Fuzzy mereology

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Abstract

Some philosophers have attempted to solve metaphysical problems about vagueness by understanding objects with vague boundaries as analogous to fuzzy sets. I formulate such a view and argue that it suffers from a serious lacuna, which I attempt to fill.

1 Fuzzy individuals and the problem of the many

This paper began life as a short section of a more general paper about non-classical mereologies. In that paper I had a mereological theory that I wanted to show could be applied to all sorts of different metaphysical positions — notably, to those positions that believe in mereological vagueness *in re* — in "vague individuals". To do that I felt I first had to dispatch the leading rival theory of vague individuals, which is due to Peter van Inwagen, and holds that the part-whole relation admits of degrees. It seemed to me that this theory had a serious technical problem, or at best a serious gap. I sat down to write a paragraph or two highlighting the gap, preferably showing that it couldn't be filled. This paper is the result.

So I will be examining, and trying to defend as best it can be defended, a view that my metaphysical prejudices are opposed to. I think that in general, in philosophy, this is a good thing to do.

1.1 What we are trying to make sense of

The theory I am trying to develop is a theory of mereological vagueness *in re*, or of vague individuals. A vague individual is a thing of which it is a vague matter which other things are part of it. Prima facie, there seem to be many such objects. Mountains are a favourite. Mt Taranaki is made out of rocks; near to the edge of the mountain there are rocks that not clearly part of the mountain, and not clearly not part, either. Call these rocks *penumbral* rocks.

If we suppose that every rock is either part of Taranaki or not part of Taranaki (i.e. that Taranaki is not a vague individual), then we get a version of the *problem of the many*. There many piles of rocks in the vicinity of Taranaki differing only in which of the penumbral rocks have been left in and which left out. Each of these seems to be an equally good candidate to be the Taranaki; each is a mountain, since only mountains can be candidates to be Taranaki. So there are as many mountains in the immediate vicinity of Taranaki as there are penumbral rocks.

This problem has many solutions. I want to focus on one, very simple minded solution that simply denies the premise that every rock is either part of Taranaki or not part of Taranaki. On this view, there are vague individuals; Taranaki is one; the penumbral rocks are not just "unclear cases", but neither part nor not part of Taranaki.

This is not a popular way to go; the main reason being, I think, that most philosophers think that there are likely to serious theoretical problems with the very concept of a vague individual; at the least, questions that need answers. A sampler: what exactly is going on when a rock fails to be either definitely part of or definitely not part of a mountain? What is the metaphysics of vague parthood? What becomes of mereological concepts such as fusion in such a setting? What criteria of identity do vague individuals have — what makes one vague individual one and two vague individuals two? A theory of vague individuals consists of a systematic answer to these questions.

There are many such theories (one of which I was attempting to put forward when I began writing this paper). For present purposes, I am going to focus on one that is suggested by Peter van Inwagen in section 17 of *Material Beings*. (van Inwagen 1990, pp. 221-223)

1.2 van Inwagen's suggestion

Van Inwagen's idea is that we should borrow some resources from fuzzy set theory. Fuzzy set theory is a generalisation of ordinary, "crisp" set theory, in which the set-membership relation admits of degrees. A degree is conventionally identified with a number in the closed interval $[0, 1]$ (but we could instead regard them as sui generis abstract objects, or equivalence classes of relations of the same degree). To be *definitely* a member of a set is to be member of that set to degree 1;

to be *definitely not* a member of a member is to be part of it to degree 0.

Van Inwagen borrows from fuzzy set theory in two respects. First, we are to imagine that the part-whole relation, like the membership relation of fuzzy set theory, admits of degrees. To be *definitely a part* of something is to be part of it to degree 1; to be *definitely not a part* of something is to be part of it to degree 0. I will say that if something is not definitely a part of something and not definitely not a part of that thing, that it is *marginally a part* of a part of that thing. Second, we are to think of vague individuals as being relevantly like fuzzy sets of mereological atoms. A mereological *atom* is a thing with no parts other than itself — perhaps the fundamental particles of microphysics are atoms. Think of Taranaki as *corresponding to* a fuzzy set of fundamental particles in the vicinity of Taranaki, with those particles being members of the set to the degree that they are part of Taranaki.

"Correspondence to" is not identity — Taranaki is not a fuzzy set, but a vague individual. The purpose of fuzzy sets here is to give us a handle for reasoning about vague individuals, not to be identified with them. This is about as far as van Inwagen goes with this — his interest is in showing that vague individuals are coherent within the context of his larger metaphysical theory, which need not detain us here. In the remainder of this section I try to make the ideas above a bit more exact and explicit (as well as reducing the dependence on fuzzy set theory). Then I will be able to state the problem (or gap) that I found in this whole approach.

1.3 A theory of fuzzy individuals

An *individual* is a thing capable of standing in the part-whole relation. Let U be the set of all individuals. An *atom* is an individual that has no parts other than itself. We are going to assume that *atomism* is true: that every individual is, in a sense to be explained, entirely made up of atoms. Atomism is not supposed to be a self-evident truth, but it is needed to get the resemblance between vague individuals and fuzzy sets to work, so assume it for the moment. Let A be the set of all atoms; atoms are individuals, so A is a subset of U.

Each individual x corresponds to a fuzzy set f of atoms, such that each is a member of f to the same degree that it is part of x. I will call that fuzzy set f the *content* of x. Let c be the *content function* — the function that maps each individual to its content, so that $c(y)$ means "the content of y". We will be able to say a lot about vague individuals by saying things about their contents.

Here is another way of thinking about the content function that gets rid of the fuzzy sets. Think of the expression c as a restricted part-whole-degree relation: $c(y, x)$ means "the degree to which the atom x is part of the individual y". From a formal point of view, these ways of thinking are interchangeable. It will help if you have them both in mind.

Almost every triple $\langle U, A, c\rangle$ is a coherent description of a world of vague individuals. In order to be so, it must meet three conditions.

First, by definition, an atom has no parts other than itself — so the content of an atom must assign that atom the degree 1 and all other atoms the degree 0. Call this the *Atomicity* principle:

$$\text{For all } x, y \in A,\ c(y, x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \qquad \text{(Atomicity)}$$

Second, since atomism is true — since every individual is made up of atoms — there can be no *null individual* — no individual whose content assigns every atom the degree 0; such an individual would not be made up of atoms. Call this the *No-null* principle:

$$\text{For all } x \in U, \text{ there is some } y \in A \text{ such that } c(x, y) > 0 \qquad \text{(No-null)}$$

Actually, (No-null) doesn't quite guarantee enough to ensure that there are no individuals that are bad in the way that null individuals are. What we need to do is ban what I will call non-normal individuals. An individual x is *normal* iff every degree less than one is less than the degree to which some atom is part of x. Normality is not easy to get your head around. For all interesting philosophical purposes in this paper, you may suppose that a normal individual is one that has some atom as a part to degree 1. The only way that a normal individual can fail to do that is if there are an infinite series of atoms that get ever closer, but never reach, being part of that individual to degree 1, and we will not be making any important philosophical use of such individuals.

Another of saying what normality is to say that an individual is normal if the supremum — the least upper bound — of its content is 1. The condition I am working towards here says that every individual is normal. So we may formulate this principle of *Normality* as follows:

$$\text{For all } x \in U,\ \sup_{z \in A} c(x, z) = 1 \qquad \text{(Normality)}$$

I have not yet said what is so bad about non-normal individuals. Nor can I — yet. To briefly anticipate the reasons that will be given later, a non-normal individual would be, to some non-zero degree (or even definitely) a part of some atom.

Atoms, by definition, however, definitely have no parts other than themselves; so Normality, like Atomicity, follows from the definition of "atom".

Third, again, since atomism is true, and every individual is made up of atoms, it should be that putting together the same atoms in the same way produces the same individual. No two individuals have exactly the same atoms as parts to the same degree; or to put it even more simply, no two individuals have the same content. This is the principle of *Extensionality*:

$$\text{If, for all } z \in A,\ c(x, z) = c(y, z), \text{ then } x = y \qquad \text{(Extl)}$$

If a triple $\langle U, A, c\rangle$ satsifies the principles of Atomicity, Normality, and Extensionality, then it is a coherent specification of a world of vague individuals. Since the three principles are obviously consistent, there need be no fear that the concept of a vague individual is covertly inconsistent.

We can begin to answer many of the questions that make philosophers suspicious about the concept of a vague individual. What makes an individual vague? Answer: a part-whole admits of degrees; a vague individual is any individual that has an atom as a part to an intermediate degree. (Note that it follows from Atomicity that no atom is vague). What becomes of merelogical fusion? Answer: the fusion of the Xes is that individual whose content is the fuzzy set-theoretic union of the contents of the Xes.[1] What are the identity criteria of vague individuals? Answer: if such criteria are needed, then they are given by the Extensionality principle.

The theory whose axioms are Atomicity, Normality and Extensionality may be called the *theory of fuzzy individuals* appropriately because (a) it is a fuzzy theory — a theory in which key concepts are regarded as admitting of degrees, and degrees have the structure of the real interval [0,1]; and (b) by analogy with the theory of fuzzy sets, except that in this case it is individuals, rather than sets, which are being said to have fuzzy boundaries.

An individual is *composite* iff it is not an atom. Nothing I have said so far guarantees that there are any composites whatever (the three theses may be satisfied if $U = A$). (Normality) and (Extl) each guarantee that there are *not* certain kinds of composites. We might consider adding some theses that do guarantee that there are certain kinds of composites — we might, for example, add a thesis of *Universalism* — the view that there are as many composites as there can possibly

[1] Actually this is more problematic than I make it sound, for reasons similar to those discussed in section 2.3. Just as there are multiple definitions of part-whole admissible in the theory of fuzzy individuals, there are multiple union-like operations in fuzzy set theory, and thus multiple admissible definitions of mereological fusion.

be consistent with Normality. This requires quantifying over fuzzy sets:

$$\text{For all } f \in \mathbb{F}(A), \text{ if } sup_{x \in A} f(x) = 1, \text{ then } (\exists x \in U)(c(x) = f) \qquad \text{(Univ)}$$

Universalism is analogous to a "general sum" or "unrestricted composition" principle; it will be attractive to those philosophers who find themselves attracted to such principles in the context of non-fuzzy mereology. There are also various restricted forms of Universalism — one could easily formulate one that guarantees only that there are as many precise composites as there could possibly be, for example. For the purposes of this paper however, I would like to remain as metaphysically neutral as possible — it seems to me that though Atomicity, Normality, and Extensionality have a claim to be conceptual truths, Universalism — even in a restricted form — is at best a metaphysical one.

2 Defining general parthood

Now for the problem. Think back to our initial examples. The aim of all this is to cast light on the relationship between (e.g.) mountains and rocks which gave rise to the problem of the many. But the theory of fuzzy individuals does not have anything to say about this. Both mountains and rocks are vague individuals; so both may have atoms as parts to some intermediate degree; but there's no way of posing the question "to what degree is this rock part of this mountain?" within the theory, for the only things that it explicitly represents as parts of individuals are atoms.

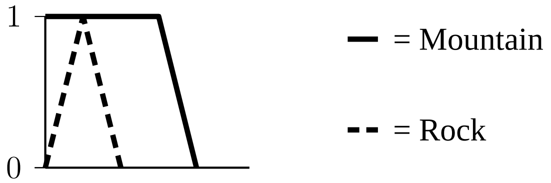
What we need to solve this problem is a definition of a *general parthood relation* — a concept of part-whole that is applicable to any pair of individuals (not just to an atom and individual, like c) — a definition in terms of U, A, and c. We can expect no help from the analogy between fuzzy individuals and fuzzy sets. Following that analogy, the part-whole relation between individuals should be relevantly like the subset relation between their contents. But in fuzzy set theory the subset relation does not admit of degrees — here the analogy between fuzzy sets and fuzzy individuals breaks down.

It is in fact possible to define a general parthood relation, and that is what I will be doing in the remainder of this section. Formally, I write $x \prec y$ for "the degree to which x is part of y". Just writing down a formal definition isn't sufficient to solve the problem — I need to show that the definition is motivated by our informal concept of parthood-to-a-degree, on the assumption that we have one.

I will proceed as follows: I'll first informally discuss some examples of rocks and mountains on which I hope you have some clear intuitions about degrees of parthood, and then present my definition of general parthood as a generalisation from these.

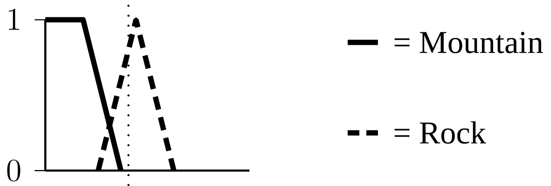
2.1 Some motivating examples

Case 1: suppose that Rock is buried a few meters beneath the summit of Mountain. Rock and Mountain are both vague individuals, but even atoms that are extremely marginal parts of Rock (we're talking about, for example, an electron located right at the edge of Rock, about to fly off into its surroundings) are definitely parts of Mountain. This scenario is shown in the diagram below:



The diagram is a way of visualising the contents of Mountain and Rock — it is a graph of degrees of parthood (on the vertical axis) against atoms (on the horizontal; in some arbitrary order). The solid line shows the content of Mountain — the degree to which each atom is part of Mountain; the dashed line the content of Rock. So in this diagram all the atoms which are part of Rock to any non-zero degree are definitely part of Mountain. This is surely a case in which *Rock is definitely part of Mountain*.

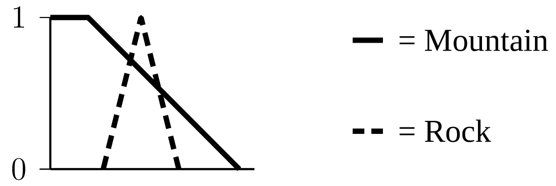
Case 2: now suppose that Rock is at the foot of Mountain, but at sufficient distance that there is some atom that is definitely part of Rock and definitely not part of Mountain, as shown below (the vertical dotted line is just a visual guide to make the relative positions of the solid and dashed lines more obvious).



In this case, I think we should say that *Rock is definitely not part of Mountain*. If this is not obvious, then notice that if Rock is very big, it could be another mountain adjacent to Mountain, in which case it seems obvious that Rock is not part of Mountain (though perhaps they overlap to some degree.

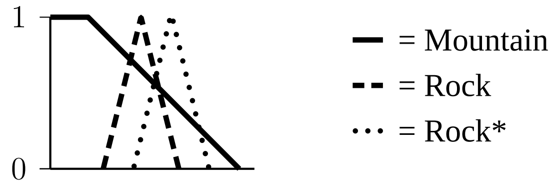
Case 3: now suppose that Rock is located on the lower slopes of Mountain, entirely within the area where all the rocks seem to be marginal parts of Mountain.

Rock is so centrally located within Mountain's prenumbra that every atom that is part of Rock to any degree is part of Mountain to an intermediate degree:



This is the kind of case we were imagining in the introduction: a paradigm case of *Rock's being a part of Mountain to some intermediate degree*.

Now suppose that there is a second rock, Rock*, of exactly the same size and shape as Rock, but which is further away from the summit than Rock. Every atom that is part of Rock* to any non-zero degree is part of Mountain to an intermediate degree, as shown below:



Rock* is also part of Mountain to some intermediate degree, but there is more we can say about the relationship between Rock and Rock*: *Rock* is part of Mountain to less of a degree than Rock is* — it is closer to the edge, made up of atoms that are part of Mountain to less of a degree than those that make up Rock.

2.2 The "more and stronger witnesses" rule

Now consider what cases 2 and 3 have in common: in both, there are some atoms that are part of Rock to a greater degree than they are part of Mountain. Let us make the following generalisation and see where it leads us: that in every case where Rock is not definitely part of Mountain, there are some atoms that are part of Rock to a greater degree than they are part of Mountain; these atoms are the *witnesses* to Rock's not being (definitely) part of Mountain.

An atom that is part of Rock to a *very much* greater degree than it is part of Mountain is a *strong* witness to Rock's not being part of Mountain; an atom that is part of Rock to a *only slightly* greater degree than it is part of Mountain is a *weak* witness to Rock's not being part of Mountain. (You can think of the strength of a witness as the vertical distance between the dashed line and the solid line on the diagrams above).

In each of the cases we have considered, the following informal general rule holds true: that the *more and stronger witnesses* to Rock's not being part of Mountain, the *lesser the degree* to which Rock is part of Mountain.

In *case 1*, there are no witnesses, so Rock is definitely part of Mountain. In *case 2*, there are a large number of witnesses, including a witness of the strongest possible kind: an atom that is definitely part of Rock and definitely not part of Mountain; that seems to be sufficient for Rock to be definitely not part of Mountain. In *case 3*, there are a smaller number than in case 2 of witnesses, all of which are weaker than some of the witnesses in case 2. So Rock is part of Mountain to a greater degree than it is in case 2. The proposed rule also explains the relationship between Rock, Rock*, and Mountain. There are more and stronger witnesses to Rock*'s not being part of Mountain than there are witnesses to Rock's not being part of Mountain. So Rock* is part of Mountain to a lesser degree than is Rock.

2.3 Standard general parthood

Here is a way of making this informal "more and stronger witnesses" rule exact. First, I give an exact definition of the strength of a witness: let the strength with which z is a witness to x's not being part of y be equal to the difference between $c(x, z)$ and $c(y, z)$, where $c(x, z) > c(y, z)$, otherwise 0 (this formalises my informal "vertical distance" suggestion above):

$$\begin{aligned} w(x, y, z) &= \begin{cases} c(x, z) - c(y, z) & \text{if } c(x, z) > c(y, z) \\ 0 & \text{otherwise} \end{cases} \\ &= \min(0, c(x, z) - c(y, z)) \end{aligned}$$

Second, we have the idea that the more and stronger witnesses there are to x's being part of y, the less the degree to which x is part of y. Take the strength of the strongest witness to x's not being part of y — where that is large, the degree to which x is part of y should be small, so let the degree to which x is part of y be 1 minus the strength of the strongest witness. Thus:

$$\begin{aligned} x \prec y &= 1 - \sup_{z \in A} w(x, y, z) \\ &= \inf_{z \in A} (1 - w(x, y, z)) \\ &= \inf_{z \in A} \min(1, 1 + c(y, z) - c(x, z)) \qquad \text{(SP)} \end{aligned}$$

I call the concept of parthood so defined standard general parthood. ("Standard" here contrasts with the "alternative" definitions of general parthood to be explored below.) Despite the complicated appearance of (SGP), it encodes a very simple idea which is easily expressed using the diagrams above: (SGP) simply says that the degree to which x is part of y is 1 minus the greatest vertical distance by which the x-line rises above the y-line (or 1 if there is no such distance). I invite you to check that this definition matches your intuitive judgements in each of cases 1 through 3.

Besides concurring with our intuitions in examples involving ordinary objects, (SGP) satisfies some essential formal criteria for a definition of part-whole. It makes $\prec$ an extension of c — it is a consequence of (SGP) that, in the case where z is an atom, z is part of x to the degree that z is a member of the content of x:

$$\text{If } x \in A \text{, then } x \prec y = c(y, x) \tag{1}$$

(SGP) also makes the relation of *being definitely part of* satisfy the axioms of a atomistic non-fuzzy mereology.[2] As a consequence of (SP), definite part-whole is a partial ordering:

$$x \prec x = 1 \tag{2}$$

$$\text{If } x \prec y = 1 \text{ and } y \prec x = 1 \text{, then } x = y \tag{3}$$

$$\text{If } x \prec y = 1 \text{ and } y \prec z = 1 \text{, then } x \prec z = 1 \tag{4}$$

And, as a consequence of (SGP), definite part-whole satisfies this atomistic supplementation principle:

$$\text{If } (\forall z \in A)(z \prec x = 1 \rightarrow z \prec y = 1) \text{, then } x \prec y = 1 \tag{5}$$

(SGP) also enables me to explain why we needed the Normality principle rather than the weaker No-null principle as an axiom of the theory of fuzzy individuals. Suppose that, as a counterexample to the Normality principle, but not to the No-null principle, there was an individual that had an atom z as a part to an intermediate degree and all other atoms to degree 0. This individual would count as part of z to degree 1, according to (SGP), contradicting the hypothesis that z is an atom. So the Normality principle is required in order to sustain atomism.

So my initial technical objection to the theory of fuzzy individuals — that it might be impossible to define part-whole within the theory — was mistaken. (SGP) is such a definition. If anything, the problem is the reverse, that there are many inequivalent definitions of which (SGP) is only one. I will briefly explain why that is, and then discuss whether that is a problem.

[2] Compare the theses (1)–(5) to the axioms of Simons' system AE. (Simons 1987, p. 51)

3 Alternative definitions

The astute reader may have noticed two features of (SGP):

First, (SGP) is logically much stronger than the informal "more and stronger witnesses" rule that it was supposed to cash out. For example, (SGP) takes the strength of a witness to be the difference between the degrees to which that witness is part of two individuals (on the diagrams: the distance between the dashed and solid lines); but why specifically the difference? All the the informal rule requires is that if the difference is greater, then the witness is stronger; and if the difference is lesser, then the witness is weaker. There are many functions on degrees, besides difference, that satsify that. For another, (SGP) interprets "more and stronger" to mean "stronger"; it pays no attention to how many witnesses there are, only to how strong the strongest is. Might not there be other, perhaps better, ways of aggregating strengths of witnesses together, so that a large number of weak witnesses could make x less of a part of y than a small number of strong witnesses would?

Second, (SGP) contains a type of mathematical function familiar from fuzzy set theory — the residuum of a t-norm (for short, a *t-residuum*). It frequently happens in fuzzy set theory that some set-theoretic concept is definable relative to a t-norm in such a way that every resulting definition is formally adequate. One might suspect that the same thing is going on here: that if we replaced the t-residuum that occurs in (SGP) with the residuum of some other t-norm, the result would be a different definition that is equally formally adequate (perhaps even equally adequate to our intuitions about rocks and mountains).

3.1 Parthood relative to a t-residuum

Indeed, there are different ways to cash out the informal rule; and the different ways correspond to different t-norms. To see how this works, let us first formulate a schema for definitions of general parthood relative to a t-residuum $\ominus$:

$$x \prec_{\ominus} y = \inf_{z \in A} \Big(c(x,z) \ominus c(y,z) \Big)$$

In the schema, $c(x,z) \ominus c(x,z)$ represents a way of measuring the strength of a witness to x's not being part of y, with 0 being maximally strong, and 1 being maximally weak. (Note that strong witnesses are represented by small numbers here — the reverse of the informal discussion in section 2.2). The schema says that the degree to which x is part of y is the strength of the strongest witness to x's not being part of y.

To be a t-residuum, $\ominus$ must satisfy the following conditions:

$$n \ominus m < 1 \text{ iff } n > m \tag{$\ominus$1}$$

$$1 \ominus n = n \tag{$\ominus$2}$$

$$n \ominus m \leq n \ominus m' \text{ if } m \leq m' \tag{$\ominus$3}$$

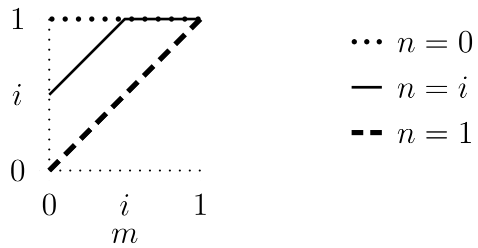
$$n \ominus m \geq n' \ominus m \text{ if } n \leq n' \tag{$\ominus$4}$$

All four of these can be motivated either by the informal concept of strength of a witness, or by the need to obtain consequences such as (1)–(5) above. ($\ominus$1) is motivated by the concept of strength of a witness: a witness to x's not being part of y is, by definition, an atom for which $c(x,z) > c(y,z)$ (recall that low numbers are strong witnesses here, and 1 represents "not a witness at all"). ($\ominus$2) is required for $\prec$ to be consistent with the special parthood relation c — for (1) to be a consequence of the schematised definition.

($\ominus$3) and ($\ominus$4) say that $x \ominus y$ increases monotonically with y and decreases monotonically with x. These can be motivated by intuition on cases such as case 3 — if z is a witness to Rock's not being part of Mountain, then the greater the degree to which z is part of Rock the stronger the witness; in contrast the lesser the degree to which z is part of Mountain the stronger the witness.

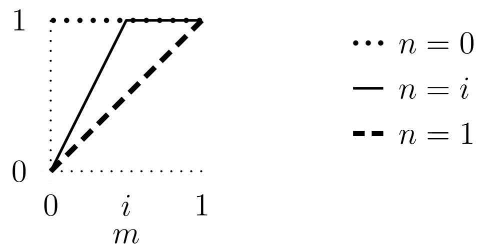
(SGP) results from instantiating the schema setting $\ominus$ to L, the residuum of the Łukasiewicz t-norm:

$$L(n,m) = \min(1, 1 + m - n)$$

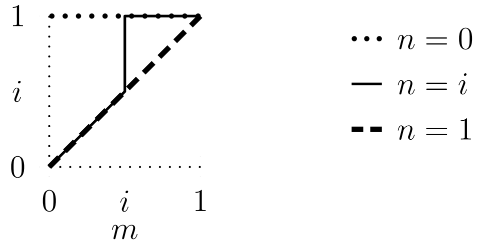


L is not, however, the only function which satisfies ($\ominus$1)–($\ominus$4). Two other such functions are P and G below, the residua of the product and Goguen t-norms respectively:

$$P(n,m) = \begin{cases} \min(1, \frac{m}{n}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}$$

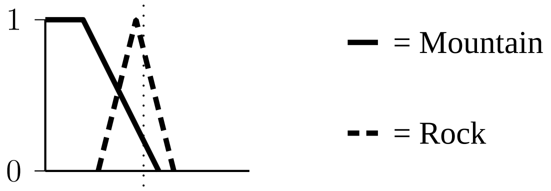


$$G(n,m) = \begin{cases} m & \text{if } n \geq m \\ 1 & \text{if } n < m \end{cases}$$



So we now have at least three possible definitions of general parthood — $\prec_L$, $\prec_P$, and $\prec_G$ — which may come apart in interesting ways. Consider the following Rock / Mountain case:

Case 4: suppose that the relationship of Rock to Mountain is similar to case 3, except that Rock is further downhill than it was in case 3. Rock is sufficiently far from the summit of Mountain that there are now some atoms that are definitely not part of Mountain but are part of Rock to an intermediate degree; but there are no atoms that are definitely part of Rock and definitely not part of Mountain, as shown below:



If by "part of", we mean $\prec_L$, then, as in case 3, Rock is part of Mountain to some intermediate degree. This is because the atoms that are definitely not part of Mountain but part of Rock to some degree are witnesses only to an intermediate degree: $L(x, 0)$ is greater than 0 where x is. This could seem odd — since these atoms are definitely not parts of Mountain, why does their existence not show that, in some sense, "some of Rock is beyond the boundaries of Mountain"?

If by "part of", we mean $\prec_P$ or $\prec_G$, on the other hand, Rock is definitely *not* part of Mountain. As soon as there is any atom part of Rock to any intermediate degree but which is definitely not part of Mountain, that atom is the strongest possible witness to Rock's not being part of Mountain: $P(x, 0)$ is always 0, as is $G(x, 0)$.

We may think of $\prec_L$, $\prec_P$, and $\prec_G$ as increasingly strict definitions of "part of". For any $x, y \in U$, $L(x, y) \geq P(x, y) \geq G(x, y)$ (as may be seen from the graphs above). It follows that $(x \prec_L y) \geq (x \prec_P y) \geq (x \prec_G y)$ — where $\prec_L$ and $\prec_P$ differ, the latter counts fewer pairs of individuals as part of one another than the former, and counts those it does to a lower degree (and likewise for $\prec_P$

and $\prec_G$). There is, in fact, no stricter definition possible than $\prec_G$ — a function $x \ominus y$ that is less than $G(x, y)$ for any x, y would fail to satisfy ($\ominus$1)–($\ominus$4).

3.2 Parthood relative to a t-residuum and an aggregator

All of the instances of the schema above still aggregate strengths of witnesses in the same way that (SGP) did — by taking the infimum. We could generalise still further by defining parthood relative to both a t-residuum and a method of aggregating strengths of witnesses, as in the schema below:

$$x \prec_{*,\ominus} y = \underset{z \in A}{*}\Big(c(x, z) \ominus c(y, z)\Big)$$

Here $*$ is a function that takes functions from atoms to degrees and returns degrees — a function of type $(A \to [0, 1]) \to [0, 1]$. Some reasonable conditions that $*$ must meet in order to fit the informal "witnesses" definition of general parthood are as follows:

$$\underset{z \in A}{*} f(z) = 1 \text{ iff } (\forall z \in A)(f(z) = 1) \tag{*1}$$

$$\underset{z \in A}{*} f(z) = 0 \text{ if } (\exists z \in A)(f(z) = 0) \tag{*2}$$

$$\underset{z \in A}{*} f(z) = x \text{ if } (\exists z \in A)(f(z) = x \text{ and } (\forall y \in A)(y \neq z \to f(y) = 1)) \tag{*3}$$

The motivation for these is as follows. x is definitely part of y iff there are no witnesses to x's not being part of y — so (*1) must be true. The existence of even a single maximally strong witness — an atom that is definitely part of x and definitely not part of y is sufficient for x to definitely not be part of y — so (*2) must be true. (*3) is required, in conjunction with ($\ominus$2), to ensure that $x \prec_{*,\ominus} y = c(y, x)$ where $x \in A$.

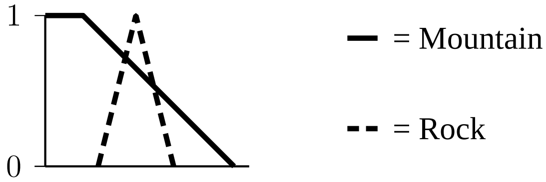
It is not easy to find any functions, besides infimum, that satisfy (*1)–(*3). But it would be good to find one. Informally, we said that "the more and stronger witnesses to Rock's not being part of Mountain, the lesser the degree to which Rock is part of Mountain" — but all the proposed definitions of general parthood so far have paid no attention to how many witnesses there are — only to how strong the strongest is. It would be nice to find a definition of parthood for which the number of atoms that are witnesses to x's not being part of y played a role as well as the strength of the strongest witness.

The difficulty with this is that we have do not have any natural way of measuring the size of a set of atoms. The set of witnesses to x's not being part of y will in many cases be infinite, and then it doesn't make clear sense to speak of "the number of atoms that are witnesses".

If, however, we assume that there are only finitely many atoms, this problem goes away, and there are a number of possible functions available for $*$, which do "pay attention to how many witnesses there are". One such is S, below.[3]

$$\underset{z\in A}{S} f(z) = \max\left(0, 1 + \sum_{z\in A} f(z) - |A|\right)$$

To illustrate how definitions of parthood involving S behave, consider again case 3:



Using $\prec_{S,L}$ as our concept of parthood, the degree to which Rock is part of Mountain is equal to 1 minus the area enclosed under the dashed line and above the solid line, with each atom counting as 1 unit wide. (If that area is greater than 1, then Rock is definitely not part of Mountain). $\prec_{S,L}$ is very strict. Since there are presumably millions of atoms in Rock, and even millions that are part of Rock to a greater extent than they are part of Mountain, this makes it very hard for Rock to be part of Mountain to an intermediate degree — most of the atoms that are part of Rock to a greater degree than the are of Mountain must be greater only by a millionth of a degree. Still, since we don't have much intuitive grip on the metric of degrees, that needn't be a problem.

A much worse problem for definitions of general parthood using S is the restriction to finite A. Though, for all we know, there are a finite number of mereological atoms in the universe, this restriction seems to me to render such definitions logical curiousities, rather than serious attempts to analyse vague parthood.

4 Conclusion

The main problems facing fuzzy theories of part-whole are not technical, but philosophical:

[3]Another possibility is the n-ary product $\prod_{z\in A} f(z)$ — the result of multiplying together all the values of $f(z)$. This also satisfies (*1)–(*3), but does not seem to have any more interesting features than S, so I will only mention it in passing here.

- *assumption of atomism*

  Metaphysically plausible, but not a conceptual truth of the concept of part-whole.

  Needed here in order to draw the distinction between vague and precise individuals.

- problem of *higher-order vagueness*

  There is a sharp distinction between those rocks that are definitely part of the mountain and those rocks that are not definitely part of the mountain. Is this any more plausible than the view that there is a sharp distinction between those rocks that are part of the mountain and those rocks that are not part of the mountain?

- problem of *excessive structure*: fuzzy theories have more structure than the vague concepts they are supposed to regiment (e.g. degrees have a metric; differences between multiple ways of defining part-whole).

- problem of *penumbral connections*: Fuzzy theories have less structure than the vague concepts they are supposed to regiment because they do not allow for "penumbral connections".

  Consider three rocks, A, B, and C, all equally marginal parts of Taranaki. A and B are close together, however, C is on the other side of the mountain.

  "If A is part of T, then B is" should be highly true.

  "If A is part of T, then C is" less true.

  "If A is part of T, then A is" definitely true!

Does fuzzy mereology offer a distinctive solution to the problem of the many?

References

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