Set Size and the Part-Whole Principle

Matthew W. Parker

Abstract. Recent work has defended "Euclidean" theories of set size, in which Cantor's Principle (two sets have equally many elements if and only if there is a one-to-one correspondence between them) is abandoned in favor of the Part-Whole Principle (if A is a proper subset of B then A is smaller than B). It has also been suggested that Gödel's argument for the unique correctness of Cantor's Principle is inadequate. Here we see from simple examples, not that Euclidean theories of set size are wrong, but that they must be either very weak and narrow or largely arbitrary and misleading.

§1. Introduction. On the standard Cantorian conception of cardinal number, equality of size is governed by what we will call…

(CP) *Cantor's Principle*: Two sets have equally many elements if and only if there is a one-to-one correspondence between them.

Another principle that is at least naively appealing has been called (Mancosu 2009)…

(PW) *The Part-Whole Principle*: If A is a proper subset of B, then A is smaller than B.

This is a variant of Common Notion 5 from Euclid's *Elements*:

(CN5) The whole is greater than the part.

Hence we will call theories and assignments of set size *Euclidean* if they satisfy PW[1] (despite differences between PW and CN5 and doubts raised about the authorship of the Common Notions[2]). It is well known that for infinite sets, CP and PW are incompatible. Galileo, for example, pointed out that the square numbers (1, 4, 9, 16,...) can be matched one-to-one with the positive integers, though the squares also form a proper subset of the positive integers (1939; see Bunn 1977, Parker 2009, and Mancosu 2009 for references to earlier, related observations). He therefore rejected the whole idea of relative size (larger and smaller) for infinite collections. Bolzano instead denied CP in favor of PW (1950, 1973).

It's widely taken for granted today that Cantor's theory is the only correct one and views like Galileo's and Bolzano's are just mistaken. Gödel argued explicitly for that position in 1947 (1990). But as pointed out in Parker 2009, Gödel's argument is only an intuition pump. He asks us to imagine transforming the physical objects in one set to resemble those in another, and infers without further argument that two such sets must have the same size. The argument is appealing, but the intuition that a set must be larger than any proper subset is also strong, at least for those not already indoctrinated into Cantorian set theory. After all, if $A \subset B$ then B contains all the elements of A *and more*. Perhaps some of the arguments below can help persuade resistant students of set theory to suspend such intuitions, but merely pumping other intuitions misses the point. CP and PW are both intuitive, so we cannot refute one just by stimulating sympathies for the other.

[1] This differs from some uses of 'Euclidean' in the related literature. Cf. Mayberry 2000; Benci, Di Nasso, and Forti 2007. Note Mayberry's theory is finitistic and takes CP for granted; it is not particularly relevant to the present discussion.

[2] Tannery 1884; cf. Heath 1956, p. 221.

Recently, several authors have developed and defended Euclidean theories of number (Katz 1981; Benci and Di Nasso 2003a; Benci, Di Nasso, and Forti 2006, 2007; Parker 2009; Mancosu 2009; Di Nasso and Forti 2010; Gwiazda 2010). Benci and Di Nasso (2003a) have even shown that Euclidean measures of set size are consistent with ZFC. Hence, the Gödelian view that Euclidean theories are false or wrong is highly questionable; it requires us to believe that, even if a concept is vague and permits different refinements (as *the size of an infinite set* did before Cantor's theory of cardinality became standard), one refinement may somehow be uniquely *correct*, and the others *mistaken*. Of course, that was very much the point of Gödel's paper (mainly as applied to the continuum), but there are problems with his view. Besides the well known epistemological objections (Benacerraf 1973), it tends to delegitimize alternative ideas that may have value. Even if Cantor's theory is by far the *best* or *most natural* theory of transfinite set size, Benci and Di Nasso's work proves that there are alternatives that inherit much from our naive notions of number, and these may have intrinsic interest or special uses, e.g., in non-standard probability theory (McCall and Armstrong 1989, Gwiazda 2010, Wenmackers and Horsten 2010), the foundations of non-standard analysis (Benci and Di Nasso 2003b), or number theory (in connection with density measures; Di Nasso 2010).

However, our main question here is whether it's possible to have a *good* Euclidean theory of set size, and my answer is, no, not really—not if that means it must be strong, general, and well motivated. I will argue that any Euclidean theory strong and general enough to determine the sizes of certain simple, countably[3] infinite sets must

[3] There is no inconsistency or question begging involved in applying Cantorian concepts like *countably infinite* here. Our question is whether there are useful Euclidean conceptions of size *in addition* to Cantor's

incorporate thoroughly arbitrary choices. One reason for this is that if sets are always larger than any of their proper subsets, set sizes are not invariant under rigid transformations. Hence, the assignment of Euclidean sizes to certain sets must be as arbitrary as a choice between equally natural coordinate systems. The failure of rigid transformation invariance for Euclidean sizes has been noted elsewhere (e.g., Di Nasso and Forti 2010), but neither its simplicity and fundamentality nor its implications have been adequately highlighted.[4] PW alone implies that very pedestrian, countable, and even bounded sets violate rigid transformation invariance. In fact, even the tiniest rotation of a point set can effect an enormous change in set size. Moreover, even *relative* sizes violate rigid transformation invariance. That is, PW implies that for some equal-sized sets A and B, the images *T*A and *T*B by a translation or rotation are not equal in size to each other. So Euclidean *relative* sizes too are as arbitrary as a choice of coordinates. The examples are very simple and the proofs extremely easy, which again illustrates how basic the arbitrariness of Euclidean sizes is. It is no merely technical snag, but a robust and general handicap.

Thus I hope to provide a more cogent and detailed account than Gödel did of just what is "wrong with" Euclidean theories, without claiming that they are *wrong* and with as little reliance as possible on either ontology or intuition. One traditional question at stake is which definitions of set size best accord with our pre-theoretic intuitions, but that is not our main concern here. What seems more important is to develop concepts and

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concepts. Recent work on Euclidean theories, such as BDF's, employs a great deal of standard set-theoretical machinery, including the Cantorian notion of cardinality.

[4] Di Nasso and Forti (2010) note that their Euclidean "numerosities" are not preserved by rigid transformations because of the Banach-Tarski paradox, but as we will see, the result is demonstrated by much simpler examples. As well, the argument from Banach-Tarski requires finite additivity, while in fact *any* Euclidean size assignment on a sufficiently broad domain violates rigid transformation invariance, whether additive or not.

theories that are *useful*, not necessarily for building bridges, but to the understanding. And while the *exploration* of Euclidean theories of size is certainly enlightening, one of the things it ultimately reveals is how limited Euclidean sizes themselves are by their unavoidable arbitrariness. This is not to say they are useless altogether; as noted they may well have special applications to probability and number theory. However, anyone who has hoped for a revolutionary new Euclidean theory of set size with breadth and informativeness approaching what we would expect from a notion of *how many* will ultimately be disappointed.

An important target of this critique is the theory of *numerosities* developed by Benci, Di Nasso, and Forti (henceforth *BDF*; see Benci and Di Nasso 2003a; BDF 2006, 2007; Di Nasso and Forti 2010; Blass, Di Nasso, and Forti 2011). Briefly, numerosities are Euclidean set sizes with the algebraic and ordering properties of the ordinary whole numbers (i.e., those of a discretely ordered semi-ring). However, I will not give a detailed account of numerosities, for the points to be made here are very general; *any* Euclidean theory that determines the sizes of the examples below will be plagued with arbitrariness.

The structure of the paper is as follows: In §2 we lay out some basic assumptions, notation, and terminology. §3 explains what I mean by arbitrariness and why it is undesirable, and tries to head off a couple of misunderstandings. In §4 we discuss the failure of *absolute* translation invariance, i.e., the fact that sets cannot in general be equal in size to their translations if PW holds. §5 concerns the failure of *relational* translation invariance, the fact that Euclidean size relations between sets are not preserved by translations. §6 concerns the failure of rotation invariance (absolute and relational) for bounded sets and the extreme sensitivity of Euclidean sizes to rotation. In §7 we consider and reject a charitable, pluralist view of the situation, and in §8 we discuss the prospects for *partial* Euclidean size assignments, and contrast the arbitrariness of Euclidean sizes with the incompleteness of the Cantorian theory. §9 concludes with brief remarks on what we should take away from all this.

§2. Notation, terminology, and background assumptions. We use **N** for the set of whole numbers (including zero), **Z** for the integers, $\mathbf{Z}^+$ for the positive integers ($\mathbf{N} \setminus \{0\}$, where '\' is set difference), and **R** for the reals. $\mathcal{P}(A)$ denotes the power set $\{B: B \subseteq A\}$.

We'll use the following expressions in connection with size concepts in general:

Definition 1. (i) A *size assignment* is a measure or a size ordering.

(ii) A *measure* (or *size function*) is just a function $[\cdot]: D \to S$ from a domain D of sets into a linearly ordered set $\mathbf{S} = \langle S, \leq \rangle$.

(iii) If $[\cdot]$ is a measure, $[A]$ is called the *absolute size* of A.

(iv) We write $[A] < [B]$ and $[B] > [A]$ if $[A] \leq [B]$ but not $[B] \leq [A]$.

(v) A measure $[\cdot]$ is *Euclidean* on D if for all $A, B \in D$, $A \subset D$ implies $[A] < [B]$.

(vi) A measure $[\cdot]$ is *total* on D if for all $A, B \in D$, $[A] \leq [B]$ or $[B] \leq [A]$.

(vii) A *size ordering* (or *size relation*) $\trianglelefteq$ is a reflexive, transitive relation (i.e., a preorder) on a domain D of sets.

(viii) The pairs (A, B) such that $A \trianglelefteq B$ are called *relative sizes*.

(ix) We write A $\boldsymbol{<}$ B and B $\boldsymbol{>}$ A if A $\boldsymbol{\leq}$ B but not B $\boldsymbol{\leq}$ A.

(x) We write A ≈ B if A $\boldsymbol{\leq}$ B and B $\boldsymbol{\leq}$ A.

(xi) A size ordering $\boldsymbol{\leq}$ is *Euclidean* on *D* if for all A, B ∈ *D*, A ⊂ B implies A $\boldsymbol{<}$ B.

(xii) A size ordering $\boldsymbol{\leq}$ is *total* on *D* if for all A, B ∈ *D*, A $\boldsymbol{\leq}$ B or B $\boldsymbol{\leq}$ A.

Note 'measure' for us is a very general term; it does not imply additivity or that [∅] = 0, though Euclidean measures always satisfy monotonicity (i.e., if A ⊆ B then A $\boldsymbol{\leq}$ B). The relations ≤, <, $\boldsymbol{\leq}$, $\boldsymbol{<}$, and ≈ are all transitive, and ≈ is an equivalence relation expressing equal size. As usual, '=' is reserved for identity: A = B if and only if 'A' and 'B' denote the very same object. Where absolute sizes are defined, we assume that [A] ≤ [B] if and only if A $\boldsymbol{\leq}$ B. Where they are not, we can always let [A] = {C ∈ *D*: C ≈ A}; then if $\boldsymbol{\leq}$ is total, letting [A] ≤ [B] if and only if A $\boldsymbol{\leq}$ B gives us a linear order on the range of [·], so that [·] is a measure.

These stipulations are inspired by an intuitive analysis of size, but they can also be motivated by utility, since relations with such properties have special applications; or we can simply take them to demarcate the scope of the present inquiry. If we must abandon such basic assumptions to defend Euclidean sizes, that in itself illustrates how limited such sizes are.

Note also that an assignment is Euclidean on *D* if PW applies to *all* proper subset/superset pairs in *D*, but this does not imply totality; we may still have sets in *D* that are not comparable at all.

It is important for us to distinguish between size *assignments*, as just defined, and *theories* of size. Assignments are mathematical relations, and we impose no constraints on their complexity. They need not be constructive, definable, or recursively enumerable. A *theory*, on the other hand, is often understood as a deductively closed set of sentences or propositions generated by some recursive set of axioms and inference rules. We need not be completely precise here; the important thing for us is that a theory, on this conception, is something we can state and work with. In the interest of usefulness, it must be somehow expressible, while an assignment need not be.

We will avoid the term 'cardinal' and its derivatives as much as possible. Cantor's adoption of that term for what he originally called "powers" reflects his later view that his concept of power was the correct analysis of the naive notion of *number of elements*.[5] We might have tried to reclaim 'cardinal' for that pre-theoretic notion without assuming that Cantor's was the uniquely correct analysis of it, but Cantor's usage is deeply entrenched, as even BDF's Euclidean papers reflect. Still, to avoid confusion and prejudice, we revert to Cantor's earlier term 'power' for his cardinals (writing $|A|$ for the power of A), and use 'count' for the general notion of number of elements, as distinguished from both non-integral magnitudes and ordinal positions (2nd, 3rd, etc.).

Central to the notion of count is the property of…

Discreteness: If $A \prec B$ then $A \cup \{x\} \preceq B$.

This just says that if one set is strictly smaller than another, then adding a single element cannot make it strictly larger—two sets can't differ in count by less than a whole element. BDF's numerosities have this property, and it's of particular interest because of the

[5] See Parker 2009, especially the footnotes, for discussion of the evolution of Cantor's thought on the number of elements of a set.

ancient project of understanding the relation of count to geometric measure and, relatedly, that of individual points to the continuum. Furthermore, for sets of numbers or points, a system of sizes that violates Discreteness would be misleading. If two sets are not vague or fuzzy and do not differ by a part of an object (as in the case of 6 apples compared to 6½ apples), then to represent them as differing by less than one whole element conveys quite the wrong idea. It suggests, for example, that we could make the sets equal by adding or removing a part of an object, or by including an object more definitely. But neither holds for the sets we will consider; they contain only whole elements and contain them wholly. So we have good reasons to assume Discreteness, but in fact most arguments here do not require it.

Finally, we will make occasional use of this commonplace notation:

Definition 2. For a transformation T: X $\rightarrow$ Y and a set A $\subseteq$ X, TA = $\{Ta\text{: } a \in \text{A}\}$. In particular, for a set A of numbers, $x\text{A} + y = \{xa + y\text{: } a \in \text{A}\}$.

So for example $2\mathbf{N} + 3 = \{3, 5, 7, 9, ...\}$.

We will need a few more expressions, concepts, and postulates, but these will be introduced as they come to bear.

§3. The trouble with arbitrariness. Here I want to say a little about what I mean by arbitrariness and why it is undesirable, and head off a couple of misunderstandings.

'Arbitrary' can mean many things, but the main thing I mean by it here is that

Euclidean size assignments are *unmotivated* in many of their specific details. Particular sizes could be chosen differently without any significant loss of utility or elegance. One might object that, in connection with Euclidean theories, arbitrariness is motivated, since it is better to have a partly arbitrary Euclidean theory than none at all.[6] But this confuses the issue. The first question is not whether arbitrariness itself is motivated but whether the particular details of a given Euclidean size assignment are well motivated. If not all of them are, that is what I call arbitrariness. Whether the benefits of such a partly unmotivated theory outweigh its detractions is another question. Perhaps a partly arbitrary theory is useful for some purposes, but it is also limited and, as I will argue shortly, misleading.

Another possible objection is that the Euclidean theories on offer *don't* determine sizes arbitrarily; they leave sizes indeterminate where well motivated principles do not decide them, and this is just what they should do.[7] Indeed, the theories of BDF and others do not determine the sizes of all sets, or even all sets of whole numbers. However, those theories are *about* Euclidean assignments that *are* total over some broad class of sets. In most of their papers, BDF define 'numerosity function' so that such a function must be total over a broad domain such as the entire power set of **N** (Benci and Di Nasso 2003a) or the class of all sets of ordinals (BDF 2006). Even where totality is not built into their definition (e.g., Di Nasso and Forti 2010), the whole point of the elaborate construction of numerosities (from selective ultrafilters) is to ensure totality. So BDF's *theory* may not itself decide sizes arbitrarily, but the assignments that the theory describes do. My claim is that Euclidean theories and assignments must be *either* arbitrary or too

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[6] One reader of an earlier draft made such a remark.

[7] This is another comment made by a reader.

weak to decide the relative sizes of some very simple sets. The theory of numerosities suffers from the latter limitation, and numerosities themselves from the former.

But what, you may ask, is so bad about arbitrariness? If there is no good reason to say that A is *not* smaller than B, for example, then why not just stipulate that it is? The problem with this is that it's misleading. Consider this analogy: You're very hungry and you want a very filling and nourishing meal. A lunch truck comes along offering three different meals: A, which is free, B, which consists of A plus a nutritious side-dish and costs a dollar, and C, which is altogether incomparable to either A or B and also costs a dollar. Since you can't compare C to A or B, you choose a partly arbitrary preference ordering under which C is better than A and at least as good as B. On that basis you pay a dollar for C, even though there is in fact no good reason to regard C as better than the free meal A. You've just wasted a dollar. You could instead have gotten A, which is neither better nor worse than C, and kept your dollar, or you could have spent your dollar on B, which is definitely better than A. Instead you've spent a dollar on C when there was no good reason to. This is how partly arbitrary size assignments are misleading and potentially costly. Of course, we are not concerned here with such quotidian matters as buying lunch, but arbitrary size assignments can be *epistemically* costly in a similar way; they obscure inherent truths by mixing them with haphazard stipulations.

One might object that language always involves haphazard stipulations, but this is mainly in the choice of symbols used. Where the concepts expressed are also somewhat arbitrary (like the culinary distinction between fruits and vegetables, for example), this again limits their usefulness and scientific interest.

The claim that arbitrariness in size assignments is misleading and obscures inherent truths suggests a degree of realism. And in a sense this is right; at least for the purposes of this paper, I do take it for granted that there are some objective logical facts that are relevant to our discussion—in particular, facts about which propositions do and don't follow from ZFC and other theories, given the standard ways of interpreting those theories. I do not suppose, as Gödel did, that there is any objective fact about what the size of an infinite set consists in, or whether set sizes satisfy CP or PW, for I think we are free to employ non-standard and even unnatural notions if we wish. But generally speaking, a *useful* size assignment is one that reflects antecedent facts about the sets measured, or the implications of other theories that are independent of the notion of size adopted. Sizes might for example reflect facts about which sets can be put in one-to-one correspondence, or which are proper subsets of others, or which ones resemble each other in the spatial arrangement of their elements. But one way or another, sizes should be *informative*. The problem with Euclidean sizes is that, if they cover even the simple examples below, then some of them are *not* informative, and this makes them misleading.

Now to the examples.

§4. Absolute translation invariance. Consider the sets $\mathbf{N} = \{0, 1, 2, \ldots\}$ and $\mathbf{N} + 1 = \{n + 1 : n \in \mathbf{N}\} = \{1, 2, 3, \ldots\} = \mathbf{N} \setminus \{0\}$. According to PW, $\mathbf{N} \succ \mathbf{N} + 1$. Yet $\mathbf{N} + 1$ is merely a translation of $\mathbf{N}$, so already we have a violation of translation invariance.

Intuitively, a translation is a transformation that preserves distances as well as directions. More precisely,

Definition 3. T: S $\rightarrow$ S is a *translation* on a metric space[8] $\langle$S, $d\rangle$ if for all $x, y \in$ S, $d(x, y) = d(Tx, Ty)$ and $d(x, Tx) = d(y, Ty)$.[9]

What we have just seen is that Euclidean sizes on the sets of whole numbers violate

(ATI) *Absolute Translation Invariance*: If T is a translation on $\langle$S, $d\rangle$ and A $\subseteq$ S then TA $\approx$ A.

But the problem is not at all restricted to whole numbers. It easily generalizes to arbitrary Euclidean spaces, in the usual geometric sense of 'Euclidean'. To be precise,

Definition 4. Let $\mathbf{S} = \langle$S, $d\,\rangle$ and $\mathbf{S}' = \langle$S$'$, $d'\,\rangle$ be metric spaces.

(i) $\mathbf{S}$ is *isometric* to $\mathbf{S}'$ if there is a one-to-one map f: S $\rightarrow$ S$'$ such that for all $x, y \in$ S, $d(x, y) = d'(f(x), f(y))$.

(ii) $\mathbf{S}$ is *Euclidean* if it is isometric to $(\mathbf{R}^n, \delta)$ for some $n \in \mathbf{Z}^+$, where δ is the standard Euclidean metric $\delta(\langle x_1,\ldots, x_n\rangle, \langle y_1,\ldots, y_n\rangle) = ((y_1 - x_1)^2 + \ldots + (y_n - x_n)^2)^{1/2}$.

Remark 1. If $\mathbf{S} = \langle$S, $d\rangle$ is a Euclidean metric space and $\trianglelefteq$ is a Euclidean size ordering on $\mathcal{P}$(S), then $\trianglelefteq$ violates ATI.

Proof. Euclidean spaces always have translations, since $\mathbf{R}^n$ does. So let T be a translation on $\mathbf{S}$. Let p $\in$ S and P = {p, Tp, TTp,…} = {T^np: $n \in \mathbf{N}$}. (See Figure 1.)

[8] A metric space is just set equipped with a measure of distance between elements. The precise definition can be found in any introductory analysis textbook or internet resource.

[9] This is not a standard definition of 'translation,' but it captures the right cases and saves us introducing more structure on the space $\langle$S, $d\rangle$. This makes it both general and easy to state.

Then TP = {Tp, T^2p, T^3p,…} ⊂ P. So by PW, TP < P, contradicting ATI.[10]

In fact, our assumptions here are stronger than necessary to get a failure of translation invariance. Besides PW, all we need is a translation with an infinite forward orbit {p, Tp, TTp,...} (cf. BDF 2007, §3.1). The Euclidean background space is just a natural way to guarantee this.

This is a trivial result, deserving only the status of a remark, but consider the implications. It implies first that the size of a set depends on its particular position in some background space. Two sets, like **N** and **N** + 1, or P and TP, can be entirely alike in structure, and yet, due only to which particular elements are contained within this structure and where they happen to be, unequal in size. And as illustrated by sets **N** and **N** + n, or P and T^nP, they may differ *vastly* in size—that is, by an arbitrarily large finite number of elements.[11]

This is not only counterintuitive but impractical. One of the very useful features of our usual notions of size is that they meaningfully classify distinct objects as *equal*. (PW only does part of this job; it tells us that certain sets *aren't* equal, but not which ones are.) Another useful feature of sizes is that they abstract away from the natures of the

[10] We could instead prove this from the existence of sets in **S** isometric to **N** and **N** + 1, but the proof from more generic point sets is illustrative. A parallel remark applies to other proofs below.

[11] P and TP also illustrate a fundamental conflict between two Common Notions from Euclid's *Elements*. Common Notion 4 is translated by Heath (1956) as,

(CN4) Things coinciding with one another are equal to one another.

If the sense of 'coinciding with' (or 'applying onto', as Heath also suggests) is the sense in which, for example, one line segment coincides with another of the same length, then surely the sets P and TP coincide as much as any two sets or figures, so by CN4 they must be equal. Yet TP is a proper part of P, so by CN5, P is larger. Below we will see examples of *bounded* figures (or point sets) for which CN4 and CN5 are incompatible. (BDF claim in 2007 that numerosities satisfy both CN4 and CN5, but this is because they interpret 'applying onto' as 'related by an isometry', and define an isometry as a numerosity-preserving transformation. This makes CN4 trivial.)

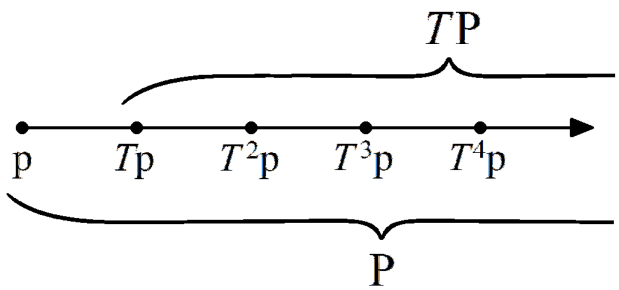

Figure 1. *T*P is just a translation of P, but given PW, *T*P is smaller than P.

natures of elements, what is left as a basis for equality but the structural features of sets?[12] So even if we reject CP, we have good motivations to hope for some weaker, more specific structural principle of equality, and ATI is an especially weak one. It proposes not that *any* bijection implies equal size (CP), not that *continuous* bijections imply equal size, not that *order* preserving bijections imply equal size, but only that bijections that preserve *all differences and directions* imply equal size. But as we have just seen, even that fails for Euclidean sizes.

One might object that transformation principles like ATI are all too Cantorian. If we are going to evaluate the competing principles CP and PW, it might be argued, we should not assume that certain mappings preserve size, as that's too prejudiced towards

[12] BDF also concern themselves with bases for equality, or as they put it (2007), the search for interesting "isometries," which they define as transformations that preserve numerosity. They point out that certain product principles imply certain limited invariance properties (2006, 2007). However, as they also acknowledge, their product principles prove problematic, and PW imposes severe restrictions on size-preserving maps: No map with an infinite forward orbit preserves Euclidean sizes (2007, §3.1). So the prospects for interesting isometries are not great.

CP.[13] But the point here is that PW does not *just* contradict CP; it also contradicts much weaker, useful principles. Even if we are prepared to drop CP, it would be useful if the size of a set indicated some antecedent fact about its structure. So preferably, transformations that preserve structure or a great deal of structure ought to preserve size. In general, one-to-one correspondence preserves very little structure, while a translation preserves as much as possible without necessarily preserving the internal structures of the individual elements. So it is one thing to say that Euclidean sizes violate CP, and quite another to say that they violate principles like ATI. Violating ATI suggests that set size is not determined by structure at all.

The violation of ATI is even worse news for point sets than for sets of numbers. Numbers arguably have distinct qualitative or structural properties, absolute positions on the number line, and special relations to each other that go beyond their relative positions. But point sets like our P and *T*P above are *only* distinguishable by their positions; the points themselves have no distinctive characteristics other than their particular positions in the space. So unless we want the size of a set to depend on its particular position (even while holding the relative positions of the elements fixed), or perhaps on the bare haecceities of its elements (if there are such things), we should like ATI to hold. And for Euclidean sizes it can't.

One consequence of this is that any assignment of *numerical* absolute sizes to sets like our P and *T*P would be arbitrary. As noted in §2, we could define the absolute size of a set as the equivalence class of sets related to it by ≈ (derived from **≤**), but there is no preferred way to associate such a Fregean "size" with any standard number applicable

[13] In fact, this argument was made by a reader of an earlier draft.

outside the particular domain of ≈, such as an ordinal or a hyper-integer.[14] After all, we might assign a particular number ν to P and $\nu - 1$ to *T*P, or $\nu + 1$ to P and ν to *T*P, or even $\nu + 243$ to P and $\nu + 242$ to *T*P. Any choice among such options would be as arbitrary as a choice between isometric coordinate systems that differ only in the locations of their origins, or between physical reference frames that are stationary relative to each other.

Further, this means we have no useful way of comparing Euclidean sizes of infinite sets (of the same power[15]) that live in different spaces. If a set Q in a metric space **T** is isometric to our set P in **S**, then it is isometric to each translation image T^iP. So why should Q be equal to P instead of, say, T^{717}P? If we could assign the sets definite numerical sizes, then sets assigned the same number could be regarded as equal. But if we have no good reason to assign a set one number rather than another, then equating sets that are assigned equal numbers is capricious, uninformative, and ultimately misleading.

So the failure of ATI might seem natural given PW, but for all of the reasons just given, it is nonetheless very limiting.

§5. Relational translation invariance. Though translations do not preserve Euclidean size, one might hope that they would at least preserve the Euclidean size *relations* between sets. That is, we would like...

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[14] The numerosities in BDF's consistency proofs are hyper-integers, i.e., equivalence classes (relative to some selective ultrafilter) of infinite sequences of integers. My present point entails that any assignment of specific hyper-integers to sets like P and *T*P is arbitrary.

[15] We might, like BDF, assume that larger Cantorian power implies larger Euclidean size, i.e., $|A| < |B| \Rightarrow A \prec B$ (the "Half-Cantor Principle"). Then we could usefully compare sets if they have different powers. But this gives us no general means of comparing sets of the *same* power while upholding PW.

(RTI) *Relational Translation Invariance.* If T is a translation on a metric space $\langle S, d\rangle$ and A, B $\subseteq$ S then A $\leq$ B if and only if TA $\leq$ TB.

But no, even this fails under mild conditions.

For a concrete example, consider EVEN = 2**N** = {0, 2, 4,…}, ODD = 2**N** + 1 = {1, 3, 5,…}, and EVEN + 2 = 2**N** + 2 = {2, 4, 6,…}.[16] Assuming PW, EVEN > EVEN + 2. But is ODD the same size as EVEN or as EVEN + 2? Or neither? Assuming PW, totality, and Discreteness, any choice violates RTI. By Discreteness, ODD can't be *between* EVEN and EVEN + 2 in size (though it lies between them in position). So by totality, ODD must be at least as large as EVEN or at least as small as EVEN + 2. If ODD is as large as EVEN, then RTI implies that EVEN + 2 is as large as ODD. But then by transitivity, EVEN + 2 is as large as EVEN, contradicting PW. Similarly, if ODD is instead as small as EVEN + 2, then RTI and transitivity imply that EVEN is as small as EVEN + 2, again contradicting PW.

As argued in §2, it would be misleading to drop Discreteness. We will consider abandoning totality in §8. But we will soon see that RTI fails under other mild conditions, without Discreteness or totality. (The failure is *robust*: It obtains from different, independent, weak assumptions.)

Again, the result generalizes easily to arbitrary Euclidean metric spaces:

Remark 2. If $\langle S, d\rangle$ is a Euclidean metric space and $\leq$ is a total Euclidean size ordering on $\mathcal{P}$(S) satisfying Discreteness, then $\leq$ violates RTI.

[16] This example is closely related to the coin flipping examples in Williamson's critique of non-standard probabilities (2007).

Proof. Let T be a translation on $\langle S, d\rangle$, $p \in S$, and $Q = \{T^{2n}p\text{: } n \in \mathbf{N}\}$. (See Figure 2.) Then repeat the preceding argument with Q in place of EVEN, TQ in place of ODD, and T^2Q in place of EVEN + 2.

Such violations of RTI are misleading. Suppose for example we have EVEN ≈ ODD but ODD **>** EVEN + 2. This would suggest that there is some special relation between the former two sets that does not hold between the latter. But aside from the postulated size relation itself, there isn't. EVEN and ODD are related in exactly the same way as ODD and EVEN + 2, namely by the translation $Tn = n + 1$. Of course, it is easy to define relations that hold between EVEN and ODD but don't hold between ODD and EVEN + 2, but by the same token, lots of relations hold between the latter two and not the former. The point is, there is no particularly *important* or *size-like* relation that holds of the former two and not the latter—nothing to warrant assigning the former sets the same size but not the latter. We could just as well stipulate that EVEN **>** ODD ≈ EVEN + 2, and adopting one of these orderings over the other obscures the fact that neither ordering is particularly privileged.

In fact, Benci and Di Nasso (2003a) show that the existence of numerosities satisfying either stipulation is consistent with ZFC. (The existence of such numerosities is implied by the existence of selective ultrafilters, which is itself independent of ZFC but implied by the Continuum Hypothesis.) So even ZFC, PW, the strong algebraic and ordering properties of numerosities, and the Continuum Hypothesis together do not resolve basic questions like whether EVEN ≈ ODD.

Benci and Di Nasso themselves suggest that the indeterminacy of numerosities

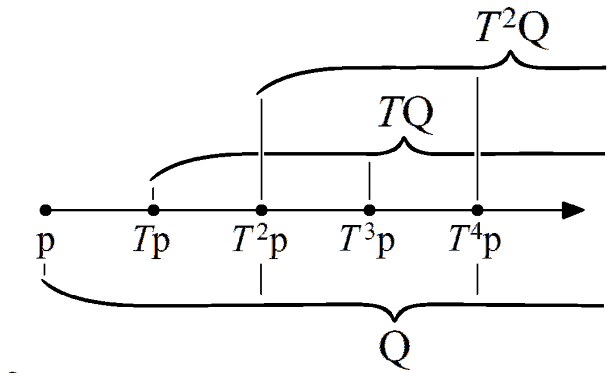

Figure 2. $T^2\mathrm{Q} \subset \mathrm{Q}$, while TQ is disjoint from both.

can be overcome by introducing further postulates, such as $n[n\mathbf{Z}^+] = [\mathbf{Z}^+]$, i.e., that the sum of n terms each equal to $[n\mathbf{Z}^+]$ is $[\mathbf{Z}^+]$. Given the algebraic properties of numerosities, this implies that EVEN **>** ODD ≈ EVEN + 2.[17]

But Benci and Di Nasso do not claim that this one postulate resolves all of the indeterminacy of numerosities, and it may well be that *no* finite or recursively enumerable set of postulates can define a unique total Euclidean size assignment on *arbitrariness* of Euclidean sizes unless the postulates themselves are non-arbitrary. And what is there to recommend Benci and Di Nasso's postulate $n[n\mathbf{Z}^+] = [\mathbf{Z}^+]$? It may look

[17] Benci and Di Nasso use '**Even**' to denote our set EVEN + 2 = {2, 4, 6,...} and '**Odd**' for our ODD = {1, 3, 5,...}. In that notation, their postulate implies that **Even ≈ Odd**, which perhaps makes it seem more natural.

natural if we are particularly interested in $\mathbf{Z}^+$, but one could just as reasonably stipulate that $n[n\mathbf{N}] = [\mathbf{N}]$ instead, and then we would have EVEN ≈ ODD **>** EVEN + 2.

The underlying difficulty is that all of these assignments violate RTI. If EVEN and ODD are equal in size, then ODD and EVEN + 2 are not, and vice versa, assuming Discreteness. So the size relation between two sets depends not only on their structural properties and relations to one another but on where in particular they happen to be on the number line. And thus the choice between stipulations like EVEN ≈ ODD and ODD ≈ EVEN + 2, or between $n[n\mathbf{Z}^+] = [\mathbf{Z}^+]$ and $n[n\mathbf{N}] = [\mathbf{N}]$, is as arbitrary as the choice of a favorite position on the number line.

Furthermore, even if one could eliminate the indeterminacy of Euclidean size assignments on $\mathcal{P}(\mathbf{N})$, there would remain many other indeterminacies to reckon with in other contexts. As before, the failure of translation invariance is even more worrying for point sets than for number sets. We might concede the possibility that 'EVEN ≈ ODD **>** EVEN + 2' is somehow a better stipulation than 'EVEN **>** ODD ≈ EVEN + 2' for reasons having to do with the distinct properties of the numbers involved. But for our point sets Q, TQ, and T^2Q, in the context of a space with no privileged origin, there is no such possibility. To stipulate that, say, Q ≈ TQ **>** T^2Q would be indisputably arbitrary, because the sets Q and TQ take the same positions in one reference frame or coordinate system as TQ and T^2Q take in another (even in a coordinate system *isometric* to the first), and there is nothing in the intrinsic natures of the points to distinguish the relation of Q to TQ from that of TQ to T^2Q.

Now let's see a violation of RTI without Discreteness or totality. It's more than enough to assume that $\mathbf{N}$ is the same size as its reflection $(-1)\mathbf{N} = \{0, -1, -2,...\}$. Then if $Ti = i + 1$ for all $i \in \mathbf{Z}$, PW implies that $T\mathbf{N} < \mathbf{N}$ and $(-1)\mathbf{N} < T(-1)\mathbf{N} = \{1, 0, -1,\ldots\}$. By transitivity, $T\mathbf{N} < T(-1)\mathbf{N}$, but since $\mathbf{N} \approx (-1)\mathbf{N}$, this contradicts RTI.

In fact, we can get a more general result with only a very weak assumption about point sets:

(WRP) *Weak Reflection Principle*. There is a translation T on $\langle S, d\rangle$ and points p, q $\in$ S such that $\{T^n\text{p}: n \in \mathbf{N}\} \approx \{T^{-n}\text{q}: n \in \mathbf{N}\}$ (where T^{-n}q is the unique point r such that Tr = q).

Remark 3. If $\leq$ is a Euclidean (partial) size ordering satisfying WRP on a Euclidean metric space $\langle S, d\rangle$ then $\leq$ violates RTI.

Proof. Again we generalize the numerical example: Let $\text{P} = \{T^n\text{p}: n \in \mathbf{N}\} \approx \text{R} = \{T^{-n}\text{q}: n \in \mathbf{N}\}$. (See Figure 3.) Then by PW, $T\text{P} < \text{P} \approx \text{R} < T\text{R}$, and hence $T\text{P} < T\text{R}$. So $\text{R} \leq \text{P}$, but it is not the case that $T\text{R} \leq T\text{P}$, contradicting RTI.

Like ATI and RTI, WRP is desirable. The sets P and R are rigid rotations of each other, so they're structurally and qualitatively alike and hence should have the same size. To regard one as larger than another would be as arbitrary as a choice between isometric coordinate systems. We will discuss rotation and reflection invariance more generally below; the present point is just that we do not have to assume Discreteness or totality to get a violation of RTI. Even for Euclidean notions of size that are *not* notions of discrete count—for continuous magnitudes,

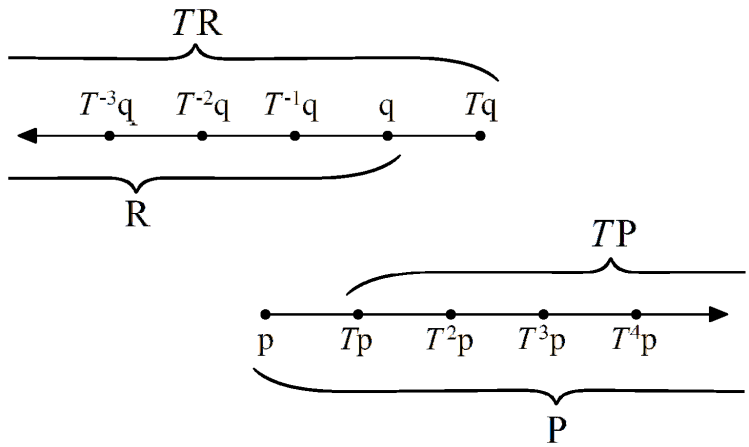

Figure 3. If P = $\{T^n\text{p}:\ n \in \mathbf{N}\}$ is the same size as R = $\{T^{-n}\text{q}:\ n \in \mathbf{N}\}$, then PW implies that *T*P < *T*R, so translations do not preserve relative size.

for example—and even for Euclidean *partial* size assignments (provided PW holds for the relevant subset/superset pairs), RTI fails under very mild and well motivated assumptions.

So in general, Euclidean size orderings are arbitrary in a clear and robust way: Given one, we can obtain an equally valid and well (or poorly) motivated one by a mere translation. The point is not that we *must* accept RTI and therefore reject PW. The point is that *if* we do not want set sizes to be as arbitrary as a choice between isometric coordinate systems, *then* we should adopt RTI (and WRP) and abandon PW.

§6. Rotation invariance. For the same reasons as we would like sizes (absolute and relative) to be translation invariant, we would also like them to be rotation invariant. In many important geometric and physical contexts, there is nothing to distinguish one direction from another, so a distinction of size that depends on the particular orientation[18] of a set is baseless and misleading. But as we will now see, Euclidean sizes *must* be sensitive to rotation, and very much so.

First, a couple more definitions. We will regard reflections as a subclass of rotations, since a reflection in n dimensions is equivalent to a rotation in $n + 1$. Hence any violation of reflection invariance implies a violation of rotation invariance in a slightly higher-dimensional space.

Definition 5. Let $\mathbf{S} = \langle S, d \rangle$ be a metric space.

(i) A map R: S $\rightarrow$ S is a *rotation* on **S** if

(a) for some $x \in$ S, $Rx = x$, and

(b) for all $x, y \in$ S, $d(x, y) = d(Rx, Ry)$.[19]

(ii) A transformation T on S is involutory if for all $x \in$ S, $TTx = x$.

So reflections in particular count as involutory rotations. Now the principles to be violated:

(ARI) *Absolute Rotation Invariance*. For any rotation R on a Euclidean metric

[18] We will use 'orientation' in the sense of direction or rotational position, rather than handedness. Orientations in this sense are what rotations alter.

[19] Like our definition of translation, this is not necessarily standard, but it is simple and adequate for our purposes.

space $\langle S, d\rangle$ and any $A \subseteq S$, $A \approx RA$.

(RRI) *Relational Rotation Invariance*. For any rotation R on a Euclidean metric space $\langle S, d\rangle$ and any point sets $A, B \subseteq S$, $A \leq B$ if and only if $RA \leq RB$.

ARI simply says that rotations preserve size, and RRI says they preserve size *relations*.

At first glance, RRI seems weaker than ARI. After all, even if a rotation changes set sizes, couldn't it always change them in the same way, and at least preserve size *relations*? But if we assume PW and totality (or weaker, that any two sets related by an involutory rotation are comparable), then RRI becomes the strongest of all the invariance principles considered here.

Remark 4. Let $\langle S, d\rangle$ be a Euclidean metric space and $\leq$ a total Euclidean size ordering on $\mathcal{P}(S)$. Then RRI implies ARI, ATI, and RTI.

Proof. Assume RRI and let R be an involutory rotation. (These exist by the isometry of $\langle S, d\rangle$ with a Euclidean space.) Then for any $A \subseteq S$, $RRA = A$. By RRI, $A \leq RA$ if and only if $RA \leq RRA = A$. By totality, one of these must hold, so both do, and $A \approx RA$. Hence RRI implies that involutory rotations preserve size. But every rotation or translation in a Euclidean space is just a composition of two reflections, which are involutory rotations by the above definition. So RRI implies ARI and ATI, and ATI implies RTI.[20]

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[20] If we exclude reflections from RRI and let it apply only to handedness-preserving rotations, we can still obtain RRI $\Rightarrow$ ATI $\Rightarrow$ RTI and ARI $\Rightarrow$ ATI $\Rightarrow$ RTI, since 180-degree rotations are also involutory, and every translation is a composition of two 180-degree rotations. However, not all *rotations* are compositions of 180-degree handedness-preserving rotations, so ARI doesn't follow from such a weakened RRI.

We already saw (Remark 1) that all Euclidean size orderings on a Euclidean metric space violate ATI, so by Remark 4, they also violate rotation invariance—both absolute and relational. For a simple example, consider the sets **N**, (−1)**N**, and (−1)**N** – 2 = {−2, −3, −4,…}. By PW, (−1)**N** – 2 **<** (−1)**N**, but both are rotations of **N** (and even if we exclude reflections, they are still rotations of **N** in the complex plane). So rotations don't always preserve size. And since **N** cannot be equal in size to both (−1)**N** and (−1)**N** – 2 **<** (−1)**N**, either **N** **<** (−1)**N** or (−1)**N** – 2 **<** **N**, assuming totality. In either case we have a set A (= **N** or (−1)**N** – 2) and a rotation *R* such that A **<** *R*A but not *R*A **<** *RR*A, contradicting RRI. So rotations can't always preserve *relative* sizes either, given PW and totality (or comparability of involutory rotation images like A and *R*A).

Now, it's pretty obvious that rigid translations don't preserve Euclidean sizes if we consider unbounded infinite sets like P and *T*P above, or half-intervals like (0, ∞) and (1, ∞), half-planes, and so on. It's also known that finitely additive measures (such as numerosities) violate rigid transformation invariance for *bounded* sets, in virtue of the very sophisticated Banach-Tarski paradox. But it is worth noting that Euclidean sizes in general, *whether additive or not*, must violate rotation invariance even for very *simple* bounded sets.

Consider an example in the Cartesian plane $\mathbf{R}^2$ with standard polar coordinates. The idea is to wrap point sets like our P = {p, *T*p, *TT*p,…} onto a circle in such a way that the arc length between consecutive points is constant and incommensurable with the circumference. Here's how:

Definition 6. For any $\theta \in \mathbf{R}$, let

$$C_\theta = \{(1, \theta + n):\ n \in \mathbf{N}\},$$

where $(1, \theta + n)$ is a polar coordinate pair with $\theta + n$ given in radians relative to the x-axis. (See Figure 4.)

So C_θ is a countably infinite set of points on the unit circle, and the arc length between $(1, \theta + n)$ and $(1, \theta + n + 1)$ is 1, which is incommensurable with the circumference 2π. Now take for example $C_0 = \{(1, 0), (1, 1), (1, 2),\ldots\}$. If we rotate this set one radian counter-clockwise, we obtain the set $C_1 = \{(1, 1), (1, 2), (1, 3),\ldots\}$. Thus $C_1 \subset C_0$, so by PW, C_1 must be smaller. By rotating C_0, say, two radians clockwise, we get $C_{-2} = \{(1, -2), (1, -1), (1, 0),\ldots\}$, which is larger.[21] So rotations in $\mathbf{R}^2$ don't preserve absolute sizes, and by Remark 4, they don't preserve relative sizes.

This generalizes easily to Euclidean metric spaces:

Definition 7. Given a metric space $\langle S, d\rangle$, $A \subseteq S$ is *bounded* if there is some $n \in \mathbf{N}$ such that for all $x, y \in A$, $d(x, y) < n$.

Remark 5. Let $\langle S, d\rangle$ be a Euclidean metric space of dimension at least two, and $\trianglelefteq$ a total Euclidean size ordering on the bounded subsets of S. Then $\trianglelefteq$ violates ARI and RRI.

Proof. Let f: $\mathbf{R}^2 \to S$ be an isometry. Then $f[C_0]$ and $f[C_1]$ are bounded and $f[C_1]$ is a rotation of $f[C_0]$, by isometry. Also, $f[C_1] \subset f[C_0]$, so by PW, $f[C_1] \vartriangleleft f[C_0]$. Thus $\trianglelefteq$ violates ARI, and by Remark 4, it also violates RRI.

[21] This example is closely related to a puzzle once presented to me by Frank Arntzenius at a restaurant. The puzzle was roughly this: Find a set of points on a sphere such that one can obtain a proper subset just by rotating the sphere. Notice the example also shows that Common Notions 4 and 5 of the *Elements* are incompatible even for bounded figures. See footnote 11.

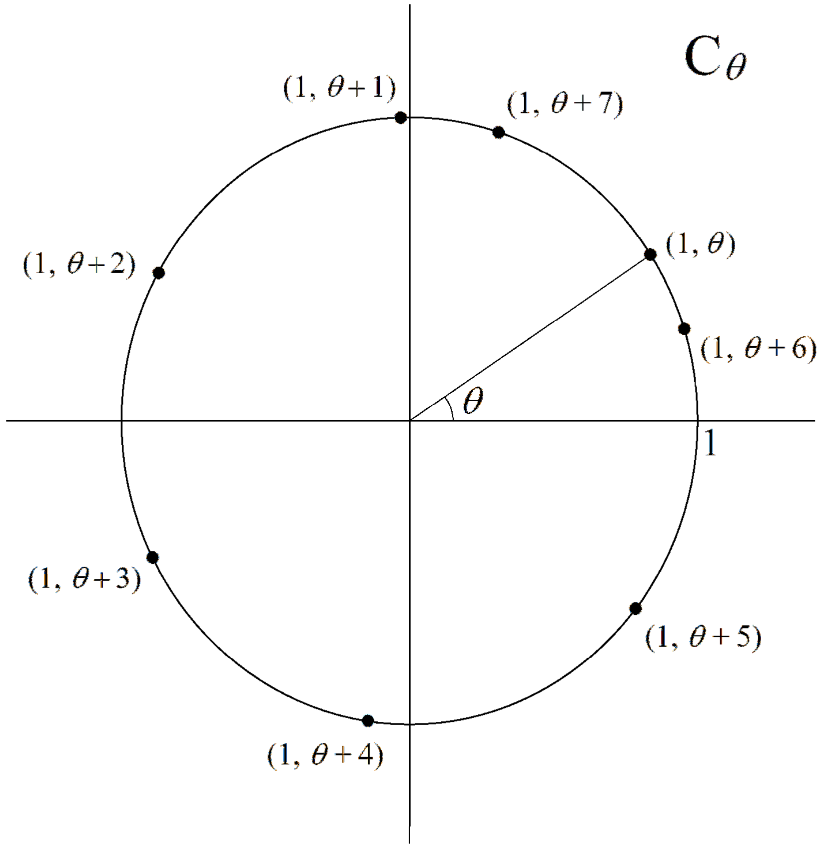

Figure 4. The first eight points in $C_\theta = \{(1, \theta + n)\colon\ n \in \mathbf{N}\}$ for an arbitrary angle θ. Note $C_{\theta+m} \subset C_\theta \subset C_{\theta-m}$ for $m \in \mathbf{N}$, and the angles $\theta \pm m$ are dense in the circle.

Note that if we exclude reflections and restrict RRI to the handedness-preserving rotations within a given space, then RRI is consistent with PW in two-dimensional spaces, but not three. In any case, such a weakened RRI is still not consistent with PW and Discreteness, even for bounded sets. Given PW and Discreteness, the size relation between C_0 and $C_{1/2} = \{(1, 1/2), (1, 3/2), (1, 5/2),\ldots\}$ cannot be the same as that between $C_{1/2}$ and C_1.

So given PW—even without Discreteness or additivity—bounded sets that are mere rotations of each other differ in size, and as well the size relation between two sets depends on their particular orientations, even while their orientations to each other remain fixed. This is again deceptive, for attributing different sizes to two sets suggests a substantive difference, while under PW it may merely reflect a difference in direction, and in contexts where there is no privileged direction or coordinate system, this means that both absolute and relative set sizes depend on an arbitrary choice of coordinate axes.

But Euclidean sizes not only vary with rotation, they are *radically sensitive* to it: Bounded sets that differ only by an arbitrarily small rotation nonetheless differ in size by arbitrarily large finite quantities.

Remark 6. For any $n \in \mathbf{N}$ and $\theta \in [0, 2\pi)$ there are angles ρ and σ arbitrarily close to θ such that $C_\rho \subset C_\theta \subset C_\sigma$ and $|C_\theta \setminus C_\rho|, |C_\sigma \setminus C_\theta| > n$.

Proof. The angles $\theta + m$ for $m \in \mathbf{N}$ are dense in the circle, so choose $m > n$ such that $\theta + m$ is arbitrarily close to θ. Then $|C_\theta \setminus C_{\theta+m}| = |\{(1, 0), (1, 1),..., (1, m-1)\}| = m > n$. Likewise choose $k > n$ such that $\theta - k$ is arbitrarily close to θ. Then $|C_{\theta-k} \setminus C_\theta| = k > n$.

So there are angles ρ and σ arbitrarily close to θ such that C_θ contains all of C_ρ as well as an arbitrarily large finite number of other points, and C_σ contains C_θ and an arbitrarily large finite number of other points. These are simple facts of geometry and set theory independent of PW or any other postulates about infinite sizes (since $C_{\theta+i} \setminus C_{\theta+j}$ is finite). But under PW, it makes sense to say that $C_{\theta-k}$ is *larger* than C_θ "by k elements," since $C_\theta < C_{\theta-1} < C_{\theta-2} < \ldots < C_{\theta-k}$. So in this sense, arbitrarily small rotations effect arbitrarily large finite changes in size.[22]

Perhaps this is unsurprising. After all, continuous, unbounded sets are even more sensitive to small rigid transformations. The set-difference between the intervals (x, ∞) and $(x + \varepsilon, \infty)$, for arbitrarily small ε, is *uncountably* infinite. So an arbitrarily small translation can induce an uncountably infinite change in size. Nonetheless, the example of the sets C_θ is striking. There we have *bounded, countable* sets whose Euclidean sizes vary *both up and down* with tiny rotations *in the same direction*. Turn C_0 a hair clockwise and it becomes hundred points larger. Turn it another hair clockwise and it becomes a million points smaller.

This just amplifies the fundamental arbitrariness implied by the failure of absolute rotation invariance. Not only do Euclidean sizes depend on an arbitrary choice of coordinate axes, but they depend very sensitively on it, so that the tiniest difference in orientation can make a vast difference in size. Similarity and proximity aren't even good guides to *approximate* size, if PW holds. And since Euclidean sizes are so sensitive to rotations in $\mathbf{R}^2$, they are similarly sensitive in any Euclidean metric space of dimension two or greater. So even in Euclidean spaces where there is no preferred direction or designated coordinate system, Euclidean set sizes depend so sensitively on direction that indiscernible differences of angle imply size differences larger than any finite number one can comprehend. In this respect, Euclidean size is a misleading and impractical way of classifying sets. However we might measure or count a point set like C_θ, we could never even *approximate* its size (beyond determining its power; see footnote 15).

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[22] For BDF's numerosities, which have a rich algebra, we can say more concretely that for any $\theta \in \mathbf{R}$ and $n \in \mathbf{N}$, there are arbitrarily small rotations R such that $[RC_\theta] - [C_\theta] > n$.

§7. Conventionalist pluralism. The central problem with Euclidean sizes is arbitrariness. Where disjoint infinite sets are concerned, there is no apparent motivation for assigning one size relation rather than another, and to choose one often involves privileging an arbitrary position or direction. One might take a pluralistic view of this situation, and respond as follows: The "size" of a set is naturally going to depend on what one means by 'size'. There are multiple theories available, including Cantor's theory and various Euclidean theories, which differ in the sizes and size relations they assign to particular sets. So yes, says the pluralist, set sizes are indeterminate until one chooses a sufficiently strong concept or theory of size, but that is no surprise. We have to define size, at east implicitly, before it has a definite meaning, and we can define it in vastly many different ways, even assuming PW. This is no mark against PW but an advantage; it means we have many Euclidean measures at our disposal to use for different purposes.

It should be clear from the introduction that I am sympathetic to this sort of pluralism in general, but there are problems with the view just stated. Firstly, we don't really have a Euclidean *theory* or *concept* of set size that determines the sizes of the simple sets discussed above. BDF have shown that Euclidean *assignments* of size exist for large classes of sets (assuming the existence of selective ultrafilters), but we have drawn a distinction between assignments and theories (§2). An assignment is any function or relation, however complex, non-computable, or undefinable, while a theory must be somewhat comprehensible and communicable, at least if it is to be useful.

Likewise, a *concept* must be something we can at least partly grasp if the word is to support its psychological applications. But it is doubtful that a satisfyingly strong and broad Euclidean size assignment can be determined by any tractable theory or intelligible concept. All known arguments for the existence of broad Euclidean assignments are non-constructive, and this suggests that there might not be any recursively enumerable theory that determines a particular Euclidean size assignment. And if not, then Euclidean size assignments are not as easy to grasp or work with as what we usually call theories or concepts. In that case, we're not just facing an arbitrary choice between different *notions* of size; there isn't anything as useful as a Euclidean *notion* of size that determines the sizes of sets like those discussed above.

Secondly (against the above-mooted pluralist position), we have seen good reasons to believe that any that a Euclidean size assignment that resolves our examples will incorporate unmotivated details. Such assignments are misleading, for while some sizes will reflect logical facts or consequences of established theories (like ZFC), others will be purely arbitrary, so that it becomes difficult to tell which results have significance and which don't. Having a variety of conceptual tools at one's disposal may be a benefit, but if each of these tools mixes meaningful content with arbitrary stipulations in a way that's difficult to extricate, that's detrimental. A random pile of arrows isn't a *theory* in any useful sense, because it's not a theory *of* anything of interest, and it's not comprehensible or useful.

So the pluralist view is correct in that a theory of size need not be a theory of *the one true* notion of size and its properties, but a useful theory is one that conveys factual or logical insights of some kind, and preferably doesn't mix these with free, unmotivated, and inexpressibly complex choices.

§8. Partial size assignments. Since Euclidean set sizes are not in general determined by well motivated principles (nor perhaps by comprehensible principles), one solution might be to adopt a *partial* Euclidean size assignment as a theory of size. EVEN, for example, might be regarded as neither smaller than ODD, nor larger, *nor* equal, on such an approach. Consider the theory consisting of just PW and BDF's Half-Cantor Principle, that $|A| < |B|$ implies $A \prec B$ (where again $|\cdot|$ is Cantorian power; BDF 2006, 2007), but without the assumption of totality. On this theory, sets of larger power are larger, and proper subsets are smaller, but sets of the same power such that neither is contained in the other have no definite size relation. Wouldn't this be better than Galileo's solution (1939), according to which *no* infinite sets have relative sizes? And wouldn't it be in some ways better than Cantor's, which fails to discriminate at all between sets of the same power, even when one contains the other?

One objection to this proposal is that it adds nothing to the familiar concepts of power and proper subset. In fact it obscures those concepts a little. To say that A is smaller than B, in this theory, is just to say that, either A has smaller power than B or A is a proper subset of B (or both). Wouldn't it be better to distinguish these cases, just as we already do in our standard language of set theory? Uniting the relations of proper subset and smaller power in one relation muddies things. But if we bring in further principles, such as Discreteness, RTI, or algebraic properties like BDF's, the union may not be so fruitless. No well motivated Euclidean theory will determine all set sizes or relative sizes, but if we open the scope of investigation to *partial* measures and *partial* size relations, perhaps some Euclidean assignment (or several) will prove to be both comprehensible and enlightening in applications.

But even partial Euclidean assignments suffer from arbitrariness. Whatever further properties we may add, PW alone implies that size is not preserved by rigid transformations, even for bounded point sets. The relative sizes of C_0 and C_1, for example, are determined by PW: $C_0 \succ C_1$. But this means that the sizes of structurally identical sets depend on their positions in space, which is clearly arbitrary, especially in contexts where there the elements are qualitatively identical and there is no privileged origin or direction. Further, size must vary radically under arbitrarily small rotations, regardless of totality, for the relations between arbitrarily near sets C_θ and $C_{\theta+i}$ are determined by PW alone. Size *relations* between sets must also vary with rotations, and if size relations satisfy Discreteness (as any notion of *number of elements* should), they must also vary with mere translations. All of this holds by PW alone, without assuming totality, so long as PW applies to the particular sets in question.

But a related objection must yet be answered. The standard Cantorian theory of set size is also indeterminate, in virtue of the independence of the Continuum Hypothesis from ZFC, and other such independence results. ZFC does not even tell us whether the continuum is the same size as the second infinite cardinal, so how are Euclidean theories any worse?[23] This seems to suggest that we must be content with partial size assignments or at least incomplete theories, whether Euclidean or Cantorian.

I do not wish our discussion to degenerate into a contest between Cantorian and

[23] Thanks to Paolo Mancosu and Leon Horsten for raising this question.

Euclidean theories; the limitations of Euclidean theories are what they are regardless of how Cantor's theory fares. But there is a difference worth noting between the independence of the Continuum Hypothesis and the arbitrariness of Euclidean sizes. The Continuum Hypothesis reduces to a statement of set theory that doesn't involve any notion of size. It says that the continuum has a one-to-one correspondence with a subset of each set that doesn't have a one-to-one correspondence with the integers. ZFC does not tell us whether *this* statement, in the language of ZFC, is true. But the Cantorian *size relation* between the continuum and sets of the second infinite power is determined by the truth or falsity of this proposition. The indeterminacy of Cantorian sizes lies not in the Cantorian notion of size itself, but in the incompleteness of the underlying size-free theory of sets. Not so for Euclidean sizes. If every fact expressible in the language of ZFC were determined, this would *still* not fix the Euclidean sizes of sets, for we have no rule for assigning sizes to sets based on their set-theoretic properties. As we have seen above, neither bijections nor even isometries can fix Euclidean sizes. To put it another way, we at least know what power consists in: One-to-one correspondence. But what constitutes Euclidean size is only partly specified; it is a mixture of common sense (PW) and chimera.

§9. Conclusion. It would be unfortunate if the observations made here had any negative impact on research programmes like BDF's. Such work has shown us that Euclidean size assignments are logically consistent, and Euclidean conceptions of size need not be mere *mis*conceptions. As well it has revealed the subtle conditions under which such assignments exist, and their connections with other deep and interesting topics. And perhaps there is some hope for *partial* Euclidean size assignments that are broadly useful and enlightening.

But the problems of arbitrariness that plague Euclidean theories are severe. Our examples show that any Euclidean theory or assignment that is strong and broad enough to relate even simple, countable sets like those discussed above will be arbitrary and misleading. Furthermore, this is no mere technicality engendered by particular devices such as the Axiom of Choice or selective ultrafilters. It is a clear, fundamental constraint on any application of PW to infinite sets, including sets of whole numbers and even bounded, countable point sets. Euclidean theories and assignments go a long way as illustrations of mathematical freedom and of interesting connections within mathematics, but as theories of size in themselves they are deeply and disappointingly limited. The problem is not that Euclidean theories are false. It is that they are either very weak and narrow or arbitrary and misleading.

Acknowledgements: I would like to thank Sylvia Wenmackers, Paolo Mancosu, Leon Horsten, Tim Button, Foad Dizadji-Bahmani, and Jeremy Gwiazda for helpful comments on earlier drafts, and the University of Bristol for a Visiting Fellowship that facilitated the research for this paper.

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