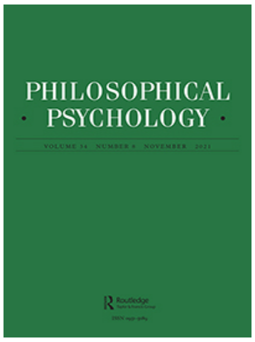

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ORIGINAL PAPER

On the development of geometric cognition: Beyond nature vs. nurture

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ABSTRACT
How is knowledge of geometry developed and acquired? This central question in the philosophy of mathematics has received very different answers. Spelke and colleagues argue for a "core cognitivist", nativist, view according to which geometric cognition is in an important way shaped by genetically determined abilities for shape recognition and orientation. Against the nativist position, Ferreirós and García-Pérez have argued for a "culturalist" account that takes geometric cognition to be fundamentally a culturally developed phenomenon. In this paper, I argue that when understood as moderate versions supported by the state-of-the-art research, the nativist and culturalist views are in fact possible to reconcile. While Ferreirós and García-Pérez present the work of Spelke and colleagues as implying that geometric cognition is genetically determined, I argue that they fail to appreciate the role that Spelke and colleagues see for cultural factors. On this basis, I provide theoretical and terminological clarifications and show that moderate versions of the nativist and culturalist view are in fact consistent with each other. I then propose a unifying theoretical framework for future study that can integrate the two accounts in ontogeny by moving beyond the crude nature (nativism) vs. nurture (culturalism) dichotomy.

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Geometric cognition; proto-geometry; core cognition; nativism; culturalism; philosophy of mathematics

1. Introduction

Geometry is a key area of mathematics in several ways. It forms an important part of mathematical education on almost every level, and geometric theories are fundamental for many scientific fields, such as physics and chemistry. Historically, geometry has served a crucial role in the development of mathematics both in terms of content and methodology, therefore shaping mathematical thinking in important ways (Boyer, 1991). Nevertheless, the development and characteristics of the cognitive processes involved in the learning and practice of geometry are not well known at the

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present time. This is the case already with the very foundation of geometric cognition. Some researchers have argued for a "core cognitivist", nativist, view according to which geometric cognition is in an important way shaped by "evolutionarily ancient" abilities for shape recognition and orientation (e.g., Spelke & Dehaene & Brannon, 2011; Spelke & Lee, 2012; Spelke et al., 2010). This approach has been contested by researchers arguing for a "culturalist" view according to which geometric cognition is fundamentally a culturally developed phenomenon (Ferreirós & García-Pérez, 2020).

In this paper, I aim to show that these two views, understood in a proper way, are not necessarily in conflict with each other. In particular, I will focus on the version of the nativist view proposed by Spelke and colleagues (Spelke et al., 2010) and the recent criticism of it by Ferreirós and García-Pérez (2020). The latter present Spelke et al. as implying that geometric cognition is in a strong way shaped by genetically stored information, and thus fails to include the way geometry is the product of a long line of cultural development. At first glance, this view is understandable, given the emphasis that Spelke and colleagues place on the influence of "evolutionarily ancient", so-called *core cognitive* systems, that we share with many non-human animals, and which are thought to function as a basis for the development of geometry and the acquisition of geometric abilities. However, I will argue that Ferreirós and García-Pérez fail to appreciate the role that Spelke and colleagues see for culture-specific factors in their account. As a consequence of this, I believe that the criticism of Ferreirós and García-Pérez is partly misplaced. Nevertheless, I believe it is fruitful criticism that will enable future research to be more attentive to the balance of genetic and cultural factors in explaining the emergence and acquisition of geometric knowledge.

The purpose of this paper is to show how this can be achieved by establishing a common ground for the nativist and culturalist views. The requirement for that is that the views in question are of moderate types that can include a plurality of influences, moving beyond a crude dichotomy between nature (nativism) vs. nurture (culturalism) in ontogeny, as well as the dichotomy between biological evolution (nativism) and cultural development (culturalism) in the phylogeny and history of geometrical knowledge. If geometric cognition is thought to be determined solely by genetically stored information, the culture-specific aspects of it are merely superficial conventions. Conversely, if geometry is thought to be exclusively a cultural development, the results about core cognition related to geometry may be inconsequential for the development of geometric cognition. I will argue that both of these radical positions are untenable in the face of the state-of-the-art of empirical data and the best philosophical understanding of them. However, I will also argue that the theory of Spelke and colleagues is not nativist in this radical sense, nor is the critical view of Ferreirós and

García-Pérez culturalist in a similarly extreme sense. While some differences remain, a common ground can be established that can work toward a synthesis of factors including products of biological evolution and cultural evolution.

In Section 1 of the paper, I will analyze the core cognitivist view and its place in the study of foundations of mathematics. I will use similar, but at present further developed, work on the foundations of arithmetic cognition as a parallel in order to establish how the core cognitivist view of geometry, what Spelke and colleagues call "natural geometry", should be understood. In Section 2, I analyze the criticism of "natural geometry" by Ferreirós and García-Pérez, focusing on their notion of "proto-geometry". I use my earlier notion of *proto-arithmetic* to illuminate the importance of distinguishing between proto-mathematical and mathematical cognition, a distinction which is important for theoretical coherence. Like the proposed core cognitive proto-geometric abilities in the case of geometry, evolutionarily developed proto-arithmetic abilities (subitizing and estimating) may form the foundation for arithmetic cognition, but should not be confused with arithmetic abilities. Based on these distinctions, I will argue that there are important similarities in the earlier discourse on arithmetic cognition and that of geometric cognition, and similar theoretical considerations should be made in both. This analysis will then be used in Section 3 to show that the account of Spelke and colleagues is in fact consistent with the culturalist view of Ferreirós and García-Pérez. In Section 4, I will propose a way forward by reconciling the two seemingly opposed views into a coherent theoretical framework. Finally, in Section 5 I will propose a way to pursue future research in such a theoretical framework, with focus on the notions of enculturation and cumulative cultural evolution.

2. Core cognition and mathematics

In the past two decades, the study of foundations of mathematics, which in the 20th century was mainly focused on approaches based on logic in the tradition of Frege (1884), Russell (1903), and others (see, e.g., Benacerraf & Putnam, 1984), has taken a significant "cognitive turn". Many researchers have aimed to explain the emergence and acquisition of mathematical knowledge in terms of evolutionarily ancient abilities that shape the development of our cognitive faculties (for overviews, see, e.g., Cohen Kadosh & Dowker, 2015; Dehaene & Brannon, 2011). For the most part, this research direction has focused on the cognitive foundations of natural numbers and their arithmetic (e.g., Butterworth, 1999; Carey, 2009; Dehaene, 1997/2011). There has been a wealth of studies on how humans (including infants and individuals in isolated cultures) and nonhuman animals process quantitative information (e.g., Gordon, 2004; Pica et al., 2004; Piazza et al., 2007; Spelke & Dehaene & Brannon, 2011; Nieder, 2016). Many researchers have argued that these processes form the cognitive basis for arithmetic knowledge (e.g., Carey, 2009; Dehaene, 1997/2011; Feigenson et al., 2004).

According to the best current understanding, there are two evolutionarily developed *core cognitive* proto-arithmetic systems for processing quantitative information (Carey, 2009): the object tracking system (OTS) and the approximate number system (ANS) (see, e.g., Pantsar, 2019, 2021a; Spelke et al., 2010). By allowing the observation of the surrounding environment in terms of discrete objects, the OTS is thought to enable the subitizing ability of determining the amount of objects without counting, up to four items (Carey, 2009; Spelke, 2010). The OTS cannot track more than four objects, but quantitative information is also processed by the ANS, which allows estimating and comparing the sizes of larger collections (Agrillo, 2015; Dehaene, 1997/2011; Feigenson et al., 2004). While the ANS is not limited to small quantities, it becomes increasingly inaccurate as the quantities become larger, thus having a logarithmic character: it is easier to distinguish the difference between, say, six and eight objects than it is between sixteen and eighteen objects. Some authors believe that the ANS is the primary system for the development of arithmetic (e.g., Dehaene, 1997/2011) while others argue that OTS is the key core cognitive system (e.g., Beck, 2017; Carey, 2009). In recent times, also hybrid models in which both the ANS and the OTS are thought to play a crucial role in the emergence and acquisition of arithmetic knowledge have been proposed (e.g., Pantsar, 2014, 2015, 2019,Pantsar, 2021b; vanMarle et al., 2018).[1] This discussion has been highly active in recent years and the analysis of the cognitive foundations of arithmetic has become an important research question both in the cognitive sciences and the philosophy of mathematics.

More recently, important results suggest that also *geometric* knowledge is based on core cognitive abilities (e.g., Izard & Spelke, 2009). Infants, animals and members of isolated cultures have been reported to be sensitive to detecting geometric shapes and orienting based on geometric structures (Dehaene et al., 2006; Izard et al., 2011; Spelke et al., 2010). For example, it has been reported that infants react to changes in angle size rather than in the orientation of the angle (Cohen & Younger, 1984), suggesting that observing shapes in terms of angles has an ontogenetically early basis. In the experiment reported by Cohen and Younger, and later replicated and extended by others (see, e.g., Lindskog et al., 2019), 6-week-old and 14-week -old infants were habituated to simple two-dimensional forms consisting of two lines forming an angle. During the test trials, the angle (either 45 or 135 degrees) remained the same while the orientation changed. The results of tracking their eye movements showed that the 6-week-olds dishabituated to a change in orientation. In other words, they were surprised by the changing orientation and had a long looking time at each new form. The 14-week-olds, on the other hand, dishabituated to the angle size and not the orientation. As long as the angle stayed the same, they were not surprised by the next form. But as soon as the angle changed, their looking times became longer.

What Cohen and Younger (1984)concluded was that there must be a developmental shift between the ages of 6 and 14 weeks, during which the infants became sensitive to recognizing the geometric property of two lines being at a certain angle. In addition to these habituation experiments on infants, similar tests have been run on older children. It has been established that at least from four years of age, children can consistently pick out a deviant geometric form (a different angle, but also others, such as different lengths) from a collection of forms (Izard & Spelke, 2009).

Results like these are often reported in terms of infants observing geometrical shapes. However, as will be seen in the next section, such evolutionarily ancient abilities should be distinguished from actual geometrical cognition. Thus the question is, is the ability to discriminate between different angles, lengths, and other properties treated in geometry indicative of a universal core cognitive ability, similar to those studied in relation with arithmetic cognition, that is the foundation of geometric cognition? Studies of the Munduruku people of Amazon suggest that the discrimination ability is indeed universal to humans. Adults and children from (at least) four years of age can make similar shape discriminations as European and North American children (Izard et al., 2011). Combining the infant data with these results, the evidence converges toward the position that as the product of biological evolution, there is an ability to observe our environments in terms of simple reoccurring shapes like angles, comparable to the abilities due to OTS and/or ANS for observing quantities.

Like in the case of proto-arithmetic cognition, there actually appear to be two distinct core cognitive systems that can feasibly form a (at least partial) basis for geometric knowledge, one for shape recognition and one for orientation. Hupbach and Nadel (2005), for example, showed that when reorienting themselves after a disorientation in a rhombic environment, children up to four years of age were not sensitive to differences in the angles of the rhombus. Instead of looking for an object in a specific corner (that they had observed before the disorientation), they looked in each four corners equally. This was unexpected since children of that age are able to distinguish between angles when they observe visual forms. It has been reported that both children and non-human animals differ also in other ways in completing detection and reorientation tasks, for example, when it comes to using the shape of a surface to find an object (Spelke et al., 2010).

Results like this have made many researchers conclude that the core cognitive system for detecting shapes must be different from the core cognitive system for orientation. Importantly, like the former system, also the latter is often considered to form a cognitive foundation for geometry (e.g., Hohol, 2019; Spelke, 2011). Experiments suggest that the orientation system, which is used by birds and other animals for navigation, works by capturing spatial relationships in an abstract manner. For example, Quirk et al. (1990) report that rats navigate according to the shape of a chamber even in the dark. It is also reported that radical changes in the texture, material and color of a chamber do not influence rats' navigation over the shape (Lever et al., 2002).

In their analysis of the core systems for quantity, shape recognition and orientation, Spelke et al. (2010) have emphasized that there is a parallel between the core cognitive origins of arithmetic and those of geometry. They argue that just like "natural number" is thought by many, most influentially by Carey (2009), to be a concept constructed based on the core cognitive abilities related to quantities, Euclidean geometry is constructed in a similar way based on the core cognitive abilities for shape recognition and orientation (Hohol, 2019; Spelke, 2011).[2] Following this analysis, Spelke et al. (2010) call Euclidean geometry "natural geometry", as a kind of conceptual counterpart of "natural number".

3. "Natural geometry" and "proto-geometry"

The claim by Spelke et al. (2010) that Euclidean geometry is "natural geometry" based on core cognitive systems has been contested recently by Ferreirós and García-Pérez (2020). They argue that what Spelke and colleagues call "natural geometry", namely Euclidean geometry, is not natural in any relevant sense, but rather a product of a long line of culturally determined development in which cognitive artifacts (e.g., ruler, compass) and external representations (e.g., pictures and diagrams) play a crucial role. To establish their view, Ferreirós and García-Pérez (ibid.) introduce a three-part distinction between the abilities due to the core cognitive systems, an intermediate stage that they label "proto-geometry", and finally proper "geometry" (or theoretical geometry). Importantly, unlike the core cognitive abilities, both proto-geometry and geometry are culturally developed. In this manner, they end up classifying the following three-level process for the emergence of geometric knowledge (Ferreirós & García-Pérez, 2020, p. 194). On Level 1 is *visuo-spatial cognition*, which includes the kind of core cognition described by Spelke and colleagues. Level 2 is the level of proto-geometry, on which basic concepts, like circle or square, are formed. On this level, artifacts and external representations play a key role, and with their help approximative results (for example, that the value of pi is approximately 3) on the basic concepts can be acquired. It is only on Level 3, geometry, that Euclidean geometry (and other properly mathematical theories of geometry) appears in the development. Euclidean geometry requires systematic refinement of proto-geometry with further cultural developments. Thus the main argument of Ferreirós and García-Pérez (2020) is that the gap between visuo-spatial cognition (which the core cognitive abilities fall under) and geometry is too wide for Euclidean geometry to be called "natural geometry".

I believe that the kind of distinction that Ferreirós & García-Pérez make between visuo-spatial cognition, proto-geometry, and geometry is important in order to avoid conceptual confusion. It is much too common in the literature to conflate different abilities related to geometric concepts. To give just one example, there is an influential paper titled "Ants Learn Geometry and Features" (Wystrach & Beugnon, 2009). With conceptual distinctions like the ones proposed by Ferreirós and García-Pérez (2020) in place, ascribing geometric abilities to ants becomes radically mistaken, as it should be.[3] I believe that establishing a coherent theoretical framework, including consistent terminology, is an important development if we wish to have fruitful and systematic interdisciplinary collaboration in the study of foundations of geometry. If the word "geometry" is used both for Euclid's system and the ability of ants, there is little hope of finding the required terminological and theoretical coherence.

To see this more clearly, let us consider the corresponding situation in the foundations of arithmetic. I have in my previous work on the epistemology of arithmetic argued for the importance of making a distinction between arithmetic and *proto-arithmetic* in order to distinguish core cognitive and other primitive abilities for treating numerosities from the actual arithmetic treatment of natural numbers (Pantsar, 2014, 2015, 2019, 2020, Pantsar, 2021b). In the case of arithmetic, the problems arising from confusing terminology are obvious. For example, in reporting perhaps the most famous experiment in the field of numerical cognition, Wynn (1992) wrote that infants can carry out simple addition and subtraction operations. Wynn observed infants reacting with surprise (i.e., longer looking time) to the "unnatural arithmetic" of 1 + 1 = 1, instantiated by the infant seeing two dolls being put behind a screen but only one being there after the screen was lifted (the other having been removed clandestinely). This is thought to be due to the OTS-based core cognitive ability of subitizing (Starkey & Cooper, 1980). From different configurations in the subitizing range (from one to four), Wynn concluded that infants are able to carry out rudimentary arithmetic operations. But as I have argued before (Pantsar, 2018), this conclusion is unwarranted. What the infants could be doing is having some kind of cognitive mechanism or procedure in place for keeping track of one quantity. When the expected quantity did not match their expectations, they were surprised. Under this explanation, nothing like an arithmetic operation is presupposed to take place in the cognitive process. What is presupposed is merely an ability to track small quantities. Nevertheless, Wynn's paper was called "Addition and subtraction by human infants", postulating arithmetic abilities to situations in which *proto-arithmetic* abilities were in fact being employed. Similar confusions have been made also in animal studies, where talk of arithmetic abilities in non-human animals is common (e.g., Agrillo, 2015).

The notion of proto-arithmetic I have proposed, when transferred to the domain of the development of geometric cognition, would include both Levels 1 and 2 of Ferreirós and García-Pérez (2020). The details of the applied taxonomy are certainly not without importance, but for the present purposes the crucial point is to distinguish between proper mathematical cognition and the proto-mathematical cognition preceding it. In this way, the approaches of Ferreirós and García-Pérez (2020) and Pantsar (2014, 2015, 2018, 2019, 2020, 2021a, Pantsar, 2021b) are compatible. The similarities go beyond the distinction between proto-mathematics and mathematics, because both approaches emphasize how modern mathematical systems have been made possible by a long line of culturally shaped development, and that we should be very careful to distinguish the cognitive abilities applied in connection with those systems from the proto-mathematical abilities that have preceded them.[4]

Directly related to this, one major problem Ferreirós and García-Pérez have with the approach of Spelke et al. (2010) is the way the latter see Euclidean concepts as being "extremely simple", in the sense that "just five postulates, together with some axioms of logic, suffice to specify all the properties of points, lines, and forms." (Spelke & Lee, 2012, p. 2785). As Ferreirós and García-Pérez (2020, p. 187) point out, this does not correspond to the modern foundational study of mathematics. First of all, basing a system of geometry on logical principles requires an axiomatic system that is much richer than Euclid's five postulates, as shown by Hilbert (1902) and others (see Manders, 2008). But more interestingly in the present context, Ferreirós and García-Pérez (2020, p. 188) continue, Euclidean geometry should not be equated with modern axiomatic Hilbert-style geometry focusing on logical step-by-step proofs. As the likes of Netz (1999) and Manders (2008) have shown, Euclidean geometry is heavily based on a particular method of proof in which diagrams (lettered diagrams, in particular, as Netz argues) play an indispensable role. Indeed, Ferreirós (2015) has argued that Euclid's postulates should not be considered to be on par with Hilbertian axioms. Rather, they should be thought of as rules for constructing diagrams. This distinction is important because, as Ferreirós and García-Pérez (2020, p. 188) argue, the different styles of geometry involve different cognitive abilities. When searching for the cognitive foundations of geometric knowledge, this is undoubtedly an important matter. Euclidean geometry may not be "extremely simple", after all, as it is closely tied to artifacts (ruler, compass) that allow the construction of diagrams, and the embodied practices involved in the successful application of those artifacts. If knowledge and skills for using such artifacts is required for developing and learning Euclidean geometry, the question is just how "natural" can it be?

4. Could Euclidean geometry still be "natural geometry"?

I agree with Ferreirós and García-Pérez (2020) that Euclidean geometry, neither in its original form nor in the modern versions (like that of Hilbert), is not as simple as it may seem. This is the case formally in terms of the logical principles required, but more importantly for the present context, also in terms of the cognitive abilities required. However, I contend that it could still be "natural geometry", in the sense meant by Spelke and colleagues (Spelke & Lee, 2012; Spelke et al., 2010). This is due to a crucial difference in what is understood by "natural geometry" by the participants of this discussion. It is important to note that Spelke et al. (2010) write about "natural geometry" as a simple abstract conceptual system comparable to "natural number". Furthermore, following Carey (2009), they emphasize that as natural as they may seem, natural number concepts do not come particularly naturally to humans (Spelke et al., 2010, pp. 863–864). They explicitly mention that the construction of integers depends on culturally developed cognitive artifacts, namely counting devices (ibid, p. 864). From this, they conclude that:

> Natural number may be therefore partly a product of human culture, built on a foundation of core systems that emerge in infancy, guide the reasoning of adults in all cultures, and are shared with other animals.. (Spelke et al., 2010, p. 864)

Similarly, moving to their treatment of the development of geometric cognition, Spelke and colleagues ask: "What are the sources of Euclidean geometrical intuitions?" (ibid.). Leaving aside potential problems with different understandings of the word "intuition", the underlying message appears to be clear. While they certainly seem to downplay the cognitive complexity (Pantsar, 2021c, 2021d) of Euclidean geometry, Spelke and colleagues are not claiming it is "natural geometry" because we have straight-forward, natural cognitive access to it. What they are claiming, to the best of my understanding, is that Euclidean geometry is in an important way based on cognitive abilities that are easily acquired due to their close relation to the cognitive core systems, which are the product of biological (rather than cultural) evolution.

Here it is again helpful to consider the case of arithmetic in parallel. As seen above, Spelke and colleagues explicitly compare the development of "natural number" to that of "natural geometry". Both, they argue, are based on cognitive core systems but require culturally determined aspects to develop. This, I assume, is the case in ontogeny (in the way individual members of a particular culture acquire arithmetic concepts), but also in phylogeny and history (in the way arithmetic concepts originally emerged within cultures). In modern mathematics, the standard way of presenting the arithmetic of natural numbers is by the *Peano* (or *Dedekind-Peano*) axioms. These are comparable to the five Euclidean postulates in that they are seemingly very simple and easy to grasp (Peano, 1889). The Peano axioms have the following informal content:

(1) Zero is number.
(2) If n is a number, the successor of n is a number.
(3) Zero is not the successor of any number.
(4) If two numbers have equal successors, they are themselves equal.
(5) If a set S of numbers contains zero, and for every number in S its successor is also in S, then every number is in S.

The last axiom is called the "induction principle" and it can be presented as a second-order axiom (as in here) or a first-order axiom schema. At least for a person with some mathematical education, the Peano axioms are likely to appear obvious. Indeed, when explained in familiar terminology, with the possible exception of the induction principle, the amount of mathematical education required to understand the axioms seems to be rather minimal.

Nevertheless, to say that the Peano axioms are *simple* would be careless in a similar way to calling the Euclidean postulates "extremely simple". They are the product of a millennia-long development in arithmetic. While their informal content is perhaps unproblematic to understand for a person educated in basic mathematics, a proper mathematical grasp of the Peano axioms may require more complex cognitive abilities than is initially apparent. Thus "natural number" as a mathematical concept may not be as "natural" as it first seems. This can be seen in the divergent paths that the development of arithmetic has taken in different cultures. The Mayans, for example, had very sophisticated systems of arithmetic that allowed calculations for extremely large numbers, yet it did not seem to include the concept of infinity, at least in the sense it exists in our arithmetic (Ifrah, 1998).[5]

This is not to suggest that the concept of natural number could not be natural in that it is in a strong way influenced by the core cognitive abilities for determining quantities. The point I want to make here is that since Spelke et al. (2010) draw a parallel between "natural number" and "natural geometry", their account of the latter already includes a strong influence by culturally developed factors. Indeed, in Spelke et al. (2010), "culturally variable counting devices" are explicitly mentioned as something that the construction of natural numbers depends upon (p. 864). Following a similar line of argumentation, it is clearly possible that diagrams and other cognitive tools (Fabry & Pantsar, 2021) play an integral role in the construction of the concepts of "natural geometry", understood as Euclidean geometry. But this matter does not concern only the difference between the Levels 1 and 3 of the three-part distinction of Ferreirós and García-Pérez (2020), i.e., visuo-spatial cognition and theoretical geometry. When analyzed in terms of the three-part distinction of the development of Euclidean geometric knowledge (and other theoretical geometry), also *proto-geometry* is the result of culturally determined developments. Following my analysis above, the account of Spelke et al. (2010) is consistent with this. In the case of arithmetic, they argue, the culturally variable counting devices are required to acquire and grasp natural number concepts. Similarly, the way I understand the proposal of Spelke and colleagues, acquiring basic concepts like circle and square (which fall under proto-geometry) is tightly connected to culturally variable devices, including artifacts like compass or ruler. Therefore, for Spelke et al. (2010), only visuo-spatial cognition appears to be free of cultural influences. This is perfectly consistent with the view of Ferreirós and García-Pérez (2020).

5. Reconciling the two approaches: Culturally determined "natural" geometry

The analysis in the previous section should make us reconsider whether the approaches of Ferreirós and García-Pérez (2020) and Spelke et al. (2010) are necessarily in conflict. If not, could the problem be more in the usage of different theoretical frameworks and the consequent clash in terminology? To start analyzing this question, it should first be noted that Ferreirós and García-Pérez (2020) do not want to deny the possibility that Euclidean geometry is natural geometry. They write:

> Perhaps it is impossible to establish conclusively that the CKS [core cognitive system] and Level 1 do not contain structures corresponding to Euclidean geometry, but here the burden of proof should be on those who make such a claim. (Ferreirós & García-Pérez, 2020, p. 200)

But why, one must wonder, should the burden of proof be on those who aim to explain geometric cognition as emerging from the cognitive core systems? Ferreirós and García-Pérez answer:

> Merely for reasons of information-theoretic plausibility, an explanation of our protogeometric knowledge that does not rely exclusively on genetically stored information seems more plausible than a nativist viewpoint. (ibid.)

Concerning this, two points should be made. First, it is important to note that neither Spelke and her collaborators nor, to the best of my knowledge, anyone else in the modern literature suggest that proto-geometric knowledge in the sense of Level 2 (let alone geometric knowledge in the sense of Level 3) of Ferreirós and García-Pérez relies *exclusively* on genetically stored information.[6] While there are different views concerning to what extent the cognitive core systems shape geometric cognition, Spelke and colleagues always include cultural influences in developing what is, in the terminology here, proto-geometric cognition. The following passage, for example, leaves little doubt about it:

> Like natural number, natural geometry is founded on at least two evolutionarily ancient, early developing, and cross-culturally universal cognitive systems that capture abstract information about the shape of the surrounding world: two core systems of geometry. Nevertheless, each system is limited: It captures only a subset of the properties encompassed by Euclidean geometry, and it applies only to a subset of the perceptible entities to which human adults give shape descriptions. Children go beyond these limits and construct a new system of geometric representation that is more complete and general, by combining productively the representations delivered by these two systems. This productive, combinatorial process, we suggest, depends in part on uniquely human, culturally variable artifacts: pictures, models, and maps. Thus, like the system of number, *the system of geometry that feels most natural to educated adults is a hard-won cognitive achievement, constructed by children as they engage with the symbol systems of their culture.* (Spelke et al., 2010, p. 865, emphasis added).

A second point should also be made about the above quotation of Ferreirós and García-Pérez. They claim that due to "reasons of information-theoretic plausibility", the nativist (i.e., core cognitivist) explanations carry the burden of proof. Perhaps this would be seen in different light once it is made clear that the core cognitivist view of Spelke and others does not require exclusive reliance on genetically stored information. But assuming that the argument of Ferreirós and García-Pérez would remain similar, it is not at all clear why the burden of proof should lie on the nativist side. As I understand information-theoretic plausibility in this context, the least plausible view is that the evolutionarily developed core-cognitive abilities do not play *any role at all* in the development of proto-geometric cognition. It seems highly unlikely that proto-geometric cognition would develop completely independently of the core cognitive abilities for orientation and shape recognition.

While I see that as unlikely, it is of course a possibility that should be analyzed. If proto-geometric cognition is indeed independent from the core cognitive abilities for orientation and shape recognition, there are two remaining options. First, proto-geometric cognition develops based on some other evolutionarily ancient abilities, but not the two that the nativists have proposed. It is certainly a conceivable scenario that other core cognitive abilities are in play. Indeed, even if we accept the core cognitivist view of Spelke and others, this possibility should be considered. But if core cognitive abilities are evoked in the explanation anyway, on what grounds do we exclude the two abilities that seem to fit proto-geometric cognition the best? I am not claiming that this kind of argument necessarily makes the nativist position stronger. But it should make us reconsider where the burden of proof lies.

The second possibility is that proto-geometric cognition develops in an important way independently of evolutionarily ancient abilities.[7] In this case, proto-geometric cognition would be essentially a matter of convention. This possibility cannot be completely ruled out, either. But again, the burden of proof surely lies now in showing that all the resemblances between proto-geometric cognition and the abilities based on core cognitive abilities are merely coincidental.

Of course, Ferreirós and García-Pérez do not claim that proto-geometric cognition is completely independent of the core cognitive abilities. But based on my analysis above, neither would Spelke and her colleagues claim that what Ferreirós and García-Pérez call proto-geometric cognition is entirely determined by the core cognitive abilities. Therefore, both views can include a role for both core cognitive abilities and cultural influences in the development of geometric cognition. Certainly there are important differences in the accounts, some of which may be irredeemable. Spelke and colleagues clearly emphasize the importance of the core cognitive abilities. For example, in Dehaene et al. (2006), – in which Spelke is one of the authors – it is explicitly argued that the data on the Munduruku is evidence of universal intuitions of basic geometric concepts (e.g., points, lines, parallelism). Even adjusting for terminology (i.e., calling the intuitions *proto-geometric*), this is very different from the kind of basic object perception that Ferreirós and García-Pérez are ready to accept.

Nevertheless, even accounting for such differences, I believe that the approaches of Ferreirós and García-Pérez and Spelke and colleagues are for the most significant part compatible. To see why, let us analyze the concluding words of Ferreirós and García-Pérez in their paper:

> [W]e aim to promote interdisciplinary work in this field by bringing in logical results and historical evidence, in a spirit of constructive cooperation with the empirical cognitive sciences. The problem of analyzing visuo-spatial cognition and proto-geometry is quite complex, and perhaps none of the disciplines mentioned can hope to succeed working in isolation. We believe that the strategy should be one of interdisciplinary cooperation, working toward *convergence* between the various fields, as well as trying to clarify and explain those issues further by "triangulating them," so to say, with information obtained from different disciplines. (Ferreirós & García-Pérez, 2020, p. 200, italics in the original).

In this spirit, I suggest that the present paper should be considered an effort to "triangulate" the issue of natural geometry in the name of exactly such interdisciplinary cooperation by applying systematic conceptual analysis on the key terms "natural geometry" and "proto-geometry". I hope to have shown that in the approaches of Spelke and colleagues on the one hand, and Ferreirós and García-Pérez on the other hand, such convergence can happen to a large degree, and the apparent conflict is mainly due to a lack of terminological and conceptual coherence between the two accounts.

6. A way forward

The reconciliation of the "nativist" and "culturalist" views I have argued for in this paper may sound like an inherently flawed pursuit. Indeed, it would be one, were we to understand nativism and culturalism as radical versions that do not allow for other types of influences in the development of geometric cognition. However, neither the view of Spelke and colleagues nor that of Ferreirós and García-Pérez, as I have argued, represents such a radical position. Instead, I interpret both ultimately as the kind of moderate accounts that leave space for a plurality of influences. Therefore, the purpose of the present account is not to argue for one of the accounts over another. Instead, I want to propose an approach in which we can move past the limiting, crude dichotomy of nature (nativism) vs. nurture (culturalism) in ontogeny. Similarly, in the proposed approach we move beyond the dichotomy between biological evolution (nativism) and cultural development (culturalism) in the phylogeny and history of geometrical knowledge. The first stage of this approach is to construct a conceptually coherent theoretical framework, including consistent inter-disciplinary terminology. As we have seen, this issue of terminology is not merely a question of establishing acceptable jargon. Inconsistent terminology can lead to erroneous positing of cognitive abilities for groups of subjects, like when ants are thought to learn geometry and infants to have arithmetic abilities.

Establishing a coherent theoretical framework is of course only a first step in explaining the emergence of geometric cognition and the acquisition of geometric knowledge and abilities. Indeed, as the work progresses, it is possible that the framework itself needs updating. Yet we must start from somewhere in trying to find basis for inter-disciplinary research on geometric cognition that can include both genetically and culturally shaped influences. Due to considerations of space, it is not possible here to give a detailed presentation of what such a theoretical framework should comprise. Nevertheless, some central characteristics of a feasible framework can be established. I submit that an acceptable framework should contain at least three key elements. First, it should be sensitive to the contribution of the core cognitive systems to the early (and possibly also later) development of geometric cognition. Second, it should be sensitive to the way cultural developments take place and how they are transmitted across generations and social groups. Finally, third, it should be able to combine the two types of contribution – and their interrelation – for a plausible explanation on a neuronal level. That is, the framework should provide theoretical grounds for an empirically informed explanation of the way human subjects develop geometric cognition in ontogeny by adopting culturally shaped information and practices.

For the first of the above key elements, the work of Spelke and colleagues, as well as others (see, e.g., Hohol, 2019), has paved the way. The empirical data on core cognitive abilities in shape recognition and orientation are increasing both in quantity and quality, and I believe that the core cognitivist view proposed by Spelke and others is fundamentally on the right track about how this data should be included in explanations of the development of geometric cognition. This work should also be combined with an account of how perception is possible in the first place, as well as an account of how the core cognitive abilities are employed in further steps of cognitive processing. Hatfield has emphasized the importance of distinguishing between "sense perception" and "cognitive perception" (Hatfield, 2009, p. 5). Sense perception refers to the mere color and spatial properties of the visual scene, while cognitive perception assigns functional significance and identities to objects. Placed within this account, the core cognitive abilities related to geometric cognition would seem to play a crucial middle part. Shape recognition is based on sense perception by applying the core cognitive systems, while cognitive perception of objects is possible by the shape recognition (and orientation) enabled by the core cognitive systems.

For the second key element, the development of culturally specific knowledge, practices and artifacts related to geometry, the kind of work presented in Ferreirós (2015) and Ferreirós and García-Pérez (2020) is of great importance (for literature on artifacts, see also Hohol & Miłkowski, 2019; Magnani, 2013, 2021; Netz, 1999). Generally, this kind of historical work fits well with work on *cumulative cultural evolution*, which refers to the way human cultures gradually develop their knowledge and skill sets (see, e.g., Boyd & Richerson, 1985, 2005; Henrich, 2015; Heyes, 2018; Tomasello, 1999). Technologies and practices are improved upon in small generational increments, and in large enough societies – or ones with extensive interactions with other societies – this process can establish a status of knowledge and skills where they are no longer tied to a small group of individuals. In addition to language and other culturally specific practices, just like Ferreirós and García-Pérez (2020) argue, a crucial aspect of mathematical practices is formed by the tools that are used (for an overview, see Robson & Stedall, 2009).

The importance of cultural evolution can also explain why the core cognitive systems have not had a universally transforming effect on human cultures, even though the core-cognitive capacities are shared universally. Just like the development of arithmetic, the emergence of geometry has required a suitable cultural setting in which language, artifacts and practices could develop. This development is connected to practical applications, educational practices, and many other cultural factors that are not universally present in human cultures (Pantsar, 2019). In this way, the evolutionarily ancient core cognitive capacities may have been a necessary condition for the development of geometry, but they are far from a sufficient condition. Geometry requires processes of cultural evolution, which take place on a much smaller time scale than biological evolution.

Finally, the theoretical framework I have proposed requires the third key element, how to combine genetic and cultural factors on the neuronal level. For this, I see great potential in the notion of *enculturation* as proposed by Menary (2015) for explaining mathematical cognition. Processes of enculturation transform our basic biological faculties through the cultural transmission of cognitive practices. In particular, enculturation refers to the transformative process in which interactions with the surrounding culture determine how cognitive practices are acquired and developed (Fabry, 2020; Menary, 2015; Pantsar, 2019, 2020). Enculturation is made possible by the mechanism Menary (2014), (2015) calls "learning driven plasticity", which enables the acquisition of new cognitive capacities due to the neural plasticity of the brain that makes both structural and functional variations possible (Anderson, 2015; Ansari, 2008; Dehaene, 2009; Fabry, 2020; Jones, 2020). The human brain is now understood to be highly plastic, developing in different ways depending on the specific experiences of different individuals, while also being constrained by genetically determined factors. Culturally developed cognitive abilities like reading and writing are thus made possible by redeploying older, evolutionarily developed neural circuits for new culturally specific functions (Dehaene, 2009; Menary, 2014). The same has been argued to be the case for the cognitive ability of arithmetic (Fabry, 2020; Jones, 2020; Menary, 2015; Pantsar, 2019). Here I want to submit that there is great potential in giving geometric cognition a similar treatment. To be more precise, I believe that close attention should be given to the hypothesis that in the development of geometric cognition, neural circuits associated with the core cognitive abilities are partly redeployed for proto-geometric and geometric functions.

However, this kind of reuse of evolutionary ancient cognitive capacities does not need to be limited to the redeployment of particular neural circuits (what is called *neuronal recycling* in the literature (Dehaene, 2009; Menary, 2014)). Anderson (2015) has argued for a more general principle of *neural reuse*, which has been proposed by Fabry (2020) and Jones (2020)as a better fit with the development of arithmetic cognition. This could be also the case for the development of geometric cognition. My proposed framework in this paper is not restricted to a particular theory concerning the mechanism of learning on the neuronal level. The important matter is that the mechanism allows for culturally-specific factors to shape the development of mathematical cognition based on proto-mathematical capacities. This can include both the redeployment of specific neuronal circuits or a more general reuse of neural resources.

There is evidence of mathematical cognition including at least some domain-general properties. Cognitive processes involving analogical reasoning, which is thought to be crucial for the kind of abstract reasoning involved in mathematics, involve a similar neural circuit as cognition of geometric relations, comprising the bilateral dorsal prefrontal and intraparietal cortices (Amalric & Dehaene, 2016; Krawczyk et al., 201c1; Watson & Chatterjee, 2012). As reported by Amalric and Dehaene (2016), fMRI studies of high-level mathematical cognition show remarkably little domain-specificity: solving problems in algebra, analysis, topology, and geometry induce activity in largely the same areas of the brain, namely the bilateral intraparietal sulci, the bilateral interior temporal regions, and the mesial prefrontal cortex. This suggests that high-level mathematical reasoning deals with similar cognitive processes regardless of the particular area of mathematics. Interestingly for the present purposes, however, geometric reasoning showed additional activation in the posterior inferior temporal and the posterior parietal cortices (Amalric & Dehaene, 2016). The inferior temporal cortex is associated with shape and object recognition (DiCarlo et al., 2012) while the posterior parietal cortex is typically activated in spatial reasoning, including movement planning (Sack, 2009). Evidence thus suggests that brain activity connected with the core cognitive processes of shape recognition and orientation occurs partly in the same regions as geometric reasoning. This supports the hypothesis that there is a connection between the core cognitive systems and the development of geometric cognition, plausibly due to a partial redeployment of the neural circuits associated with the core cognitive processes for new, culturally specific purposes. However, at present the empirical data is far from conclusive and further study is needed especially on how activity in different areas of the brain is associated with cognitive tasks in different developmental stages and levels of expertise.

Although we still need to learn much more about the neuronal-level mechanism of culturally shaped learning, I believe that the framework of enculturation provides a highly promising platform on which to proceed. In this manner, by explaining how cultural practices, which are the product of cumulative cultural evolution, transform our basic biological core cognitive faculties, introducing the notion of enculturation can complete the three-part theoretical framework suggested above. How this process exactly happens is something that requires the kind of inter-disciplinary triangulation suggested by Ferreirós and García-Pérez. Granted, the rough outline of a framework I have suggested here is only the very beginning. But I believe it is the beginning of a highly important process: finding a coherent way to get the best out of the results offered by different disciplines involved in the study of the foundations of geometric cognition.

Notes

1. In addition, it has been recently proposed that the distinction between the OTS and the ANS is not required to explain many empirical data (Cheyette & Piantadosi, 2020).
2. Euclidean geometry refers to the system of geometry presented by Euclid in his textbook *The Elements* around the year 300 BC. This system is the basis for the geometry taught in schools. It should be noted that there are also many other types of geometric systems studied in mathematics, including two so-called non-Euclidean geometries, namely hyperbolic and elliptic geometries. These geometries reject the Euclidean postulate that given a line on a plane and a point not on the line, at most one line that is parallel to the given line can go through the point. See (Coxeter, 1998) for more.
3. To be clear, I do not want to suggest that Wystrach and Beugnon (2009) believe that ants learn geometry in the sense human beings do. The point I want to make is that the two abilities are so distant from each other that a single term should not cover both.
4. In Section 5 I will discuss the importance and the mechanisms of culturally shaped factors on the development of mathematical cognition.
5. There could be some debate whether such systems should be called "arithmetic". However, as I have argued before (Pantsar, 2014, 2015, 2018), limiting the word "arithmetic" to Western modern axiomatic system is problematic. While, to the best of our knowledge, the Mayan arithmetic did not include the kind of proof procedures familiar to us, the calculating ability and practical applications were highly sophisticated and general, comparable to Western pre-modern arithmetic (Ifrah, 1998; Pantsar, 2019). I have proposed that both systems should be called arithmetic, while modern axiomatic arithmetic should be characterized as "formal arithmetic" (see Pantsar, 2018 for more).
6. It should be noted that "genetically stored information" is a potentially problematic notion here, as the same genes can be expressed in different phenotypes. Instead of the genetic information itself, it would seem that it is the phenotype that is – under the nativist account – important for the development of geometric cognition. I thank Regina Fabry for pointing this out.
7. I say "in an important way" here because certainly *some* genetically determined abilities, like vision, always play a role.

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