's theorem we get,

$$(AP + BP) = EP \ (is\ a\ minimum) \qquad (5)$$

However, according to FLTP

$$(AC + BC)\ (is\ a\ minimum) \qquad (6)$$

$$But, \quad (AP + BP) \neq (AC + BC) \quad since\ C\ is\ not\ on\ the\ circle \qquad (7)$$

From Eq. (7) we see that FLTP (6) violates Ptolemy's theorem (5). Since, FLTP violates a fundamental theorem of geometry, viz., the Ptolemy's theorem, FLTP is invalid.

Least distance path is the least time path

In the case of reflection, the speed of travel is constant throughout the path. Therefore, we get the result that if the distance of travel is a minimum then the time of travel is necessarily a minimum.

Let the constant speed of travel be v. If s_1 and s_2 are the distances of travel before and after reflection, then, we get.

$$\left(\frac{s_1}{v} + \frac{s_2}{v}\right) = (t_1 + t_2) = \left[\frac{(s_1 + s_2)}{v}\right] \qquad (8)$$

If $(s_1 + s_2)$ is a minimum then a constant multiple (or fraction) of $(s_1 + s_2)$ is also minimum. We demonstrate this geometrically below.

Construction of the equilateral triangle and its circum circle for travel time

Let (s_1/v) = A'C. We draw a parallel line A'B' to AB. A'B' represents the time of travel along AB, A'C the time of travel along AC and CB' the time of travel along CB . We draw an equilateral triangle A'E'B' with A'B' as the side length. Draw the circumcircle of the triangle A'E'B'. Let P' be any point on this circle. Join P' to A', E' and B'.

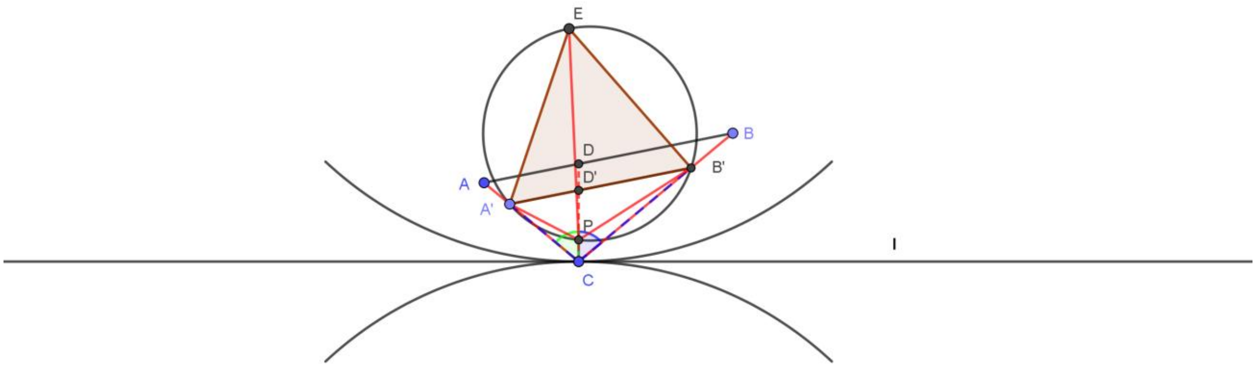

Fig. 5. The figure showsthe equilateral triangle A'E'B' (shaded) on the side A'B' and the circum circle of the equilateral triangle A'E'B'. A'C and B'C represent times of travel along AC and BC respectively.

A'B', A'C (=AC/v) and B'C represent times of travel along AB, AC and BC respectively. According to Ptolemy's theorem we get,

$$A'P + B'P = E'P \quad is\ a\ minimum. \qquad (9)$$

However, according to FLTP

$$A'C + B'C = EC \qquad (is\ a\ minimum) \qquad (10)$$

$$But, \quad (A'P + B'P) \neq (A'C + B'C) \quad since\ C\ is\ not\ on\ the\ circle \qquad (11)$$

From Eqs (9), (10) and (11) we see that FLTP (10) violates Ptolemy's theorem (9). Thus, since, FLTP violates a fundamental theorem of geometry, viz., the Ptolemy's theorem, FLTP is invalid.

If light were to take the least time path between any two points in a given medium, then, when it undergoes reflection on its paths, the point of incidence must lie on the circumcircle of the equilateral triangle drawn with the line segment joining the end points of the path, as the side.

This completes the demonstration of the invalidity of FLTP in reflection phenomena.

We now proceed to the demonstration of the invalidity of FLTP in refraction phenomena.

Demonstration of the invalidity of FLTP in refraction phenomena

It is very easy now for us to demonstrate the invalidity of FLTP in refraction phenomena, because the argument follows similar lines as in the case of reflection.

Let m_1, m_2 be two media through which light passes from a point A in m_1 to a point B in m_2 by refraction at a point C on the interface l, of the two media (see Fig. 6). Let us assume m_1 be the rarer medium and m_2 be the denser medium. Let n be the normal to l at C. It is well known that light bends towards the normal when it goes from a lighter to a denser medium.

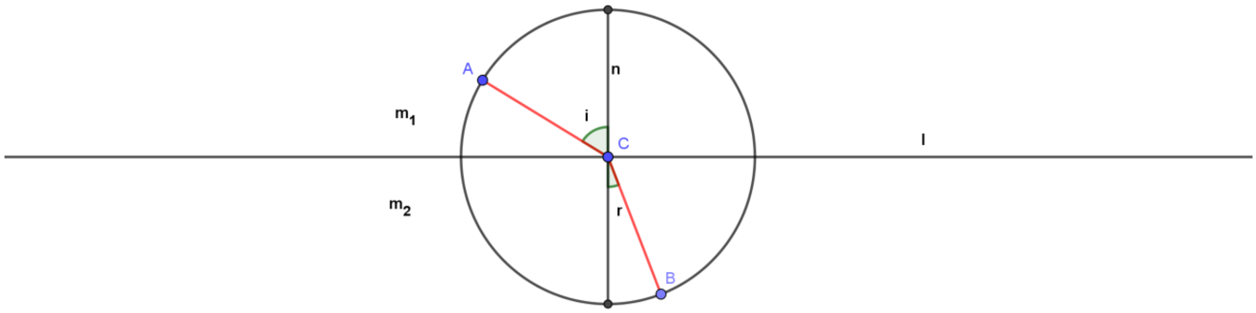

Fig. 6 The figure shows the path ACB of a ray of light refracted at the point C on l, the surface of separation of two media m_1, m_2. n is the normal to l at C.

Let AC = CB = s. Let v_1 and v_2 be the speeds of travel of light in the two media m_1, m_2 respectively. Let i, r be the angles of incidence and refraction respectively. According to the sine law of refraction, we get,

$$\frac{\sin(i)}{\sin(r)} = \frac{v_1}{v_2} = \text{constant} \qquad (12)$$

Let t_1, t_2 be the times if travel along AC and CB respectively. Then we get,

$$\frac{s}{v_1} = t_1, \qquad \frac{s}{v_2} = t_2 \qquad (13)$$

According to FLTP $(t_1 + t_2)$ is a minimum.

Geometric demonstration of the invalidity of FLTP for refraction phenomena

Without loss of generality we can assume $v_2 = 1$. Then A'C (=AC/v_1) and CB respectively represent times of travel t_1, t_2 in m_1, m_2 (see Fig.7). Then, according to FLTP (A'C + CB) is a minimum. We can test if this is true, using PT.

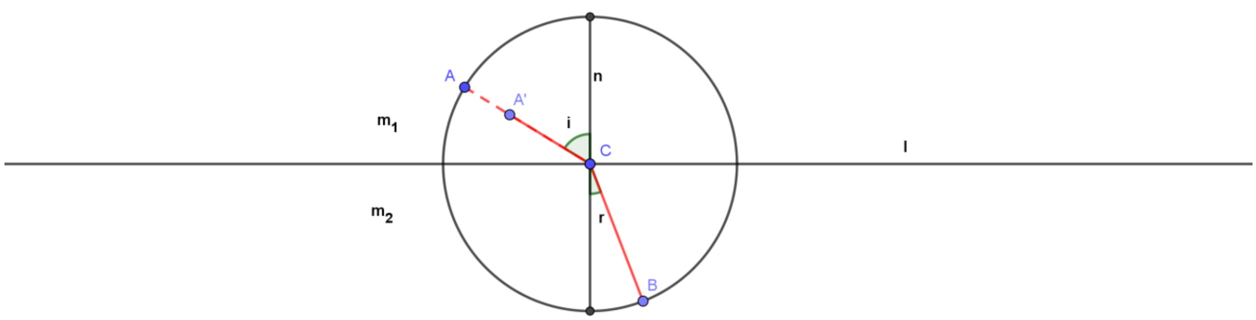

Fig. 7 The figure shows the relative times of travel A'C, in medium m_1 and CB in medium m_2.

Join A’and B (see Fig.8). Construct the equilateral triangle A’EB (shaded) on the side A’B and the circum circle of the equilateral triangle A’EB. Let P be any point on the circle. Join P to A’, E and B.

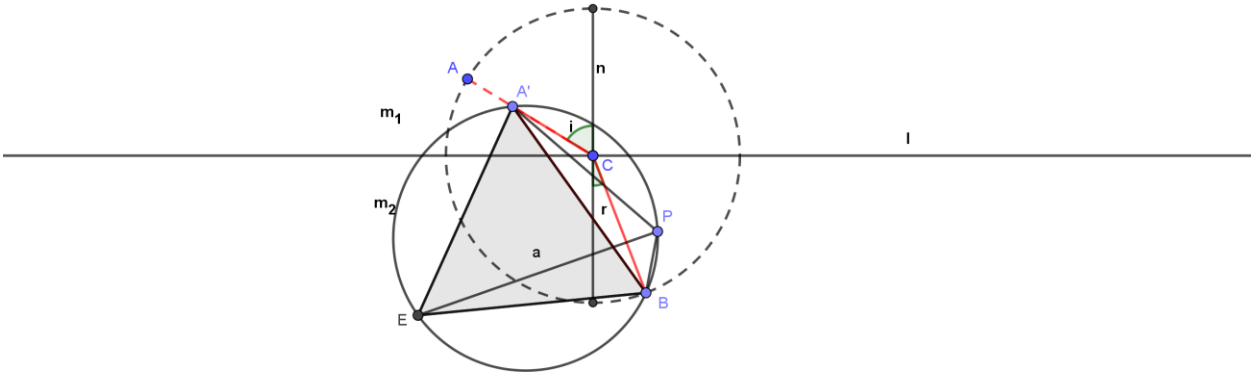

Fig. 8 The figure shows an arbitrary point P on the circumcircle A’BE. According to PT (A’P + PB) is a minimum. According to FLTP (A’C + CB) is a minimum.

According to PT, (A’P + PB) is a minimum.

$$(A'P + PB) \quad \text{is a minimum (PT)} \qquad (14)$$

However, according to FLTP we get

$$(A'C + CB) \quad \text{is a minimum (FLTP)} \qquad (15)$$

Since P can be anywhere on the circle, we choose it as the intersection point of the circle and the line through E and C.

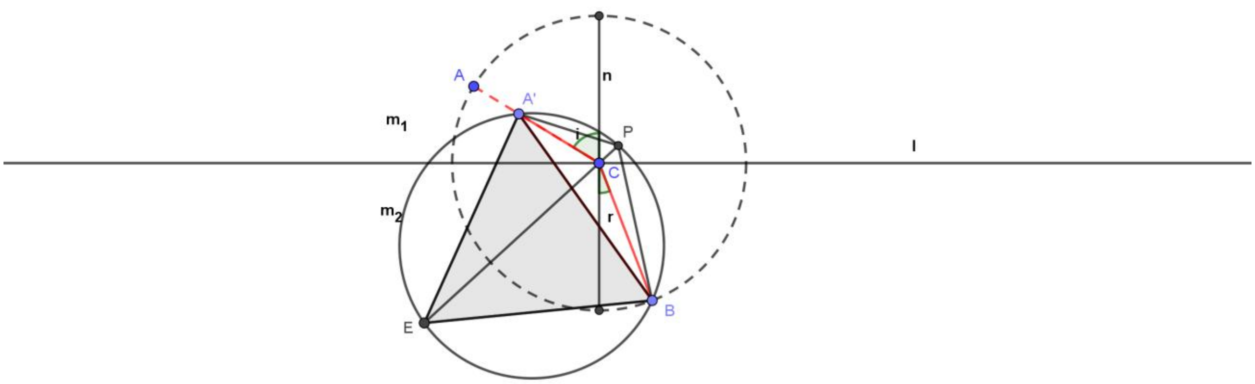

Fig. 9 The figure shows P chosen conveniently on the line through E. C to show the impossibility of (A’C + CB) being a minimum as demanded by FLTP.

Clearly, A’P > A’C, PB > CB. Therefore, (A’P + PB) > (A’C + CB). Or in simple terms, since C does not lie on the circle. This violates PT. It is impossible for (A’C + CB) to be a minimum without violating PT.

Thus we prove that if the time of travel is a minimum for refraction, then it violates PT. Therefore, we conclude that FLTP is invalid as it contradicts PT– a fundamental geometric theorem.

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