Synthese (2014) 191:4201–4229
DOI 10.1007/s11229-014-0526-y

An empirically feasible approach to the epistemology of arithmetic

Markus Pantsar

Received: 2 December 2013 / Accepted: 10 July 2014 / Published online: 27 August 2014

Abstract Recent years have seen an explosion of empirical data concerning arithmetical cognition. In this paper that data is taken to be philosophically important and an outline for an empirically feasible epistemological theory of arithmetic is presented. The epistemological theory is based on the empirically well-supported hypothesis that our arithmetical ability is built on a protoarithmetical ability to categorize observations in terms of quantities that we have already as infants and share with many nonhuman animals. It is argued here that arithmetical knowledge developed in such a way cannot be totally conceptual in the sense relevant to the philosophy of arithmetic, but neither can arithmetic understood to be empirical. Rather, we need to develop a contextual a priori notion of arithmetical knowledge that preserves the special mathematical characteristics without ignoring the roots of arithmetical cognition. Such a contextual a priori theory is shown not to require any ontologically problematic assumptions, in addition to fitting well within a standard framework of general epistemology.

Keywords Arithmetical cognition · Philosophy of mathematics · Epistemology · Empirical study · Contextual a priori 1

1 Introduction: a wish-list for an epistemological theory

In this paper I wish to propose a framework for an empirically feasible epistemological theory of arithmetic. The bulk of the paper focuses on explaining what that empirical feasibility entails, but we should first establish the context by specifying what we generally desire of an epistemological theory of mathematics. While there is bound to be little consensus over any comprehensive and detailed list of criteria, it seems that

M. Pantsar (✉)
Department of Philosophy, History, Culture and Art Studies, University of Helsinki, P. O. Box 24, 00014 Helsinki, Finland
e-mail: markus.pantsar@gmail.com

at least the following five conditions should be fulfilled by an epistemological theory of mathematics:

(1) It should not require any unreasonable ontological assumptions.
(2) It should be epistemologically feasible as part of a generally empiricist philosophy.
(3) It should be able to explain the apparent objectivity of at least some mathematical truths.
(4) It should not make the applications of mathematical theories in empirical sciences a miracle.
(5) It should not rid mathematics of its special character.

A short description of each criterion is in place. The condition (1) is particularly important. It may seem trivial since there are such wide differences between the views that different philosophers consider to be ontologically reasonable. But this lack of consensus actually gives us a useful criterion for an epistemological theory of mathematics. Taking into account all the ontological difficulties, it makes sense to avoid ontological assumptions as far as possible when we are primarily concerned with epistemology. For many philosophers, a platonist ontology would be unreasonable. But for some, denying the existence of mathematical objects would be equally unacceptable. Having such ontological assumptions play key roles in an epistemological theory could be needlessly limiting, and thus the approach here is to formulate as ontologically neutral an epistemological theory as possible.

The purpose of criterion (2) is to include the epistemology of mathematics as part of general epistemology. This does not mean that mathematical knowledge could not be essentially non-empirical. Instead, the main idea is that such non-empirical character can never be simply assumed. As a guideline, the study of mathematical knowledge should not include assumptions we would not be ready to accept for *other* modes of knowledge. While there is much disagreement in epistemology, it should be safe to say that it is now widely accepted that a general philosophical theory of knowledge should be fundamentally empiricist.

Criterion (3) deals with the simple fact that mathematicians as well as laymen often get the impression that mathematics deals with objective truths. Whether or not this is an illusion—and here different areas of mathematics must be dealt with independently—it is something that an epistemological theory should be able to explain.

With criterion (4) we move on to the subject of applicability. The various scientific applications of mathematical theories are often (e.g. Field 1980) considered to be the strongest argument for mathematical realism. Whether or not we agree with that, applications are an important problem in epistemology. There is obviously some way in which knowledge of mathematics enhances our knowledge of the world, yet there are common fictionalist and conventionalist understandings according to which mathematical knowledge is not *about* the world in any substantial sense. This connection has to be somehow included in a satisfactory epistemological theory.

Condition (5) focuses on mathematics as an intellectual discipline. As such, it is clearly special. Regardless of its ultimate subject matter, mathematical research appears to have an a priori character that must be explained by an epistemological theory.

These five conditions are not meant to be a comprehensive list based on the philosophical literature, although they can be recognized in the goals of most epistemological accounts. Further criteria may be required, but at the very least they should provide us with a good starting point. The conditions also point out difficulties in particular epistemological theories. It is clear that, for example, platonist epistemology faces serious issues with criteria (1) and (2). Criterion (5) is a challenge for empiricist approaches, whereas a conventionalist will need to explain questions concerning criteria (3) and (4) with particular care.

However, rather than go into such considerations in detail, the purpose of the criteria here is to test the outline for an epistemological theory presented in this paper: that arithmetical knowledge is contextual a priori in character. But that theory is based on fulfilling an additional criterion, one that is starting to be increasingly accepted in philosophy of mathematics, although still often ignored: empirical feasibility.

Empirical feasibility must not be confused with empiricist epistemology. If the best empirical data on mathematical cognition implies that we should pursue, for example, a platonist epistemology, then that is what an empirically feasible epistemological theory should be like. The motivation for including empirical feasibility as a criterion is thus not connected to any particular epistemological theory. It depends only on whether we consider the empirical data on mathematical cognition to be strong and reliable enough to warrant including it as a criterion. I believe that this point has been reached. In the next section I will present an overview of some of the philosophically relevant empirical results on arithmetical cognition. But from now on, we work with a sixth criterion for an epistemological theory of mathematical knowledge:

(6) It should be empirically feasible: the best scientific data about mathematical cognition should not be in conflict with philosophy.

Before we move on to the scientific data, a short disclaimer is in place. I do not believe that the current set of data establishes conclusively the results I will later use as the criteria of empirical feasibility. While some of the data is indeed very strong and the theories used to explain them widely supported, at the moment we are still in a phase in which new and surprising discoveries are made constantly.

If one is highly skeptical about the empirical data, perhaps this paper is not best understood as arguing for the correctness of a certain epistemological theory. Instead, it should be seen as a fundamentally hypothetical pursuit: if the best-supported theories in psychology, cognitive science and neurobiology are correct, what kind of an epistemological theory of arithmetic do we need? This by itself should be worth studying, especially since we have so little other data to base epistemology of mathematics on.

2 Empirical data

The state of the art in psychology, cognitive science and neurobiology is that human beings are not the only animals able to deal with numerosities.[1] In the words of the neurobiologist Andreas Nieder (2011, p. 107):

[1] Here is a case where the empirical scientists often are less than careful about equivocating terminology. To talk about natural numbers is premature at this primitive stage, as many scientists note. Hence the term

> Basic numerical competence does not depend on language; it is rooted in biological primitives that can already be found in animals. Animals possess impressive numerical capabilities and are able to nonverbally and approximately grasp the numerical properties of objects and events. Such a numerical estimation system for representing number as language-independent mental magnitudes (analog magnitude system) is thought to be a precursor on which verbal numerical representations build, and its neural foundations can be studied in animal models.

This passage captures well the current situation in the empirical study of proto-mathematical thinking. It has been established time and again that many nonhuman animals and human infants have a basic ability to recognize numerosities. Instead of simply focusing on duration, size or other magnitudes, experiments have shown a common propensity toward processing observations based on the quantity of the objects. This ability was detected long ago in primates like chimpanzees, but also in rats and even small fish (Rumbaugh et al. 1987). The ability of chimpanzees may not be that surprising given their relatively developed linguistic abilities, but one would not expect rodents or goldfish to be able to deal with numerosities. However, rats can learn to use small numerosities quite impressively. They can learn to press a lever a certain amount of times to get food, also when the duration of the presses is varied. They can distinguish the numerosity of tones from the total duration of them. The experiments have been controlled for other variables and there is little doubt that rats not only have the ability, but also a natural tendency to process observations in terms of numerosities (Mechner 1958; Mechner and Guevrekian 1962; Church and Meck 1984). The case of fish is even more surprising, yet the evidence is strong also for their ability to deal with numerosities. Small fish can learn to choose the right hole to go through based on the quantity of objects drawn above the hole. Even when the combined surface area and illumination of the objects was the same, mosquitofish were able to make the distinction, clearly pointing toward ability to process observations in terms of numerosities (Agrillo et al. 2009).

The ability to recognize numerosities, which many nonhuman animals have, is an intriguing phenomenon, but it has also been found that newborn babies at very early stages show a similar ability. The groundbreaking work when it comes to infants was Wynn (1992), in which she showed that babies reacted to instances of the unnatural arithmetic of $1 + 1 = 1$ in experimental settings. When the infants saw one plus one dolls placed behind a screen, they expected to see two dolls and were puzzled when there was only one. This tendency has later been established to occur in many variations of the experiment: including ones in which the size, shape and location of

Footnote1 continued
"numerosity", referring to a primitive conception of a discrete quantity. In the literature, numerosities are often referred to as *cardinalities*, but this can also be confusing since "cardinality" has an established meaning as a technical term in set theory. Another common confusion is to call primitive processing of numerosities "arithmetic". In this paper, arithmetic means a system with explicit rules and number symbols (or number words). For the more primitive processing of numerosities, it is better to talk of *proto-arithmetic*.

the dolls are changed. The infants still found the changing quantity most surprising (Simon et al. 1995 and Dehaene 2011, pp. 40–48).[2]

There are many more experiments which point toward the primitive ability to deal with numerosities, and I will not present further examples at this point. The recent second edition of Stanislas Dehaene's classic *Number Sense* is a good source to the early experimental material, also providing updated references to subsequent studies. Dehaene and Brannon (eds.) (2011) is a comprehensive collection of recent advances in the field. But the empirical study of proto-arithmetical cognition is a growing and highly active area of research. New relevant data is being published constantly and it is not possible to give here a comprehensive account of the current state of the art in the field.

It is beyond doubt, however, that during the past decade the amount of research on these subjects has developed in immense steps. Recently, researches have even been able to locate specific neurons in the monkey brain that activate when the animal is processing objects in terms of a certain numerosity. The same neurons activate independently of the sensory modality, i.e., regardless of whether the monkey sees or hears three things, or indeed sees the number symbol 3 (Nieder et al. 2002; Nieder and Miller 2003; Nieder and Dehaene 2009; Nieder 2012, 2013). Such supramodality of numerosities is a highly interesting result and it could be at the very root of what we understand as the abstractness of natural numbers (Nieder 2012, 2013).

At this point it seems hardly worthwhile to contest these findings. Rather, we should accept that processing observations in terms of numerosities is a much wider phenomenon than scientists and philosophers used to believe. Previously, even primitive forms of mathematics were thought to be an exclusively human affair, and one that takes years to develop in children.[3]

---

[2] I believe that the many replications of the Wynn experiment quite clearly show that the infants acted based on the quantity of objects they saw. But at the same time, it seems that Wynn and others are hasty in making the conclusion that the infants are adding and subtracting. They could equally well just be keeping track of the quantity of the objects they expect to see, thus holding only one numerosity in their minds. Postulating the ability to do arithmetical (or even proto-arithmetical) operations is not necessary, and in fact quite problematic.

[3] While the nonhuman and infant abilities with numerosities are widely accepted in modern cognitive science and psychology, not everybody subscribes to it. Kelly Mix, in particular, has been a prominent critic of the ability to deal with numerosities in infants, claiming that it is continuous magnitudes rather than discrete numerosities that the infants respond to. I cannot go here into the details (for one part of them, see Mix et al. 2002) but it should be noted that many experiments seem extremely unlikely to fall into such equivocations. For example, experiments are often controlled so that the total visible surface area of objects remains the same when their quantity is changed. One criticism by Mix (2002), however, is philosophically particularly interesting and should be addressed here:

> Just as animal researchers are at risk for anthropomorphizing their non-human subject, infant researchers are at risk for overlaying adult reasoning on basic perceptual processes. Longer looking times are commonly interpreted as evidence that infants have formed an abstract representation, compared it to a test stimulus and effectively said to themselves, 'Hey, that's different!' or 'How surprising!'

While it is indeed important not to postulate excessive cognitive capability to the subjects, what Mix criticizes is the exact thing that proponents of proto-arithmetical ability in infants and animals *deny*. They do not claim that infants make abstract representations which they discuss within their minds. Rather, this process happens automatically.

In hindsight, such findings should not have been particularly surprising. The modern history of biology reminds us constantly that human beings are much less special than previously thought. If dealing with numerosities is useful for us, it makes sense that it is beneficial for other animals, as well. Is there any better way, for example, to keep track of one's offspring than somehow recognizing the quantity of them? And if a human ability is present in animals, it is likely to be present in some form already in human infants.

Looking at such empirical results, we can immediately see potential philosophical relevance. If there are ways of dealing with quantities independently of language, in the philosophy of mathematics we should be interested in such findings. After all, although it has been losing popularity recently, one of the most prominent philosophical theories concerning mathematics has been conventionalism. Strict conventionalism takes mathematical statements to consist ultimately of human conventions and as such not to be about the world in any substantial sense. But it would appear that any strong form of conventionalism must have a tight connection to language-dependency of mathematics. The nature of conventions is a complex issue, but in no reasonable account can rats and infants be understood to process numerosities based on conventions.

However, the basic ability to estimate and process quantities is of course nothing like the arithmetic of natural numbers as we understand it. The relation between results of psychology and neurobiology and the philosophy of mathematics is not a straight-forward one. In the above quotation of Nieder, we can locate perhaps the most important question in importing such empirical results into philosophy:

> Such a numerical estimation system for representing number as language-independent mental magnitudes [...] is thought to be a precursor on which verbal numerical representations build

It is important to remember that we are indeed dealing with an estimation system. In the jargon of cognitive science this is often called the *approximate number system*, or the ANS. The ANS can be thought to consist of two main parts. First is the ability to *subitize*, that is, to immediately determine, without counting, the numerosity of objects in one's field of vision. Second is the "analog magnitude system" , a way of keeping track of numerosities in the working memory (Dehaene 2011; Brannon and Merritt 2011; Nieder and Dehaene 2009).

While the existence of such abilities in nonhuman animals and infants (as well as in adult human beings) has been an important discovery, it should not be confused with discovering mathematical thinking in animals. The estimation system is very limited: when dealing with more than three or four objects, the ability quickly starts to lose accuracy (Dehaene 2011, pp. 17–20). In addition, this ability is best described as quasi-analog: the representation of quantities is only discrete with small numerosities and gets more and more continuous as the quantities get bigger, thus making the primitive ability to deal with numerosities fundamentally distinct from developed arithmetic. All this is clearly different from actual arithmetical thinking, which deals (at least for a very large part) with the verbal numerical representations that Nieder mentions. And while Nieder is no doubt correct in stating that the primitive ability is among the empirical researchers generally thought to be a precursor to the actual mathematical ability, at this point this is still in considerable part a conjecture.

Philosophically, this conjecture can be absolutely crucial. Take the above sketch of an argument against conventionalism, based on the non-linguistic character of the ANS. Clearly that argument would be refuted if it turned out that our developed mathematical ability is *independent* of the primitive non-linguistic numerosity system. This way, the question here is not whether ANS exists or whether it can be *described* as proto-mathematical ability. The question is whether it is plausible that our arithmetical thinking actually *develops* from the ANS.

How could such a question be studied? The first idea is of course to track the development of arithmetical thinking and examine whether it is continuous in the way that Nieder assumes. This, however, includes an important problem. As it happens, the primitive ability does not disappear when our mathematical thinking develops. This is best seen in the phenomenon of subitizing. Adult human beings subitize and their ability closely resembles the primitive ability of animals and infants. Thus it appears that adult subjects have two systems of dealing with numerosities: one an approximate estimation tool of magnitudes, the other a language-dependent system that we use in learning arithmetic.

That, however, cannot be thought to be a refutation of the argument against conventionalism. The existence of the two systems should not be confused with the latter not developing out of the former. If the verbal numerical presentations are indeed built on the primitive system of mental magnitudes, it is irrelevant to the conventionalist debate that both systems continue to exist. In fact this would hardly be surprising: people continue to have all kinds of intuitions and primitive conceptions even when they develop a better knowledge about the subject. Optical illusions are one case in point. Even though we perfectly well know that two lines are of equal length, in some circumstances we cannot help seeing one as shorter. Yet we would never claim that our ability to establish the length of the lines was not built on seeing the lines. We simply accept that our impressions of objects can be different when the circumstances change. Obviously the same thing can happen with numerosities: we can continue to have rough estimates of quantities even when we have developed tools to process them in an exact manner.

Fortunately, the questions presented above have been studied empirically—and the results are philosophically highly relevant. What happens in the brain when we observe objects? The full story is too long to be told here, but the philosophically important main idea is that the brain is full of different types of "neural filters" . When we see something, for example, an enormous amount of activity goes on in the brain so that we can gather the relevant information from our field of vision. This is why it has been tremendously difficult to develop visual recognition in robots. In order to separate the relevant parts of the visual field from the irrelevant ones, the robot has to be programmed in excruciating detail. Our brain—and for that matter, the rat or even the fly brain—does such things automatically because it is accommodated to recognize the aspects that are important. Crucially to the matter at hand, part of this activity has to do with quantities (Nieder 2011).

A lot is known about the primitive ability to deal with numerosities. It is known, for example, that it is much more accurately modeled logarithmically than linearly. It is easy to tell the difference between 1 and 2, but harder between 4 and 5. At 19 and 20 monkeys (and us) are hopeless at recognizing the difference without using more sophisticated tools, i.e., counting. The difference between 10 and 20, however, is easily established. Thus the ratio of numerosities models our natural ability better than the difference between them and hence the ability is better described as roughly logarithmical.[4]

That much we know from observing the behavior of subjects in the experiments. Now the interesting question is whether that is mirrored also in the way the brain processes numerosities. As it turns out, there is a remarkable resemblance. We can know that because, as described in Nieder (2011), it is now established that not only are there distinct areas of the brain where quantities are processed, but also specific neurons which represent certain quantities. When the monkey is presented with two objects, a specific group of neurons activate. When the numerosity of objects is three, a (partly) different group is activated. The experiments have been controlled for other variables, and the scientists have been able to tease out the effect of a particular quantity in the monkey brain. The brain, however, is a complex organ, and while there are specific neurons for each small numerosity, those neurons do not activate completely discretely. When the neurons for the numerosity "two" are activated, so is a small part of the neurons for "one" and "three" . And just like the behavior of monkeys predicts, as the numerosities become larger, the bigger the "noise" is between the different groups of neurons. Distinguishing between four and five is much more difficult than between one and two because in the former case more of the same neurons activate. Our natural capacity to deal with numerosities is one of approximate estimations that loses accuracy as the quantities become larger. What happens in the brain mirrors this.[5]

That is not all. Unlike many other animals, monkeys have (through extensive learning) the ability to understand symbols assigned to concepts, including numerosities. Diester and Nieder (2007) established that to large part the same neurons in the prefrontal cortex were activated regardless of whether the monkey saw two objects or the symbol 2.[6]

So far we have been dealing with primitive forms of dealing with numerosities based on subitizing and the analog magnitude system. But what happens when we are counting? Undoubtedly this is fundamental to our developed capacity to deal with numerosities and enables us to formulate the exact nature of numbers. In Nieder et al. (2006), monkeys were presented objects one by one, to simulate a non-verbal account of counting. As expected, there were differences in the parts of the brain that activated compared to the task of seeing a group of objects at once. However, the study found that at the end of the enumeration, a large part of the activated neurons were the same as with the ANS. In short, when counting, the monkey was dealing (partly) with the same representations for numerosities as with subitizing.

[4] The ability is not properly logarithmical, however, as distinguishing between two and four objects is easier than between 8 and 16 objects. See Dehaene (2011, chap. 4).

[5] In addition to numerosities, similar results have been acquired in the study of proportions (Nieder 2011).

[6] Interestingly, in the intraparietal sulcus, which is another part of the brain associated with numerosity, the number signs triggered much less activity. This suggests that while there are clear connections between the number symbols and the numerosities of the ANS, these are different in the two areas of the brain.

What do these results suggest our ability with numerosities to be? The best hypothesis is that our basic experiences about small quantities are given by the approximate number system. As we develop the linguistic ability to count, we no longer require the ANS for our numerical ability. Indeed, even though we never completely lose the ANS, it could be said that when it comes to mathematical thinking, the primitive ability is replaced by a language-based one. This gives us a lot of added expressive power, which makes arithmetic as we know it possible.

Obviously these subjects need a lot of further study. But the data we currently have is already relevant for the philosophy of mathematics. Let us consider the earlier ANS-based argument against conventionalism and the difficulty it faced on the basis that mathematical ability may be distinct from the primitive numerosity system. We now know that there are important connections between the ANS, the symbolic presentation of numerosities, and counting. There is starting to be way too much correlation to be explained away simply as a coincidence. The data clearly points to the direction that our verbal ability to deal with numerosities was built to accommodate the primitive non-verbal system. That is of course nothing extraordinary. Although there are different areas of the brain involved in subitizing, counting and recognizing symbols for numerosities, it would be surprising if there were no connections between them at all. The brain is in general built to facilitate learning and the existing information is used to assimilate new data. If we have one mode for dealing with numerosities, as we initially do, it would seem highly unlikely that the brain starts to build another one completely from scratch instead of utilizing the existing connections.

In conclusion, the current set of empirical data points to philosophically interesting theories about our capacity to deal with numerosities. At this last stage, however, we skipped past an important point. One crucial question is of course whether these results concerning the monkey brain can be applied to the *human* brain. Evidence points strongly toward this indeed being the case. In Piazza et al. (2007), it was found that when observing numerosities, the same areas of brain are activated in humans as in monkeys. Given the similarity between monkey and human brains, such results are hardly surprising. Naturally, differences should be expected when the primitive numerosity system is little by little supported and finally largely replaced by the verbal capacity to deal with numerosities. But the primitive approximate number system does not vanish. There have been many experiments that show college students to have similar patterns as monkeys and rats when solving number-ordering and quantity estimation tasks. It seems that even people educated with discrete arithmetic often cannot help reverting to the primitive approximate estimation system (Cantlon and Brannon 2006).[7]

With all the above data, it seems highly plausible that our capacity to process quantities has a shared origin with many nonhuman animals, and that it is connected to an ability that we have already as newborn infants. Of course there are many questions that still need to be answered about these origins and their development, and the

[7] See also Brannon and Merritt (2011) for a good overview of the subject. Research on an Amazon tribe with limited numerical lexicon has also established that education in verbal numerical thinking enhances the acuity of the ANS, thus suggesting that the connection between the ANS and language-based numerical thinking goes both ways (Piazza et al. 2013).

answers should not be treated as a foregone conclusion. Still, I believe it is definitely worthwhile to consider the philosophical consequences of the modern developments in psychology, neurobiology and cognitive science. The questions that interest philosophers most in this context concern the development from the approximate number system to axiomatic theories of arithmetic. At this stage, aside from the empirical work, we must start asking fundamentally philosophical questions. No amount of work on even the most brilliant monkeys or infants will present us with anything close to the human adult ability in mathematics. We now have a good idea what arithmetical knowledge is initially based on. What we need to answer accordingly is what arithmetical knowledge *is*.

3 Empirically feasible epistemology

Based on the data presented in the previous section, what kind of an epistemological theory of arithmetic is empirically feasible? The only clear-cut requirement would seem to be that the theory cannot contradict with the proto-mathematical ability of ANS. If we accept that ANS is indeed the origin of our arithmetic, then arithmetic knowledge cannot be completely conventional. I propose here that it also follows that arithmetical knowledge cannot be completely *conceptual* in the language-dependent sense relevant to the current context.

There is much debate on what we should understand by the concept of "concept". It is not possible to go into that discussion here, but for the purpose of this paper it makes sense to distinguish between concepts in a *primitive* sense and a *linguistic* sense. The simple argument for that position is that whatever capacity the goldfish, for example, uses in its ability to distinguish between small numerosities, it is not the same as the capacity that human mathematicians use in practicing arithmetic. The primitive proto-arithmetical numerosity concepts given by the ANS should not be confused with properly mathematical concepts, which are more well-defined and precise, something that require a language to process.[8]

With this understanding of properly mathematical concepts as language-dependent, based on the considerations in the previous section, an empirically feasible epistemology would seem to require that arithmetical knowledge is not *only* about those concepts. Although there may be a great amount of purely conventional content in arithmetical concepts, the data on ANS suggests that our conventions cannot explain *all* there is to arithmetical knowledge. Arithmetical concepts seem to be at least partly determined by the proto-arithmetical capacity we share with many animals. Thus, to get a full understanding of the nature of arithmetical knowledge, we cannot limit ourselves exclusively to the developed language-dependent arithmetical concepts.

However, the claim that arithmetical knowledge is not only about language-dependent concepts obviously prompts the question what *else* it is about. One natural

[8] It is a matter of debate whether we should accept there being non-linguistic concepts in the first place. Fortunately, the particular definition of "concept" does not change the essential argument here. The key question is whether we accept that there is a relevant difference between dealing with language-dependent mathematical concepts and the proto-arithmetical ability. I am confident that few are ready to deny the existence of such difference.

alternative is of course that it is also about *objects*. Along these lines, Burge (2007, 2010) has recently evoked empirical data in support of his argument that the objects of arithmetic can be represented in a *de re* fashion, i.e., the reference for numbers in arithmetical sentences is not totally linguistically conceptual, *de dicto*. He uses forms of subitizing as an example of the way we use numbers without using numerals (2007, pp. 74–75):

> …these types of *de re* representation are associated with special immediate, non-descriptive powers - understanding and immediate perceptual applicability.

However, it seems that Burge's argument uses a mistaken interpretation of the empirical data. Whereas I do agree that the empirical data presents a serious challenge for any philosophical account that takes mathematics to be purely language-dependent, this does not imply that in subitizing we refer to numerosities as *objects*. Surely there are various ways in which processes like subitizing can be explained without there being such objects as numbers. Azzouni (2010, p. 33) suggests, for example, that we may have a natural tendency to put objects into one-to-one comparison and thus be able to subitize between small quantities. There is some empirical evidence that such a process indeed forms at least part of our ability to represent objects in the working memory (Leslie et al. 2008; Feigenson 2011). But even if this were false, it is easy to propose other plausible explanations. What if instead of numbers, whether *de dicto* or *de re*, we think of the primitive numerosities as dealing with comparative concepts like *equal*, *more* and *fewer*?

In this mode of explanation, when we subitize a group of, say, two objects, we do not arrive at an abstract object, the number two. Rather, we form a kind of baseline numerosity. When then presented with a group of three objects, we naturally subitize the concept "more". With four objects, "a lot more", and with one object, "fewer". As we keep subitizing, these comparative concepts start to form into numerosity concepts in the brain, leading to ultimately there being specific groups of neurons corresponding to each small numerosity.

This is one very hypothetical explanation and I do not claim that these concepts correspond to anything that actually happens in the process of subitizing, although there is evidence that something like that is actually going on (Lourenco and Longo 2011). The point is that such simple comparative concepts would be *enough* to explain the ability to subitize, remembering that it starts to radically weaken for groups of more than four objects. There is no need to evoke numbers as objects.

Such arguments, however, do not imply that there is no objective basis for our ability to deal with numerosities—only that the objects that we are concerned with in arithmetic (or proto-arithmetic) are not necessarily numbers. In the philosophy of mathematics, this kind of argumentation is of course nothing new, as there is a well-established distinction between the objectivity of truth-value and the existence of mathematical objects. The former, endorsed by, e.g., Kreisel[9] is what I mean by

[9] Dummett (e.g. 1973, p. 228) has made this "Kreisel's dictum" famous: "The point is not the existence of mathematical objects, but the objectivity of mathematical truth," although the quote is not known to be found in Kreisel's own writings. The quotation as attributed to Kreisel can only be found in Putnam (2004, p. 67).

arithmetical objectivity here and ANS would seem to be a good fit with such a truth-value-based conception of objectivity. It is a common position in philosophy to believe that statements like "$2 + 1 = 3$" are true without there existing such things as natural numbers. Based on the rejection of the existence of mathematical objects, some philosophers have argued that the truths of such statements are merely a matter of convention—in which case they would clearly not be objective in an epistemologically relevant sense.

If there is, however, good reason to believe that the truth of arithmetical statements is something more than a human convention, the truths would seem to be objective in a sense highly relevant to philosophy. But this is exactly what the empirical data suggests. Statements like "2 + 1 = 3" are true not only based on convention, but also because the statement corresponds to something in how our brains are hardwired to process numerosities. This way, I argue, the research on ANS clearly points toward an objective basis for arithmetic, even if we do not evoke numbers as objects.[10]

The question of objects and objectivity is related to various basic epistemological problems. Objectivity comes in many forms and it is not clear that all philosophers would accept such physiological propensities toward processing quantities as enough to give arithmetic an objective basis. Such thinking, however, seems quite problematic. If there is a basis for arithmetical knowledge that we already have as infants and share with many nonhuman animals, it is hard to see what more we could reasonably require in terms of objectivity from an epistemological theory of mathematics. Many mathematicians are convinced that they reach truths about platonic objects with their research and the present understanding of objectivity is clearly weaker. However, the platonist will have a hard time persuading more formalistically minded people of that kind of objectivity. After all, the formal part of mathematics, i.e., proving theorems from axioms, can be explained also from a strictly conventionalist basis.[11] But if there are features in our brain structure that make us process observations in certain proto-mathematical ways, and mathematics is based on those processes, the strict conventionalist case is suddenly much weaker. Infants and animals cannot be said to process numerosities based on conventions. If the ANS is behind our arithmetical ability, we should think of arithmetic as having an objective basis, one determined by our physiological structure which we share with many animals.[12]

Above I have argued that if arithmetical knowledge is based on ANS, we should consider it to be objective, albeit in a weaker sense than understood in the platonist tradition. But so far we have only discussed ANS and the philosophical consequences of ANS-based arithmetic. Before we go any further, we should ask *how* arithmetical knowledge develops based on ANS. It seems clear that in order to get a conclusive

[10] In the philosophy of mathematics, such objectivism without objects is not a new suggestion in mathematical ontology. If we take a structuralist approach to mathematics, the focus can be turned from objects to characteristics that arithmetical structures have. Among structuralists, however, there is great disagreement about the ontological status of mathematical structures, ranging from the objectively existing *ante rem* structures of Resnik (1981) and Shapiro (1997) to the modal constructions of Hellman (1989).

[11] As before, by strict conventionalism I understand the position that mathematical statements are ultimately mere human conventions not based on any stronger objective basis.

[12] I acknowledge that this understanding of objectivity is somewhat non-standard, but it seems to be the relevant one when it comes to the present arithmetical context. See Footnote 20 for further analysis.

answer, we would need to have a much better empirical understanding of the development of mathematical thinking than there currently is. However, there are already some empirical results which suggest an answer. The hypothesis that ANS is the foundation for arithmetical thinking includes several predictions which have received corroboration from experiments. For instance, studies have shown that we do not lose ANS when we develop symbolic means to deal with quantities (Butterworth 2010; Spelke 2011). In both older children and adults there is a strong correlation between improved performance in symbolic mathematics and the non-symbolic processing of quantities. Furthermore, it is known that the same brain areas activate when dealing with symbolic and non-symbolic quantities, and that the corresponding areas also activate in the brains of non-human primates (Piazza 2010). Much of the current data suggest a fundamentally simple and coherent picture. Starting very soon after our birth, we have a non-linguistic ability to deal with small quantities. This ability is then developed to actual arithmetical ability, in a process where the development of language is likely to play an important role (Spelke 2011).

Exactly how this happens is still largely unknown, but there is evidence that grasping the general idea of *successor* is central in this development. Based on subitizing we have the ability to distinguish between small quantities, usually from one to four. This means that we have different neural representations in our brain for those small numerosities. Learning number words for the small cardinalities comes naturally under such circumstances, as they correspond to existing neural representations. For understanding larger number words, a highly plausible hypothesis is that children grasp the idea that these numbers form a progression that can be continued (Butterworth 2010). When one is added to three, even an infant can tell that the numerosities are different. But this ability comes from the approximate number system and it is only later that the child learns that adding one to larger numerosities is similar; that there also is a distinct numerosity for the end product of that process (Feigenson 2011). From that there is a short way to addition of two numerosities, which in turn enables the process of multiplication and so on.[13]

The above general hypothesis for the development of arithmetical knowledge seems to be highly plausible and receives substantial backing from empirical data. But just how relevant should such a hypothesis be in the philosophy of mathematics? We should be particularly careful at this point, because building epistemological theories of arithmetic based on empirical data is an extremely tricky endeavor. Carey (2009a), for example, has proposed such a theory in which she distinguishes between the "logical" and "ontogenetic" programs of accounting for the origin of arithmetic. Roughly put, the distinction is that there are two kinds of building blocks. The logical building blocks are the logically necessary prerequisites in order to have some arithmetical capacity. The ontogenetic building blocks are the actual capacities used in the historical development of arithmetic. Importantly, Carey holds that the ontogenetic blocks depend on the

[13] It should be noted that having some grasp of the idea of successor does not necessarily mean understanding the formal consequences of defining natural numbers in terms of a successor function. When a child learns to count beyond the first few number words, she clearly understands something about numerosities forming a succession. Yet at that stage she is unlikely to grasp the full meaning of the successor-based nature of numerosities, e.g., their infinity.

logical ones on characterizing the capacity. We can use, for example, the Dedekind–Peano axioms to characterize what is meant by numerosities in the ontogenetic building blocks—presumably also in the hypothesis based on the ANS.

This kind of argumentation, however, is extremely problematic since it seems to simply assume what is the most important matter that an epistemological theory should argue for: that we are speaking of numerosities in essentially same sense in the ANS-based developmental theories as we do in the developed arithmetical theories. Clearly the primitive capacity to deal with numerosities is very different from our developed arithmetical ability, and the key question in the whole matter of empirically feasible epistemology of arithmetic is whether there is a continuous development from the former to the latter—and if so, what this development entails. I do not claim that we know the full answer, and as long as we do not, we should be careful about making the kind of assumptions that Carey seems to make.[14]

Moreover, I believe that we should as a methodological guideline refrain from mixing the logical and ontogenetic aspects. Even if we were convinced that the ANS is the basis for our arithmetical ability, we should be interested in how it develops into accepting systems like the Dedekind–Peano axioms. When we simply draw properties from the latter to describe the former, we might be seriously distorting the ontogenetic explanations. Since, presumably, many people are not convinced that there is a continuous development from the ANS to the Dedekind–Peano axioms, this is all the more important. What we should do is take the best empirical knowledge we have and try to figure out what kind of an epistemological theory of arithmetic can be plausibly developed based on it. If it resembles closely our developed theories, we are likely to be on the right track with our ontogenetic explanations. As I have argued in this part of the paper, I believe this is quite feasible with the current set of data. I could be wrong: it needs to be stressed that there is no conclusive evidence yet. But even with the existing evidence, we should be interested in what kind of an epistemological theory we need in order to account for the empirical findings.

It should be noted that this is not just a philosophical exercise: as long as the empirical evidence underdetermines the epistemological conclusions, we need philosophical theories to fill out the picture. Of course these should not be confused with hard evidence. But if we can formulate a plausible epistemological outline of an ANS-based theory of arithmetical knowledge, at the very least we know that such accounts are philosophically feasible. In philosophy we should hardly wait for a complete neurobiological description of the development of arithmetical thinking.

[14] Carey (2009b, chap. 8) also presents an account of children learning the inductive nature of natural numbers that is based on the successor function. While I believe such an account is highly plausible, we must be careful not to postulate the inductive ability before we know when it is actually acquired. Interestingly, Carey recounts experiments with nonhuman animals which fail to make the inductive step, thus giving further evidence that while ANS may be the basis for our arithmetical ability, it is not enough by itself.

4 Contextual a priori

In the previous section I have tried to argue that in order to support the objectivity of arithmetical knowledge, an empirically feasible epistemological theory of arithmetic does not need to evoke independently existing mathematical objects. The physiological similarity of brain structures is enough to give our ability with numerosities an objective basis. It is on this similarity that arithmetic as we know it can be developed upon. Rather than being a contentious philosophical claim, this is actually such a trivial fact that it is often presupposed without argument. When children are first introduced to number words and basic arithmetic, we simply assume that they observe the world as distinct objects, as well as have the ability to process these observations in terms of discrete quantities. This assumption has been made as long as arithmetic has been taught, but only in the past few decades have we found an explanation for its success. In arithmetic, we do not introduce children to a new way of thinking. We give them the necessary conceptual knowledge to expand and make exact a way of thinking they already possessed in a primitive form.

The philosophically interesting question is, if this account based on ANS is correct, what type of knowledge do we reach this way? In particular, how does this type correspond to our prior understanding of the nature of mathematical knowledge? Is it, for example, a priori in character, as mathematical knowledge is often understood to be? Or do the primitive origins mean that it is essentially a posteriori, tied to empirical aspects?

I propose that mathematical knowledge is best understood outside of this Kantian dichotomy, as it includes characteristics from both but as a whole fits neither concept. My rejection of the dichotomy is not Quinean in character, as I believe that there are fruitful distinctions to be made between a priori and posteriori modes of knowledge. Rather, I believe that the Kantian dichotomy does not necessarily exhaust the field of such fruitful distinctions.

The question of a priori knowledge is of course actively studied and given remarkably different treatments among philosophers.[15] Since Quine (1951), it has been by no means obvious that the dichotomy between a priori and posteriori should even exist. In the Quinean web of beliefs, traditional candidates for a priori knowledge, such as arithmetic and logic, are taken to be empirically falsifiable due to their connections to scientific theories. The Quinean theory is often understood to take no beliefs as separate from the web. In the case of logic, for example, Putnam (1968) famously argued that a revision is already at hand with quantum logic.

It is not possible to go here into such general questions about the possibility of a priori knowledge. In any case, the characterization of arithmetical knowledge I defend in this paper is not a strictly a priori one. Whether other forms of knowledge may be purely a priori is another question, but here I claim that arithmetical knowledge has characteristics that do not fit what I see as a feasible notion of a priori.

Let us consider the classic Kantian characterization of a priori as knowledge that can be gained essentially independently of sense experience. Under what kind of

15 See Boghossian and Peacocke (eds.) (2000) for a nice collection of the kind of problems that engage modern philosophers in the study of a priori.

interpretation would the present ANS-based theory of arithmetical knowledge count as a priori? It cannot be denied that an arithmetical statement like $2 + 1 = 3$ undoubtedly has a strong *prima facie* appearance of being an a priori truth. The way we define the symbol "3" in axiomatic arithmetic is with the help of a successor function S. In a standard definition, we define 0 as the first natural number S(0) as the next number, then S(S(0)), S(S(S(0)))) and so on. Then we name S(0) as 1, S(S(0)) as 2, S(S(S(0)))) as 3, etc. The operation of addition we also define with the help of the successor function, in that the result of addition of $X + Y$ is applying the successor function to the number X for Y times. In the addition $2 + 1 = 3$ this simply means that we first use the function S two times on the number 0 to get the number X and then once more to add Y to it. Thus constructed, it is easy to get the impression that the symbol 3 means the same thing as $2 + 1$, because both mean applying the successor function S on the number 0 three times. Arithmetical knowledge thus presented has a distinct appearance of being a priori.

However, the matter is not that simple when we look at arithmetical knowledge in a wider context. We must remember that axiomatization was a very late development in the history of arithmetic. When we have such sophisticated systems in place, it is natural to understand arithmetic as a priori. Indeed, I will argue that once the context for arithmetical knowledge is set, it is best described as a priori. We can develop a context in which arithmetical truths are derived essentially independently of sense experience and cannot be falsified by empirical results. But in order to be strictly a priori, arithmetical knowledge would have to be independent of experience in a stronger sense. It could not depend on the empirical context set by ANS.

Earlier I distinguished between the primitive ANS-based numerosity concepts and the linguistic arithmetical number concepts. If arithmetic were only about the latter, there would be a strong case for arithmetical knowledge being a priori. But since the content of arithmetic is at least partly determined by the primitive ability to deal with numerosities, we must include them in a full account of arithmetical knowledge. Now the question is whether we can have a priori knowledge of the primitive numerosity concepts. It seems that if we can, we are opening up too many possibilities for a priori knowledge. There is strong evidence that the ability to acquire primitive numerosity concepts is innate and thus underwritten by our brain structures. If we consider knowledge about those concepts to be a priori, what is there to stop us from considering knowledge about *any* ability underwritten by brain structures as a priori? Under such interpretation, arithmetical knowledge could indeed be classified as strictly a priori. But it would seem to follow that so could knowledge about many basic facts about our linguistic and cognitive abilities, which is clearly unacceptable. Through introspection we may have privileged access to some insights about language and cognition, but it would be hopelessly antiquated to insist that these subjects can be studied without empirical means and that our knowledge of them does not consist primarily of empirical *facts*.

The fact that we should study a type of knowledge by empirical means, however, does not imply that it could not be a priori in character. Furthermore, clearly also a priori modes of knowledge can be connected to empirical aspects. Although a feasible definition of a priori knowledge would seem to require it to be essentially independent of empirical experience, this is not to suggest that no sense experience at all is required of acquiring the knowledge. Obviously we first need to hear or read about concepts in order to acquire knowledge of them. We would not be able to know that, say, all bachelors are unmarried, if we had no empirical access whatsoever to the concepts involved.

But this is clearly different from the kind of basis that the ANS gives for our arithmetical concepts. We have proto-arithmetical concepts long before we are able to understand number words. Indeed, the proto-arithmetical ANS shapes the very way we experience the world. If the ANS-based theory for arithmetical knowledge is correct, arithmetical concepts are constrained empirically by the ANS in a way which is essentially different from the way the concepts involved in statements like "all bachelors are unmarried" are. Even if the latter could be feasible characterized as strict a priori knowledge, the former could not.

It may seem plausible that we could gain arithmetical knowledge outside of the empirical context of ANS, but this is due to the traditional philosophical understanding of the subject, rather than any evidence or argument. The modern evidence suggests a different picture where ANS is fundamentally linked to the ability to develop the linguistic concept of natural number. When we learn arithmetic, we do it after years of experience in dealing with numerosities in a primitive way, based on an ability we share to a large extent with other animals. Arithmetically speaking, of course, the origins of arithmetical knowledge are irrelevant. But a full *philosophical* analysis of the arithmetical fact, say, that $2 + 1 = 3$ would seem to require including the ANS-based origins. For all the appearance of being a priori, I will argue, arithmetical knowledge seems to be better understood as a priori only in that context. Since the context is set by the ANS-based experience of observing the world, it should be understood as essentially empirical.

It should be noted that I do not claim to *refute* the possibility of strictly a priori arithmetical knowledge, just like the epistemological account based on ANS is not meant to refute platonism. Even if the way we acquire mathematical knowledge is not purely about the linguistic concepts, there remains the possibility that we have a special mathematical a priori faculty for gaining knowledge of non-linguistic conceptual truths.[16] However, the purpose of this paper is to propose an explanation of arithmetical knowledge on ontologically and epistemologically minimal grounds. If left unanalyzed, in such a pursuit special faculties for mathematical knowledge are no more acceptable than platonic objects.[17]

In questioning the strict a priori nature of arithmetical knowledge, I am taking arithmetic as something wider than modern axiomatic systems. But there is a potential problem with such an approach. Above I have argued that ANS is enough to give an objective basis for arithmetical knowledge. But if we consider arithmetic from a wider

[16] Such a faculty is most often associated with Gödel (1947) in the literature.

[17] This should not be confused with informal mathematical insights, which undoubtedly serve an important purpose in mathematical thought process. Mathematicians often report, for example, that they know a theorem to be true before they can prove it. The psychological processes involved in such cases are an interesting subject, but it cannot be treated here. But as a *prima facie* explanation, it seems more likely that mathematicians are so familiar with their subject matter that they can recognize patterns and lines of arguments before they are fully articulated—rather than having a special epistemic connection to a world of abstract mathematical ideas.

perspective than systems like Peano arithmetic, there is the potential problem that too many different systems can be considered to be arithmetical. This is a potential argument against the objective basis for arithmetical knowledge.

The counter-argument naturally concerns the different axiomatizations of arithmetic —if they are not arithmetically equivalent—but also the diverging ways in which different peoples have developed arithmetic. In the philosophy of mathematics, more attention is usually given to mathematically revisionist approaches like ultra-finitism (Essenin-Volpin 1961), which denies the possibility of infinity of the natural numbers.[18] But equally relevant are cultures which have considerably different systems of dealing with numerosities, sometimes not developed much beyond the proto-arithmetical experiences given to us by ANS. The Pirahã people, for example, have the number words corresponding to "one", "two" and "many". They can do the corresponding arithmetic in which "one" plus "one" is "two" and "one" plus "two" is "many". However, "two" plus "two" is also "many", even though "one" is different from "two" (Gordon (2004)). The resulting system of numerosities is clearly very different from our arithmetic. In order to have our knowledge of arithmetic, we need to have developed the idea that we can add one to any number and the result is always (for finite numbers) *another* number.

Most people would—quite understandably—not want to consider the Pirahã system to be arithmetic at all, but there has also been at least one culture which was arithmetically developed but yet their arithmetic took a considerably different route from ours. The Mayans were sophisticated mathematicians and could calculate numerosities up to billions (Ifrah 1998). They could also use arithmetic to great success in predicting astronomical events. In many ways, their ability with numerosities was comparable to ours, and it would be problematic to claim that they were dealing with a fundamentally different conception of arithmetic. In most practical applications of arithmetic, their system was equivalent to ours. At the same time, however, their arithmetic seems also to have been remarkably different from ours. While they could calculate with extremely high precision, they did not prove general truths about numerosities. Moreover, they did not seem to have a concept of the infinity of the them.[19] All in all, the Mayan arithmetic at the same time closely resembled our arithmetic and was fundamentally different from it.

Should we consider the Mayan arithmetic to be arithmetic in the way we understand the word? In the literature, there are many different ways of defining arithmetic. To take a common understanding, the natural numbers form a discrete sequence (the so-called ω-sequence) that is linearly ordered, each number has finitely many predecessors and exactly one, unique, successor, and the sequence does not have a largest element. Under such understanding, neither the Mayan system nor the ultra-finitist approaches—not to mention the Pirahã system—can be considered to be arithmetic. In the context of this paper, however, that seems needlessly limiting. There is an extremely relevant difference to be made between proto-arithmetic and arithmetic that is more relevant

[18] It should be mentioned that there are also finitist approaches which aim to be non-revisionist mathematically. See Lavine (1994) for an example of non-revisionist set-theoretical ultra-finitism.

[19] Ifrah (1998, p. 298) writes that Mayans had some notion of infinity, but in the arithmetical writings that remain, that is not clear.

here. When we develop our theory of numerosities to include number symbols (or words) and explicit rules for operations on them, as well expand it beyond the small numerosities of ANS to apply to large quantities, I believe it makes sense to speak of an arithmetic.

This makes sense particularly when we consider developmental aspects. While there is an important move from school arithmetic to formal systems like Peano arithmetic, developmentally the more crucial jump is from the ANS-based proto-arithmetic to the language-based arithmetic. Hence, the most appropriate distinction between the arithmetical and the unarithmetical is not in the jump to formal axiomatic systems. It is at an earlier stage, when exact number symbols or words and explicit rules are introduced. Thus understood, the use of the term "arithmetic" here may be somewhat non-standard in the philosophy of mathematics. But that is mostly because in philosophy the developmental angle has been largely neglected in favor of the logical and formal aspects.

With this understanding of arithmetic, it is possible that arithmetic can develop considerably differently in different cultures. Instead of proving things like the infinity of numerosities, a culture can use arithmetic exclusively as a tool of explaining the physical world. While the more radically different cases like the Pirahã system may be explained away as unarithmetical, differences in the more developed theories suggest that the shared initial concept of discrete numerosity underdetermines the development of arithmetic. It seems that we shared the concept of numerosity with the Mayans when it came to calculations, but in the end developed arithmetic differently. But dismissing the Mayan arithmetic as unmathematical would seem to be too limiting. If we want to understand arithmetical knowledge philosophically, we should also consider such closely related systems of dealing with discrete numerosities.

The different ways of dealing with numerosities do suggest that the development from the proto-arithmetical experiences given to us by the ANS to our axiomatic systems of arithmetic was hardly inevitable. But does this suggest that we are in danger of losing the *objectivity* of arithmetical knowledge? While I do want to advocate taking a wider approach to arithmetic than the axiomatic system, I do *not* want to suggest that any system that deals with discrete quantities is arithmetical. The criterion I have proposed is that an ANS-based system becomes arithmetical once it has explicit number symbols or words and rules in place. But this implies that the shared basis between two arithmetical systems can be ultimately just the approximate number system. Now the big question is whether ANS is enough of a reason to speak of an objective basis for mathematics.[20]

[20] The matter of objectivity was already mentioned in Footnote 12 and should now be dealt with in more detail. The sense I have understood "objectivity" in this paper is obviously much weaker than some standard understandings of objectivity in the philosophy of mathematics, including the objective existence of natural numbers or the natural number structure. But in the light of the empirical data reviewed in this paper, that strong kind of objectivity seems too much to require. If there are general physiological features responsible for at least some of the content of arithmetical theories, it seems clear that arithmetical statements have content that is in a relevant sense objective. Although it is not possible here to go deeper into the philosophical question of objectivity, I believe that any arithmetically relevant understanding of "objectivity" should include this sense. That is why many important writings about mathematical objects and objectivity, e.g. Putnam (1980) and Field (1998), do not touch the argument given here. While questions treated in those writings—such as whether the continuum hypothesis has an objective truth-value—clearly are central to

Interestingly, there is a related debate concerning moral realism. The argument for the objectivity of arithmetic based on the ANS is clearly an evolutionary one: we share the ANS with other people and animals because it was developed into the brain structure by evolution. But ever since Ruse (1986), evolutionary explanations for moral values have been considered to be a problematic fit with moral realism. Street (2006), for example, argues that we cannot reconcile the position that there are objective moral truths with the fact that evolutionary forces have deeply influenced our values. Clarke-Doane (2012) has made the connection to mathematical objectivism explicit by arguing from an epistemological basis that it may not be possible to be a moral anti-realist but a mathematical realist.

The evolutionary arguments against moral realism are quite forceful because there is a clear possibility of an epistemological gap between moral truths and the moral beliefs that natural selection favors. In Clarke-Doane's radical example, even if killing our offspring were morally good, it would still be evolutionary advantageous to believe it is bad. Could the case with ANS-based arithmetic be similar? If there were objective arithmetical truths, do we have any reason to believe that evolution has made it possible for ANS—and the arithmetic developed on it—to capture them? Clarke-Doane argues for a strong similarity between moral and mathematical realism in this sense, even contending that $1 + 1 = 0$ could, realistically construed, be a mathematical truth while the first-order (with identity) logical truth equivalent to $1 + 1 = 2$ would be the evolutionary advantageous one. Even if that example seems far-fetched, the above examples of ultra-finitism and Mayan mathematics surely give relevant examples of alternative arithmetics. Clearly not all arithmetical theories can give us accurate knowledge of objective truths. How could we tell which one does? This is the first important question. The second one is whether *any* arithmetical theory can give us objective truths.

How does the present ANS-based theory of arithmetical knowledge fare with such epistemological challenges? The first question seems less problematic. Compared to moral disagreements, the arithmetical ones seem quite minor. The modern axiomatizations of arithmetic agree for the most part and the different characteristics of Mayan arithmetic, for example, are perhaps more likely due to underdevelopment of the arithmetical ideas than any substantial disagreement.

The second question is more interesting, but while the epistemological gap may indeed arise with a platonist understanding of mathematical objectivism, there is no such problem with the present account. Clarke-Doane's example of $1 + 1 = 0$ is inconceivable if we consider arithmetical truths to arise from the ANS. If the content of arithmetical knowledge is in some way based on evolutionary processes, there is naturally no epistemological gap involved in gaining that knowledge. But now the question is whether such a basis for arithmetic can be considered objective under any

Footnote 20 continued
the question of mathematical objectivity, so is the question whether that is the case for $1+1 = 2$. To discuss the objectivity of this latter question without taking into account the empirical data reviewed in this paper would seem to be taking a needlessly limiting view of objectivity and thus ending up ignoring important evidence.

relevant understanding of objectivity? What is the difference to the problematic notion of moral objectivity?

I hope to have answered this latter question already, starting from the round-up of the empirical data in Section 2. ANS is something we share with fish and rats, and although further analysis is not possible here, it should be clear that the roots of arithmetic go much deeper, and are much more specific, than those of human moral values. Indeed, that is where the analogue from moral values to mathematics no longer applies. The arguments of Street and Clarke-Doane may be powerful against moral realism and some of that power may apply to the argument against mathematical realism. But we do not need to assume the existence of mathematical objects in order to support the objectivity of mathematical truths. There may or may not be such things as natural numbers, but the empirical data strongly suggests that some of the basic rules concerning numerosities are objective in a highly relevant sense, i.e., they are underwritten by shared brain structure and not mere human conventions.

While I believe that ANS is enough to give an objective basis for arithmetic, it is clear from the different arithmetical systems that ANS underdetermines our arithmetical theories. There are conventional and cultural factors involved, which can be independent of the content determined by ANS. That raises an important question: to what level do conventional aspects determine our arithmetical conceptualizations? With the difference in Pirahã and Peano systems of dealing with numerosities, for example, it seems that the conventional part can be huge. Indeed, that should be expected based on the very primitive nature of ANS. Just like the first attempts at explaining empirical data can give completely different results, explaining and expanding on our proto-mathematical ability should hardly be expected to be something that happens uniformly.

The ratio of conventional and biological elements in particular theories of numerosities is an intriguing question, but the philosophical key point is that based on the empirical data, it appears that arithmetical knowledge is never *completely* conventional. There is an objective element to our arithmetic, one determined by our biological structure. Arithmetical knowledge concerns the conventional factors, but never only them. Thus arithmetical knowledge is not only about the meanings of certain linguistic number concepts and cannot be given a satisfactory philosophical explanation based only on human conventions.

The case with nonhuman animals and infants should make this clear. How could arithmetical knowledge be totally conceptual in the language-dependent sense if subjects that have little or no linguistic ability can have proto-arithmetical abilities? The way arithmetical knowledge seemed to be strictly a priori, in that $2 + 1$ and 3 both consisted of applying the successor function three times, has been revealed to involve a limited understanding of arithmetic. That account only works if we take arithmetic to be completely about the meanings of symbols in language-dependent formal systems. But philosophically, that is not sufficient.

I hope that the above considerations have illuminated the difference between strict a priori knowledge and arithmetical knowledge. The statement “all bachelors are unmarried” is often presented as a standard case for strictly a priori knowledge. “$2 + 1 = 3$” is another one often suggested. But there is an important difference between the two. The latter formulates a way of observing the world we have as infants and share with many animals. The content of "2 + 1 = 3" is used successfully by agents who do not have linguistic number concepts. The former deals exclusively with the meanings of certain words. While arithmetic can be *presented* as relations of certain concepts in our language - as in the example with the successor function—that does not mean it is only *about* these concepts and relations.

If arithmetical knowledge is not strictly a priori, would it not make sense to characterize it as a posteriori, captured by empiricist theories? In the account of Kitcher (1983), mathematical knowledge concerns generalizations of operations that we do in our environment. For instance, we play around with collections of pebbles and learn that the operations obey certain rules. We can then generalize on these rules and end up with rules of arithmetic applicable not only to small collections of pebbles, but to larger quantities of any kind of objects, even up to infinity. Without going into the details, it seems clear that this is indeed essentially how we first learn about numerosities. Empirical aspects play an important role in most people's learning of arithmetic. It is also plausible that the development of arithmetic was initially tightly connected to empirical aspects.

However, there are two differences that make such an empirical account problematic as an explanation of arithmetical knowledge. First, after the context is set, the empirical methods are not indispensable, which makes arithmetical knowledge different in character. I do not want to go too deep into thought experiments here, but simply by understanding the axioms and rules of PA, it is possible to build (with the Gödelian restrictions) a complete knowledge of arithmetic. In actuality, most children use their fingers to learn to count as well as add, but this should by no means be thought to be necessary.

Second, there is the rather obvious point that while arithmetical statements may not be purely conceptual knowledge in the linguistic sense, they simply are not empirically confirmable or falsifiable under any relevant reading of "empirical". Arithmetical calculations are neither corroborated nor refuted by direct experiment. Mathematical proof has its own special character and an epistemological theory must include it.

There is, however, a more sophisticated argument for the empirical nature of arithmetic. If ANS is indeed the basis for our arithmetical ability, clearly we must experience the world as discrete objects. But although ANS gives us only the ability to determine the quantity of objects up to small numerosities, we also experience pluralities of objects much greater than that. This may be seen as putting into doubt my earlier contention that we need language-dependent concepts to develop genuinely arithmetical thinking. Is there not a possibility that we can non-linguistically grasp the successor relation and thus ground the epistemology of arithmetic on a much stronger empirical context than what is provided by the ANS?

While that is indeed a possibility, there is empirical evidence against it. The underdetermination of arithmetical theories is one argument to that effect, but against that it can be argued that only sufficiently developed systems should be considered to be arithmetical. However, a much stronger argument can be found in the empirical data about the ANS. The roughly logarithmic nature of ANS suggests that we do *not* observe large pluralities in the same way as we do small ones. As described in Dehaene 2011 (Chap. 4), our mental arithmetic is a constant duel between two systems in the brain. Dehaene locates our "number sense" (i.e., the non-linguistic ability to deal with quantities) to a part of the intraparietal sulcus, what he and his colleagues call "hIPS" (horizontal part of the intraparietal sulcus). Language-processing, on the other hand, is located in regions of the left hemisphere. Now the question is, what happens in the brain when we are given simple arithmetical tasks? What Dehaene and his colleagues found out was that the more exact we need to be in our calculations, the more we rely on the regions of the left hemisphere and less on hIPS. When asked, for example, whether $4 + 5 = 7$ or 9—when we need to do exact calculations—we need to process the arithmetic linguistically in the left hemisphere. But when asked whether $4+5 = 8$ or 3—when approximation is more important—the hIPS activates more.

Such data is important because it is strong evidence for the position that the development of language-dependent concepts is the key factor in moving from the proto-arithmetical numerosity estimation of ANS to actual arithmetical thinking. While it does not *prove* that developed arithmetic is language-dependent, it clearly points toward that. Two main areas of activity in the brain have been found when given arithmetical tasks. One, the hIPS, deals with small numerosities after which it becomes increasingly approximate. The other, the left hemisphere regions, deals with exact operations on larger numerosities—what we usually understand by actual arithmetical cognition. So far there is no evidence of there also being a non-linguistic capacity for those latter operations. It seems to be the case that we do not experience large pluralities in the same way as we do small ones. Based on the current data, the best hypothesis is the one proposed earlier: while ANS is at the root of our arithmetical ability, it takes language-dependent concepts to develop it into actual arithmetical ability.

If arithmetic is neither a priori nor empirical, then what is it? I propose it is actually a combination of the two, best described as *contextual a priori*. Contextual or relativized notions of a priori are not new in philosophy. The most famous such conceptions in the literature are perhaps by Putnam and Kuhn. The famous concept of paradigm in Kuhn can be interpreted as being a notion of contextual a priori. In his later work, Kuhn (1993, pp. 331–332) describes this connection:

> [the concept of paradigm resembles] Kant's a priori when the latter is taken in its second, relativized sense. Both are constitutive of *possible experience* of the world, but neither dictates what that experience must be. Rather, they are constitutive of the infinite range of possible experiences that might conceivably occur in the actual world to which they give access. Which of these conceivable experiences occurs in that actual world is something that must be learned, both from everyday experience and from the more systematic and refined experience that characterizes scientific practice.

Putnam's (1976) characterization of his concept of contextual a priori also sounds Kuhnian in spirit:

> there are statements in science which can only be overthrown by a new theory–sometimes by a revolutionary new theory–and not by observation alone. Such statements *have* a sort of 'apriority' prior to the invention of the new theory which challenges or replaces them: they are *contextually a priori.*

Putnam's favored example is Euclidean geometry, which he holds to be a false theory about the world, but nevertheless contextual a priori knowledge. The concept of contextual a priori that I have in mind, however, is fundamentally different. I do not want to propose an epistemological theory in which false beliefs can be considered to be knowledge. Instead, I define contextual a priori knowledge to mean a priori in a *context set by empirical facts*. By empirical facts I do not mean facts about mathematical cognition that can be empirically studied, or any such connection to empirical methodology. What I mean is that we learn the basic rules concerning numerosities empirically, by observation. But this can only happen because of the ANS. When we see two apples and one apple grouped together, we see three apples. We could see a lot of things in that settings, but experiments have shown that numerosities are central to the way our brain processes such observations. This way, the ANS imposes on us a group of empirical facts. When children start learning arithmetic, they quickly understand the addition "$2 + 1 = 3$". It seems unlikely that this is due to some special mathematical insight that deals with a priori truths. A more plausible explanation is that by the time they are learning arithmetic, they have seen empirical evidence of that addition countless times.

That, however, does not mean that arithmetical knowledge is essentially empirical. The empirical instances corresponding to "$2 + 1 = 3$" work as evidence for children because ANS forces them to process observations in terms of numerosities. In this way, the empirical instances are essentially a case of the ANS imposing the proto-arithmetical content on the child. I have argued that when understood in this ANS-based manner, arithmetical statements have objective truth-values. But they are not necessarily true about an objective world with no agents observing it. Rather, the best explanation is that simple arithmetical sentences are true when they correspond to the experiences that our primitive ability to deal with numerosities gives us.

These experiences, usually dealing with small numerosities, are quite rudimentary. For the numerosities from one to four, the conception that our brain structure forces upon us is that there are four discrete quantities, forming a structure in which one succeeds the other. Even without knowing the rules of addition and subtraction, we can tell when something goes wrong in such operations on small quantities. When we move to actual arithmetical thinking, we use language-based concepts in order to generalize on these conceptions. This is not only a plausible philosophical picture, but as we have seen, also supported by empirical evidence.

This characterization of arithmetical knowledge as contextual a priori may resemble the above description by Kuhn, but in my view arithmetic is not just another paradigm. Arithmetic certainly gives us a possible way of experiencing the world, but based on the empirical data, the proto-arithmetical ability seems to give us much more than that: an effectively *inevitable* way of experiencing it. We cannot help categorizing observations in terms of numerosities. Not only is this inevitable for adult human beings, but also for infants and many nonhuman animals. This way arithmetic is not relativized a priori knowledge in Kuhn's sense. It is not a priori within some paradigm that could be completely overthrown by another paradigm. Of course a new paradigm may prevail over an older one, as has happened with the introduction of axiomatic systems. But for us to use the new paradigm as an arithmetic of numerosities, it cannot conflict with the proto-arithmetic that ANS gives us. How the proto-arithmetic develops into actual arithmetical thinking based on the primitive origins is an interesting question. It is plausible that a considerable part of arithmetic is essentially conventional. But since arithmetical knowledge is always based on something non-conventional—the ANS—arithmetic is more than simply another paradigm of categorizing our observations. It is a paradigm we can never totally abandon, nor alter to a degree where it conflicts with the experiences given by ANS.

In this way, arithmetical knowledge can be objectively true, but it is true as a theory of what follows from one characterization (based on the successor function) of our primitive ability to deal with numerosities. The primitive ability sets the context and the characterization sets the theory. After this is done, arithmetic is not empirical as a theory of that process. It is essentially a priori in character. As a theory of the *world* it could even be seen to be false. It could be the case, for example, that there is not an infinite amount of things in the universe. Any ontological theory postulating arithmetical objects as things in the universe would thus render infinitary arithmetic false. But that does not need to be what arithmetic is about. Rather than a theory about the world—whatever we mean by that—we can think of arithmetic ultimately as a theory of experience or the categorization of observations. As such it seems to be best described as contextually a priori. Of course this does not mean that arithmetic could not be a theory about the world, as well. But that is not a necessary step to take in order to accept the present epistemological theory of arithmetic.

Above is of course only a very rough outline of an epistemological theory of arithmetic. The mere idea of generalizing on the application of the successor function, for example, leaves many arithmetical details open. However, in philosophy, I believe we should be careful about filling in these details. There is no doubt that we can build a detailed picture in the manner of *Where Mathematics Comes from* by Lakoff and Núñez (2000). Their purpose is to formulate a scenario of rigorous development of mathematics, starting from primitive proto-arithmetical ability. The main idea (p. 53) is that mathematical ideas are metaphorical. Basic arithmetical operations are what Lakoff & Núñez call *grounding* metaphors, corresponding to simple operations on physical objects. Addition is grounded on object collection, subtraction on taking objects away from a collection, etc. On top of the simple grounded ideas there are *linking* metaphors, which form abstract ideas. Examples are numbers as points on a line, algebraic treatment of geometrical figures, etc. This way, they propose a cumulative development that our mathematics has had from its primitive origins. For the most part, their account seems plausible.

However, it should be remembered that what Lakoff & Núñez draw is for a large part a possible picture without empirical support. Such work can of course be quite illuminating: it shows that from very humble empirical origins we can step by step build sophisticated mathematical theories. No doubt something like that has actually happened in the development of mathematics. Nevertheless, there is limited value in such explanations as long as they are not backed by hard empirical data. After all, from the fact that we have sophisticated mathematical theories, we *know* that they can be developed from extremely simple ideas. It is how this process actually happens in individuals (as well as historically within cultures) that interests us. We now have a very good idea about the first steps toward arithmetical knowledge. For the rest, empirical researches, philosophers and mathematicians should work together to build as strong theories as possible.

In the beginning of this paper, I composed a wish-list for an epistemological theory of mathematics. It is now time to see how the outlined theory of contextual a priori epistemology of arithmetic fares.

(1) It should not require any unreasonable ontological assumptions.

Regardless of what one's ontological leanings may be, the assumptions needed for the present approach are hardly unreasonable. No mind-independent existence of mathematical objects is presupposed, but neither is that denied. Ontologically, the contextual a priori model is highly versatile. What are often seen as basic arithmetical intuitions are based on experience and hypothesized to arise from the structure of our brain. Whether that structure has developed to mirror some feature of the world is another question, and one definitely worth asking. One proposed explanation is that evolution will favor developments that correspond to the structure of the world, and thus the proto-arithmetical structure of our observations can mirror the arithmetical structure of the world. I find such speculations unsatisfactory in many ways, but they are compatible with the epistemological theory proposed here. Indeed, the same goes for forms of platonism: nothing in the current approach prevents the possibility that the brain structure has developed to get information about numbers as independently existing objects. But one strength of the contextual a priori model is that no such assumptions are needed. There are evolutionary advantages in processing observations in terms of quantities and that by itself can be enough for the proto-arithmetical structures to arise. The only ontological assumption needed for that is the existence of the required biological organisms.

(2) It should be epistemologically feasible as a part of a generally empiricist philosophy.

One problem many philosophers have with platonism is that it seems to require an epistemic access unlike all the other known psychological processes. The approach here leads to no such problems. Modern psychology and cognitive science have shown that the brain structure plays a tremendously important role in constructing our immediate experiences. That some of the brain filters deal with quantities is perfectly feasible as part of a modern empiricist epistemology and gets solid support from the empirical data.

(3) It should be able to explain the apparent objectivity of at least some mathematical truths.

Since the empirical data strongly suggests that we have our proto-arithmetical ability to process numerosities already as infants and share it with many nonhuman animals, it should be safe to say that there is very little that is subjective in the ability. Whether we want to call the basic proto-arithmetical processes objective or something along the lines of "maximally intersubjective" is a moot point: the important matter is that they are not completely culture-dependent and conventional. If arithmetic is developed to accommodate these proto-arithmetical processes, there is no great mystery in that it appears to be objective. In the relevant sense, it is. Of course there are also culture-dependent aspects to arithmetic, and in other fields of mathematics they are more prominent. It should certainly be interesting philosophical work to consider the culture-dependency of other areas of mathematics based on the present model, but that will have to wait for another occasion.

(4) It should not make the applications of mathematical theories in empirical sciences a miracle.

This is a very tricky issue that we have not discussed so far. Indeed, it would require (at least) another paper to tackle. Generally speaking, it should be plausible that some mathematical theories have scientific applications since they are at the very core of our explanations of the world. If our understanding of the world is build in part on proto-arithmetical experiences, it is no wonder that we benefit from applying them. And if there is a continuous development from arithmetic to, say, complex analysis or probability theory, we can expect that these more specialized fields will also have some applications. But this is considering the problem on a very general level. The fact is that many times empirical applications are surprising and demand independent explanations. Moreover, there is also the question why physical objects conform to such complex mathematical laws at all. Unless we accept the theory-dependency of observations in its extreme formulations, this is a *bona fide* question to ask. Here I do not want to propose an answer. Rather, I will have to be content with remarking that the problem is not any worse with the present approach than in competing epistemological theories. Indeed, I believe it is much less damaging than, say, in strict conventionalism or classical platonism. In the former we assume that ultimately arbitrary conventions can help explain the world, while in the latter we assume that some abstract world of objects helps us explain the physical world. In platonism this just raises a new question about the connection between these two worlds. In strict conventionalism we are left with no explanation.

(5) It should not rid mathematics of its special character.

Since the model here is a priori in the empirical context, after the proto-mathematical context is set, all arithmetical knowledge can be (in principle) acquired in an essentially a priori manner. The method of proving theorems is retained exactly as it is in traditional a priori explanations and none of the special character of mathematics is lost.

(6) It should be empirically feasible: the best scientific data about mathematical cognition should not be in conflict with philosophy.

If the representation of the empirical data here is correct, and the epistemological theory has been developed to accommodate the data as planned, this last point is achieved by default.

Acknowledgments This work has been made possible by the generous support of the Academy of Finland and the University of Bucharest. I am in great gratitude to the many colleagues who have provided valuable comments on the manuscript at some point, especially Daniel Cohnitz and Dirk Schlimm.

References

Agrillo, C., Dadda, M., Serena, G., & Bisazza, A. (2009). Use of number by fish. *PLoS One*, *4*(3), e4786.
Azzouni, J. (2010). *Talking about nothing—Numbers, hallucinations and fiction*. Oxford: Oxford University Press.

Boghossian, P., & Peacocke, C. (2000). *New essays on the a priori*. Oxford: Oxford University Press.
Brannon, E., & Merritt, D. (2011). Evolutionary foundations of the approximate number system. In Dehaene (Ed.), *Space, time and number in the brain* (pp. 107–122). London: Academic Press.
Burge, T. (2007). *Postscript to "Belief de re" in foundations of mind*. Oxford: Oxford University Press.
Burge, T. (2010). *Origins of objectivity*. Oxford: Oxford University Press.
Butterworth, B. (2010). Foundational numerical capacities and the origins of dyscalculia. *Trends in Cognitive Sciences*, *14*, 534–541.
Cantlon, J. F., & Brannon, E. M. (2006). Shared system for ordering small and large numbers in monkeys and humans. *Psychological Science*, *17*(5), 402–407.
Carey, S. (2009a). Where our number concepts come from. *The Journal of Philosophy*, *106*(4), 220–254.
Carey, S. (2009b). *The origin of concepts*. Oxford: Oxford University Press.
Church, R., & Meck, W. (1984). The numerical attribute of stimuli. In H. L. Roitblat, T. G. Bever, & H. S. Terrace (Eds.), *Animal cognition* (pp. 445–464). Hillsdale: Erlbaum.
Clarke-Doane, J. (2012). Morality and mathematics: The evolutionary challenge. *Ethics*, *122*(2), 313–340.
Dehaene, S. (2011). *Number sense* (2nd ed.). New York: Oxford University Press.
Dehaene, S., & Brannon, E. (Eds.). (2011). *Space, time and number in the brain*. London: Academic Press.
Diester, I., & Nieder, A. (2007). Semantic associations between signs and numerical categories in the prefrontal cortex. *PLoS Biology*, *5*, e294.
Dummett, M. (1973). *Frege: Philosophy of language*. London: Duckworth.
Essenin-Volpin, A. (1961). Le Programme Ultra-intuitionniste des fondements des mathématiques. *Infinistic methods* (pp. 201–223). New York: Pergamon Press.
Feigenson, L. (2011). Objects, sets, and ensembles. In Dehaene & Brennan (Eds.), *Space, time and number in the brain* (pp. 13–22). London: Academic Press.
Field, H. (1980). *Science without numbers: A defense of nominalism*. Princeton, NJ: Princeton University Press.
Field, H. (1998). Mathematical objectivity and mathematical objects. In S. Laurence & C. Macdonald (Eds.), *Contemporary readings in the foundations of metaphysics* (pp. 387–403). Oxford: Basil Blackwell.
Gödel, K. (1947). What is Cantor's continuum problem? *The American Mathematical Monthly*, *54*, 515–25.
Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*, 496–499.
Hellman, G. (1989). *Mathematics without numbers*. Oxford: Oxford University Press.
Ifrah, G. (1998). *The universal history of numbers: From prehistory to the invention of the computer*. Great Britain: The Harvill Press.
Kitcher, P. (1983). *The nature of mathematical knowledge*. New York: Oxford University Press.
Kuhn, T. (1993). Afterwords. In Paul Horwich (Ed.), *World changes* (pp. 331–332). Cambridge, MA: MIT Press.
Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from*. New York: Basic Books.
Lavine, S. (1994). *Understanding the infinite*. Cambridge, MA: Harvard University Press.
Leslie, A. M., et al. (2008). The generative basis of natural number concepts. *Trends in Cognitive Science*, *12*, 213–218.
Lourenco, S., & Longo, M. (2011). Origins and development of generalized magnitude representation. In Dehaene & Brennan (Eds.), *Space, time and number in the brain* (pp. 225–241). London: Academic Press.
Mechner, F. (1958). Probability relations within response sequences under ratio reinforcement. *Journal of Experimental Analysis of Behavior*, *1*, 109–21.
Mechner, F., & Guevrekian, L. (1962). Effects of deprivation upon counting and timing in rats. *Journal of Experimental Analysis of Behavior*, *5*, 463–66.
Mix, K. (2002). Trying to build on shifting sand: Commentary on Cohen and Marks. *Developmental Science*, *5*, 205–206.
Mix, K., et al. (2002). Multiple cues for quantification in infancy: Is number one of them? *Psychological Bulletin*, *128*, 278–294.
Nieder, A. (2011). The neural code for number. In S. Dehaene & E. Brannon (Eds.), *Space, time and number in the brain* (pp. 107–122). London: Academic Press.
Nieder, A. (2012). Supramodal numerosity selectivity of neurons in primate prefrontal and posterior parietal cortices. *Proceedings of the National Academy of Sciences of the United States of America*, *109*, 11860–11865.
Nieder, A. (2013). Coding of abstract quantity by 'number neurons' of the primate brain. *Journal of Comparative Psychology A*, *199*, 1–16.

Nieder, A., et al. (2006). Temporal and spatial enumeration processes in the primate parietal cortex. *Science*, *313*(2006), 1431–1435.
Nieder, A., & Dehaene, S. (2009). Representation of number in the brain. *Annual Review of Neuroscience*, *32*, 185–208.
Nieder, A., Freedman, D., & Miller, E. (2002). Representations of the quantity of visual items in the primate prefrontal cortex. *Science*, *297*, 1708–1711.
Nieder, A., & Miller, E. (2003). Coding of cognitive magnitude. Compressed scaling of numerical information in the primate prefrontal cortex. *Neuron*, *37*, 149–157.
Piazza, M. (2010). Neurocognitive start-up tools for symbolic number representations. *Trends in Cognitive Sciences*, *14*, 542–551.
Piazza, M., et al. (2007). A magnitude code common to numerosities and number symbols in human intraparietal cortex. *Neuron*, *53*, 293–305.
Piazza, M., et al. (2013). Education enhances the acuity of nonverbal approximate number system. *Psychological Science*, *24*, 1037–1043.
Putnam, H. (1968). Is logic empirical? In *Boston studies in the philosophy of science, vol. 5, D (pp. 216–241). Dordrecht: Reidel.*
Putnam, H. (1980). Models and reality. *Journal of Symbolic Logic*, *45*, 464–482.
Putnam, H. (1983). 'Two dogmas' revisited. In *Realism and reason*: *Philosophical papers* (Vol. 3, pp. 87–97). Cambridge: Cambridge University Press.
Putnam, H. (2004). *Ethics without ontology*. Cambridge, MA: Harvard University Press.
Quine, W. V. O. (1951). Two dogmas of empiricism. In *From a logical point of view*. Cambridge, MA: Harvard University Press.
Resnik, M. (1981). Mathematics as a science of patterns: Ontology and reference. *Nous*, *15*, 529–550.
Rumbaugh, D., Savage-Rumbaugh, S., & Hegel, M. (1987). Summation in the chimpanzee. *Journal of Experimental Psychology: Animal Behavior Processes*, *13*, 107–115.
Ruse, M. (1986). *Taking Darwin seriously: A naturalistic approach to philosophy*. New York: Prometheus Book.
Shapiro, S. (1997). *Philosophy of mathematics: Structure and ontology*. New York: Oxford University Press.
Simon, T. J., Hespos, S. J., & Rochat, P. (1995). Do infants understand simple arithmetic? A replication of Wynn (1992). *Cognitive Development*, *10*, 253–269.
Spelke, E. (2011). Natural number and natural geometry. In S. Dehaene & E. Brannon (Eds.), *Space, time and number in the brain* (pp. 287–318). London: Academic Press.
Street, S. (2006). A Darwinian dilemma for realist theories of value. *Philosophical Studies*, *127*, 109–166.
Wynn, K. (1992). Addition and subtraction by human infants. *Nature*, *358*, 749–751.