1–35
10.1093/philmat/nkab010
Philosophia Mathematica

Objectivity in Mathematics, Without Mathematical Objects[†]

Markus Pantsar[*]

Department of Philosophy, History and Art, University of Helsinki, 00100 Helsinki, Finland.

ABSTRACT

I identify two reasons for believing in the objectivity of mathematical knowledge: apparent objectivity and applications in science. Focusing on arithmetic, I analyze platonism and cognitive nativism in terms of explaining these two reasons. After establishing that both theories run into difficulties, I present an alternative epistemological account that combines the theoretical frameworks of enculturation and cumulative cultural evolution. I show that this account can explain why arithmetical knowledge appears to be objective and has scientific applications. Finally, I will argue that, while this account is compatible with platonist metaphysics, it does not require postulating mind-independent mathematical objects.

1. INTRODUCTION

Many of the most crucial questions in the philosophy of mathematics concern what mathematical knowledge is *like* (see, *e.g.*, [Benacerraf and Putnam, 1983; Shapiro, 2005]. Is it analytic or synthetic? Is it necessary or contingent? Does it have content or is it purely formal? And finally, closely related to all of the above, the question this paper will focus on: is mathematical knowledge *objective*? That question, like the others, is tightly connected to the question what mathematical knowledge is *about*. For the area of mathematics that children

[†]This research was made possible by grants from the Alfred Kordelin Foundation and the Finnish Cultural Foundation. I would like to thank Regina E. Fabry for providing many helpful comments on an earlier version of the manuscript. I also want to thank the two anonymous reviewers for their comments. An early version of this work was presented at the workshop 'Ontology in arithmetic and logic' at l'Université du Québec à Montréal; I am grateful to the organizer Jean-Charles Pelland and all the participants at the workshop. Special thanks are in place to Bahram Assadian, Stefan Buijsman, Øystein Linnebo, Christopher Peacocke, Dirk Schlimm, and Andrea Sereni for helpful discussions.
[*]Orcid.org/0000-0001-8572-1453. E-mail: markus.pantsar@gmail.com

are taught first, namely arithmetic, this latter question seems like a simple one to answer. Arithmetic is about numbers, more specifically the ordered set of natural numbers $N = 0, 1, 2, 3, \ldots$. But the question what natural numbers are has puzzled philosophers since antiquity (see, *e.g.*, [Shapiro, 1997] for a review). Clearly numbers are not physical objects; so they must be abstract. But if they are abstract, in what sense do numbers exist? Are they just the result of human conventions, as suggested by, *e.g.*, Field [1980]? If not, how can we, as Benacerraf [1973] famously asked, as physical subjects get knowledge of abstract non-physical objects? If mathematical objects exist independently of human conventions and practices, how do we get epistemic access to them, *i.e.*, how can we establish that our mathematical theories accurately capture the characteristics of mathematical objects?[1] In order to answer that question, we need to specify how mathematical knowledge is possible for human subjects. Thus the question what mathematical knowledge is about is ultimately tightly linked to the question how mathematical knowledge can be *acquired*.

The foundation of platonist epistemology is that since sense perception can never be about the mathematical objects themselves, knowledge about mathematical objects must be gained through *reason* (*The Republic*, 527a–b).[2] With some exceptions (*e.g.*, Mill [1843], intuitionists like Heyting [1931] and Brouwer [1948], Wittgenstein [1976], Field [1980], and Kitcher [1983], this platonist view ruled the epistemology of mathematics well into the second half of the twentieth century. But more than that, for both platonists and non-platonists, it seemed that there was a largely uniform way of approaching the epistemology of mathematics. Mathematics was thought of exclusively as a sophisticated, (relatively) mature human ability, and the epistemology of mathematics as mainly a topic for philosophy, with its usual *a priori* methodology.

Then some unexpected results started to surface from empirical research. It was discovered that agents that were previously not thought to have sufficient reasoning ability, *i.e.*, non-human animals and human infants, were also able to process quantitative information (see, *e.g.*, [Starkey and Cooper, 1980; Wynn, 1992; Dehaene, 1997/2011; Hauser *et al.*, 2000; Cantlon and Brannon, 2006; Hunt *et al.*, 2008; Pepperberg, 2012; Agrillo, 2015]). With paper titles like 'Addition and subtraction by human infants' [Wynn, 1992], 'Arithmetic in newborn chicks' [Rugani *et al.*, 2009], and 'Numerical and arithmetical abilities in non-primate species' [Agrillo, 2015], there appeared to be two clear emerging messages from empirical cognitive scientists. First, arithmetical ability is not only the domain of sufficiently mature human agents. Second, for

[1] Following a common custom in the literature (*e.g.*, [Dummett, 2006]), I will use the term 'mind-independent' to refer to something that is not dependent on human conventions, practices, languages, and thoughts.

[2] Here I proceed with the custom that Platonism with capital 'P' refers specifically to Plato's philosophy whereas platonism with a lower case 'p' refers to a more general realist metaphysical position on mathematics. Tait [2001] has suggested that instead of platonism, it would be clearer to talk about 'realism' than platonism. For the sake of terminological congruity with the relevant literature, however, I will follow the more common custom.

progress in explaining the acquisition of arithmetical knowledge, we need to be well-informed concerning the empirical research on quantitative cognition. As I have argued before [2014; 2015a; 2018a; 2019a], and will be discussed in detail in Section 5, I find the first conclusion unwarranted. The second conclusion, on the other hand, is something I agree with. The account I will be proposing in Section 6 will be heavily based on the state-of-the-art empirical research on quantitative cognition. Indeed, I believe that the quantitative abilities we have as infants and share with non-human animals form a foundation for the development of arithmetical knowledge. But, importantly, those abilities themselves must not be considered to be arithmetical.

This distinction between arithmetical and what I call *proto-arithmetical* abilities is important to make when we consider the question whether mathematical knowledge is objective. As I will argue, as bad a fit as the idea of infant and (non-human) animal arithmetic seems to be with platonism, the two views share one important characteristic. On both accounts, mathematical knowledge can be considered to be essentially objective. In Section 2, I will provide specific criteria for objectivity, but initially objectivity can be understood as the position that mathematics is not about human conventions. Objectivity is of course central to platonist philosophy of mathematics: the set of truths we accept in mathematics is not a matter of convention (see, *e.g.*, [Panza and Sereni, 2013]). In one way or another, they are determined by objective, mind-independent criteria. Perhaps less obviously, objectivity of arithmetical knowledge is also an essential characteristic of the nativist view that infants and non-human animals possess arithmetical abilities.[3] But this kind of cognitive nativism appears to be as committed to objectivism as platonism is. Infants and most non-human animals cannot be feasibly thought to adhere to conventions; so whatever their putatively arithmetical ability is, it must be due to biological evolution, 'hard-wired' into their cognitive architecture. The resulting arithmetical knowledge therefore has to be objective in a strong sense, since it is not determined by conventions or customs. Rather, it is dictated by cognitive capabilities that are independent of the particular surroundings where the infant or animal has been born and raised.[4]

In Sections 4 and 5, I argue that both views fail to give a satisfactory explanation why mathematical knowledge is seen to be objective. However, it is at least *prima facie* plausible that mathematical knowledge *is* objective. As I show in Section 3, we cannot simply dismiss the widely shared impression among mathematicians (as well as scientists and laypeople) that mathematical knowledge is objective, nor can we dismiss that there is a strong philosophical argument for the objectivity of mathematical knowledge based on mathematical applications in science. What I aim to do in this paper, however, is to show that both the apparent objectivity and the mathematical applications can be

[3] I call this view both 'nativism' and 'cognitive nativism' in this paper.

[4] I assume here that it is accepted that true arithmetical beliefs constitute knowledge. Of course not all beliefs based on evolutionarily developed capacities are knowledge.

explained without assuming that mathematical knowledge is indeed objective in the strong sense of the platonist and nativist views. To establish this, I will argue that we need a finer-grained understanding of the development of mathematical knowledge, with sufficient emphasis on its culturally determined character. The biological origins underdetermine the development of arithmetical and other mathematical knowledge, which makes it crucial to identify and analyze the cultural conditions conducive to the development of mathematics. While this might seem to be a step toward the kind of cultural relativism that conflicts with the apparent objectivity (*e.g.*, [Bloor, 1976]), I will argue that this is not the case. Instead, with a proper understanding of the cultural factors in developing early quantitative abilities, I argue that we can explain the reasons behind the apparent objectivity of mathematical knowledge and the applications in science. In particular, I will apply the framework of *enculturation* as developed by Menary [2015] in explaining how culturally dependent factors influence the development of mathematical cognition and mathematical knowledge.

2. WHAT IS OBJECTIVITY?

Before analyzing the characteristics and foundations of mathematical knowledge, it is important to explicate what is meant by 'objective'. This is a widely studied philosophical question and it is not possible here to go into the details, but it is important to note that mathematical objectivity in modern philosophy has been primarily an epistemological rather than an ontological matter. Tait [2001] traces this to Cantor who wrote:

> First, we may regard the whole numbers as real in so far as, on the basis of definitions, they occupy an entirely determinate place in our understanding, are well distinguished from all other parts of our thought and stand to them in determinate relationships, and thus modify the substance of our minds in a determinate way. ([Cantor, 1883], quoted in [Tait, 2001, p. 22]).

It is thus not the existence of some platonic world of mathematical objects, but the characteristics of mathematical *thinking* that should be the standard for objectivity. If we interpret Cantor's idea in epistemological terms, mathematical knowledge is objective (if it indeed is objective) because of the determinate role that knowledge about mathematical objects plays in our thinking. As Tait [2001] points out, Cantor's formulation is heavily psychologistic, and in contemporary literature it is more common to talk about the objectivity of mathematical *discourse*. But the crucial point is that Cantor moves the focus on objectivity away from being primarily about the existence of mathematical objects.[5]

[5] This move from the existence of mathematical objects to the objectivity of mathematical discourse is often referred to as 'Kreisel's Dictum' in the literature. This is due to

When is a form of discourse objective? According to Tait, 'Objectivity in mathematics is established when meaning has been specified for mathematical propositions, including existential propositions $\exists x F(x)$' [2001, p. 22]. But as he also points out here, philosophers have understood this 'meaning' in different ways. For Hilbert, the meaning of mathematical propositions was fixed entirely by the axioms of the particular mathematical system [Zach, 2019]. But ever since Gödel [1931] proved the incompleteness of all formal axiomatic systems of arithmetic, this approach has been untenable. In order to establish meaning, and hence objectivity, something more than agreeing on axioms is needed. Importantly, in the above manner specified by Tait, it would require establishing the meaning of propositions that make *existence* claims concerning mathematical objects, *i.e*, we would need to explain how mathematical objects exist.

However, in this paper I take a different argumentative path. My main purpose is not to argue for the objectivity of mathematics, nor will I aim to explain the way in which mathematical objects exist.[6] Instead, I will show how we can make sense of the two main *reasons* for believing in the objectivity of mathematics: the apparent objectivity of mathematics and mathematical applications in the sciences. Ultimately, I will argue that whether one accepts mathematics as being objective or not, there is a better argument for the putative objectivity of mathematical discourse than either platonism or nativism. Thus, even though I will not be arguing for the objectivity of mathematics *per se*, I consider its *apparent* objectivity to be a strong reason for considering the possibility that mathematics, and mathematical knowledge, are objective.[7]

First, however, we need to establish criteria for objectivity in order to determine whether the apparent objectivity of mathematical discourse is of the type that we would generally accept as being objective. Shapiro [2007] has used Wright's [1992] influential criteria for the objectivity of discourse for a similar purpose, arguing that mathematical discourse is indeed objective. The three criteria are *epistemic constraint*, *cognitive command*, and *wider cosmological role*.[8] The criterion of epistemic constraint states that non-objective matters are always knowable, *i.e.*, for non-objective matters it is always the case that:

$P \leftrightarrow$ (P may be known). [Wright, 1992, p. 75].

Dummett, who alludes to Kreisel's review of Wittgenstein's *Remarks on the Foundations of Mathematics* in which he supposedly points out that 'the problem is not the existence of mathematical objects but the objectivity of mathematical statements' [Dummett, 1978, p. xxxvii]. However, in that review Kreisel does not express this view explicitly and it is unclear where, and indeed if, he ever did that.

[6] Although both of those questions will be treated in the final section.

[7] Like Tait [2001], in this paper I treat objectivity of mathematics and objectivity of mathematical discourse in an interchangeable manner. Furthermore, I assume that successful mathematical discourse can achieve knowledge.

[8] [Shapiro, 2007] also includes the criteria of *response dependency* and *judgment dependency* in his analysis of Wright's criteria for mathematical discourse. Here I do not include them as I do not believe they add anything substantial to the analysis of the criteria of cognitive command in the present context.

While this touches upon some of the most fundamental questions in philosophy of mathematics (*e.g.*, the question of provability vs truth; see [Pantsar, 2009] for more), I take it is as uncontroversial that mathematical truths can at least *appear* to be unknowable. For one thing, there are well-defined mathematical problems that are generally considered to be too complex computationally to be solved for large enough inputs [Arora and Barak, 2007; Pantsar, 2019b; 2021a]. There is also the Gödelian [1931] restriction that no consistent formal system strong enough to express arithmetic can be complete, *i.e.*, prove all true sentences in the system. If we accept that proof from axioms is our way of knowing things in mathematics, this alone implies that there are unknowable truths in our mathematical systems.[9] But even disregarding such limitations of formal axiomatic systems, it is clear that formal (as well as informal) mathematical languages can be complex enough to make the prospect of knowing every truth expressible in them unfeasible. This is enough to accept that mathematical knowledge at least appears to be such that it does not meet the epistemic constraint, which suffices for the present purposes.

Wright's [1992, p. 92] criterion of *cognitive command* means that a discourse concerns something objective if and only if we can *a priori* rule out the possibility that there are disagreements that Shapiro [2007, p. 356] calls 'blameless'. In a blameless disagreement, there is no reason such as divergent information or different conditions for explaining the disagreement. Roughly put, blameless disagreements are seen as differences in opinion that cannot be solved, akin to matters of taste. A case could be made that in mathematics there appear to be just such disagreements. Should we accept the continuum hypothesis? The axiom of choice? The parallel axiom? In different times, different mathematicians have held different views on them and many other mathematical statements. It could be argued that whether we include the axiom of choice in the axioms of set theory, for example, gives us two different systems of mathematics, neither of which is more correct than the other. In other words, a blameless disagreement is reached and hence in such cases it would appear that the criterion of cognitive command fails. Here I cannot analyze that question in detail (see [Tait, 2001] and [Shapiro, 2007] for interesting treatments), but as far as *apparent* objectivity is concerned, it seems clear that at least much of mathematics fulfills the criterion of cognitive command. There is no (blameless) disagreement over, for example, what should be acceptable theorems in basic arithmetic. If, like Dostoevsky's character in *Notes from Underground* [1864], somebody claims that $2 + 2 = 5$, it would standardly not be considered to be grounds for a blameless disagreement.

Finally, the criterion of *wider cosmological role* demands that in order to concern something objective, a discourse must feature also in explanations that

[9] For the present purposes this exposition is sufficient, but it should be noted that the matter is complicated. It does not follow that any particular mathematical truth would be unknowable, since the unprovable sentences depend on the particular axiomatizations and encodings. Nevertheless, Gödel's [1931] first incompleteness theorem implies that no single formal axiomatic system proves all mathematical truths (see [Pantsar, 2009]).

are not exclusive to that discourse [Wright, 1992, p. 198]. This criterion would seem to be straightforwardly met by mathematical discourse due to the rich variety of applications it has in science. However, as we will see in the next section, the matter is not quite that simple. It could be that the mathematical applications are *dispensable* from the scientific explanations, in which case the wider cosmological role of mathematical discourse could be put into question. Yet it can hardly be contested that mathematics at least *appears* to play an explanatory role in sciences. For the present purposes, that is enough. We can safely conclude that the appearance of mathematical objectivity is indeed of the kind of objectivity that Wright [1992] has in mind.

3. TWO REASONS FOR BELIEVING IN THE OBJECTIVITY OF MATHEMATICAL KNOWLEDGE

As stated in the introduction, I see two main reasons for believing in the objectivity of mathematical knowledge. First is that mathematical knowledge appears to be objective. Second is that mathematical knowledge is intertwined in a significant way with scientific knowledge that is generally accepted to be objective. While the latter is a much-discussed philosophical argument, which I will explore in more detail below, the first reason may initially seem feeble, since appearances can be misleading in philosophy. Shapiro [1997], for example, has written about *working realism*, the way in which most mathematicians work *as if* mathematical objects were real. Mathematical language is full of existential expressions with apparent commitment to objects. But for a working realist, writing that 'there exists a number' may not entail any metaphysical commitment to the existence of numbers. It might be simply a convenient way of expressing something metaphysically far more economical, like 'from generally agreed upon axioms, with generally accepted rules of proof we can deduce that . . . '.

Nevertheless, I contend that the apparent objectivity of mathematical knowledge demands an explanation. The first question to ask, however, is how pervasive a phenomenon is the apparent objectivity? There are regrettably few studies on the topic and, to the best of my knowledge, there are none that reliably and systematically explore views on mathematical objectivity.[10] However, based on the data presented by Müller-Hill [2009], we can proceed under the assumption that at least for a significant portion of mathematicians (as well as non-mathematicians), mathematical knowledge appears to be objective. In any case, in the present context it is important to distinguish the question

[10] One potential exception is a questionnaire that revealed 82.4% of 'professional mathematicians' believed in the objectivity of mathematical knowledge [Müller-Hill, 2009]. But this figure was based on an open Internet questionnaire and we cannot know for sure that the participants were in fact mathematicians. Moreover, other data gathered from the same questionnaire suggest that the same mathematicians were in fact inconsistent in their knowledge ascriptions. For a critical review of the study and its interpretation, see [Pantsar, 2015b].

whether mathematical knowledge is objective from the question why mathematical knowledge *appears* to be objective. With this distinction in place, we arrive at three philosophically interesting positions. First is that mathematical knowledge appears to be objective because it *is* objective in essentially the way it appears on a literal reading. In short, mathematical statements are true because they state facts about mind-independent objects and relations between them. I take this to be a common position among platonists, although the *apparent* objectivity could be independent of the *actual* objectivity.[11] The second position is that the apparent objectivity is indeed due to actual objectivity, but it comes from other sources, and not mind-independent mathematical objects. I take cognitive nativism to adhere to this view. The third view is that mathematical knowledge appears to be objective even though it is in fact *not* objective.

The first two positions will be analyzed in detail in the rest of this paper, but let us first focus on the third option. This last position implies that mathematical knowledge is ultimately about human conventions and nothing else. I call this view *strict conventionalism*, to distinguish it from more moderate positions that take conventions to play an important part in mathematical discourse.[12] For strict conventionalists, mathematics is about arbitrary rules of symbol manipulation and it cannot be tied to anything more robust. This is the view of at least Field [1980] and Balaguer [2009], and it is commonly attributed also to Wittgenstein [1976].[13] Furthermore, this is also the view often associated with mathematical *formalism*, although this issue is trickier due to the different meanings of formalism in the philosophy of mathematics (see, *e.g.*, [Linnebo, 2017]). But combining conventionalism and formalism, mathematics could be likened to a game like chess in that it is about clearly defined, well-established rules, and these rules are human inventions.

However, as one of the pioneers of formalism (but certainly not conventionalism), Frege already noted in his *Grundgesetze*, there is a very important difference between chess and mathematics. Unlike the rules of chess, the rules of mathematics have important applications outside their own domain [Frege, 1893/1903]. This brings us to the second main reason why mathematical knowledge is considered to be objective. Aside from its apparent objectivity, mathematical knowledge has widespread applications in science, and there are few philosophers who are ready to question the objectivity (at least as an ideal)

[11] For example, it could be that the appearance of objectivity comes from educational practices.

[12] From here on, when I write about conventionalism, I am always referring to its strict version.

[13] An interesting question is how intuitionist philosophy of mathematics should be understood in this regard. According to Shapiro [2007], the original intuitionists like Heyting [1931] and Brouwer [1948] did not believe in the objectivity of mathematical knowledge, whereas at least some modern intuitionists, like Tennant [1997], do.

of scientific knowledge.[14] In contemporary philosophy of mathematics, applications have become the main argument for the objectivity of mathematical knowledge [Colyvan, 2001; Brown, 2008; Lange, 2017]. How could mathematics be an arbitrary game when it is integral to modern science which can explain all kinds of non-mathematical matters? This argument carries significant power for philosophers with radically different views. Field [1980], for example, considers it the strongest argument for mathematical realism (which he argues against).

The topic of scientific applications of mathematics is particularly important because it moves the focus from the appearance of objectivity to something more substantial. After all, the appearance of objectivity could be simply due to firmly established customs. We know that how the queen moves in chess is a matter of convention, but we might be unable to detect more fundamental and general rules as conventions. Indeed, mathematical rules can be such deeply entrenched conventions. That multiplying two negative numbers produces a positive number, for example, is a mathematical convention that is difficult to see as anything other than convention. There are good algebraic reasons for this custom, but what could be the non-conventional understanding of a negative times a negative quantity? However, does $(-4) * (-4) = 16$ appear any less objective than $4 * 4 = 16$ to most people with basic mathematics education?[15] And if a convention like that appears to be objective, could it be that all of mathematics is actually similar?

However, even if strict conventionalists could explain the apparent objectivity of mathematics, mathematical applications in science provide a much more difficult challenge. Conventionalists are of course aware of this problem. Indeed, applications outside mathematics are often evoked as the only reason why we prefer some mathematical theories over others. Field [1980, p. 15], for example, has argued that we prefer mathematical theories over their alternatives not because they are true, but because they are more *useful*. Since mathematical theories are not about anything objective, no theory is truer than its alternatives. They can only be more useful. However, this is a problematic state of affairs, even if we bought into Field's [1980] claim that mathematics can be taken out of scientific theories without losing any non-mathematical explanatory power, *i.e.*, mathematics is not *indispensable* for scientific explanations.[16] An inevitable follow-up question is *why* certain mathematical theories are more

---

[14] This is not to say that in post-modernist philosophy such points of view are not endorsed, but I see no reason to engage with views such as those of Feyerabend [1975] here.

[15] This point about multiplying two negative integers is made by various mathematicians, *e.g.*, [Stewart, 2006].

[16] This is a highly contested claim indeed. Field's [1980] argument is that Newtonian mechanics can be reconstructed without appeal to mathematical notions. The book is called *Science without Numbers*, but Shapiro [2000] and others have claimed that what Field actually does in his nominalization project is remove the mathematical structure of numbers and replace it with another mathematical structure of Newtonian space and time. For more details, see [Pantsar, 2009; Shapiro, 2000].

useful than others. Field's theory seems to imply that this is an arbitrary matter, but if we can find reasons to believe that the matter is not arbitrary, clearly the alternative explanation would be preferable.

Volumes have been written about mathematical explanations in science and it is not possible to go into the full range of details of the topic here (for overviews, see [Colyvan, 2001; Lange, 2017; Pincock, 2012; Reutlinger and Saatsi, 2018; Pantsar, 2018b]). I will only mention one much-discussed example that will make it sufficiently clear what the problem is. Baker [2005; 2009] brought to the attention of philosophers a phenomenon from biology [Yoshimura, 1997], in which species of cicadas have life cycles that include nymphal stages of either 13 or 17 years, depending on the geographic area. The long nymphal stage by itself poses an interesting question, but particularly intriguing is why 13 or 17, but not 15 or 16 years? The only proposed explanation so far that makes sense appeals to the arithmetical fact that 13 and 17 are prime numbers whereas 15 and 16 are not. Prime number nymphal periods mean fewer intersections with the periods of other insects with multi-year nymphal stages, giving the cicadas an evolutionary advantage.

This explanation of the prime nymphal periods is clearly partly biological (concerning, *e.g.*, the life cycles of the cicadas), but prime numbers seem to play a crucial role in the explanation. However, it is a highly debated matter whether prime numbers are indispensable for the explanation, making the explanation 'genuinely mathematical'.[17] In the conventionalist position of Field, no mathematical explanation of a non-mathematical phenomenon is indispensable, but we choose some theories because they are useful in explanations. Yet it is difficult to see how the move from indispensability to usefulness changes the fundamental problem. Rather than asking why a mathematical theory is indispensable, we should now ask why a mathematical theory is useful. As I see it, the issue itself remains equally important, and the strict conventionalist can only propose that the matter is arbitrary. Thus the strict conventionalist position has a serious weakness. It cannot (nor does it aim to) provide a feasible *non-arbitrary* explanation why some mathematical theories have knowledge-inducing applications and others do not. For this reason, I will move the focus in the next two sections to two alternative explanations of the usefulness of mathematics, namely platonism and cognitive nativism, before developing my own view in Section 6.

4. PLATONISM AND THE OBJECTIVITY OF MATHEMATICAL KNOWLEDGE

Linnebo [2018b] defines mathematical platonism as the conjunction of three claims. *Existence* states that there are mathematical objects, *abstractness* that the objects are abstract, and*independence* that mathematical objects are independent of our languages, thoughts, and practices. Following this definition,

[17] See [Lange, 2013; 2017] for a defence of this view and [Saatsi, 2011; 2016] for a contrary position. For a commentary on the debate, see [Pantsar, 2018b].

any epistemological theory associated with platonism must take mathematical knowledge to be objective. As initial presuppositions of platonist epistemology, I take it as uncontroversial that expressions used in mathematical theorems refer to mathematical objects, and that knowledge of mathematical theorems requires the theorems to be *true*. In platonist philosophy, the class of true mathematical statements is determined by the characteristics of mathematical objects which, due to the independence claim, do not depend on our particular mathematical languages, thoughts, and practices. Consequently, Wright's [1992] criteria of epistemic constraint and cognitive command would seem to be immediately met.[18]

Platonism about mathematics, as understood above, is not a uniform view, but a heterogeneous collection of various epistemological and metaphysical positions. The common thread to platonism is that it endorses realism about mathematical statements or objects, but this can mean radically different things. Traditionally, platonism referred to the view that mathematical objects have a mind-independent existence. However, modern platonists, like Shapiro [1997; 2000], are more likely to argue that it is about mathematical *structures* rather than objects like numbers. Also epistemologically, hard-line positions like Gödel's [1983] 'special epistemic faculty' are rare in modern platonism. Not to attack a straw man, here I focus on the type of platonism that aims to minimize epistemological and ontological commitments. In particular, I will restrict my analysis to the diverse arguments according to which mathematical objects do have a mind-independent existence, but it is a 'thin' existence as a referent of true statements, and nothing more.

As Linnebo [2018a, p. 4] has put it, 'thin objects' are such that their existence 'does not make a substantial demand of the world'. More precisely, in Linnebo's [2018a] account the notion of thin objects is understood in the Fregean, maximally general, sense, as referents of singular terms. Frege's [1884] famous definition of number was based on the following *abstraction principle*:

> The number of Fs is equal to the number of Gs if and only if Fs are equinumerous with Gs.

The neo-Fregean position of Hale and Wright [2001; 2009], among others, is based on the same idea: the only role played by numbers is as referents of singular terms, *i.e.*, for 'the number of Fs is equal to the number of Gs' to be true it only needs to be the case that Fs are equinumerous with Gs.[19] This is the fundamental idea behind referring to objects in terms of abstraction principles. They give a criterion of identity for new concepts by 'carving up' previous propositional content. Frege's [1884] famous example was the abstraction principle:

[18] As shown in the end of this section, the matter is not straightforward for all accounts of platonism. However, I will also show that for all platonist accounts, the criterion of wider cosmological role poses more problems.

[19] This is called *Hume's principle* in the literature, following Boolos [1998, p. 181].

> The direction of line A is the same as the direction of line B if and only if A and B are parallel.

By carving up the propositional content of two lines being parallel, new abstract objects (directions) are introduced. This is what Linnebo [2018a] means by thin objects: numbers and directions are referents of singular terms in the abstraction principles, and that is all they are required to be. Thus, while numbers and directions are abstract objects as referents of singular terms, their existence does not make any substantial demands of the world.[20]

In the introduction, I mentioned the epistemological problem of Benacerraf [1973] as perhaps the main difficulty that a platonist account in the philosophy of mathematics faces. For the platonists, as Benacerraf saw it, the epistemological problem was in explaining a connection between humans as physical subjects and causally inert abstract mathematical objects. Both Linnebo and the neo-Fregeans aim to escape this problem by arguing that the thin notion of mathematical objects does not require any epistemological connection that is not present in the abstraction principles. Since the accounts involving thin objects do not require ontology beyond the abstraction principles, they also avoid Benacerraf's epistemological problem.

However, those accounts are now faced with the problem of explaining what the existence of thin objects amounts to. Most importantly, one can ask whether the thin objects exist in the mind-independent manner required of platonism under Linnebo's definition. Rayo [2013; 2015] has argued for a theory he calls 'subtle platonism' as a way of escaping the problem. The fundamental idea of this subtle reading of platonism is that mathematical objects like numbers exist, but they do so in a *trivial* sense. Essentially this refers to the triviality of the compositional semantics for mathematical statements. Standard semantics would imply that an arithmetical statement like '$1 + 1 = 2$' can only be true if the world contains numbers. If the world does not contain numbers, as Field [1980], for example, has argued, any existential mathematical statement ($1 + 1 = 2$ can be formulated as one) is always false. After all, for a mathematical statement to be true, under this conception, there have to exist mathematical objects. One popular solution to this problem is that existential mathematical statements should be paraphrased in nominalistic or modal terms (*e.g.*, [Putnam, 1967; Chihara, 1973; 1990; 2005; Hellman, 1989; 1905]). In this approach, phrases like 'there exists x' are replaced by phrases like 'it is possible to construct x' [Chihara, 2005]. Shapiro [1997, p. 228] has argued against such paraphrasing strategies by pointing out the infeasibility of solving ontological problems by a mere change of vocabulary. More recently, also

[20] For present purposes, it is not necessary to go into the differences of the accounts of Linnebo and the neo-Fregeans, but it should be noted that the two accounts are not equivalent. One of the fundamental differences Linnebo sees is that his account does not require the 'syntactic priority thesis' stating a type of priority of syntactic categories over ontological ones [Linnebo, 2018a, p. xii].

Rayo [2015] has argued that the different ways of paraphrasing mathematical statements to avoid existential quantification do not solve the problem. However, he contends that his trivial semantics does, because in it the truth conditions for mathematical statements do not require substantial existence claims; hence the characterization of them as trivial. Let us take a closer look at what this means. Rayo [2013; 2015] does not aim to avoid commitment to numbers. Rather, he accepts that arithmetical statements commit to numbers, but for the statements to be true, there is no need for numbers to exist in a non-trivial manner. Consider his example: 'the number of dinosaurs is zero'. According to Rayo, for this sentence to be true it is only required that there are no dinosaurs, *i.e.*, the sentence is true in a possible world w if and only if there are no dinosaurs in w. The truth condition (the right-hand side of the biconditional) says nothing about numbers. As he formulates it:

> For the number of the dinosaurs to be zero *just is* for there to be no dinosaurs. [Rayo, 2015, p. 81; emphasis original]

Hence, Rayo points out, there is no difference between there being no dinosaurs and the number of dinosaurs being zero. Generally:

> 'For the number of the Fs to be n *just is* for it to be the case that $\exists!_n x(Fx)$' (*ibid.*, emphasis original).[21]

In other words, there is no difference between there being n things that have the property F and the number of Fs being n. Moving to purely arithmetical sentences like '1+1 = 2', Rayo argues, the truth conditions are satisfied trivially regardless of what the world is like, since the right-hand side of the biconditional

$$1 + 1 = 2 \text{ is true if and only if } 1 + 1 = 2$$

is fulfilled in all possible worlds. Thus, arithmetical sentences commit to numbers, yet by trivial semantics they escape Benacerraf's problem. After all, Rayo [2015] argues, the commitment to numbers does not have anything to do with the truth conditions of arithmetical sentences; so there is no epistemological cost.

Rayo's subtle platonism shows promise in offering an epistemologically feasible platonist alternative to conventionalism, since it does what platonists generally aim to do: instead of advocating a paraphrasing or revisionist alteration of mathematics, it takes mathematical practice as it is and maintains that commitment to numbers need not carry any epistemological problems. Yet it is not clear that this commitment to numbers amounts to anything more than applying a particular use of language. For all there is to its credit, Rayo's account appears to take such a general way of understanding the commitment

[21] The expression $\exists!_n x(Fx)$ means that there exist exactly n values of x for which it holds that Fx.

to numbers that it is questionable whether it can be called platonism, even as a subtle variant. It is possible to entertain an alternative view, according to which the only commitment being made is to a linguistic practice, which may not be anything more than a conventional shorthand. And if this is the case, Rayo's subtle platonism will end up with the same potential problem of arbitrariness as the conventionalist accounts.

One might wonder how this could be the case since the truth conditions of arithmetical statements are clearly not arbitrary. The number of dinosaurs is zero just in case there are no dinosaurs. But there being no dinosaurs is not something arbitrary, it is a fact concerning dinosaurs. That the number of Earth's moons is one is not arbitrary, and so on. This is clear enough, but what is there to prevent statements of *pure* arithmetic like $1+1=2$ from being arbitrary? Rayo [2015] discusses also pure arithmetic in terms of triviality: the truth conditions of $1+1=2$ are trivial because they are satisfied regardless of what the world is like, and hence the existence of numbers is also trivial. In this picture, there is no room for non-trivial truth-makers for arithmetical statements. Yet I can envision a possible world where the truth conditions of $1+1=2$ are *not* trivial: a world with no subjects who process their observations, thoughts, or ideas in terms of numbers. In such a world the truth condition of $1+1=2$ would not be trivial, since any proposition involving natural numbers would require new conceptual thinking. This is not to say that $1+1=2$ would be *false* (or without truth value) in such possible worlds. Rather, the argument is that in such possible worlds there would be no cognitive agents who possess number concepts. Consequently, the truth conditions of $1+1=2$ are *not* satisfied regardless of what the world is like. For them to be satisfied, there need to be cognitive agents who possess number concepts, or at least some form of equivalent numerical ability.[22]

Neither would it be trivial that the number of Earth's moons is one, because it would not be trivial that things are discussed in terms of their number. Understanding such statements and establishing their truth would require the subjects first to learn about natural numbers, or at least a small finite subset of them. This brings us to what I see as the real crux of the matter: just how do we initially learn about natural numbers and the way they can be associated with objects in the world? Rayo's account does not get involved with this issue. In the context I have brought up, however, what appears to be trivial may not be so trivial after all. And if we can identify non-trivial truth-makers for arithmetical statements, we must reconsider how Rayo's account should be interpreted — including the triviality of the existence of numbers.

In a similar way, I see a potential problem with the thin objects of Linnebo and the neo-Fregean account of abstract objects. Abstraction principles may indeed carve out the propositional content so that reference to new abstract

[22] Another possibility is that numbers exist in the possible world independently of cognitive agents, but assuming a platonist ontology of mind-independent objects would appear to go against the spirit of the subtle version of platonism and trivial existence of numbers that Rayo argues for.

objects is possible, and manage to do that in an epistemologically unproblematic way. But why do we end up endorsing particular abstraction principles like Hume's principle? Is it really the case that we have an understanding of the principle of equinumerosity which we then use to introduce the concept of number? Or did the concept of number exist beforehand, and equinumerosity is used to characterize it? This is at least a plausible possibility, given how numbers have been referred to for millennia without explicit recourse to abstraction principles like Hume's principle. Now the question is, could it be the case that there has been reference to numbers even without *implicit* use of equinumerosity? Could there be reference to numbers due to some other kind of epistemological access besides abstraction principles? If so, would this reference still make numbers 'thin' objects in the sense of Linnebo's account?

In the next two sections, I will discuss that possibility and its philosophical consequences in detail. For now, however, let us assume that the account of thin objects is correct for mathematical objects. Can this account escape the conventionalist threat mentioned above in reference to the trivial truth conditions of Rayo's account? If mathematical objects only exist as referents of abstraction principles, is there any reason to believe that mathematical *knowledge* concerns anything more than conventions? If not, is there any reason still to believe that mathematical knowledge is *objective*? Returning to Wright's [1992] criteria of objectivity, it is not clear that any of them are necessarily met by the accounts of Linnebo and Rayo, which makes it necessary to reassess how platonist those views are. Epistemic constraint seems questionable, given that the abstraction principles may be just conventions; a further argument would be needed to establish that they are more than that. Similarly, the criterion of cognitive command may not be met if we do not have any non-conventional reasons to choose some abstraction principles over others.

However, the most serious problem concerns the criterion of wider cosmological role. How can we justify that mathematical discourse, as envisioned in the accounts of Linnebo and Rayo, is used outside mathematics? If the abstraction principles or the trivial mathematical truths are only conventions, I cannot see any such justification. The accounts would run into the most serious problem of conventionalism as discussed in the previous section: mathematical applications in science appear to be arbitrary. If the abstraction principles are based on something more than conventions, however, we can no longer assume — at least without argument — that mathematical objects only have a thin or trivial existence. In other words, the epistemologically unproblematic versions of platonism proposed by Linnebo and Rayo face a new challenge: they must explain why they should still be associated with platonism, instead of conventionalism. For Linnebo [2018b], platonist mathematical objects are abstract, and they exist in a manner independent of thought, language, and practice. In order for the account of Linnebo to remain platonist, since the propositional content in the abstraction principles is responsible for the existence of mathematical objects, it must be independent of our thoughts, languages, and practices.

As I will argue in the concluding section, there is indeed a non-conventionalist way of understanding thin objects. But before we can establish that, we should know just how mathematical thoughts, languages, and practices come to exist. I will propose an explanation in Section 6. For now, however, let us return briefly to traditional forms of platonism, according to which mathematical objects have an existence that is not thin. Clearly they fulfill Wright's criteria of epistemic constraint and cognitive command for objectivity. But are they in a better position to explain the wider cosmological role of mathematics, namely the existence of mathematical applications in science? The best-known answer is provided by Quine [1966; 1969], according to whom we cannot separate the mathematical content from the non-mathematical content in scientific theories, and therefore both should have the same ontological status.[23] If we accept, for example, that prime numbers are indispensable for the best explanation of the cicada nymphal stages, we must accept that prime numbers are as real as the insects or the molecules they consist of. In this holistic picture of scientific explanations, we cannot cherry-pick the objects that we make ontological commitment to.

Yet it seems clear that abstract mathematical objects must be in some way very different from physical objects. In this respect, the Quinean account does not provide us with many answers. Even if we agreed that there should be ontological commitment to mathematical objects, just as there is to non-mathematical objects, we are still left with the question how abstract and non-abstract objects are connected in scientific explanations. I see this as a basic problem of platonism about mathematics with regard to scientific explanations. Granted, Quine's argument gives uniformity to the explanations in terms of ontological commitments: physical objects have a mind-independent existence and so do mathematical ones. But this alone does little to explain the wider cosmological role of mathematics. For that, there would need to be an explanation of how abstract mathematical objects (or structures) and physical objects are related to each other. While such accounts have been discussed (see, *e.g.*, [Brown, 2008]), none of them seems to bridge the gap between the abstract and the physical in a satisfactory way. Unfortunately, analyzing them goes beyond the scope of this paper. Instead, I will move on to a different type of account of mathematical knowledge and see how they fare with the questions of objectivity and applications. If successful, such accounts can potentially solve the problem of integrating abstract and non-abstract objects in scientific theories.

5. EVOLUTIONARY NATIVISM AND THE OBJECTIVITY OF MATHEMATICAL KNOWLEDGE

As we saw in the previous section, platonist approaches have developed into increasingly light ontological and epistemological variations, up to the point that it becomes necessary to ask what distinguishes them from conventionalist

[23] To be precise, as mentioned earlier, this is the case if mathematics is *indispensable* for the scientific explanation in question, *i.e.*, the application of mathematics makes it possible to explain non-mathematical scientific facts that cannot be explained without mathematics.

positions about mathematical knowledge. But in one way these new variations of platonism, such as those of Rayo [2015] and Linnebo [2018a], but also others like Peacocke [1993], Shapiro [1997], and Brown [2008], are very much part of the older platonist tradition: their methodology takes epistemology and ontology of mathematics to be an *a priori* pursuit in which we can elucidate the nature of mathematical knowledge through logical and conceptual analysis.

As was mentioned in the introduction, however, recently this approach has been challenged by researchers who believe that we can explain mathematical knowledge on the basis of the evolutionarily determined ability to treat quantitative information that we already possess as infants and share with many non-human animals (*e.g.*, [Dehaene, 1997/2011; Butterworth, 1999; Lakoff and Núñez, 2000; Carey, 2009]). On this view, the *a priori* approach to philosophy of mathematics is largely replaced by empirical research on our cognitive capacities, which can reveal how mathematical knowledge is possible, and what it is about.[24] An important part of this work is empirical research on young children (often infants) and non-human animals. Numerous authors have concluded that infants and many non-human animals, including primates and parrots, but also fish, have numerical abilities [Starkey and Cooper, 1980; Dehaene, 1997/2011; Hauser *et al.*, 2000; Cantlon and Brannon, 2006; Pepperberg, 2012]. Indeed, many influential authors explicitly call these animal and infant abilities arithmetical (*e.g.*, [Wynn, 1992; Dehaene, 2001; Rugani *et al.*, 2009; Agrillo, 2015]. As mentioned in the introduction, I call this account *nativism* (or cognitive nativism) about arithmetical knowledge.

Both platonism and nativism would appear to be committed to the position that mathematical — or at least arithmetical — knowledge is essentially objective. On the nativist account, which takes arithmetical ability to be the product of biological evolution, no existence of mathematical objects needs to be assumed. If it is accepted that the evolutionarily developed arithmetical ability constitutes knowledge and it is (at least partly) shared with infants [Wynn, 1992] and newly hatched chicks [Rugani *et al.*, 2009], there would seem to be a strong case for its objectivity. Rather than being in some essential way dependent on the content of our thoughts, languages, or conventions, arithmetical knowledge would be determined by evolutionary processes that have led to biological characteristics we possess already as infants and share with a wide range of non-human animals.

This way, platonism and nativism both have a straightforward explanation for the apparent objectivity of mathematical knowledge. But given that in nativism no mathematical objects are presumed to exist, with an Occam-type parsimony principle, this appears to give the nativist position an edge over at least the traditional 'non-thin' versions of platonism: it does not have to face Benacerraf's epistemological challenge, as it does not require the existence of

[24] For overviews of the development in recent years, both empirical and philosophical, see *e.g.* [Cohen Kadosh and Dowker, 2015; Bangu, 2018; Pantsar and Dutilh Novaes, 2020].

mind-independent abstract objects.[25] Neither does nativism succumb to the arbitrariness of mathematics that threatens conventionalist theories. The class of sentences considered to be mathematical truths would not be arbitrary, since it would be (at least partly) determined by biological characteristics due to evolutionary processes.

However, as I have argued elsewhere in detail [Pantsar, 2014; 2015a; 2018a; 2019a; 2020], the problem with the above line of argumentation is that it assumes that the infant and animal quantitative ability is indeed arithmetical. This is often explicitly claimed. For example, one of the most important and influential researchers in the field, Dehaene, writes:

> In the course of biological evolution, selection has shaped our brain representations to ensure that they are adapted to the external world. I have argued that arithmetic is such an adaptation. [2001, p. 31]

That arithmetic is such an evolutionarily determined adaptation is a very problematic assumption and unsupported by empirical evidence. To see this, let us consider the famous infant experiment reported by Wynn [1992]. It had been previously established that infants can distinguish between small numerosities [Starkey and Cooper, 1980]. This ability, called *subitizing*, generally stops working after three or four objects, but until then it allows determining the number of objects in our field of vision without counting. There are good empirical reasons to believe that we already possess the subitizing ability as infants and share it with many non-human animals (see, *e.g.*, [Dehaene, 1997/2011; Spelke, 2000]). What Wynn argued was that, based on the subitizing ability, infants can in fact carry out simple addition and subtraction operations. She observed infants reacting with surprise (*i.e.*, longer looking time) to the 'unnatural arithmetic' of $1+1=1$, instantiated by the infant seeing two dolls being put behind a screen but only one being there after the screen was lifted (the other having been removed clandestinely). From different configurations in the subitizing range (from one to four), she concluded that infants are able to carry out rudimentary arithmetical operations. But as I have argued before [2018a], this conclusion is unwarranted. The infants' behavior could be explained by their having some kind of cognitive mechanism or procedure in place for keeping track of one small quantity at a time. When the exposed quantity of objects did not match their expectations, they were surprised. Under this explanation, nothing like an arithmetical operation is presupposed to take place in the cognitive process. What is presupposed is merely an ability to track small quantities, which gets wide support from empirical data.

This distinction is crucial to make. We must not confuse developed arithmetical ability with primitive quantitative abilities, nor assume that the difference

[25] Of course the possibility of mind-independent mathematical objects is not denied in evolutionary nativism. Platonism and nativism are not necessarily metaphysically contradictory positions, even though they seem to clash in terms of epistemology.

is only gradual. In my previous work, I have distinguished between *proto-arithmetical* quantitative ability and actual arithmetical ability for exactly this purpose [2014; 2018a; 2019a].[26] It could be that the subitizing ability and the *object-tracking system* that makes it possible form the cognitive foundation for the development of arithmetical ability ([Carey, 2009; Beck, 2017]; see also [Pantsar, 2021b]). At least a partial cognitive foundation for arithmetic could also be found in another evolutionarily developed quantitative ability, the estimation ability due to the *approximate number system* that we also possess as infants and share with non-human animals [Dehaene, 1997/2011].[27] But even if this were the case, the proto-arithmetical ability, or abilities, need to be treated as conceptually distinct from arithmetical ability. The proto-arithmetical subitizing and estimation abilities may be related to arithmetical knowledge, but to what extent and how is something that needs to be carefully analyzed. What the infants are doing in Wynn's experiment may be *described* with the help of arithmetical language (such as $1 + 1 = 2$). But this must not be confused with the infants' actually carrying out addition (or subtraction) operations, even in rudimentary forms. Therefore, I am ready to accept the above quotation of Dehaene only after one crucial amendment. I agree with the plausibility of the general principle that 'In the course of biological evolution, selection has shaped our brain representations to ensure that they are adapted to the external world'. However, rather than arithmetic, it is the *proto-arithmetical* ability (or abilities) that are such an adaptation.

With the focus moved to proto-mathematical abilities, the question becomes whether the nativist account can still account for the apparent objectivity of mathematical knowledge, as well as mathematical applications. There are arguments both for and against the apparent objectivity. Support for the apparent objectivity comes from history. Unlike many other crucial innovations, like the alphabet, which is generally thought to have developed only once during known human history (see, *e.g.*, [Sampson, 1985]), arithmetic is known to have developed several times independently [Ifrah, 1998]. The particular systems of arithmetic have had different characteristics but they have also shown great similarity both in terms of counting and operations (addition and multiplication). This suggests that the development of arithmetic taps into the proto-arithmetical abilities, which determine in an important way the content of arithmetical theories. Indeed, most known languages have some kind of a numeral system, and many of these systems show recursivity in some numeral base [Ifrah, 1998]. If the proto-arithmetical abilities are the reason for this kind of independent development of numeral systems and arithmetical knowledge in

[26] I have also proposed [2018a] that the word 'number' should be reserved for arithmetic, while it would be clearer to speak of 'numerosities' for the proto-arithmetical abilities. A similar distinction is suggested by De Cruz *et al.* [2010], as well as Núñez [2017] who proposes the term 'quantical' ability for what I call proto-arithmetical.

[27] Carey [2009] calls the two systems *core cognitive* to emphasise their independence from other cognitive systems.

different cultures, there is a strong case to be made that arithmetical knowledge is objective in a relevant sense. At the very least, it would *appear* to be objective.

However, there are also human cultures, such as the Amazonian tribes of Pirahã and Munduruku, whose knowledge and skills with quantities are not considerably above the primitive quantitative systems shared with many non-human animals [Gordon, 2004; Pica *et al.*, 2004]. The Pirahã numeral system, for example, includes words roughly for 'one', 'two', and 'many', which are not used in a consistent manner. Yet both the Pirahã and the Munduruku have the same (or similar) proto-arithmetical subitizing and estimation abilities as we do [Gordon, 2004; Pica *et al.*, 2004; Dehaene *et al.*, 2008]. This tells us that having proto-arithmetical abilities is not sufficient for developing even rudimentary numeral systems, let alone proper arithmetic. There appears to be a connection between the proto-arithmetical abilities and arithmetic, but clearly the former underdetermine the development of the latter.

The upshot of this is that we cannot directly establish proto-arithmetical abilities as the foundation of arithmetic, which in turn implies that arithmetical knowledge could be in an important way dependent on human conventions. This possibility would be highly problematic for the nativist position, since it could be the case that arithmetical knowledge is in fact to a large extent *independent* of the proto-arithmetical abilities. After all, with the distinction between proto-arithmetical and arithmetical abilities in place, it becomes clear that it is nativism over the *proto-arithmetical* abilities that the empirical evidence supports. While the apparent objectivity of proto-arithmetical abilities would not be in question, the apparent objectivity of arithmetical (and presumably other mathematical) knowledge could not be exclusively, or even mainly, due to the evolutionarily developed abilities that are the foundation of the nativist position.

The role of mathematics in the sciences would suffer the same destiny in that scenario: if mathematics is not fundamentally a product of the evolutionarily developed proto-arithmetic, and possibly other proto-mathematical abilities, the success of mathematical applications in science could be either a coincidence or based on the conventionalist character of both mathematics and science. No more substantial link between the two could be drawn.

However, I do not believe that to be the case. There is no reason to believe that infants and non-human animals have arithmetical (or other mathematical) abilities, but as I will argue in the next section, there are very good reasons to think that the proto-arithmetical (and possible other proto-mathematical) abilities form a partial cognitive foundation for arithmetic (and other areas of mathematics). Arithmetic has developed in different cultures, *e.g.*, the Chinese, Mayans, and Greek, independently in ways that converge in terms of counting and basic operations like addition and multiplication [Ifrah, 1998]. However, in terms of numeral systems, arithmetical practices and applications, the different cultures show also a great deal of divergence [Ifrah, 1998; Pantsar, 2019a]. Thus I will argue in the next section that cultural factors play an integral role in the development of arithmetic, making it underdetermined by the proto-arithmetical abilities. Hence the nativist position is not plausible in the face of the best data and theoretical understanding we have about arithmetical and proto-arithmetical cognition. But with a proper understanding of the development of arithmetic from its proto-arithmetical origins, we can still trace the cognitive foundations of mathematics at least partly to evolutionarily developed abilities.

6. ENCULTURATED MATHEMATICS

In the previous section, I criticized Dehaene [2001] for not distinguishing between proto-arithmetic and arithmetic, and consequently for claiming that arithmetic is an evolutionarily developed adaptation. I believe that this equivocation is much more than a mere terminological issue, and both in empirical research and philosophy there should be an aim to rid the literature of such possible confusions. However, I do not want to claim that Dehaene (or others making a similar equivocation) sees arithmetic as entirely a product of biological evolution. Indeed, Dehaene explicitly recognizes also another type of evolution crucial for the development of mathematics:

> Specific to the human species, however, is a second level of evolution at the cultural level. As humans, we are born with multiple intuitions concerning numbers, sets, continuous quantities, iteration, logic, or the geometry of space. Through language and the development of new symbols systems, we have the ability to build extensions of these foundational systems and to draw various links between them. [Dehaene, 2001, p. 31]

Again, it is important to be careful with terminology like 'multiple intuitions'. I would not be prepared say, for example, that we have intuitions about numbers and sets. I do think we have *proto-arithmetical* abilities that can *lead* to something that may feasibly be called 'intuitions' about numbers. I also believe that it is possible that we have similar proto-geometrical, as well as possibly proto-set-theoretical and proto-logical abilities. But it is important that these be distinguished from any genuinely mathematical intuitions. When it comes to the latter, however, Dehaene points out one crucial aspect. Language and symbol systems allow extending and drawing links between intuitions, although, as we will see, there are also other factors involved. Nevertheless, having made these clarifications, I believe Dehaene is fundamentally correct about the most important aspect. It is indeed the evolution on the *cultural* level that we need to include in order to explain how the proto-arithmetical abilities can develop into mathematical knowledge and skills.

The variation in quantitative abilities between different cultures is enormous. From research on cultures like Pirahã and Munduruku that appear to have no arithmetical skills and knowledge [Gordon, 2004; Pica *et al.*, 2004] to cultures like the Mayans, ancient Chinese and Greek, which independently made great advances both in arithmetic and its applications, it becomes clear that arithmetic is the product of a long line of development in specific cultural circumstances [Ifrah, 1998]. In the Western tradition heavily developed by the ancient Greek, but following a longer tradition that can be traced at least to Babylon and Egypt,[28] we can see that arithmetic slowly developed into the subject we are currently familiar with. Some important advances have happened quickly due to specific innovations, but in the bigger picture the development of arithmetic has been a long gradual process from the introduction of numerals to modern formal systems [Boyer, 1991]. This is consistent with the theory of *cumulative cultural evolution* for explaining the development of human knowledge and skills [Boyd and Richerson, 1985; 2005; Tomasello, 1999; Henrich, 2015; Heyes, 2018]. According to cumulative cultural evolution, technologies and other cultural innovations are improved upon in small (trans-)generational increments. As these improvements are used widely enough, knowledge and skills based on them can reach a status where they are no longer tied to small groups of individuals. New improvements are adopted widely within cultures and passed on to subsequent generations, enabling a cumulative evolution of culturally developed artifacts, knowledge, and skills. This passage of the knowledge and skills can also extend across cultures through regular interactions.

Basic arithmetic became such a skill in many cultures and arithmetical knowledge was spread increasingly widely. Large groups of children were starting to be educated first in counting processes, then moving on to arithmetical operations. Applications of arithmetic were invented, further strengthening the place of arithmetic within educational and other cultural practices.[29] Ultimately this cultural development made arithmetical knowledge and skills widely possessed, thus facilitating the refinement and development of them, as well as that of the educational practices for distributing them [Pantsar, 2019a]. This is the history (and pre-history) behind arithmetic as we currently know it: a subject that the vast majority of children in the world acquire basic knowledge and skills in.[30]

Conventionalists might at this point see an argument to support their position. Few culturally developed practices are as widely shared in the modern world as basic arithmetic. If we want to explain the apparent objectivity of arithmetical knowledge, is this widely spread cultural background not enough? If most of the world learns largely similar arithmetical content, is it any wonder

[28] D'Errico *et al.* [2018] argue that this development can in fact be tracked back at least to roughly 70,000–62,000 BC, to artifacts such as a notched hyena femur bone found at the Les Pradelles Mousterian site in France.

[29] Unfortunately, very little has remained to tell us how this process has taken place. From ancient Greece, higher-level mathematical works by the likes of Archimedes and Euclid have survived but although they were used for educational purposes, not much is known about early mathematical education and what role it played [Morgan, 1999]. Mueller [1991] notes that arithmetic was apparently widely known in fifth century (BCE) Athens but it is not clear that this knowledge was acquired in schools. See [Fried, 2012] for more.

[30] While the arithmetical content is largely equivalent cross-culturally, there are many differences in educational practices, including the kind of tools that are used (pen and paper, abacus, calculators, *etc.*). See [Fabry and Pantsar, 2019] for more on the impact of cognitive tools on mathematical problem-solving processes.

that we start seeing arithmetical knowledge as objective? While I believe that the question is well placed, it immediately prompts a further question: why are there such large similarities in basic knowledge and skills when it comes to arithmetic? Is this simply the result of the domination of certain cultures in the modern world, or are there culture-independent factors that determine, at least partly, the similarities in arithmetical knowledge and skills? In order to answer such questions, we need to understand better how *cultural learning* [Heyes, 2018] takes place, *i.e.*, how cumulative cultural evolution is possible by transmitting new knowledge and skills to subsequent generations.

In this paper, I apply Menary's [2014; 2015] notion of *enculturation* in formulating a theoretical framework for analyzing cultural learning in the field of arithmetic (for more details, see [Pantsar, 2019a; b; 2020; 2021b]. Enculturation refers to transformative processes in which interactions with the surrounding culture determine the way cognitive practices are acquired and developed [Menary, 2015; Fabry, 2018]. The fundamental idea behind this framework is that our genetically determined, evolutionarily developed, biological faculties are transformed through the cultural transmission of cognitive practices. The enculturation account is thus a potentially important improvement over crude dichotomies like 'nature versus nurture', or as in the present context, conventionalism versus nativism. Instead of focusing on biologically determined abilities or culturally developed abilities separately, the enculturation framework can provide a link between the two. Indeed, in the enculturation framework a strict dichotomy between biology and culture should no longer be made.[31] In the case of arithmetical cognition, researchers have recently seen enculturation as a way to connect research on proto-arithmetical abilities with research on arithmetic as a culturally developed phenomenon [Menary, 2015; Pantsar, 2019a; Jones, 2020; Fabry, 2020].

Most researchers of numerical cognition agree that the evolutionarily developed core cognitive quantitative abilities form a cognitive basis for the development of arithmetical abilities. However, as mentioned in the previous section, there is disagreement over which cognitive core system is central in the developmental process. Dehaene [1997/2011] and Halberda and Feigenson [2008], for example, argue that the approximate number system is the primary core cognitive resource in the development of number concepts and therefore also in the development of arithmetical ability. Many others see the object-tracking system that allows subitizing as the prevalent system in that development (*e.g.*, Carey [2009]; Izard *et al.* [2008]; Sarnecka and Carey [2008]; Carey *et al.* [2017]; Beck [2017]; Cheung and Le Corre [2018]). There are also researchers, Spelke [2011], Pantsar [2014; 2015a; 2019a; 2021b], vanMarle *et al.* [2018], who argue that both core cognitive systems play a crucial role in the process.

While there is still considerable disagreement over the roles that the different cognitive core systems play in the development of arithmetical cognition, with

[31] This dichotomy can still be fruitful for explanatory purposes, however, which is why I refer to it at times in the rest of the paper.

the enculturation framework we can nevertheless propose an explanation of how such development from proto-arithmetical core cognitive abilities to culturally developed arithmetical abilities can take place. The well-established phenomenon of plasticity of the brain is fundamental to the enculturation framework. Through the mechanism Menary [2014] calls *learning driven plasticity*, the neural plasticity of the brain enables the acquisition of new cognitive capacities. The neural plasticity of the brain allows for structural and functional variations based on the experiences of individuals in their ontogeny [Dehaene, 2009; Ansari, 2008; Anderson, 2015]. This enables the acquisition of new cognitive abilities such as reading and writing by redeploying older, evolutionarily developed neural circuits for new, culturally specific functions [Dehaene, 2009; Menary, 2014].[32] When it comes to arithmetical cognition, the hypothesis many researchers accept is that the evolutionarily developed proto-arithmetical ability for processing numerosities in the intraparietal sulci, together with linguistic abilities for number words and symbols, is deployed resulting in two different (although partially overlapping) systems for processing numerosities in the brain, thus explaining how adult subjects have both arithmetical and proto-arithmetical abilities [Dehaene and Cohen, 2007; Nieder and Dehaene, 2009; Dehaene, 1997/2011; Menary, 2015].

The enculturation framework can then provide the link between the proto-arithmetical abilities, which are the product of *biological evolution*, and the arithmetical ability that is the product of cumulative *cultural* evolution.[33] By redeploying proto-arithmetical neural circuits for arithmetical abilities, individuals are able to acquire culturally developed cognitive practices. This, in turn, enables them to develop further or modify cognitive practices, which can then be adopted more widely in the culture. Therefore enculturation and cumulative cultural evolution form a feedback loop which explains the acquisition and development of cognitive practices.[34] This is consistent with the arguments of Fabry [2017], who employs Laland's [2017] notion of *cognitive innovation* to analyze the introduction and acquisition of new cognitive practices. Fabry emphasizes the importance of the interplay between cognitive innovations, enculturation, and cumulative cultural evolution in transforming our cognitive capacities. I find the resulting theoretical framework fruitful for explaining how proto-arithmetical abilities are employed in developing arithmetical knowledge and skills, because it connects the phylogenetic and ontogenetic levels of

[32] This principle is called *neuronal recycling*. Anderson [2015] has proposed the more general principle of *neural reuse* instead, which has been argued by Jones [2020] and Fabry [2020] to be a better fit for understanding the development of arithmetical cognition.

[33] This should not be confused with the claim that cultural evolution is not fundamentally a biological phenomenon. The two terms are used here as different categories of explanation without any metaphysical claims associated.

[34] This idea of a feedback loop is a generalization of the feedback loop that according to Fabry [2018] connects enculturation and 'epistemic engineering' [Sterelny, 2014; Menary, 2014; 2015], which she sees as one constitutive process of cumulative cultural evolution.

development. On the level of individual ontogenetic development, the proto-arithmetical neural circuits are (partially) redeployed to acquire new numerical cognitive practices. On the phylogenetic level, contributions by individuals make group-level innovation possible, resulting in cumulative cultural evolution that gives rise to new practices, which are acquired by new generations of individuals in ontogeny through the process of enculturation.[35]

Many details of this process are as yet unknown, but I contend that the above account provides a feasible framework for explaining how arithmetical knowledge has developed. It is of course possible that there has been at least some genetic and cultural co-evolution in the development of numerical abilities, and the order of development presented above is not fully accurate.[36] However, from what we know about the proto-arithmetical abilities, they are largely universal and do not show significant cultural variation [Dehaene, 1997/2011].[37] While there may have been some cultural co-evolution of numerical abilities taking place also during the period of the genetic evolution of proto-arithmetical abilities, this effect appears to have been minor.

Importantly, in contrast to the nativist account described in the previous section, the present account does not take arithmetical knowledge to be essentially a product of biological evolution. The cultural aspects of the development of arithmetic play a crucial role in this enculturated epistemological theory of arithmetic. In contrast to strict conventionalist accounts, however, the cultural aspects do not entirely determine arithmetical content. Through the process of re-deploying proto-arithmetical abilities in acquiring and innovating new cognitive practices, arithmetical knowledge is partly determined by the core cognitive, evolutionarily developed, quantitative systems. Not only do I contend that this is the most feasible epistemological account of arithmetic, but it can also provide the best explanation for the two reasons for believing that arithmetical knowledge is objective. It can explain the apparent objectivity of arithmetical and other mathematical knowledge, and it can explain arithmetical and other mathematical applications in science.

On the present epistemological account, there can be two reasons for the apparent objectivity of mathematical knowledge. First, it could be that mathematical knowledge appears to be objective simply because it is based on deeply-entrenched conventions, as discussed in Section 3. However, unlike in the conventionalist theories, on the present account there is also a second plausible reason. Basic arithmetic, at least, appears to be objective to us because it is based on evolutionarily developed proto-arithmetical abilities. These abilities are universal to human beings, except for cases of developmental dysfunctions or injuries. In [Pantsar, 2014], I have called arithmetical knowledge *maximally intersubjective* because of this biological foundation. From data on different

[35] How individual contributions are introduced and accepted by mathematical communities, thus leading to mathematical innovation, is a highly interesting research question. See, *e.g.*, [Wagner, 2017] for discussion on how mathematical practices are established.

[36] I thank an anonymous reviewer for pointing this out.

[37] Indeed, they are shared (at least to a large degree) with many non-human animals.

cultures, it appears that we share the proto-mathematical abilities as widely as we share any cognitive or physical abilities [Dehaene, 1997/2011]. They are not dependent on our languages or practices. If arithmetical knowledge and skills are in an important way determined by proto-arithmetical abilities, it is quite understandable that many would in fact consider such maximally intersubjective knowledge to be objective.

I believe that also the wide applicability of mathematics in science can be explained with the present enculturated account of arithmetical and other mathematical knowledge. There are two potential explanations for this, the first of which has been suggested by Bloom [2000] and Maddy [2014] in a slightly different context. Maddy argues that the foundations of arithmetic are a combination of set theory and logic. In particular, logic on her account is based on our evolutionarily developed cognitive abilities. In contrast to the present account, Maddy's account of arithmetic is thus based on proto-logic rather than proto-arithmetic. I cannot enter the debate between the two positions here (see [Pantsar, 2016] for more), but for the question of applications, Maddy's explanation is also applicable to my account. She argues that the proto-logical ability has evolutionarily developed to mirror how the world is structured:

> Much as our primitive cognitive architecture, designed to detect [the logical structure of the world], produces our firm conviction in simple cases of rudimentary logic, our human language-learning device produces a comparably unwavering confidence in this potentially infinite pattern. [Maddy, 2014, p. 234]

In a similar manner, it could be argued that our primitive cognitive architecture is 'designed' to detect the *arithmetical* structure of the world as discrete objects.[38] One argument of that type has been proposed by De Cruz [2016], who argues that anti-realist accounts fail to explain how the proto-arithmetical ability has evolved. In particular, she asks what the adaptive behavior central to the development of proto-arithmetical ability is based on, if not some type of realist ontology of mathematics. This would then provide a seemingly straightforward explanation for the existence of arithmetical and other mathematical applications in science. If arithmetic, and by extension other fields of mathematics, are based on our evolutionarily developed ability to detect the structure of the world, there is hardly any mystery to the fact that mathematics has applications in explaining that world.

The second explanation for mathematical applications in the present account puts a Kantian twist into the above views of Maddy and De Cruz. Rather than presume that proto-logic or proto-arithmetic detect objective features of the world *an sich*, it is possible to take the weaker position that they detect the

[38] I add the scare-quotes to 'designed' in paraphrasing Maddy for a good reason: on the present account, there is no justification for claiming that proto-mathematical abilities are designed in any way. Indeed, I see the kind of teleological explanation that Maddy proposes as quite problematic.

structure our cognitive architecture imposes on our observations and thoughts. Perhaps this also mirrors the structure of the world, but we may never be able to establish that. If all our observations and thoughts are structured by our cognitive architecture, we cannot hope to access the world beyond that structure. This is another debate that I cannot enter here (see [Pantsar, 2014] for more), but this latter 'Kantian' account would appear to provide an equally good explanation for the existence of mathematical explanations as the realist accounts of Maddy and De Cruz. To answer De Cruz's question above, it could be that the evolutionary endowments are simply vindicated by their adaptative success. Observing the world in terms of countable macro-level entities (such as objects and other animals) has clear advantages in acquiring food, avoiding predators, raising offspring, and many other crucial behaviors. However, I do not see why this would imply a realist ontology of numbers either as objects or placeholders in a structure.

It is of course possible that through neural plasticity we develop numerical or logical abilities that do not detect the structure of the world but neither do they detect some inevitable conditions that our cognitive architecture imposes on our observations and thoughts. However, it is at least a plausible hypothesis that the proto-arithmetical abilities are evolutionarily developed in tandem with observing the world in terms of discrete macro-level objects, being thus tied to the general way we experience our surroundings. But it is not my purpose here to argue that this is indeed the case. Instead, I have wanted to establish that the resulting argument is similar both in the scenario in which we assume that the proto-mathematical abilities detect the structure of the world and the scenario in which we believe them to detect inevitable conditions of our observations. If mathematics is in an important way determined by proto-mathematical abilities, mathematics is connected to science through a great variety of crucial applications because (at least part of) mathematics deals with the very foundational structures that science is also based on. In the spirit of Occam, we may want to restrain from making the realist assumption of Maddy or De Cruz, and be content that the relevant foundational structures concern our observational and cognitive capacities. But this does not make mathematical applications in science more problematic in any way, since the connection between mathematics and science is established in a similar manner.

Of course behind these considerations lies the question whether our proto-mathematical abilities are evolved in a 'truth-tracking' manner so as to ensure that our cognitive capacities provide us with accurate information about the structure of world. There are ways in which such truth tracking seems plausible. Subitizing and estimating can give animals important information that can help with survival and success in life. Being able to establish the numerosity of prey animals or predators, for example, can be the difference between thriving and perishing. Not being able to track the quantity of offspring can be disadvantageous to protecting them and thus prevent the offspring from reaching maturity.

It should also be noted that in such cases processing quantities can increase the *complexity* of the cognitive process, thus making it slower

[Pantsar, 2019b; 2021a]. Experiments show that even in cases where other cognitive processes would save time in making the right decision, the proto-arithmetical abilities are triggered.[39] This speaks for the general evolutionary advantage of possessing proto-arithmetical abilities. After all, given that applying the proto-arithmetical abilities makes decisions slower, there is likely to be some pay-off for the decrease in speed. Given the early evolutionary origins of the object-tracking system and the approximate-number system, it is tempting to explain this advantage by the proto-arithmetical abilities' helping to detect the structure of the world in terms of macro-level objects. However, I recognize that it is at least a feasible possibility that such truth tracking fails. Ultimately, if all observations of our surroundings are filtered through cognitive core systems, we may not be able to determine the structure of the world entirely independently of those systems. Indeed, if my argument in this section is along the right lines, this extends from everyday experience of our local surroundings to modern scientific theories. By including mathematics in our scientific explanations, we are always working in a context influenced by the proto-mathematical abilities. In this context, it is to be expected that mathematics has scientific applications, and that mathematical knowledge appears to us as objective.

7. CONCLUSION: IS MATHEMATICAL KNOWLEDGE OBJECTIVE?

The purpose of this paper has been to explain from an epistemologically feasible basis why many mathematicians, philosophers, and laypeople believe in the objectivity of mathematical knowledge. However, one crucial question remains: is mathematical knowledge in fact objective? Conforming to Wright's [1992] analysis of objectivity, I believe there are good reasons to consider at least basic arithmetical knowledge to be objective. The enculturated development of proto-arithmetical abilities into arithmetic makes it possible that there are unknown mathematical truths, thus fulfilling the criterion of epistemic constraint. This of course depends on the specific cultural context. The arithmetical truth $2+3 = 5$ appears to be such an unknown truth for cultures like the Pirahã.[40] One could

[39] This has been confirmed in adult humans by tests in which the subjects where asked whether two number symbols are the same. This requires no ability with quantities (it is enough to recognize whether the shapes of the symbols are the same), but the data show a clear 'distance effect' in the reaction times: for example, the pair 71 65 takes more time than the pair 79 65 [Hinrichs *et al.*, 1981; Pinel *et al.*, 2001]. The best explanation for this is that even though we know the task, we cannot help processing the number symbols as quantities, which triggers the approximate-number system. Since there's more distance between 79 and 65 than 71 and 65, the characteristics of the approximate-number system predict that the former case is processed faster. See [Pantsar, 2019b] for more.

[40] This is still debated. Overmann and Coolidge [2013], for example, have argued for the possibility that number concepts could be independent of language, which would imply that the Pirahã could possess number concepts even though their language does not contain corresponding numerals. If that is indeed the case and the truth of $2 + 3 = 5$ is thought to be within the grasp of monolingual Pirahã, the sum in this example can be replaced by one involving larger numbers. Overmann and Coolidge (*ibid.*) accept that learning exact large numbers demands language.

claim that since their language does not contain the necessary numerals to express such truths, their cultural context is not relevant here. But this is a problem only for the conventionalist position. If they *did* develop arithmetic based on their proto-arithmetical ability, they would end up holding $2+3=5$ as an arithmetical truth. For our culture, unsolved problems like the Goldbach conjecture can be unknown truths.[41]

Also the criterion of cognitive command seems to be fulfilled, since there are disagreements over arithmetical truths that are not blameless. If basic arithmetical truths are determined by proto-arithmetical abilities, Dostoevsky's character believing in $2+2=5$ would be mistaken and blameworthy: his belief would not follow the proto-arithmetical foundation of arithmetic, namely, his belief in $2+2=5$ would conflict with his experiences in subitizing and/or estimating groups of objects. Finally, the wider cosmological role is fulfilled since arithmetic clearly has applications in science and everyday life. Whether these applications are ultimately indispensable is not crucial, since their fruitfulness in explanations is not tied to indispensability.

However, even if basic arithmetic can be considered to be objective, this is not to say that *all* mathematical knowledge is objective in the same sense. Members of a certain culture might not end up believing that, say, $(-4)*(-4)=16$, which seems to be only partly determined by proto-arithmetical abilities. And when we extend the account beyond arithmetic, the part played by conventions can become increasingly prominent. To the extent that mathematics is based on proto-mathematical cognition (and in addition to proto-arithmetic, this can include feasibly at least proto-geometry, proto-set-theory, and proto-logic), there would always seem to be a partly objective foundation for it. But as the role of conventions becomes more important, can we still call such knowledge objective in the same sense as we would call arithmetical knowledge objective? Such an analysis is beyond the scope of this paper, but I do not see any fundamental difficulties in parts of mathematical knowledge being considered more objective than others.

Finally, if it is accepted that some part of mathematical knowledge is objective, we should consider the ontological status of the mathematical objects postulated in it. As I see it, the account proposed in this paper is consistent with different forms of platonism, without requiring any of them. I have argued that arithmetical truths are at least partly determined by our proto-arithmetical abilities, but of course this does not rule out the possibility that numbers or structures as arithmetical objects could exist independently of the thoughts, languages, and practices of human subjects. What I have argued is that no such assumption needs to be made to provide a feasible epistemology of arithmetic.

How about the ontologically lighter versions of platonism, such as the accounts of Rayo and Linnebo? I contend that my account is consistent also

[41] The Goldbach conjecture states that every even integer greater than two can be expressed as the sum of two prime numbers.

with mathematical objects existing in this 'thin' sense, because the existence of abstract objects is not ruled out. My account does not require postulating mathematical objects even in the thin sense, but it can explain why the discourse involving abstract mathematical objects makes sense. In particular, the present account of mathematics as the product of the enculturated development based on proto-mathematical abilities can feasibly complement the accounts of both Rayo and Linnebo, since it can provide an explanation for why we end up accepting some mathematical truths and adopting some abstraction principles. Indeed, this seems to give a promising account of what the 'thin' existence of objects could mean in a non-conventionalist framework.

REFERENCES

Agrillo, C. [2015]: 'Numerical and arithmetic abilities in non-primate species', in [Cohen Kadosh and Dowker, 2015], pp. 214–236.

Anderson, M. [2015]: *After Phrenology: Neural Reuse and the Interactive Brain.* Cambridge, Mass.: MIT Press.

Ansari, D. [2008]: 'Effects of development and enculturation on number representation in the brain', *Nature Reviews Neuroscience* **9**, 278–291.

Arora, S., and B. Barak [2007]: *Computational Complexity: A Modern Approach.* Cambridge University Press.

Baker, A. [2005]: 'Are there genuine mathematical explanations of physical phenomena?', *Mind* **114**, 223–238.

______ [2009]: 'Mathematical explanation in science. *British Journal of Philosophy of Science* **60**, 611–633.

Balaguer, M. [2009]: 'Realism and anti-realism in mathematics', in A. Irvine, ed., *Philosophy of Mathematics*, pp. 35–101. Handbook of the Philosophy of Science. North Holland.

Bangu, S., ed. [2018]: *Naturalizing Logico-Mathematical Knowledge.* New York: Routledge.

Beck, J. [2017]: 'Can bootstrapping explain concept learning?', *Cognition* **158**, 110–121.

Benacerraf, P. [1973]: 'Mathematical truth', *The Journal of Philosophy* **70**, 661–679.

Benacerraf, P., and H. Putnam, eds [1983]: *Philosophy of Mathematics: Selected Readings,* 2nd ed. Cambridge University Press.

Bloom, P. [2000]: *How Children Learn the Meanings of Words.* Cambridge, Mass.: MIT Press.

Bloor, D. [1976]: *Knowledge and Social Imagery,* London: Routledge and Kegan Paul.

Boolos, G. [1998]: *Logic, Logic and Logic.* R. Jeffrey and J.P. Burgess, eds. Cambridge, Mass.: Harvard University Press.

Boyd, R., and P.J. Richerson [1985]: *Culture and the Evolutionary Process.* University of Chicago Press.

______ [2005]: *Not by Genes Alone.* University of Chicago Press.

Boyer, C. [1991]: *A History of Mathematics.* 2nd ed. Hoboken, N.J.: Wiley.

Brouwer, L.E.J. [1948]: 'Consciousness, philosophy and mathematics', in [Benacerraf and Putnam, 1983], pp. 90–96.

Brown, J.R. [2008]: *Philosophy of Mathematics: A Contemporary Introduction to the World of Proofs and Pictures.* 2nd ed. New York: Routledge.

Butterworth, B. [1999]: *What Counts: How Every Brain is Hardwired for Math*. New York: The Free Press.

Cantlon, J.F., and E.M. Brannon [2006]: 'Shared system for ordering small and large numbers in monkeys and humans', *Psychological Science* **17**, 402–407.

Cantor, G. [1883]: 'Über unendliche, lineare Punktmannigfaltigkeiten, 5', *Mathematische Annalen* **21**, 545–586.

Carey, S. [2009]: *The Origin of Concepts*. Oxford University Press.

Carey, S., A. Shusterman, P. Haward, and R. Distefano [2017]: 'Do analog number representations underlie the meanings of young children's verbal numerals?', *Cognition* **168**, 243–255.

Cheung, P., and M. Le Corre [2018]: 'Parallel individuation supports numerical comparisons in preschoolers', *Journal of Numerical Cognition* **4**, 380–409.

Chihara, C. [1973]: *Ontology and the Vicious Circle Principle*. Ithaca, N.Y.: Cornell University Press.

______ [1990]: *Constructibility and Mathematical Existence*. Oxford University Press.

______ [2005]: 'Nominalism', in S. Shapiro, ed., *The Oxford Handbook of Philosophy of Mathematics and Logic*, pp. 483–514. Oxford University Press.

Cohen Kadosh, R.C., and A. Dowker [2015]: *The Oxford Handbook of Numerical Cognition*. Oxford University Press.

Colyvan, M. [2001]: *The Indispensability of Mathematics*. Oxford University Press.

De Cruz, H. [2016]: 'Numerical cognition and mathematical realism', *Philosopher's Imprint* **16**, 1–13.

De Cruz, H., H. Neth, and D. Schlimm [2010]: 'The cognitive basis of arithmetic', in B. Löwe and T. Müller, eds, *PhiMSAMP. Philosophy of Mathematics: Sociological Aspects and Mathematical Practice*, pp. 59–106. London: College Publications.

d'Errico, F., L. Doyon, I. Colagé, A. Queffelec, E. Le Vraux, G. Giacobini, B. Vandermeersch, and B. Maureille [2018]: 'From number sense to number symbols: An archaeological perspective', *Philosophical Transactions of the Royal Society B: Biological Sciences* **373**, Article 20160518. doi.org/10.1098/rstb.2016.0518.

Dehaene, S. [1997/2011]: *The Number Sense: How the Mind Creates Mathematics*. 2nd ed. Oxford University Press.

______ [2001]: 'Précis of the number sense', *Mind and Language* **16**, 16–36.

______ [2009]: *Reading in the Brain: The New Science of How We Read*. London: Penguin.

Dehaene, S., and L. Cohen [2007]: 'Cultural recycling of cortical maps', *Neuron* **56**, 384–398.

Dehaene, S., V. Izard, E. Spelke, and P. Pica [2008]: 'Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures', Science **320**, 1217–1220.

Dostoevsky, F. [1864]: *Notes from Underground*. R. Pevear and L. Volokhonsky, trans. London: Vintage Classics, 1994.

Dummett, M. [1978]: *Truth and Other Enigmas*, Cambridge, Mass.: Harvard University Press.

______ [2006]: *Thought and Reality*. Oxford University Press.

Fabry, R.E. [2017]: 'Cognitive innovation, cumulative cultural evolution, and enculturation', *Journal of Cognition and Culture* **17**, 375–395.

______ [2018]: 'Betwixt and between: The enculturated predictive processing approach to cognition', *Synthese* **195**, 2483–2518.

______ [2020]: 'The cerebral, extra-cerebral bodily, and socio-cultural dimensions of enculturated arithmetical cognition', *Synthese* **197**, 3685–3720.

Fabry, R., and M. Pantsar [2019]: 'A fresh look at research strategies in cognitive science: The case of enculturated mathematical problem solving', *Synthese*. doi.org/10.1007/s11229-019-02276-9.

Feyerabend, P. [1975]: *Against Method*. London: Verso.

Field, H. [1980]: *Science Without Numbers*. Princeton University Press.

Frege, G. [1884]: *Foundations of Arithmetic*. J.L. Austin, trans., 1953. Oxford: Blackwell.

——— [1893/1903]: *Grundgesetze der Arithmetik*. Jena: Pohle. Published in English as *Basic Laws of Arithmetic*, Philip A. Ebert and Marcus Rossberg, trans. Oxford University Press, 2013.

Fried, M.N. [2012]: 'Euclid's book on the regular solids: Its place in the *Elements* and its educational value', in B. Sriraman, ed., *Crossroads in the History of Mathematics and Mathematics Education*, pp. 173–196. Charlotte, N.C.: Information Age Publishing.

Gödel, K. [1931]: 'On formally undecidable propositions', in *Collected Works, Volume I*, pp. 145–195. Oxford University Press, 1986.

——— [1983]: 'What is Cantor's continuum problem', in [Benacerraf and Putnam, 1983], pp. 470–485.

Gordon, P. [2004]: 'Numerical cognition without words: Evidence from Amazonia', *Science* **306**, 496–499.

Halberda, J., and L. Feigenson [2008]: 'Set representations required for acquisition of the natural number concept', *Behavioral and Brain Sciences* **31**, 655–656.

Hale, B., and C. Wright [2001]: *The Reason's Proper Study*. Oxford: Clarendon Press.

——— [2009]: 'The metaontology of abstraction', in D. Chalmers, O. Manley, and R. Wasserman, eds, *Metametaphysics: New Essays on the Foundations of Ontology*, pp. 178–212.. Oxford University Press.

Hauser, M. D., S. Carey, and L.B. Hauser [2000]: 'Spontaneous number representation in semi-free-ranging rhesus monkeys', *Proceedings of the Royal Society of London B: Biological Sciences* **267**, 829–833. doi.org/10.1098/rspb.2000.1078

Hellman, G. [1989]: *Mathematics Without Numbers*. Oxford University Press.

——— [2005]: 'Structuralism', in [Shapiro, 2005], pp. 536–562.

Henrich, J. [2015]: *The Secret of Our Success: How Culture Is Driving Human Evolution, Domesticating Our Species, and Making Us Smarter*. Princeton University Press.

Heyes, C. [2018]: *Cognitive Gadgets: The Cultural Evolution of Thinking*. Cambridge, Mass.: Harvard University Press.

Heyting, A. [1931]: 'The intuitionistic foundations of mathematics', in [Benacerraf and Putnam, 1983], pp. 52–61.

Hinrichs, J.V., D.S. Yurko, and J.M. Hu [1981]: 'Two-digit number comparison: Use of place information', *Journal of Experimental Psychology: Human Perception and Performance* **7**, 890–901.

Hunt, S., J. Low, and K.C. Burns [2008]: 'Adaptive numerical competency in a food-hoarding songbird', *Proceedings of the Royal Society of London B: Biological Sciences* **275**, 2373–2379. doi.org/10.1098/rspb.2008.0702

Ifrah, G. [1998]: *The Universal History of Numbers: From Prehistory to their Invention of the Computer*. London: Harville Press.

Izard, V., P. Pica, E.S. Spelke, and S. Dehaene [2008]: 'Exact equality and successor function: Two key concepts on the path towards understanding exact numbers', *Philosophical Psychology* **21**, 491–505.

Jones, M. [2020]: 'Numerals and neural reuse', *Synthese* **197**, 3657–3681.

Kitcher, P. [1983]: *The Nature of Mathematical Knowledge.* New York: Oxford University Press.

Lakoff, G., and R. Núñez [2000]: *Where Mathematics Comes From.* New York: Basic Books.

Laland, K.N. [2017]: *Darwin's Unfinished Symphony: How Culture Made the Human Mind.* Princeton University Press.

Lange, M. [2013]: 'What makes a scientific explanation distinctively mathematical?', *British Journal of Philosophy of Science* **64**, 485–511.

______ [2017]: *Because Without Cause: Non-causal Explanations in Science and Mathematics.* Oxford University Press.

Linnebo, Ø. [2017]: *Philosophy of Mathematics.* Princeton University Press.

______ [2018a]: *Thin Objects.* Oxford University Press.

______ [2018b]: 'Platonism in the philosophy of mathematics', E.N. Zalta, ed., *The Stanford Encyclopedia of Philosophy* (Spring, 2020). https://plato.stanford.edu/archives/spr2018/entries/platonism-mathematics. Accessed March 2021.

Maddy, P. [2014]: 'A second philosophy of arithmetic', *The Review of Symbolic Logic* **7**, 222–249.

Menary, R. [2014]: 'Neuronal recycling, neural plasticity and niche construction', *Mind and Language* **29**, 286–303.

______ [2015]: *Mathematical Cognition: A Case of Enculturation.* Frankfurt am Main: Open MIND, MIND Group.

Mill, J.S. [1843]: 'A system of logic', in J.M. Robson, ed., *Collected Works of John Stuart Mill*, Vols 7 and 8. University of Toronto Press.

Morgan, T.J. [1999]: 'Literate education in classical Athens', *The Classical Quarterly* **49**, 46–61.

Mueller, I. [1991]: 'Mathematics and education: Some notes on the Platonic program', *Apeiron* **24**, 85–104.

Müller-Hill, E. [2009]: 'Formalizability and knowledge ascriptions in mathematical practice', *Philosophia Scientiae* **13**, 21–43.

Nieder, A., and S. Dehaene [2009]: 'Representation of number in the brain', *Annual Review of Neuroscience* **32**, 185–208.

Núñez, R. [2017]: 'Is there really an evolved capacity for number?', *Trends in Cognitive Science* **21**, 409–424.

Overmann, K.A., and F.L. Coolidge [2013]: 'On the nature of numerosity and the role of language in developing number concepts: A reply to Everett', *Current Anthropology* **54**, 83–84.

Pantsar, M. [2009]: *Truth, Proof and Gödelian Arguments: A Defence of Tarskian Truth in Mathematics*. Philosophical Studies from the University of Helsinki, Vol. 23.

______ [2014]: 'An empirically feasible approach to the epistemology of arithmetic', *Synthese* **191**, 4201–4229.

______ [2015a]: 'In search of aleph-null: How infinity can be created', *Synthese* **192**, 2489–2511.

______ [2015b]: 'Assessing the empirical philosophy of mathematics', *Discipline filosofiche* **25**, 111–130. doi.org/10.1400/236780.

______ [2016]: 'The modal status of contextually *a priori* arithmetical truths', in F. Boccuni and A. Sereni, eds., *Philosophy of Mathematics: Objectivity, Cognition, and Proof*, 67–81. Springer. doi.org/10.1007/978-3-319-31644-4_5.

______ [2018a]: 'Early numerical cognition and mathematical processes', *Theoria* **33**, 285–304.

______ [2018b]: 'Mathematical explanations and mathematical applications', in B. Sriraman, ed., *Handbook of Mathematics of the Arts and Sciences*. Springer. doi.org/10.1007/978-3-319-70658-0_30-1

______ [2019a]: 'The enculturated move from proto-arithmetic to arithmetic', *Frontiers in Psychology* **10**, 1454.

______ [2019b]: 'Cognitive and computational complexity: Considerations from mathematical problem solving', *Erkenntnis*. doi.org/10.1007/s10670-019-00140-3.

______ [2020]: 'Mathematical cognition and enculturation: Introduction to the *Synthese* special issue', *Synthese* **197**, 3647–3655. doi.org/10.1007/s11229-019-02478-1.

______ [2021a]: 'Descriptive complexity, computational tractability, and the logical and cognitive foundations of mathematics', *Minds and Machines* **31**, 75–98. doi.org/10.1007/s11023-020-09545-4.

______ [2021b]: 'Bootstrapping of integer concepts: The stronger deviant-interpreation challenge (and how to solve it)', *Synthese* doi.org/10.1007/s11229-021-03046-2.

Pantsar, M., and C. Dutilh Novaes, eds [2020]: 'Mathematical cognition and enculturation'. *Synthese* Special Issue. *Synthese* **197**. doi.org/10.1007/s11229-019-02478-1.

Panza, M., and A. Sereni [2013]: *Plato's Problem: An Introduction to Mathematical Platonism.* Springer.

Peacocke, C. [1993]: 'How are *a piori* truths possible?', *European Journal of Philosophy* **1**, 175–199.

Pepperberg, I.M. [2012]: 'Further evidence for addition and numerical competence by a grey parrot (Psittacus erithacus)', *Animal Cognition* **15**, 711–717.

Pica, P., C. Lemer, V. Izard, and S. Dehaene [2004]: 'Exact and approximate arithmetic in an Amazonian indigene group', *Science* **306**, 499–503.

Pincock, C. [2012]: *Mathematics and Scientific Representation.* Oxford University Press.

Pinel, P., S. Dehaene, D. Riviere, and D. LeBihan [2001]: 'Modulation of parietal activation by semantic distance in a number comparison task', *Neuroimage* **14**, 1013–1026.

Plato [1992]: *The Republic.* 2nd ed. G.M.A. Grube, trans. Indianapolis: Hackett Publishing Company.

Putnam, H. [1967]: 'Mathematics without foundations', *Journal of Philosophy* **64**, 5–22.

Quine, W.V. [1966]: 'The scope of language of science', in W.V. Quine, *The Ways of Paradox and Other Essays*, pp. 215–232. New York: Random House.

______ [1969]: 'Existence and quantification', in W.V. Quine, *Ontological Relativity and Other Essays*, pp. 91–113. New York: Columbia University.

Rayo, A. [2013]: *The Construction of Logical Space.* Oxford University Press.

______ [2015]: 'Nominalism, trivialism, logicism', *Philosophia Mathematica* (3) **23**, 65–86.

Reutlinger, A., and J. Saatsi, eds [2018]: *Explanation Beyond Causation.* Oxford University Press.

Rugani, R., L. Fontanari, E. Simoni, L. Regolin, and G. Vallortigara [2009]: 'Arithmetic in newborn chicks', *Proceedings of the Royal Society B: Biological Sciences* **276**, 2451–2460. doi.org/10.1098/rspb.2009.0044

Saatsi, J. [2011]: 'The enhanced indispensability argument: Representational versus explanatory role of mathematics in science', *British Journal of Philosophy of Science* **62**, 143–154.

______ [2016]: 'On the "indispensable explanatory role of mathematics" ' *Mind* **125**, 1045–1070.

Sampson, G. [1985]: *Writing Systems: A Linguistic Introduction.* Stanford, Calif.: Stanford University Press.
Sarnecka, B.W., and S. Carey [2008]: 'How counting represents number: What children must learn and when they learn it', *Cognition* **108**, 662–674.
Shapiro, S. [1997]: *Philosophy of Mathematics: Structure and Ontology.* Oxford University Press.
______ [2000]: *Thinking About Mathematics.* Oxford University Press.
________, ed. [2005]: *The Oxford Handbook of Philosophy of Mathematics and Logic.* Oxford University Press.
______ [2007]: 'The objectivity of mathematics', *Synthese* **156**, 337–381.
Spelke, E.S. [2000]: 'Core knowledge', *American Psychologist* **55**, 1233.
______ [2011]: 'Quinean bootstrapping or Fodorian combination? Core and constructed knowledge of number', *Behavioral and Brain Sciences* **34**, 149–150.
Starkey, P., and R.G. Cooper [1980]: 'Perception of numbers by human infants', *Science* **210**, 1033–1035.
Sterelny, K. [2014]: *The Evolved Apprentice: How Evolution Made Humans Unique.* Jean Nicod Lectures; 4. Cambridge, Mass.: MIT Press.
Stewart, I. [2006]: *Letters to a Young Mathematician*, New York: Basic Books.
Tait, W.W. [2001]: 'Beyond the axioms: The question of objectivity in mathematics', *Philosophia Mathematica* (3) **9**, 21–36.
Tennant, N. [1997]: *The Taming of the True.* Oxford University Press.
Tomasello, M. [1999]: *The Cultural Origins of Human Cognition.* Cambridge, Mass.: Harvard University Press.
vanMarle, K., F.W. Chu, Y. Mou, J.H. Seok, J. Rouder and D.C. Geary [2018]: 'Attaching meaning to the number words: Contributions of the object tracking and approximate number systems', *Developmental Science* **21**, e12495.
Wagner, R. [2017]: *Making and Breaking Mathematical Sense: Histories and Philosophies of Mathematical Practice.* Princeton University Press.
Wittgenstein, L. [1976]: *Lectures on the Foundations of Mathematics, Cambridge 1939*, C. Diamond, ed. University of Chicago Press.
Wright, C. [1992]: *Truth and Objectivity.* Cambridge, Mass.: Harvard University Press.
Wynn, K. [1992]: 'Addition and subtraction by human infants', *Nature* **358**, 749–751.
Yoshimura, J. [1997]: 'The evolutionary origins of periodical cicadas during Ice Ages', *The American Naturalist* **49**, 112–124.
Zach, R. [2019]: 'Hilbert's program', in E.N. Zalta, ed., *The Stanford Encyclopedia of Philosophy* (Spring, 2019). https://plato.stanford.edu/entries/hilbert-program. Accessed March 2021.