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Fishbones, Wheels, Eyes, and Butterflies: Heuristic Structural Reasoning in the Search for Solutions to the Navier-Stokes Equations

Lydia Patton

Abstract Arguments for the effectiveness, and even the indispensability, of mathematics in scientific explanation rely on the claim that mathematics is an effective or even a necessary component in successful scientific predictions and explanations. Well-known accounts of successful mathematical explanation in physical science appeals to scientists' ability to solve equations directly in key domains. But there are spectacular physical theories, including general relativity and fluid dynamics, in which the equations of the theory cannot be solved directly in target domains, and yet scientists and mathematicians do effective work there (McLarty 2023, Elder 2023). Building on extant accounts of structural scientific explanation (Bokulich 2011, Leng 2021), I argue that philosophical accounts of the role of equations in scientific explanation need not rely on scientists' ability to solve equations independently of their understanding of the empirical or experimental context. For instance, the process of formulating solutions to equations can involve significant appeal to information about experimental contexts (Curiel 2010) or of physically similar systems (Sterrett 2023). Working from a close analysis of work in fluid mechanics by Martin Bazant and Keith Moffatt (2005), I propose an account of heuristic structural explanation in mathematics (Einstein 1921, Pincock 2021), which explains how physical explanations can be constructed even in domains where basic equations cannot be solved directly.

Key words: explanation, equations, fluid dynamics, Navier-Stokes, simulation, physics

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1.1 The Argument from Successful Theories

Contemporary structuralist and structural realist accounts analyze the stability of the formal relations within a theory, arguing from the persistence and success of an entity or relation ('structure') in scientific theories to the objectivity, or even the reality, of those elements. According to Hilary Putnam, if Newton simply thought up the law of gravity, and that law, with its consequences, deep embedding in other relations, and extraordinary fruitfulness sprang into existence at Newton's whim, that would be a miracle (Putnam 1975, 73). Instead, the best explanation of the success of the law is that Newton caught on to something, and the fruitfulness and depth of the relation he caught on to is evidence that his theory consists of statements about the world with "objective content" (73). Putnam went on to argue that certain formal relations and elements of mathematics are not only stable across theory change, not only fruitful and deep as a theory develops over time, but also are indispensable to the results of physical theories.[1] Resnik makes a pragmatic indispensability argument on this basis for the existence of mathematical objects and the truth of mathematical statements:

1. In stating its laws and conducting its derivations science assumes the existence of many mathematical objects and the truth of much mathematics.
2. These assumptions are indispensable to the pursuit of science; moreover, many of the important conclusions drawn from and within science could not be drawn without taking mathematical claims to be true.
3. We are justified in drawing conclusions from and within science only if we are justified in taking the mathematics used in science to be true. (Resnik 1995, 169–70)

One can turn such arguments around and apply them to the physical theories in which mathematical reasoning is applied. According to "methodological" arguments for scientific realism, those elements essential to successful scientific theories warrant realist commitment. Methodological realism about the mathematics employed in scientific theories is closely related to indispensability, and works roughly as follows:

1. Scientists work with formal posits (e.g., equations, variables and relations), and with mathematical reasoning in general, in proving results.
2. Some such formal relations persist, and are indispensable to the success of our best scientific theories.

[1] Putnam's original 'no miracles' argument was made in the context of mathematics, not physics. The indispensability argument is sometimes associated with Quine as well. Colyvan (Colyvan 2015, §1) emphasizes that Quine and Putnam took this position to be a matter of intellectual honesty. We are committed to the existence of quarks, electrons, neutrinos, and so on. In the case of abstract entities or relations, this commitment may only be because they are employed in successful scientific theories. But, in that case, we must also be committed to the existence of the real number line, as well.

3. In such cases (2), it is warranted to be committed to the reality of the entities and relations that are indispensable to the success of our theories.

The key argument here is from the success of a theory to realism. To run an indispensability argument requires determining which elements of a theory are required to make the results of the theory come out true.

The 'Divide et Impera' ('divide and conquer' or DEI) strategy is a version of methodological realism.[2] DEI argues from the necessary employment of a structure or posit in a theory to its objectivity or reality.[3] Structural realism is a version of the DEI strategy (Psillos 2018, §2.7).[4] Mathematical structure is necessary to derive the results of a theory, and that structure persists over theory change.

All these accounts rely on a set of assertions which add up to a proposed argument, which can be called "The Leading Argument".

The Leading Argument

1. Theory change is a Darwinian process in which the most successful theories win out over the less fruitful or explanatory ones.
2. Theories are successful if they are able to generate more predictions or explanations in the target domain than rivals. [5]
3. Theories that "latch on" to objective relations or to real elements will be more successful than rivals that do not. [6]
   a. The elements that are truly necessary to prove the results of the theory should be isolated from the 'idle wheels' that do not contribute to the theory's success. (Methodological realism)
   b. The mathematical structure of a theory is likely to be deployed in successful theories, to be clarified and refined as theories develop, and to be an effective instrument for constructing scientific explanations. [7]
4. The best explanation for the persistence of structures or elements over theory change is that the entities, processes, or structures to which the theory refers are objective or real. (The no miracles argument)
5. Mathematical structure persists over theory change.

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[2] See Kitcher 1993, Psillos 1999, Leplin 1986. See Lyons 2006, Cordero 2011, Patton 2015 for responses.

[3] "The underlying thought," as Psillos notes, "is that the empirical successes of a theory do not indiscriminably support all theoretical claims of the theory, but rather the empirical support is differentially distributed among the various claims of the theory according to the contribution they make to the generation of the successes" (Psillos 2018, §2.5).

[4] "In opposition to scientific realism, structural realism restricts the cognitive content of scientific theories to their mathematical structure together with their empirical consequences. But, in opposition to instrumentalism, structural realism suggests that the mathematical structure of a theory represents the structure of the world (real relations between things)... structural realism contends that there is continuity in theory-change, but this continuity is. . . at the level of mathematical structure" (§2.7).

[5] A main theme of Kitcher's *The Advancement of Science* (Kitcher 1993).

[6] Worrall (Worrall 1989) famously argues for this claim; a version of this argument can be found in Peirce and the pragmatist tradition.

[7] The indispensability or 'unreasonable effectiveness' argument (the latter due to Eugene Wigner). A classic assertion of structural realism, but found in other approaches as well.

Conclusion Mathematical structure that persists over theory change is more likely to be objective or to refer to reality.

A number of challenges have been posed to this leading argument.[8] W. v. O. Quine's 'web of belief' view denies that mathematical or logical claims are more epistemically fundamental than empirical assertions based on observation. Instead, Quine argues, abstract mathematical or logical statements simply become more central to our web of belief: more statements depend on them than vice versa.[9] The centrality of mathematical and logical claims within our conceptual schemes makes us less inclined to give them up, and we might be less likely to question them than we are to question other beliefs. But that does not make mathematical or logical claims more epistemically certain or more likely to be true than claims based on observation. When a theory makes a false prediction, ultimately the only evidence about how to revise the theory comes from "our own direct observations". [10]

One might question whether the employment of abstract entities and mathematics in physical reasoning provides them with a warrant of objectivity or reality at all. Recent accounts emphasize that models can be deliberately counterfactual, or even fictional, and can still serve their desired epistemic roles. Alisa Bokulich (Bokulich 2017) emphasizes that idealized, even counterfactual models can support successful scientific reasoning, which may challenge the argument from the use of models in successful theories to a warrant of those models' truth or reference to reality. [11]

Both these challenges are articulated against the background assumption that successful theories or models are successful for epistemic reasons. That assumption itself can be challenged. In analyzing episodes such as the Chem-

[8] Laudan's well-known argument against convergent realism has been covered extensively, so I won't discuss it in detail here.

[9] "Mathematics and logic, central as they are to the conceptual scheme, tend to be accorded [...] immunity, in view of our conservative preference for revisions which disturb the system least; and herein, perhaps, lies the 'necessity' which the laws of mathematics and logic are felt to enjoy" (Quine 1950, xiii). Thus, while Quine is often associated with the 'indispensability' argument, he did not argue that indispensability made mathematics more likely to be necessary or epistemically certain beyond the possibility of revision.

[10] When faced with a theory that is newly in conflict with observation, "Toward settling just which beliefs to give up we consider what beliefs had mainly underlain the false prediction, and what further beliefs had underlain these, and so on [...] We stop such probing of evidence, as was remarked, when we are satisfied. Some of us are more easily satisfied than others. [...] But there is a limit: when we get down to our own direct observation, there is nowhere deeper to look. [...] the ultimate evidence that our whole system of beliefs has to answer up to consists strictly of our own direct observations" (Quine and Ullian 1978, 13).

[11] To be sure, accounts that untether models from their targets have deep questions to answer. As Knuuttila notes, "Any account approaching scientific models as fictions faces at least the two following challenges. First, if scientific models are considered as fictions rather than representations of real-world target systems, how are scientists supposed to gain knowledge by constructing and using them? And, second, how should the ontological status of fictional models be understood?" (Knuuttila 2021, 5078).

ical Revolution, Hasok Chang notes that claims that one theory is more epistemically successful than another may be based on epistemic grounds, or on something "cruder": "an unreflective triumphalism that celebrates the winning side in an episode, whichever it may happen to be" (Chang 2009, 240). Chang argues that, from a contemporary perspective, Lavoisier's theory "was just as wrong as the phlogiston theory in its advanced versions" (240). It makes little sense to argue that Lavoisier's theory was more successful because it was truer or more epistemically justified: from a modern perspective, that is not the case.

We will focus on the question: What counts as success for a scientific theory? The leading argument relies on the empirical claim that mathematics must be deployed in successful theories and explains their success. According to this argument, mathematical and logical structures are necessary to the success of physical theories in generating predictions and explanations. When we focus on the use of differential equations, a problem leaps out at once: differential equations are not directly solvable in many domains of interest.[12] Given that fact, how are we to evaluate claims that the mathematics in question is 'unreasonably effective'[13] in those domains?

1.2 Are Theories Based on Differential Equations Successful?

The leading argument relies on a history of successful scientific theories in which mathematical structures are indispensable, that is, are necessary to deriving predictions and explanations within a theory.[14] Premises 2 and 3b of the leading argument rest on a claim about the development of science: that theories are successful if they predict and explain, and that mathematical structure is more likely to be deployed in successful theories. What happens, then, if precisely the problem that faces scientists is the inability to solve the equations, the inadequacy of current theory to make predictions and support explanations in the domain of interest on the basis of solved equations?

In that case, indispensability arguments lose much of their force for these theories. If we cannot derive the desired results directly from the equations,

[12] My thinking on this issue has deep roots in long conversations with Erik Curiel about this topic. To learn more about Curiel's view, I warmly recommend Curiel, 2010 (preprint), Curiel, 2016 (preprint). After giving this paper as a talk in Indiana, discussions with Colin McLarty and Susan Sterrett have been indispensable to developing these views further. Their own contributions are found in McLarty 2023, Sterrett 2023.

[13] To use Wigner's phrase.

[14] Structuralist and methodological realist accounts may appeal to posits about entities as well, which is not necessary for some versions of structural realism. For instance, Ladyman, Ross, and John Collier 2007 defends a version of ontic structural realism without (necessary) commitment to scientific entities.

which are central to a theory's mathematical structure, then how can we argue that structural mathematical reasoning is effective, much less necessary or indispensable? How can we compare theories to each other on the basis of which results are provable in which, which is necessary to decide which is more successful?

No problem, we might assert confidently. The leading argument runs for successful theories. But theories in which direct solutions to a theory's equations are available only in a limited domain surely are unsuccessful theories. Thus, there is no challenge to the usual arguments, and they may go on as before.

There is bad news for this response. General relativity is one of the most successful theories on the contemporary scene. But the field equations of general relativity are non-linear, and there is no general formula for their solution in a vast range of cases. Einstein's field equations, which are PDEs, determine how the spacetime metric tensor changes with respect to changes in massenergy.[15] Even when the equations are linearized, they cannot be solved directly in some of the most significant scientific contexts, famously including the inspiral and merger stages of a binary black hole ringdown. [16].

The Navier-Stokes equations[17] are partial differential equations describing the behavior of a fluid.[18] Contemporary applications describe the behavior of a fluid by solving the equations to find a velocity vector field. This results, usually, in a set of nonlinear PDEs, which are notoriously difficult to solve. John Wheeler has remarked on a kinship in this sense between the field equations of GR and the Navier-Stokes equations:

> An objection one hears raised against the general theory of relativity is that the equations are non-linear, and hence too difficult to correspond to reality. I would like to apply that argument to hydrodynamics - rivers cannot possibly flow in North America because the hydrodynamical equations are non-linear and hence much too difficult to correspond to nature! (Wheeler 2011, n.p.)

Both are sets of differential equations with deep grounding in physical laws.[19] Both include non-linear terms, and thus do not have straightforward solutions in the physical domains of interest: the strong field regime for GR, and most physical fluids for the Navier-Stokes equations.

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[15] Differential equations relate functions to their rates of change. Partial differential equations (PDEs) do so for multiple variables. The field equations of GR and the Navier-Stokes equations are both sets of partial differential equations.

[16] See Jamee Elder's paper for this volume (Elder 2023)

[17] Some refer to the Navier-Stokes "equation", and some to plural equations. It depends on how they are formulated.

[18] For a detailed presentation of the equations, see McLarty 2023. For the "dispute over the viability of various theories of relativistic, dissipative fluids" see Curiel, 2010 (preprint).

[19] The Navier-Stokes equations are effectively formulations of conservation of mass and Newton's second law in the fluid domain. McLarty's paper for this volume (McLarty 2023) makes this point lucid.

Like the field equations, the original equations of fluid dynamics are not directly solvable in most physical contexts of interest. Wheeler's point was that, nonetheless, the Navier-Stokes equations are used daily in practical applications.[20] The field equations are used in practice, just as the Navier-Stokes equations are. But both sets of equations lack direct solutions in many applied contexts of interest. Creative techniques including linearization; partial, constrained, or idealized solutions; and simulation are used to mediate between the equations and the natural phenomena being studied.

Summing up: There are two widely accepted, empirically well founded theories, fluid dynamics and general relativity, which rest on systems of partial differential equations that are not directly solvable in most domains of interest. These theories enjoy considerable experimental support and are able to generate effective explanations and predictions in their target regimes. We cannot account for their effectiveness by appealing to the ability to generate predictions and explanations directly from solutions to the equations in the primary domains of interest: the theories are not 'successful' in this sense. Thus, we need to explain how scientists and mathematicians can make progress with physical explanations in cases where reasoning based on the relevant equations alone is neither successful[21] nor complete.

We might marshal the no miracles argument to argue that the empirical success of these theories is best explained by the fact that they 'latch on' to real or objective relations in nature. Or we could formulate an empiricist account, explaining the theories' success by the contact they make with observable phenomena. Both of these accounts sound reasonable, but they float above the rough practical terrain they're supposed to explain. To really navigate this rocky terrain, we'd need to explain: How do theories of this kind make contact with the target phenomena?

One philosophical explanation of how theories reach out to objects emphasizes the role of models, conceiving them as neither the target phenomenon, nor part of the theory proper.[22] Models are constructed deliberately to build

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[20] See Darrigol (Darrigol 2002, 95): "The Navier-Stokes equation is now regarded as the universal basis of fluid mechanics, no matter how complex and unpredictable the behavior of its solutions may be. It is also known to be the only hydrodynamic equation that is compatible with the isotropy and linearity of the stress-strain relation. Yet the early life of this equation was as fleeting as the foam on a wave crest. Navier's original proof of 1822 was not influential, and the equation was rediscovered or re-derived at least four times, by Cauchy in 1823, by Poisson in 1829, by Saint-Venant in 1837, and by Stokes in 1845. ... All of these investigators wished to fill the gap they perceived between the rational fluid mechanics inherited from d'Alembert, Euler, and Lagrange, and the actual behavior of fluids in hydraulic or aerodynamic processes."

[21] In a very specific sense of 'successful': independently generating explanations and predictions.

[22] Some accounts allow for models to be objects, but I am focusing on those that see models as distinct: "what makes something a model explanation is that the explanans in question makes essential reference to a scientific model, and that scientific model (as I believe is the case with all models) involves a certain degree of idealization and/or fictionalization" (Bokulich 2011, 38).

connections between theories and phenomena according to this account: the "models as mediators" view (Morrison 1999, Morgan and Morrison 1999, Knuuttila 2005).[23] For models to work as mediators does not require that they be taken to be true or objective representations. Niels Bohr worked from the lines of the emission spectra of substances to construct a model involving discrete orbits of electrons around the nucleus. The development of the Bohr model is closely related to his formulation of an equation that correctly describes the relation between energy levels and spectral lines for the hydrogen atom (Bokulich 2011, §4). As Bokulich notes, the Bohr model is not an accurate depiction of the atom itself, but it allowed for the derivation of correct predictions via the formulation of equations. Bohr was able to relate observational data (emission lines and atomic spectral analysis) with theory (equations and predictions) by means of his atomic model.

Bokulich (2011) has proposed an account of 'structural model explanation', building on work by Morrison (1999) and Woodward (2003). Structural model explanations capture causal relations within target systems: a model "explains the explanandum by showing how the elements of the model correctly capture the pattern of counterfactual dependence of the target system. More precisely, in order for a model M to explain a given phenomenon P, we require that the counterfactual structure of M be isomorphic in the relevant respects to the counterfactual structure of P".[24] Once such relations of structural isomorphism are demonstrated to be valid, there is a further 'justificatory step', "specifying what the domain of applicability of the model is, and showing that the phenomenon in the real world to be explained falls within that domain" (39).

An idealized or even counterfactual model can be used to mediate between theory and target system, as long as (1) an isomorphism can be proven to exist between the relations of the model and the relations of the target system, and (2) it can be shown that the target system falls within the valid domain of application the model. Leng (Leng 2021) observes that there is no reason structural model explanations of this kind could not apply to mathematical explanations within physical theories. Leng presents an account of "mathematical explanations as structural explanations", in which mathematical explanations "can be presented as deductively valid arguments whose premises include a mathematical theorem expressed modal structurally, together with

[23] This account tacitly accepts a distinction between theory and observation, since otherwise there is nothing to 'mediate' (no gap to close). This distinction does not need to be as strong as that defended by some members of the Vienna Circle. It need only be made in practice.

[24] Bokulich 2011, 39. A precedent to this account can be found in Heinrich Hertz's Bild theory; see Eisenthal (Eisenthal 2018.

empirical claims establishing that the conditions for the mathematical theorem are instantiated in the physical system under consideration" (10417).[25]

Leng's extension of the structural model explanation account to mathematical explanations is plausible. After all, Bokulich's account deliberately allows for highly idealized and even counterfactual explanations, as long as the required isomorphism between model and target system is achieved. But there are a few puzzling aspects of this extension prima facie. First, if mathematical explanations can be structural model explanations, then this pushes against the view that models are not part of a background theory. Many structural mathematical explanations are based on equations that are counted as part of a theory.

The 'models as mediators' view involve at least prima facie commitment to a gap between theories (which are idealized and abstract) and target systems (observable material things and their relations). If there is no gap, at least in practice, then there is nothing between which to mediate. Equations can, themselves, be regarded as models.[26] It may seem, then, as if we don't need anything beyond equations to generate mathematical structural explanations in key cases, like the basic equations of many physical theories.

Another prominent account of models thus has it that they simply are equations, or, better, that some models are abstract systems that can ultimately be reduced to equations. For instance, according to this view the ideal gas law is not just an equation, but also a model from which can be derived predictions concerning the behavior of possible physical systems. More broadly, as Morrison notes, "We frequently refer to abstract formal systems as models that provide a way of setting a theory in a mathematical framework; gauge theory, for example, is a model for elementary particle physics insofar as it provides a mathematical structure for describing the interactions specified by the electroweak theory" (Morrison 2005, 145).

A system of partial differential equations could serve as a model in this sense. However, that would require being able to formulate relevant problems about the behavior of a system, and find solutions to those problems, in the domain of interest. Those solutions should be the source of structural descriptions that allow us to learn about the target systems.

1.3 Working to Formulate and Solve an Equation

An equation can provide "a mathematical structure for describing" the physical interactions in question, and thus can generate structural descriptions and explanations (145). A structural mathematical explanation does not necessar-

[25] Importantly, Leng adds, "these explanations, which can be understood in modal structural terms, involve no commitment to mathematical objects platonistically construed" (Leng 2021, 10417).

[26] For discussion see Sterrett (Sterrett 2023.

ily have to 'mediate' between theory and observation, then. A mathematical structure itself can be shown to be isomorphic to the relations of interest in the target system, and can be shown to be valid when applied to that system, thus satisfying Bokulich's criteria for structural explanation.[27]

It is particularly tempting to apply such an analysis to systems of partial differential equations. PDEs provide exactly the kind of causal, counterfactual explanation of the behavior of a system that grounds structural explanations. Systems of PDEs describe how one variable increases while another decreases, under precisely which conditions a system will remain at equilibrium, how a target system evolves over time, and so on.

Should we then conclude that the equations are the source of the relevant counterfactual, causal information, and of the resulting structural explanations? In that case, we wouldn't need a model separate from the equations. It is quite possible. But we need to think about why and how that might be. And, in particular, we need to consider what it takes to formulate, and then to solve, an equation or system of equations.[28]

In the case of the Navier-Stokes equations (and the field equations of GR), to reiterate: direct solutions are not available in many contexts of interest, because the original form of the equations is nonlinear. Nonlinear terms complicate the basic method for the solution of differential equations: finding the existence and uniqueness of solutions in a domain. Under certain conditions, given background constraints, the fluid or field equations can be reduced to linear PDEs. Many researchers use linearized forms of the equations to find solutions in a restricted domain.

Working with weak, regularized, or linearized forms of the equations can be a tool for learning about the potential for solutions in the strong or nonlinear cases. Jean Leray "was the first to study the Navier-Stokes equations in the context of weak solutions" (Ozanski and Pooley 2018, 114). Leray studied the equations on the "whole space R^3".[29] Leray's work allowed for the study of the equations from the perspective of weak solutions. By working with these solutions, he was able to derive "lower bounds on various norms of the strong solutions", and to "indicate the rate of blow-up" of strong solutions, among other results (115). Leray's strategy was to work with a regularized form of the equations, derive weak solutions, and arrive at conclusions about the form and limits of the strong solutions. Weak solutions become instruments for learning about the strong solutions.[30]

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[27] If the structural explanation is not in terms of a model distinct from theory, it does not meet Bokulich's criteria for a structural model explanation.

[28] This is a central theme of Susan Sterrett's contribution to this volume (Sterrett 2023.

[29] He studied the "regularized form" of the equations, "which are obtained by replacing the nonlinear term $(u \cdot \nabla)u$ by $J_\epsilon(u \cdot \nabla)u$ (where J_ϵ is the standard mollification operator)" (Ozanski and Pooley 2018, 115). A mollification operator is a smooth function applied to a nonsmooth one, which has the result of 'regularizing' or smoothing the function.

[30] McLarty 2023, §2.3 - §2.5 discusses weak solutions to Navier-Stokes.

Another strategy for learning about possible solutions to the Navier-Stokes equations is to generate solutions using simulations. Linear PDEs with boundary conditions, however complex, can be solved with a computer to yield physical solutions that predict the behavior of fluids (or of the metric, in GR). These solutions often require stipulating hypothetical values for key parameters, or generally making assumptions that allow for simulation of the target systems.

Using creative engineering, a simulation of a relevant class of phenomena might be provided.[31] The equations might then be shown to apply to the simulated phenomena.[32] If the right kinds of mediated connections between theory, simulation, and physical context could be found, then we can still argue that the equations are driving the structural explanations provided. One way to establish such a mediated connection would be to provide exact solutions to the equations for simulated situations.

1.3.1 Finding Exact Solutions to the Navier-Stokes Equations

This section examines a set of exact solutions to the Navier-Stokes equations that generate broader structural descriptions. Martin Bazant (MIT) and Keith Moffatt (Cambridge) have provided solutions that "describe steady vortex structures with two-dimensional symmetry in an infinite fluid" (Bazant and Moffatt 2005, 55). They describe "two classes of exact solutions of the Navier-Stokes equations":

1. The first is a class of similarity solutions obtained by conformal mapping of the Burgers vortex sheet to produce wavy sheets, stars, flowers and other vorticity patterns. (Bazant and Moffatt 2005, 55)
2. The second is a class of non-similarity solutions obtained by continuation and mapping of the classical solution to steady advection–diffusion around a finite circular absorber in a two-dimensional potential flow, resulting in more complicated vortex structures that we describe as avenues, fishbones, wheels, eyes and butterflies. These solutions exhibit a transition from 'clouds' to 'wakes' of vorticity in the transverse flow with increasing Reynolds number. (Bazant and Moffatt 2005, 55)

The similarity solutions are stable solutions, while the non-similarity solutions model systems that evolve beyond equilibrium with an increasing Reynolds number. The second class of solutions is particularly interesting

[31] It is important to consider how engineering factors into heuristic reasoning. Sterrett has done significant work on reasoning about physically similar systems (Sterrett 2017b), analogue models (Sterrett 2017a), and dimensional analysis (Sterrett 2009).

[32] The LIGO detection of gravitational waves used simulations to excellent effect. See Elder 2023).

for the insights it promises into the idea of vorticity (the rotation of the fluid).[33] Turbulence is characterized by increasing vorticity, and increasing turbulence results in the evolution of systems into nonlinear fluid systems. Thus, studying turbulence is central to understanding the nonlinearity of some fluid motion.

Bazant and Moffatt analyze vorticity beginning with a vortex that maintains a steady rotation. A steady "Burgers vortex sheet" is a vortex in which the diffusion of the vortex, its spreading outward and thus lessening, is counterbalanced exactly by convection (inward rotation) and stretching of the vortex.[34] In a Burgers vortex, diffusion and convection are balanced so that the vorticity of the fluid remains stable.[35] The formal relationship Burgers identified, emphasized here, turned out to be the basis of a number of exact solutions to the Navier-Stokes equations. These solutions in turn allowed for analysis of certain kinds of turbulence in terms of vortex shapes.[36]

Bazant and Moffatt first obtain a class of similarity solutions via conformal mapping (Figure 2). This mapping technique involves preserving local angles, but generating distinct solutions or maps.[37] They begin with a contour map of the steady Burgers vortex sheet (Bazant and Moffatt 2005, fig. 1, 56). The initial class of solutions can be transformed back to that contour map. This is a class of distinct physical solutions, "self-similar" solutions generated by Möbius transformations. The solutions do not depend in any complex way on the Reynolds number, which is the ratio of inertial to viscous forces in the fluid. In fact, that dependence can be removed entirely for these solutions (59).

Using these techniques, it is also possible to obtain a set of solutions based on a dynamic process. These solutions involve generating "avenues" through a fluid by solving the "canonical problem of steady advection-diffusion around an absorbing circular cylinder in a uniform background flow of constant velocity and constant concentration" (59). In other words, a cylinder that can

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[33] 'Vorticity' is often evaluated formally as the curl of the vector velocity field of the fluid.

[34] In a car wash, water is thrown outward so that it diffuses entirely. But when you pull the plug in a sink, the water is constrained by the shape of the sink and by the force that causes it to move inward. This is referred to as convection or advection (the latter if a material is transported by the fluid). A diagram is provided in (Jumars et al. 2009, 1).

[35] "In 1938 Taylor also recognised the fact that the competition between stretching and viscous diffusion of vorticity must be the mechanism controlling the dissipation of energy in turbulence. A decade later Burgers obtained exact solutions describing steady vortex tubes and layers in locally uniform straining flow where the two effects are in balance. Burgers introduced this vortex as 'a mathematical model illustrating the theory of turbulence', and he noted particularly that the vortex had the property that the rate of viscous dissipation per unit length of vortex was independent of viscosity in the limit of vanishing viscosity (i.e. high Reynolds number). The discovery of the exact solutions stimulated the development of the models of the dissipative scales of turbulence as random collections of vortex tubes and/or sheets" (Tryggeson 2007, 14).

[36] Usually cylinders, because Burgers gave his solution in cylindrical coordinates.

[37] See Sterrett 2023 for detailed analysis.

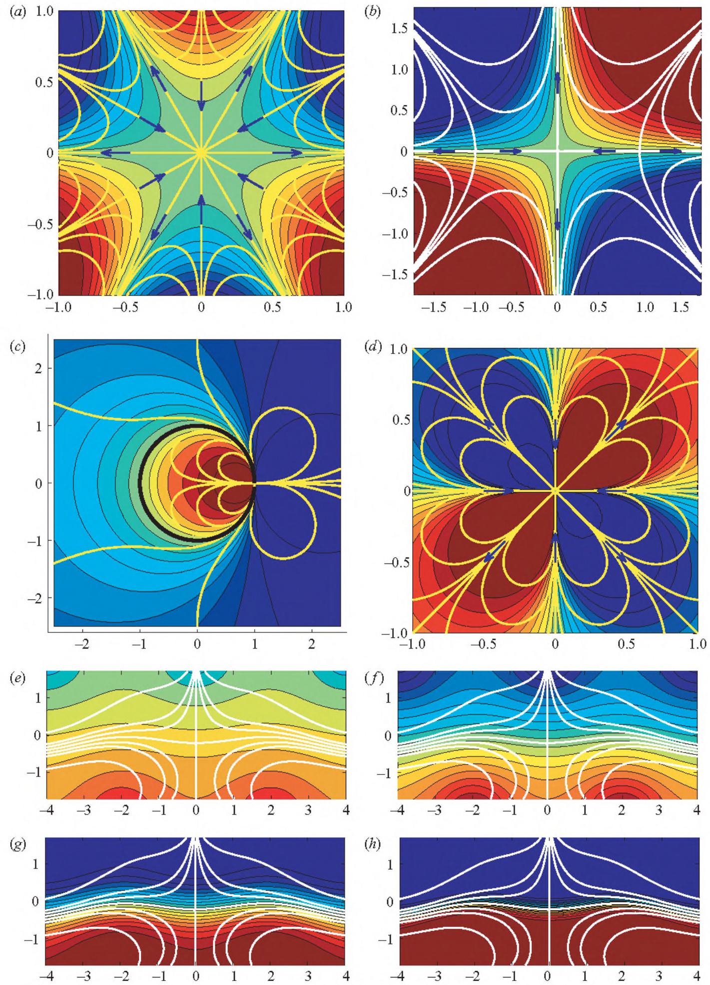

FIGURE 2. Solutions obtained by conformal mapping, $w = f(z)$, to Burgers' vortex sheet: (*a*) a vortex star, $f(z) = z^3$; (*b*) a vortex cross with stagnation points on its arms, $f(z) = 1 - z^2$; (*c*) a circular vortex sheet, $f(z) = (1+z)/\mathrm{i}(1-z)$ (where the black circle indicates the separatrix of the transverse flow); (*d*) a vortex flower, $f(z) = z^{-2}$; and (*e–h*) wavy vortex sheets, non-uniformly strained by dipole singularities, $f(z) = z + \sum_{j=1}^{3}(z - z_j)^{-1}$, $z_1 = 2.5\mathrm{i}$, $z_2 = 2 - 2.5\mathrm{i}$, $z_3 = -2 - 2.5\mathrm{i}$, for (*e*) $Re = 0.01$, (*f*) $Re = 0.1$, (*g*) $Re = 1$, and (*h*) $Re = 10$.

take on water is dragged through a uniformly flowing and uniformly dense background fluid. The cross-section of the results is displayed in Figure 3.

60 *M. Z. Bazant and H. K. Moffatt*

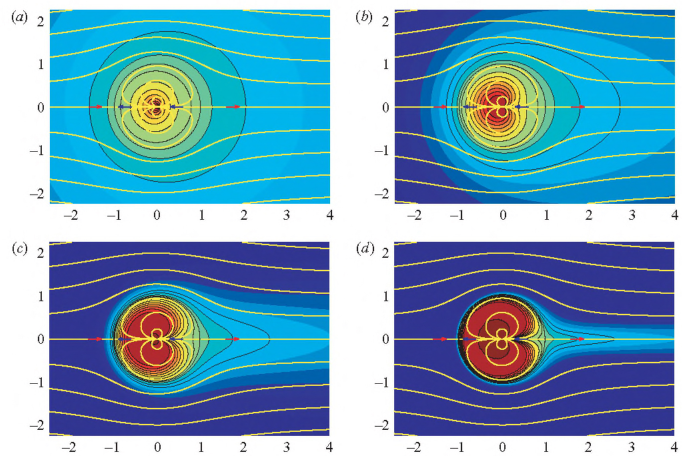

FIGURE 3. Steady vortex avenues confined by transverse flow with a dipole source inside and a uniform background flow outside (a dipole at ∞). These transverse jets exhibit a non-trivial dependence on the Reynolds number with a transition from 'clouds' to 'wakes' of vorticity: (*a*) $Re=0.1$, (*b*) $Re=1$, (*c*) $Re=10$, (*d*) $Re=100$.

Once these avenues have been generated, conformal mapping allows for the generation of non-self-similar but exact solutions to the Navier-Stokes equations (Figure 4).

These are a set of vortex structures or 'avenues', described as 'fishbones, wheels, eyes, and butterflies', that constitute dynamical solutions to the equations in simulated situations. Bazant and Moffatt thus obtain a different set of solutions, generated by applying conformal mapping to vortex avenues. Vortex avenues have a "non-trivial dependence" on the Reynolds number, which is a dimensionless parameter that depends on the viscosity of the fluid. The second set of solutions cannot be mapped directly back on to the Burgers vortex sheet. They are built from simulations of vortex avenues that are complicated by increasing vorticity (possibly, turbulence), which is not a symmetrical evolution of the fluid system.

The authors conclude,

> We have presented two classes of exact solutions of the Navier–Stokes equations representing steady vortex structures with two-dimensional symmetry, confined by transverse potential flows. These solutions provide mathematical insights into the Navier–Stokes equations and physical insights into ways that vorticity may be confined. They also provide stringent tests for the accuracy of numerical simulations. (Bazant and Moffatt 2005, 63)

62 *M. Z. Bazant and H. K. Moffatt*

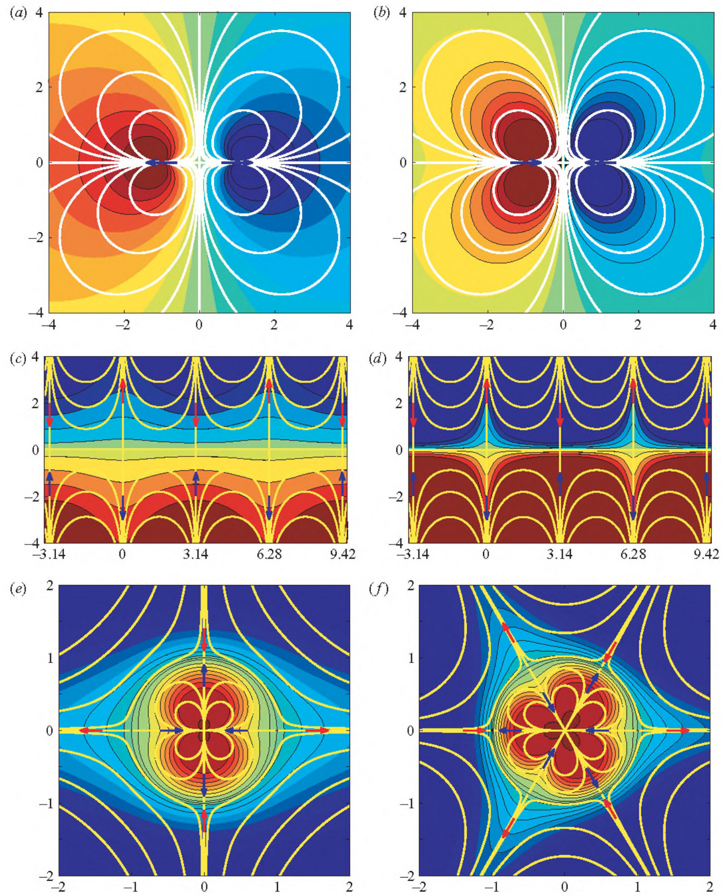

FIGURE 4. Solutions obtained by conformal mappings, $w = f(z)$, to the vortex avenues in figure 3: (*a*) vortex eyes, $f(z) = (1+z)/(1-z)$, a pair of avenues stabilized by transverse diverging dipoles at $Re = 1$; (*b*) a vortex butterfly, $f(z) = -(1+z)/(1-z)$, stabilized by converging dipoles at $Re = 10$; vortex fishbones, $f(z) = \mathrm{i}\log z$, at (*c*) $Re = 0.1$ and (*d*) $Re = 10$; and vortex wheels, (*e*) $f(z) = z^2$ and (*f*) $f(z) = z^3$, at $Re = 1$.

From the perspective of the account of mathematical structural explanation discussed above, these results are intriguing. The solutions derived by Bazant and Moffatt construct "steady vortex structures" that allow for descriptions of the strucural properties of fluid systems and their evolution over time, and insights into the mathematics involved. Moreover, the vortex avenues generated by Bazant's and Moffatt's methods are based on simulated systems. The solutions they provide are thus valid for those systems, which is how it can be proven that the solutions apply to physical phenomena.

The solutions would apply to any physical systems with sufficient similarity to the simulated ones they employ. Still, Bazant and Moffatt warn that the set of physical situations to which these exact solutions will be applicable is very thin: "Our solutions are likely to be difficult to observe in the laboratory because they involve carefully placed singularities and/or stagnation points in the in-plane potential flow field, which must be achieved while also setting the appropriate out-of-plane shearing flow, which provides the vorticity" (Bazant and Moffatt 2005, 63).

The fluid flows in the second set of solutions depend on jets of water that are flowing in different directions (arrows mark this). The avenues generated involve carefully placing hypothetical jets of water in ways that stabilize turbulence of the flow. Placing the jets and stabilizing the turbulence will be much more difficult in reality than in a simulated situation.

The generation of exact solutions to the equations in these simulated situations validates the solutions for physical fluids, however hypothetical. This accomplishes one of Bokulich's criteria for structural explanation, that the structures generated must be shown to apply to the physical domain of interest. In fact, Bazant's and Moffatt's solutions meet all of Bokulich's requirements for structural explanations. They allow for the articulation of fluid systems as structures (vortex avenues) that can be shown to be isomorphic to physical target systems, and they can be shown to validly apply to such systems. While the solutions are illustrated with diagrams above for clarity, the diagrams represent quantitative solutions to the Navier-Stokes equations using complex analysis and conformal mapping.

Do these solutions to the Navier-Stokes equations constitute structural explanations derived directly from the mathematics? The structural explanations in question are not generated by the equations alone. The solutions are achievable only in simulated situations. The simulations make solutions to the equations possible in idealized physical contexts. The solutions are thus based on empirical *understanding* of the physical situations that are likely to arise, and of the phenomena (like vorticity) that are likely to complicate the solutions in applications.

It is important not to generalize too widely from this example. Most work in fluid dynamics on the basis of the Navier-Stokes equations does not consist of giving exact solutions to the equations: in fact, such work is rare. It is more common to see work building on approximations (e.g. linear or Galerkin approximations), or on the Leray-Hopf (weak general) solutions.

But Bazant's and Moffatt's work is philosophically interesting precisely for this reason. Much of the technical work done in fluid dynamics does not touch on the question of whether the Navier-Stokes equations can be the basis of structural explanations. Instead, much current work uses approximation and creative modeling techniques to solve problems in the context of the equations, but somewhat independently of them. By "independent of the equations", I mean that the work done assumes the equations as a mathematical and physical framework, but does not put the equations themselves into question. Bazant's and Moffatt's work thus has heuristic potential: it allows us to investigate how the equations themselves can be the source of structural explanations, and what the scope and limits of those explanations can be.[38]

1.4 Heuristic Reasoning: What's In an Equation?

In this concluding section, I want to return to the question of what makes a 'successful' theory. The development of fluid dynamics, from the earlier work of D'Alembert, Euler, Lagrange, the Bernoullis, and Helmholtz to the contributions of contemporary researchers, provides every reason to conclude that fluid dynamics is a successful theory.[39] The formal background in analytical fluid mechanics was provided early on. Its extension to modeling was provided by the partial differential equation formulation by Navier and Stokes.[40] The contemporary theory of fluids is used in multiple contexts to achieve practical and theoretical results.

The theory of fluid dynamics is not 'successful' in one traditional sense: it does not support the derivation of predictions and explanations in relevant domains directly from the equations of the theory.[41] But that traditional sense of 'success' is much too limited. It focuses on one method: deriving predictions, representations, and results from basic equations in isolation. A more accurate picture of the use of equations would also take into account another process: how formulations of and solutions to equations build on scientists' understanding of empirical and simulated situations to generate structural explanations.

I would call this a 'heuristic' approach to scientific explanation.[42] The term 'heuristic' is sometimes used in a narrow sense, e.g., in decision theory, to describe mental shortcuts for solving problems. That is very far from the sense I want to identify. Instead, I want to use the sense of 'heuristic' that Einstein used when he called the principle of covariance a 'heuristic aid': structural epistemic reasoning that informs the search for scientific explanations and

[38] More work in fluid dynamics could be analyzed using this framework, of course.

[39] As Olivier Darrigol notes, the origins of the Navier-Stokes equations were in the attempt to mediate between fluid mechanics and actual fluid behavior: "All of these investigators wished to fill the gap they perceived between the rational fluid mechanics inherited from ďAlembert, Euler, and Lagrange, and the actual behavior of fluids in hydraulic or aerodynamic processes" (Darrigol 2002, 95).

[40] The particular techniques used here owe a debt to Burgers, who identified a number of formal relationships that do not depend on the Reynolds number (indexed to the viscosity of the fluid), which allowed for leaps forward in the modeling of fluids.

[41] This is, of course, the sense of scientific explanation corresponding to the Deductive-Nomological Model.

[42] There are heuristic accounts of scientific models which are clearly related, but modeling is not the focus here.

the assessment of whether the phenomena are in agreement with a theory, especially in novel experimental contexts.[43]

Elaborating this stronger sense of heuristic reasoning allows for a better understanding of how equations function in the development of scientific explanations over time. Given the account of structural explanation elaborated above, oe might conclude that there are two ways to account for the role of equations. We might argue that equations cannot generate structural explanations on their own: that independent models are needed to mediate between abstract 'background' theories and physical observable phenomena. On the other hand, we might argue that abstract equations on their own yield explanations in physical domains of interest directly, without any mediation needed.

The third account that I want to articulate is a heuristic account of structural explanation.[44] Solutions to equations can yield structural explanations that have epistemic, heuristic value even before they have been experimentally tested. These solutions need not be derived from the equations in isolation, however. Rather, they involve (and support) significant understanding of the empirical, experimental, and mathematical situation in which the equations were formulated. That understanding may be mobilized through simulation of the relevant phenomena - for instance, by the use of hypothetical general cases. In some situations, the necessary framework for solutions is achieved by models used as mediators. But in other situations, the techniques may not involve models, and here we need a broader account.

First, the movement between data (e.g., measurements) and the formulation of the relevant problem is far from trivial. In the context of differential equations, the problem, as Curiel has noted, is how to characterize "the tran-

[43] "The law of transmission of light, the acceptance of which is justified by our actual knowledge, played an important part in this process of thought. Once in possession of the Lorentz transformation, however, we can combine this with the Principle of Relativity, and sum up the theory [. . .] in brief: General laws of nature are co-variant with respect to Lorentz transformations. This is a definite mathematical condition that the theory of relativity demands of a natural law, and in virtue of this, the theory becomes a valuable heuristic aid in the search for general laws of nature. If a general law of nature were to be found which did not satisfy this condition, then at least one of the fundamental assumptions of the theory would have been disproved" (Einstein 1921, 50–51). An earlier, quite different treatment of the heuristic role of mathematics in relativity, as well as how Einstein's heuristic reasoning developed over time, is found in section 3 of **zahar1980**. Recently, Pincock has traced the derivation of Poiseuille's law and explicitly analyzes "how an experimental discovery can prompt the search for a theoretical explanation and also how obtaining such an explanation can provide heuristic benefits for further experimental discoveries" (Pincock 2021, 11667). Pincock's paper appears in a topical collection edited by Sorin Bangu, Emiliano Ippoliti, and Marianna Antonutti, "Explanatory and Heuristic Power of Mathematics" (Bangu, Ippoliti, and Antonutti 2021). An earlier collection focuses on heuristic reasoning in particular (Ippoliti 2015).

[44] Existing accounts of heuristic explanation include Einstein 1921, Eisenthal 2018, Ippoliti 2015, and Obradović and Ninković 2009. Pincock 2021 deserves particular mention, since Pincock here develops a heuristic interpretation of the derivation of Poiseuille's law in fluid mechanics.

sitions to and fro between, on the one hand, inaccurate and finitely determined measurements, and, on the other, the mathematically rigorous initial-value formulation of a system of partial-differential equations" (Curiel, 2010 (preprint), §1). Once an initial value problem has been formulated and initial and boundary conditions found, it is more straightforward to find a solution. [45]

Heuristic solutions like Bazant and Moffatt's can play the role of epistemic lighthouses. They define, not just classes of models associated with a theory, but broader depictions of how the theory could be instantiated: how the equations of the theory could be solved in well classified physical situations. Even if the physical solutions thus achieved are unlikely - e.g., scientists do not expect to encounter the exact physical situations in nature - they provide standards or benchmarks that give working scientists pointers about how to proceed and how to evaluate the results they do achieve.[46] A key heuristic value of simulated solutions is to set standards for actual physical solutions. Bazant's and Moffatt's results provide exactly such a check, for a broad domain of physically meaningful situations.[47]

The heuristic account of structural explanation provides a nice explanation of how scientific theories can be successful, and yet the equations of those theories not be solvable in target domains of the theory. The equations can be given simulated, idealized, or limited solutions that are the basis of structural explanations in those domains. Structural isomorphisms between the simulated or idealized solutions to the equations and the real-world systems in question yield structural explanations that work well in practice.

It is key to note, though, that the process of finding solutions to equations in specific physical contexts - even if those contexts are simulated - is not reducible to deriving predictions based on the abstract equations alone. Finding a solution to an equation in a particular context may involve substantial empirical and experimental reasoning, as well as creative mathematical reasoning, as Sterrett (Sterrett 2023) argues convincingly. The point is made well in the concluding section of Erik Curiel's 2010 preprint on this subject:

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[45] Solving the initial value problem alone, however, does not prove a solution applies everywhere in a domain. In fact, that solution process does the opposite: it involves precisely determining how to find a solution to the equation in a specific physical situation. For instance, a solution or set of solutions may split up the domain into simulated cases, as Bazant and Moffatt do in the paper examined above. Simulations have multiple roles, including providing methods for data assimilation and ways to relate measurements to equations (Parker 2017).

[46] As Sterrett observes, simulations can extend reasoning to physically similar or analogue systems (Sterrett 2017b, Sterrett 2017a).

[47] Bazant and Moffatt's solutions can even be regarded as epistemic 'artifacts', and thus are potentially consistent with the artifactual account of scientific modeling articulated in Knuuttila Knuuttila 2021. However, Knuuttila emphasizes the independence of artifactual models from 'real-world systems' (S5079). The heuristic account emphasizes the isomorphisms between structural explanations and real-world systems.

> one also needs all the collateral knowledge, both theoretical and practical, not contained in the equation, in order to apply the equation to the modeling and comprehension of all the phenomena putatively treated by the theory. To put the matter more vividly: the equation as a result of a (profound) investigation of the physical phenomena at issue, of all the empirical data and attempts to model that data heretofore, a teasing apart and characterization of the maximally common structure underlying the system of relations that obtains among them (supposing there is such a thing)—the sort of investigation that Newton and Maxwell and Einstein did and Hilbert did not accomplish—the equation as a result of that is the theory. (Curiel, 2010 (preprint), §7)

Equations are the result of substantial structural reasoning, built up over time and involving empirical and experimental investigation. Abstract equations can yield predictions and explanations, even when they can't be directly solved in a specific domain. But it is crucial, in such cases, to scrutinize what it takes to find simulated, limited, or idealized solutions.[48] That scrutiny may reveal limitations to the scope of the equations. The creative process of finding solutions in such cases can provide substantial heuristic information about the value of the equations and the strength of the theory.

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