Arithmetic, Logicism, and Frege's Definitions[1]

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Abstract: This paper describes both an exegetical puzzle that lies at the heart of Frege's writings—how to reconcile his logicism with his definitions and claims about his definitions—and two interpretations that try to resolve that puzzle, what I call the "explicative interpretation" and the "analysis interpretation." This paper defends the explicative interpretation primarily by criticizing the most careful and sophisticated defenses of the analysis interpretation, those given my Michael Dummett and Patricia Blanchette. Specifically, I argue that Frege's text either are inconsistent with the analysis interpretation or do not support it. I also defend the explicative interpretation from the recent charge that it cannot make sense of Frege's logicism. While I do not provide the explicative interpretation's full solution to the puzzle, I show that its main competitor is seriously problematic.

Key Words: Frege; Logicism; Definitions; Analysis Interpretation; Explicative Interpretation

> "It is really a scandal that science is still in the dark about the nature of number. It would not be so bad if there were no generally recognized definition of number, provided that there were at least agreement in substance. But science has not even decided whether a number is a group of things or a figure drawn with chalk on a blackboard by a human hand, whether it is something mental so that it is up to psychology to inform us of its origin or whether it is a logical construction, whether it is created and may eventually perish or whether it is eternal. Is this not a scandal? …[Science] does not know what thought content is associated with its theorems or what subject matter it deals with: it is completely in the dark about its own nature. Is this not a scandal? And is it not a scandal that a string of thoughtless utterances can succeed in making the claim to be representing the latest state of the science?" –Gottlob Frege (1899: III).

I. An Exegetical Puzzle

Frege's writings present his readers with an exegetical puzzle that lies at the heart of his work. On one hand, Frege is clearly a logicist. He claims that the "ultimate ground" that justifies our assertions of arithmetical truths was logic (see, *inter alia*, Frege (1882: 163-4; 1884: §14, §17, §87; 1885: 94-6; 1893: VII, 1)). He attempted to prove this by providing gapless proofs of

[1] I cite Frege's published work by original publication date followed by original page numbers/sections. I cite unpublished work by supposed composition date followed by page numbers in translations. I use the unitalicized 'Begriffsschrift' to refer to Frege's formal language(s) and the italicized '*Begriffsschrift*' to refer to his book.

arithmetical truths, using his Begriffsschrift script,[2] using only laws of logic and definitions. On the other hand, the definitions that Frege provides for well-known mathematical terms—both the informal definitions in the *Foundations* (1884) and formal ones in *Basic Laws* (1893, 1903)[3]—do not seem to be about well-known mathematical objects, but involve concepts and their extensions. Further, Frege's own view in his later "Logic in Mathematics" (1914) was that definitions are *merely* stipulations, having only psychological value.

The puzzle is that these two features seem to be in tension. After all, if his definitions aren't, as they seem not to be, about familiar mathematical objects but merely stipulations, how can Frege claim, by providing his proofs, to be providing the ultimate grounds for the truths of *arithmetic* and not some other, new set of truths? But if his definitions aren't mere stipulations but attempts to "get things right," why are they so different from what one might expect? And why would he write that definitions are mere stipulations?

This puzzle is well-known among Frege scholars. There are two competing interpretations that try to resolve it. According to the "explicative interpretation," chiefly defended by Joan Weiner (1990, 2004a, 2007, 2010), Frege's definitions were not attempts to capture the sense or reference of the arithmetical terms of his time. (I'll refer to these terms as the "pre-proof" terms.) Rather, they were stipulations to be used to construct a systematic science that would allow him to retain the "well-known properties" (Frege (1884: §70)) of numbers, while simultaneously removing various pitfalls and problems—gestured to in the epigraph—of mathematic work at that time. On this interpretation, crudely put, Frege was *explicating* the mathematical work at the time; his definitions were stipulations to help achieve that explication.

By contrast, according to the "analysis interpretation," chiefly defended by Michael Dummett (see his (1991a, b, 1981) and Patricia Blanchette (see her (1994, 2012)),[4] Frege's definitions were attempts to either capture "as close as possible" (Dummett (1991a: 34; 1991b: 152)) the sense of pre-proof arithmetical terms or reduce those pre-proof terms into their simpler components (Blanchette (1994; 2012: 23)). In other words, Frege's definitions were *analyses* that were to help establish that the pre-proof arithmetical claims of that time were ultimately grounded in logical laws. Frege's later comments on definitions reveal either a change of heart or the belief that definitions can serve multiple purposes (e.g. both stipulations and analyses).[5]

I defend the explicative interpretation. Whereas other works defend it by providing exegetical support from Frege's writings for it, this paper takes aim at the analysis interpretation directly. Specifically, I argue that Frege's writings do not support the analysis interpretation. In sections II-IV, I criticize the two most promising ways the analysis interpretation has been developed. First, I criticize Dummett's view that Frege's conception of definitions changed over time by showing, in section II, that it did not. In section III, I show *contra* Dummett, Benacerraf and others, that Frege's comments about "fruitful definitions" in the *Foundations* do not show that this conception did shift. In section IV, I criticize Blanchette's interpretation, on which

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[2] The Begriffsschrift undergoes some changes that won't matter for purposes here.

[3] There are some technical differences between the definitions in the *Foundations* and *Basic Laws*, but that won't matter for purposes here.

[4] See also Shieh (2008: 1008). Other authors have either (i) endorsed this interpretation (see, e.g., Jeshion (2001: 964 fn. 49), Nelson (2008: 161)) or (ii) would likely endorse this interpretation, given their views about Frege's definitions as not mere stipulations and otherwise being the "correct" or "right" definitions (e.g., Kneale and Kneale (1962: :461), Noonan (2001: 117-8), Reck (2007: 36)).

[5] Benacerraf straddles these interpretative camps. He holds that Frege's definitions were not meant to capture the sense or reference of pre-proof arithmetical terms (1981: 28-9), but they were also not mere stipulations (1981: 28) because they should be "fruitful." I discuss the issue of "fruitfulness" in section III.

Frege's definitions were stipulations used to prove analyses of arithmetical truths. I argue, first, that her interpretation does not sit well with Frege's logicism, and second that the passages she cites do not favor her interpretation over the explicative one. Finally, in section V, I respond to Blanchette's charge that the explicative interpretation does not itself sit well with Frege's logicism.

II. Did Frege's Conception of Definitions Change Over Time?

The defender of the analysis interpretation might try to resolve our exegetical puzzle by holding that Frege's conception of definition changed, specifically, that the definitions Frege provides earlier in his career were not meant to conform to the stipulative conception provided in "Logic in Mathematics." Dummett adopts this position, holding that the definitions Frege provided in the *Foundations* were not stipulative, but what Frege later calls "analytic definitions." (1991a: 21; cf. 1991b: 176, 179).[6] If Dummett is correct that Frege's conception of definition changed, and the earlier *Foundations* conception was not stipulative, then he's provided a way of resolving our exegetical puzzle. But Dummett is wrong; Frege's conception of definitions did not change.

Before arguing that, it will be useful to describe the conception of definitions in "Logic in Mathematics" more fully. In "Logic in Mathematics," Frege writes repeatedly that definitions are stipulative (1914: 208, 211, 230, 240, 244). To define a sign, one introduces a "simple" or shorter sign and stipulates that the simple sign is to have the same sense as that which one is defining. Consequently, a definition "is really only a tautology and does not add to our knowledge" (1914: 208). Definitions are not "absolutely essential to a system. We could make do with [the original things being defined]" (1914: 208). That is, definitions are eliminable in our proofs; we could using the same premises arrive at the same conclusions without those definitions. The use of definitions is thus "for ease and simplicity of expression" (1914: 208). After all, a simple sign can be easier to use in a proof than a more complex string of signs. However, merely because a definition lacks "logical significance" it does not mean that it lack "psychological significance." For even "if from a logical point of view definitions are at bottom quite inessential, they are nevertheless of great importance for thinking as this actually takes place in human beings" (1914: 209). Frege calls these stipulative definitions "constructive definitions," but says "we prefer to call it a 'definition' *tout court*" (1914: 2010).

Definitions, on this conception, have four key features: definitions are (i) stipulative; (ii) eliminable; (iii) possessing no logical significance, only psychological significance; (iv) introductions of signs that gives those signs a sense. I call this the "stipulative conception of definitions;" when I speak of "stipulative definitions," I have in mind definitions, according to the stipulative conception.[7]

Nevertheless, in "Logic in Mathematics," Frege recognizes that there is another kind of case when one might speak of definitions (1914: 209ff.). Suppose a sign, say 'A', has a long established use. To define A in terms of other signs, we can attempt to provide a logical analysis of A, where we attempt to provide a complex expression, C, that has the same sense as A. If we have an "immediate insight" that A and C has the same sense, then Frege suggests we call 'A =

[6] Frege's discussion of analytic definitions in "Logic in Mathematics" makes use of the sense/reference distinction, which Frege had yet to draw when he wrote the *Foundations*. Thus, a more cautious formulation of Dummett's view may be that Frege's conception of definition changed from a "proto"-analytic conception in the *Foundations* to a stipulative conception in later works. I'll ignore this more cautious formulation, since what matters for this interpretation is that Frege's conception of definition changed.

[7] Frege sometimes describes stipulative definitions as 'constructive' definitions; but I'll only use the former term to avoid confusion.

C' an "analytic definition." But he thinks "it is better to eschew the word 'definition' altogether in this case, because what we should here like to call a definition is really to be regarded as an axiom" (1914: 2010). For axioms are "truths for which no proof can be given in our system and for which no proof is needed" (1914: 205). As analytic definitions are really just axioms, Frege's discussion here does not provide a second, alternative conception of definition.

What if we lack an immediate insight that A = C? We cannot regard 'A = C' as an axiom, since it would be in need of proof. However, we can introduce a new term, N, and stipulate that N have the same sense as the complex expression C. In this case, 'C = N' is a stipulative definition. And "if we have managed in this way to construct a system for mathematics without any need for the sign A," then we can "bypass" the question of whether N or C have the same sense as A, as we only use the stipulative definition (1914: 211).

Once again, Frege's discussion does not furnish a second conception of definition. The definition that results from this process—C = N—is a stipulative definition. And while Frege does not require that the new sign, N, is to have the same sense as the long established sign, A, this is consistent with the stipulative conception because N is not being defined as A *but C*. Thus, the definition that results from this process, C = N, is *not* an analytic definition, that is, an axiom. Frege never uses the term 'analytic definition' to describe such a definition. Indeed, he concludes his discussion of this kind of case by reaffirming the stipulative conception: "We stick then to our original conception: a *definition is an arbitrary stipulation* by which a new sign is introduced to take the place of a complex expression…" (1914: 211). So there is a single conception of definitions in "Logic in Mathematics," the stipulative conception.[8]

Frege uses the stipulative conception throughout his writings. In *Begriffsschrift* (1879), Frege provides an example of a definition and writes, regarding it, that

> It does not say "The right side of the equation has the same content as the left," but "it is to have the same content." …But we can do without the notation introduced by this proposition and hence without the proposition itself as its definition; nothing follows from the proposition that could not also be inferred without it. Our sole purpose in introducing such definitions is to bring about an extrinsic simplification by stipulating an abbreviation. (1879: §24).

In the forward to the first volume of *Basic Laws* (1893), Frege writes that "definitions themselves are not creative, and in my view must not be; they merely introduce abbreviated notations (names), which could be dispensed with were it not for the insurmountable external difficulties that the resulting prolixity would cause" (1893: VI). A decade later, in "On the Foundation of Geometry: First Series," Frege writes that "in mathematics, what is called a definition is usually the stipulation of the meaning of a word or sign. A definition differs from all other mathematical propositions in that it contains a word or sign which hitherto has had no meaning, but which now acquires one through it" (1903b: 319-20) and "…[definitions] are arbitrary stipulations and thus differ from all assertoric propositions. And even if what a definition has stipulated is subsequently expressed as an assertion, still its epistemic value is no

[8] Horty (2007: 34-40) claims that in "Logic in Mathematics" Frege permits a kind of definition— "analytic definitions"—that are not required to be eliminable. Horty's interpretation goes wrong at two places. First, he thinks that Frege permits an analytic definition of A = C where we lack an immediate insight that A = C. But Frege never permits that or even uses that term to describe such a case. Second, Horty infers from the fact that the defining symbol, N, need not have the same sense as the analyzed symbol, A, that the definition of N = C need not meet be eliminable in proof (2007: 38-9). But this inference is invalid, since the eliminable requirement is that the definition N = C could be eliminated from proofs, and that definition could be eliminable in proofs, even if the sense of N is not identical to the sense of A.

greater than that of an example of the law identity a = a. By defining, no knowledge is engendered" (1903a: §66, cf. §93). And in the second volume of *Basic Laws* (1903), Frege writes, "every definition must… contain a simple sign whose reference it stipulates" (1903a: §66, cf. §93).[9]

In addition to explicitly discussing his conception of definitions, Frege also comments on his own usage in comparison to others. In "On the Foundations of Geometry: First Series," he indicates that the stipulative conception of definitions is the one used *in mathematics* generally (1903b: 319-320). This thought is repeated in "On the Foundations of Geometry: Second Series," where he writes that "I believe that my exposition about the use of the words 'axiom' and 'definition' I move within the bounds of traditional usage, and that I may justifiably demand that one not cause confusion by a completely new usage"(1906: 295). Indeed, Frege has even stronger words for those who deviate from the traditional usage without clearly saying what they mean: "Whoever willfully deviates from the traditional sense of a word and does not indicate in what sense he wants to use it… should not be astonished if he causes confusion. And if this occurs deliberately in science, it is a sin against science" (1906: 301).

Frege saw himself as employing the traditional conception of definitions, and held that one should clearly indicate if one intended to use definitions in a non-traditional way. Further, Frege nowhere indicates that he *is* using definitions in some non-traditional way. Given that Frege consistently and repeatedly endorses the stipulative conception of definitions, he must have seen it as the traditional one. At no point do we see a change in his conception of definitions. Thus, since he nowhere indicates that the definitions in the *Foundations* or *Basic Laws* were different from traditional definitions, we have good reason for thinking that Frege's definitions in the *Foundations* and *Basic Laws* were stipulative.[10]

Dummett claims, not only that Frege's *Foundations* definitions weren't stipulative, but more strongly that they were analytic definitions. There is good reason for rejecting that stronger claim. For the only place that Frege contrasts analytic definitions with stipulative definitions is in "Logic in Mathematics," where he describes them as axioms which we recognize as correct by "an immediate insight." But as Dummett recognizes (1991b: 176ff)., Frege did not think his *Foundations* definitions would be recognized as correct by an immediate insight (cf. Frege (1998: §69). So, the only description Frege gives of analytic definitions does not match his description of the definitions in *Foundations*. Thus, it is unlikely he would have regarded his *Foundations* definitions as analytic definitions.

Dummett might try to defend his stronger claim by arguing that after Frege formulated the sense/reference distinction he changed his view about what constituted an adequate analytic definition. Specifically, he might claim that Frege eventually rejected the requirement that an analytic definition should capture the same sense as the defining term, and point to Frege's (1892) review of Husserl's *Philosophy of Arithmetic* in support of this interpretation.[11] More recently, Sanford Shieh has argued that the definitions Frege provides in the *Foundations* are not stipulative, and after Frege formulated the sense/reference distinction, his views change on what he regards as an adequate requirement on a definition ((2008: 1000-1, 1003-4), cf. Nelson (2008: 162 fn. 4)). The key passage both cite in Frege's review reads:

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[9] For brevity concerns, I've omitted other relevant passages; see additionally (1893: §27; 1893: §33; 1903b: 320;1906: 294).

[10] Weiner (1990: 87ff.) also argues for this conclusion.

[11] Dummett (1991a: 23f.) mentions this possible interpretation without clearly endorsing it; see also (1991b: 31-2).

> For the mathematician, it is no more right and no more wrong to define a conic as the line of intersection of a plane with the surface of a circular cone than to define it as a plane curve with an equation of the second degree in parallel coordinates. His choice of one or the other of these expressions or of some other one is guided solely by reasons of convenience and is made irrespective of the fact that *the expressions have neither the same sense nor evoke the same ideas*. (1894: 320, italics mine).

This defense fails. First, let us suppose that Dummett and Shieh are right that this passage shows that Frege did not require for an analytic definition that it preserve sense. Then it does not follow that after formulating the sense/reference distinction Frege's view about the adequacy of an analytic definition changes. For on Dummett's interpretation, Frege's definitions in the *Foundations* were analytic ones, and *they do not (and were not intended to) preserve sense*. So Dummett must be forced to claim that Frege changed his mind about the adequacy of analytic definitions *even after* he formulated the sense/reference distinction. It's unclear what would cause such a change.

Second, there is no reason to read this passage as either speaking of analytic definitions or as articulating a requirement inconsistent with the stipulative conception. Indeed, given the stipulative conception, one would even expect what Frege writes. For Frege's point is not that an expression to-be-defined (in this case 'conic') can have a different sense than the defining expression. Rather, his point is that there is no barrier—no rightness or wrongness—to defining an expression using one complex expression or, instead, to define it using some other complex expression. This is exactly what one would expect on the stipulative conception for defined expression *lack* a sense and are *given* a sense by defining them.[12] What Frege couldn't consistently say, but also what he does *not* say, is that one could define 'conic' twice, with two expressions having different senses. As he writes, "…I require that each sign be defined *just once*" (1897: 366). Indeed, Frege is explicit on the inadmissibility of "redefining" or defining a sign twice using different complex signs in §58 of the second volume of *Basic Laws*, where he gives this example again (1903b: §58). So there is no need to read this passage as rejecting the requirement that a defined expression must have the same sense as the defining one.

Thus far, I've argued directly against Dummett's interpretation by showing that Frege displays, throughout his career, a single conception of definition as stipulative, and what Frege says about analytic definitions provides good reason for rejecting Dummett's claim that the definitions in *Foundations* are analytic definitions. Nevertheless, I've not discussed Dummett's positive reason for thinking that the *Foundations* definitions are not stipulative. I turn to that in the next section.

III. Can Fruitful Definitions Be Eliminable?

Dummett argues that Frege's conception of definitions changes because, in the *Foundations*, Frege describes some definitions as being fruitful, but in subsequent works ceases to describe definitions in this way (1991b: 23). Similarly, Benacerraf holds that the stipulative conception of definitions is incompatible with Frege's description of "fruitful" definitions in the *Foundations*. More recently, Sanford Shieh has also claimed that Frege's discussion of fruitful definitions in the *Foundations* is inconsistent with the conception of definitions in "Logic in Mathematics" (2008: 996-7, 1002).

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[12] Frege does recognize it is sometimes "expedient" to use well-established signs (1914: 211). But even then the sense the sign had prior to the definition is "no longer of any concern to us."

According to these authors, stipulative definitions cannot be fruitful because stipulative definitions are eliminable in our proofs—for every proof of a statement from some premises that uses a definition, there is another proof of that same statement from those same premises that does not use it. But a definition is fruitful only if we can "prove things with it that we could not have proved without it" (Benacerraf (1981: 28)). That is, fruitful definitions are *ineliminable.* But if fruitful definitions are ineliminable, then insofar as we have reason for thinking Frege thought his definitions in *Foundations* were fruitful,[13] we have reason for thinking he did not regard those definitions as stipulative.

However, a careful consideration of Frege's discussion of fruitful definitions in the *Foundations* does not reveal anything inconsistent with his stipulative conception of definitions. There are three relevant passages about fruitful definitions. The first two are as follows. Speaking of the practice of most mathematicians, Frege writes:

> If a definition shows itself tractable when used in proofs, if no contradictions are anywhere encountered, and if connexions are revealed between matters apparently remote from one another, this leading to an advance in order and regularity, it is usual [for mathematicians] to regard the definition as sufficiently established, and few questions are asked as to its logical justification. This procedure has at least the advantage that it makes it difficult to miss the mark altogether. Even I agree that definitions must show their worth by their fruitfulness: it must be possible to use them for constructing proofs. (1884: ix).

And later:

> Definitions show their worth by proving fruitful. Those that could just as well be omitted and leave no link missing in the chain of our proofs should be rejected as completely worthless. (1884: §70)

Both of these passages are consistent with the stipulative conception of definition. The first passage requires that it is possible to use definitions in proofs, which is compatible with the stipulative conception. The second passage states that definitions which are not actually used in a proof—so that their omission leaves "no link missing," i.e. does not produce a "gappy" proof—are worthless. This statement is also compatible with the stipulative conception of definitions. In fact, in "Logic in Mathematics," Frege makes similar remarks:

> When we look around us at the writings of mathematicians, we come across many things which look like definitions, and are even called such, without really being definitions. Such definitions are to be compared with those stucco-embellishments on buildings which look as though they supported something whereas in reality they could be removed without the slightest detriment to the building. We can recognize such definitions by the fact that no use is made of them, that no proof ever draws upon them. (1914: 212).

Thus, neither passage provides reason for thinking that Frege's conception of definitions in the *Foundations* is inconsistent with the stipulative conception.

There is one final passage in *Foundations* were Frege discusses fruitful definitions. It begins with Frege criticizing Kant's division of analytic and synthetic judgments. On Kant's view, a statement is analytic only if the predicate is "contained" in the subject. If the predicate of a statement is not contained in the subject, Kant labels it synthetic and says that such statements "extend our knowledge" (1781/1787: A7/B11-A10/B14).

[13] Frege never claims his definitions are fruitful, though he does suggest it at (1884: §79).

Frege criticized Kant's division for not being exhaustive. For it only applies to statements with a subject and predicate, but not all statements can be analyzed in that way. Consequently, Frege re-draws the division between analytic and synthetic so that a statement is analytic only if there is a proof of it from logical laws and definitions alone. If there is no such proof, then a statement is synthetic (1884: §3). This way of drawing the division is, unlike Kant's, exhaustive.

Frege then continues his criticism of Kant:

> [Kant] seem to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful. If we look through the definitions given in the course of this book, we shall scarcely find one that is of this description. The same is true of the really fruitful definitions in mathematics, such as that of the continuity of a function. What we find in these is not a simple list of characteristics; every element in the definitions is intimately, I might almost say organically, connected with others.
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> What we shall be able to infer from it [a "more fruitful type of definition"] cannot be inspected in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought, therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic. The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. Often we need several definitions for the proof of some proposition, which consequently is not contained in any one of them alone, yet does follow purely logically from all of them together. (1884: §88)

While this is a difficult passage, it is not unnatural to read it as Dummett *et al.* do—as revealing a conception of definitions inconsistent with the stipulative one.

Nevertheless, despite first appearances, this passage does not show that fruitful definitions cannot be eliminable definitions. After all, the example of a fruitful definition that Frege gives—the continuity of a function—is eliminable. So, a definition can be both fruitful *and* eliminable.[14] But even if fruitful definitions can be eliminable, how do we reconcile this passage with Frege's other texts where he *denies* that definitions extend knowledge?

We can reconcile them by distinguishing between two different ways a statement might be thought to extend knowledge. First, a statement, specifically a definition, might be thought to extend knowledge by stipulating that something has a particular property that it did not have before the definition. Frege is deeply opposed to definitions extending knowledge in this sense; for him, *the act of defining* does not extend knowledge. "*By defining*, no knowledge is engendered" ((1903b: 320, italics mine), cf. (1890-2)) and "…no definition is creative in the sense of being able to endow a thing with properties that it has not already got…" (1891: 132).[15] Indeed, it is important to notice that in the above passage, Frege never claims that *definitions* extend knowledge, but rather the *conclusions we draw* from definitions can.

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[14] Weiner (1990: 90ff.) makes this point. Tappenden (1995: 429-31) claims that Weiner attributes to Frege Cauchy's definition, whereas Frege had in mind Weierstrass'. But even supposing that, Weierstrass' definition is also eliminable in proofs, a point Tappenden does not dispute (cf. (1995: 432, 456)). So this point does not help the interpretation according to which the definitions in the *Foundations* are not eliminable.

[15] Frege sometimes refers to these as "creative definitions" and consistently lambasts them; cf. (1885: 98-9; 1893: XIII-XIV; 1895: 448-9).

A second way in which a statement might extend our knowledge is Kant' sense, according to which if the predicate of a statement is not contained in the subject, then the statement extends our knowledge. Given that, for Kant, if the predicate of a statement is not contained in the subject, then that statement is synthetic, it follows that for Kant all statements that extend our knowledge are synthetic. One of Frege's points in this passage is that a definition can allow us to prove a claim that extends knowledge *in this Kantian sense*. For in using a definition, one might prove a claim for which the predicate is not contained in the subject—perhaps because the claim lacks a subject/predicate structure. Such a claim "ought, therefore, on Kant's view, to be regarded as synthetic." But for Frege that claim could be analytic—indeed, the only specific example Frege gives, in the *Foundations*, of an analytic truth that extends knowledge is a claim that lacks a simple subject/predicate structure (1884: §91). Thus, Frege is contrasting his way of drawing the analytic/synthetic distinction by discussing Kant's views on what extends knowledge. Any statement that extends knowledge (in Kant's sense) will be synthetic, on Kant's way of drawing the distinction, whereas some statements that extend knowledge (in Kant's sense) will be analytic, on Frege's way of drawing the distinction.

IV. Do Fregean Proofs Provide Analyses?

The defender of the analysis interpretation may offer an alternative way to resolve our exegetical puzzle than previous ones. She might concede that the definitions Frege provides for his proof-sketches (in the *Foundations*) and proofs proper (in the *Basic Laws)* are stipulative definitions. Nevertheless, she might argue that this does not preclude that the claims Frege proves, using those stipulative definitions, are good analyses of pre-proof arithmetical claims. In this way, she might resolve our exegetical puzzle.[16]

Patricia Blanchette has provided the most careful and sophisticated defense of this interpretation. She claims that Frege repeatedly used "conceptual analysis," which she describes as:

> Beginning with an ordinary arithmetical truth of the kind typically expressed via a familiar arithmetical sentence, Frege subjects that truth to careful analysis, usually by breaking down its important components into what he takes to be simpler ones. The new, highly analyzed version of the original truth is expressed either by a sentence whose syntactic complexity is greater than that of the familiar sentence, or by a definitional abbreviation thereof. Frege's strategy is then to derive that sentence using only logic, which is to say that he proves the highly analyzed version of the arithmetical truth from purely logical principles. Finally, Frege's central claim is that the proofs of his highly analyzed versions of arithmetical truths demonstrate the purely logical grounding of arithmetic, and thereby substantiate his thesis that arithmetic is a branch of logic. (2012: 23)

Blanchette provides textual evidence intended to show that Frege did regard some of the derived claims of his Begriffsschrift proofs as good analyses of some pre-proof arithmetical claims.

Here I critique Blanchette's interpretation. I argue that Blanchette's interpretation does not sit well with Frege's explicit description of his project. That is a major disadvantage to her interpretation. Additionally, I examine the textual evidence she adduce. I argue that Frege's texts do not support Blanchette's interpretation or do not support it over the explicative one.

A. The Logicist's Lacuna

[16] One way to do this would be to claim that the stipulative definitions themselves provide good analyses; for a stipulative definition could also be a good analysis (cf. Horty (2007: 35), Shieh (2008: 1002)). But this is not the only way.

To see an important difference between Blanchette's interpretation and the explicative one, consider the role of successful Begriffsschrift proofs on each. Suppose that Frege has constructed gapless Begriffsschrift proofs from basic logical laws and definitions that show that the numbers (so defined) have the relevant properties. Would such proofs vindicate his logicist ambitions? On the explicative interpretation, they would. But on Blanchette's interpretation Frege would have to provide additional arguments. Specifically, Frege would have to argue, first, that his derived claims provided good analyses of pre-proof arithmetic claims. (We'll consider whether he thought this below.) But, secondly, Frege would have to argue that those pre-proof arithmetical claims are logical truths, not because they were derived from logical laws and definitions, but *because other claims that are an analyzed form of them were so derived*. Let's call 'the Logicist's Lacuna' the claim that if a claim that is an analyzed form of a pre-proof arithmetical claim is derived from logical laws and definitions alone, then the pre-proof arithmetical claim is itself a logical truth.

On Blanchette's interpretation, the Logicist's Lacuna is important for Frege's methodology. If it were false, then Frege's entire argumentative strategy for logicism would be defective. Thus, the success of Frege's logicism, and the significance of his Begriffsschrift proofs, rests on it. It is no wonder Blanchette describes it as Frege's "central claim" (2012: 23).[17]

A major disadvantage of Blanchette's interpretation is that Frege nowhere defends the Logicist's Lacuna, and it is in tension with what he writes. It is in tension with what he writes because when Frege explicitly describes what it takes for an arithmetical truth to be a logical truth, he does not mention anything about analyses of pre-proof claims; for instance, in *Foundations*, he merely writes "Arithmetic thus becomes simply a development of logic, *and every proposition of arithmetic a law of logic, albeit a derivative one*" (1884: §87, italics mine). He does not write here "every proposition of arithmetic is a law of logic, albeit related to a derived proposition of logic that is an analysis of that arithmetical proposition." Likewise, in the introduction to the first volume of *Basic Laws*, where Frege describes what is necessary to convincingly show that "arithmetic is a branch of logic," he mentions higher demands on proofs but nothing about successful analyses of pre-proof arithmetical claims (1893: 1).

Some might think that what Frege writes in (1884: §4) is not in tension with Logicist's Lacuna and, in fact, lends credence to it. Speaking of the "demand" to prove, if possible, the "fundamental propositions of arithmetic" with "the utmost rigour," Frege writes,

> …we very soon come to propositions which cannot be proved so long as we do not succeed in analyzing concepts which occur in them into simpler concepts or reducing them to something of greater generality. Now here it is above all Number which has to be either defined or recognized as indefinable. This is the point which the present work is meant to settle. On the outcome of this task will depend the decision as to the nature of the laws of arithmetic.

Here Frege refers to propositions that contain the concept of Number but are yet to be proven or analyzed. Further, he seems to suggest that whether the foundations of arithmetic are analytic or synthetic depends upon whether those "pre-proof" propositions can themselves be proven or analyzed. Since he thinks they can, this suggests that he does think that a "pre-proof" arithmetical claim can be shown to be a logical truth by deriving an analyzed form of that claim from logical laws and definitions alone.

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[17] This is what separates the truth/falsity of Logicist's Lacuna and Basic Law V. If Basic Law V is false, then the *success* of Frege's proofs are brought into question; if the Logicist's Lacuna is false, then the *significance* of his proofs are brought into question, at least, on Blanchette's interpretation.

The idea that this passage supports Logicist's Lacuna is based on a confusion. In this passage, Frege is concerned with claims that *already appear in proofs*. He is wondering whether those proofs can be extended by (e.g.) using an analysis of a concept that appears in these claims or subsuming them under some "greater generality." Thus, the sense in which these claims are "pre-proof" or "pre-analysis" is merely a causal one: they have yet to be proven or analyzed. But the kind of "pre-proof" claims that matter to Logicist Lacuna, and Blanchette's interpretation more generally, are ordinary claims that *do not appear in Frege's proofs*. Blanchette's claim is that *those* claims can be shown to be logical truths by providing a proof in which claims that are highly analyzed versions of them are derived from logical truths. Since this passage is not concerned with those claims, but rather claims involving Number that can be defined in the act of providing proofs, this passage does not support Logicist's Lacuna.

In short, given the importance of the Logicist Lacuna, it is very implausible that Frege would spend no time defending the "central claim" of his project. After all, Frege went to great lengths, including creating and developing a new and unwieldy formal language, to make sure that there is no place at which Kantian intuition make an appearance in justifying arithmetical truths. It is implausible that he would have felt no similar compulsion for this other integral part of his project.

To all this, Blanchette might respond that we shouldn't expect Frege to have defended the Logicist's Lacuna because it was simple an unwitting assumption. That response is implausible. While all authors make assumptions, it is implausible to suppose that Frege made an assumption so integral to his project's methodology, given how otherwise careful and explicit he is about that methodology. Nevertheless, I do not intend to rest my criticism of Blanchette's interpretation simply on this—after all, if it very obvious that Frege *did* utilize conceptual analysis as Blanchette describes and *did* regard his Begriffsschrift proofs as proving claims that were good analyses of pre-proof claims, then we might have to attribute to Frege an unacknowledged assumption.

B. Fregean Analyses?

Blanchette's interpretation and the explicative one agree that Frege's definitions shouldn't allow him to prove claims that we take to express false arithmetical claims. Thus, on both interpretations, if Frege could construct a Begriffsschrift proof that 0 = 1, or that 13 lacked a successor, there would be a problem. On both interpretations, we should expect that Frege would be proving things that closely mirrored well-recognized pre-proof arithmetical claims.

One of the things that separates the two interpretations is the relationship between those Begriffsschrift-proved claims and the pre-proof claims.[18] Blanchette claims that Frege regarded the results of certain Begriffsschrift derivations as providing good analyses of pre-proof claims. However, she repeatedly laments that Frege never clearly states what constitutes a *good* analysis—see, *inter alia*, (2012: 24, 77-8, 104)—and perhaps for that reason she never clearly states sufficient and/or necessary conditions either.[19] While this makes evaluating her interpretation difficult, she is nevertheless quite clear that Frege repeatedly used conceptual analysis to break down logical and pre-proof arithmetical claims into components parts and that the purpose of this procedure was ultimately to show something about the pre-proof arithmetical claims. However, the texts Blanchette cites in support of her interpretation do not support it over

[18] This isn't the only thing. On the explicative interpretation, the pre-proofs claims made by mathematicians are problematic themselves (recall the epigram), whereas on Blanchette's interpretation, they seem not to be.

[19] Though she does spend some time describing what *isn't* necessary; see (2012: 77-89).

and above the explicative interpretation.[20] Frege does not follow the general pattern of conceptual analysis she describes, and nowhere clearly indicates that the Begriffsschrift proofs provide claims that are good analyses of pre-proof claims.

1. Begriffsschrift (1879)

According to Blanchette, Frege first uses conceptual analysis in the *Begriffsschrift*, where he "provides the promised "reduction" of the relation of *ordering of a sequence* and of subsidiary notions" (2012: 13).[21] To support her interpretation, she quotes Frege:

> Through the present example, moreover, we see how pure thought, irrespective of any content given by the sense or even by an intuition a priori, can, solely from the content that results from its own constitution, bring forth judgments that at first sight appear to be possible only on the basis of some intuition. (2012: 14), quoting Frege (1879: §23).

She comments, "That is to say, a handful of judgment-contents that look "at first sight" as if they would be knowable only via sensation or a priori intuition will be shown here to be in principle knowable without appeal to intuition" (2012: 14). They will be shown this way by "…a proof from principles of logic and definitions of analyzed versions of the judgment-contents in question" (2012: 14). So, Blanchette claims, this is a case where there are some pre-proofs claims concerning ordering of a sequence which Frege provides an analysis in Begriffsschrift notation, and then shows how that claim is not derived from intuition at all.

The problem is that the passages Blanchette cites do no show that Frege regarded his Begriffsschrift proofs as providing a conceptual analysis. First, Frege does not promise a reduction of ordering of a sequence. The passage Blanchette has in mind comes from the opening sections of the *Begriffsschrift*. But nowhere in that passage does Frege promise a reduction of ordering of a sequence. Rather, he first distinguishes between "how we have gradually arrived at a given proposition" and "how we can finally provide it with the most secure foundation" (1879: 5). He then writes,

> Now, when I came to consider the question to which of these two kinds the judgments of arithmetic belong, I first had to ascertain how far one could proceed in arithmetic by means of inference alone, with the sole support of those laws of thought that transcend all particulars. My initial step was to attempt to reduce the concept of ordering in a sequence to that of *logical* consequence, so as to proceed from there to the concept of number. To prevent anything intuitive from penetrating here unnoticed, I had to bend every effort to keep the chain of inferences free of gaps. In attempting to comply with this requirement in the strictest possible way I found the inadequacy of language to be an obstacle… This deficiency led me to the idea of the present ideography. (1879: 5-6).

Nowhere in this passage does Frege promise a reduction of ordering of a sequence or provide any natural breaking down of a concept.

Second, Blanchette takes Frege's prefatory comments in section III about "judgments" to be about pre-proofs judgments. But Frege is referring to his formal Begriffsschrift results. For the "judgments" Frege refers to are what are "developed in what follows" and what follows are claims in Begriffsschrift notation with occasion natural language glosses. So, when Frege

[20] In additional to the three passages I cover here, she also includes one from the *Basic Laws*. Since her discussion of it is very short and builds on her discussion of the *Foundations*, I won't cover it.

[21] Blanchette is writing loosely; Frege will not reduce the relation, but a series of non-proof claims about the relation.

indicates that “pure thought… can… bring forth judgments” that are not based on Kantian intuition, he does not mean that his proofs show some antecedent pre-proof claim to be free of Kantian intuition, but that by following his proofs we’ll arrive *at Begriffsschrift claims* that we can see to be free of Kantian intuition.

Finally, if, as Blanchette claims, the claims involving properties being hereditary in the *f*-sequence that Frege proves using his Begriffsschrift are meant as analyses of the claims about ordering in a sequence, then we would expect him to begin or end his proofs with such an analysis. He does not. Rather, he merely states the proved claim in a natural language and moves on. Indeed, in later work, when Frege refers to this part of the *Begriffsschrift*, he never states that he provides an analysis of ordering in a sequence. So there is no exegetical evidence that Frege would have regarded his Begriffsschrift proofs as proving analyses of pre-proof claims. Frege’s *Begriffsschrift* does not display the method of conceptual analysis Blanchette describes.

2. *Boole’s Logical Calculus and the Concept Script (1881)*

Blanchette argues that Frege also utilizes conceptual analysis in his unpublished paper “Boole’s Logical Calculus and the Concept-Script.” She writes that Frege’s purpose there is not to demonstrate “the logical grounding of an arithmetical truth, but the preliminary result that a given relatively complex arithmetical truth is grounded in illuminatingly simpler ones” (2012: 18). The “relatively complex arithmetical truth” to be proved is a natural language claim, which Blanchette labels “SUM”: “the sum of two multiples of a number is in its turn a multiple of that number” (1881: 27). The “simpler” arithmetic truths are (in modern notation):

(P1) $(\forall m)(\forall n)(\forall p)((m + n) + p) = (m + (n + p))$

(P2) $(\forall n)(n = n + 0)$

After providing “a careful analysis of the relation “multiple-of,” Blanchette writes, Frege goes on to provide a “highly-analyzed version of what is expressed by (SUM),” which she calls (SUM*), a claim in the Begriffsschrift notation (2012: 18). Since Frege proves (SUM*) from (P1) and (P2), he can claim (SUM) is grounded in “illuminatingly simpler” arithmetical truths.

There are multiple problems with Blanchette’s discussion of this passage which together show that Frege was not performing conceptual analysis. First, this example was part of a string of examples to show how Frege’s Begriffsschrift is superior to Boole’s calculus in that it can “achieve the more far-reaching goals it sets itself” of expressing thoughts when combined with arithmetical signs. Indeed, immediately prior to the example Blanchette focuses on, Frege writes that “it is of little significance which topic I choose” as his purpose is to illustrate the superiority of his method to Boole’s (1881: 27). So Frege’s overall purpose is not to provide a reduction of one arithmetical truth to another. Indeed, this is not even a secondary purpose. To see that, consider what Frege says about (P1) and (P2). He does not describe them, as Blanchette does, as “simpler truths” from which to derive (SUM*). Rather, he regards them as the only constraints on “the numbers whose multiples are to be considered” (1881: 27).

Second, Frege does not provide a “careful analysis of the relation “multiple-of.” He begins by reproducing a theorem from the *Begriffsschrift* concerning hereditary properties in an *f*-sequence. But he nowhere claims that the theorem “breaks down” the multiple-of relation. Indeed, he regards the natural language prose he provides as unnecessary for this proof; the formulae “really ought to speak for themselves without continual prose glosses,” but he provides those glosses anyway because he is trying to “make myself understood without recourse to [the *Begriffsschrift*]” (1881: 27). Indeed, after proving (SUM*), he claims one can prove another theorem, but he simply lists a sequence of Begriffsschrift formulae without any natural language glosses.

Finally, if Blanchette's interpretation were correct, we would expect that Frege would argue that (SUM*) is a good analysis of (SUM). As she writes, "[o]n the basis of the derivation of (SUM*), Frege takes himself to have established that the theorem expressed by (SUM) follows logically from those premises [(P1) and (P2)]" (2012: 18). But Frege never says this or something like it. He simply writes, "this gives us [SUM*], the theorem to be proved" (1881: 31). He then states (SUM*) and moves on. Thus, there is no textual evidence that Frege regarding his Begriffsschrift proof as proving a claim, i.e. (SUM*), that provides a good analysis of a pre-proof claim, i.e. (SUM).

Blanchette might reply that Frege does not need to argue that (SUM*) is a good analysis of (SUM) because they are so similar, assuming it would be obvious to his readers that, upon deriving (SUM*), he's provided a proof for (SUM). This reply gets the aim of the passage wrong. Frege's purpose is to show how *his* Begriffsschrift can present thoughts, even when combined with arithmetical signs. The purpose is not to vindicate the status of non-Begriffsschrift claim like (SUM), but to illustrate the expressive resources of his Begriffsschrift. Like the *Begriffsschrift* before it, this work does not display the conceptual analysis that Blanchette describes.

3. *Foundations of Arithmetic (1884)*

Blanchette claims that Frege used conceptual analysis in the *Foundations*. She claims that after criticizing alternative analyses, Frege gives analyses of zero, successor, and cardinal number (2012: 19). He gives these analyses in terms of first-level concepts, second-level concepts, and extensions of concepts, among other things. Frege stipulates that the ordinary terms (e.g. "zero") are to mean the same as his analyses (2012: 21). Then, using his new stipulative definitions, Frege sketches in §79-83 proofs of various arithmetical claims. What he's really shown with all this, Blanchette claims, is not that it is probable that (as he would have put it) "the laws of arithmetic are analytic judgments" (1884: §87) but that (as she would put it) "a collection of truths about cardinality-concepts and their extensions are 'laws of logic, albeit…derivative ones" (2012: 22). On her view, Frege is merely assuming that "the logical grounding of these truths, i.e. of the output of his careful analyses, suffices to establish the logical grounding of what he primarily cares about, namely the truths of arithmetic" (2012: 22).

Once again, there are several problems with Blanchette's interpretation. The first we saw earlier regarding the Logicist Lacuna. In her discussion of Frege's conclusion in §87, she must interpret Frege as either (i) misspeaking, in so far as he claims to have shown probable that the truths of arithmetic are analytic, or (ii) not misspeaking, but tacitly assuming the Logicist's Lacuna. Neither is plausible.

There are two other problems. None of the passages Blanchette discusses gives any reason for preferring her interpretation over the explicative interpretation. It is consistent with both interpretations that Frege would criticize other accounts of numbers that do not secure obvious truths about numbers, that he would criticize definitions of number that would not exclude well-known falsehoods about numbers (e.g. Caesar = 0), and that he would sketch proofs of the well-known properties of numbers. The issue between the two interpretations is whether he would have regarded his proof sketches as providing something that was a good analysis of some pre-proof claim. Blanchette gives no textual evidence that he would have.

Finally, Frege does explicitly address the issue of the correctness of his definitions. In §69, he writes "That this definition is correct will perhaps be hardly evident at first. For do we not think of the extensions of concepts as something quite different from numbers?" This leads him to a section entitled "Our Definition Completed and Its Worth Proved." However, his defense of his definition does not turn on its preserving either sense or reference or otherwise being a good analysis. He begins the section with a passage seen earlier:

> Definitions show their worth by proving fruitful. Those that could just as well be omitted and leave no link missing in the chain of our proofs should be rejected as completely worthless.
>
> Let us try, therefore, whether we can derive from our definition of the Number which belongs to the concept *F* any of the well-known properties of numbers. (1884: §70)

Notice that he does not suggest, as one would suspect on Blanchette's interpretation, that his definition must be a good analysis. But this is exactly the place where one would expect him to do so, as he is considering that kind of objection. The issue of whether his definitions are good analyses does not come up at all. Thus, the *Foundations* does not show that Frege used the method of conceptual analysis.

In conclusion, I've argued that Blanchette's interpretation problematically attributes to Frege a claim, the Logicist's Lacuna, which Frege never discusses or defends, but which is integral to the success of his basic argumentative strategy. I conceded that perhaps we must attribute to Frege the Logicist's Lacuna as an unacknowledged assumption if Frege's texts show that he used conceptual analysis as Blanchette describes. But none of the texts Blanchette cites show that he *did* use the method of conceptual analysis. There is no textual evidence that Frege regarded his Begriffsschrift proofs as proving good analyses of pre-proof claims.

V. Logicism and the Explicative Interpretation

In previous sections, I've criticized prominent and important defenses of the analysis interpretation. I've argued that Frege saw his definitions as mere stipulations that need not capture as close as possible the existing sense of pre-proof arithmetical terms or otherwise be good analyses of those terms.

Nevertheless, on the explicative interpretation, it does not follow that Frege placed *no* constraints on definitions. He did place at least the following constraints. First, definitions were not to lead to contradictions (1884: ix). Second, definitions were to be used in proofs (1884: §70; 1914: 212). Third, his definitions were to help prove that numbers have their "well-known properties" (1884: §70). This third constraint has two elements. First, his definitions were to allow him to prove those truths that we take sentences of arithmetic to express (e.g. '1 has a successor'). Second, for those "gappy" inferences of arithmetic that we take to be correct, his definitions were to allow him to render those inferences into gapless Begriffsschrift proofs. As Weiner writes commenting on this constraint,

> Faithful definitions must be definitions on which those sentences that we take to express truths of arithmetic come out true and on which those series of sentences that we take to express correct inferences turn out to be enthymematic versions of gapless proofs in the logical system. (2007: 690)

Even with these constraints, one might argue that the explicative interpretation fails to resolve our exegetical puzzle because it fails to do justice to Frege's logicism. Blanchette has recently leveled this charge. She argues that this interpretation "does not cohere well with Frege's general view about the point and the implications of his Logicist reduction" (2012: 84). For this interpretation merely requires Frege's project "to preserve the (apparent) truth-values and derivability-relations of the sentences of arithmetic as ordinarily understood" (2012: 84). But this requirement is too weak, as the empiricist account of arithmetic that Frege criticized could meet it:

> Starting with some mundane facts about pebbles—e.g., that a pebble can always be moved 1 inch to the right—we can construct an ω-sequence of pebble-positions. It's now a simple matter to use pebble-positions to define arithmetical terminology, and to prove the resulting contents of, e.g., "every natural number has a successor" by appealing to pebble-axioms. We can easily preserve (apparent) truth-values and derivability relations under this interpretation. (2012: 85)

If the empiricist account of arithmetic could meet the requirements for Frege's project that the explicative interpretation maintains, then it is hard to see how meeting those requirements could vindicate Frege's *logicism*. Thus, the explicative interpretation cannot account for Frege's logicism, and we should reject it.[22]

This is an important objection. I argued that Blanchette's interpretation has difficulty making sense of Frege's logicism because it must attribute to him a claim—the Logicist's Lacuna—that he never articulates or defends. Blanchette's objection here is more powerful because it claims the explicative interpretation cannot make sense of *what Frege actually writes*, specifically, his criticisms of alternative views of arithmetic.

The objection fails for two reasons, one technical, one more fundamental. Blanchette is correct that if a "reduction" of one formal language to another merely requires preservation of truth-values and derivability-relations, then "reductions are relatively easy to come by" (2012: 84). But, on the explicative interpretation, it is not merely the inferences in a formal language are to be preserved in the Begriffsschrift but that the gappy inferences made in pre-proof contexts must be also be represented as gapless proofs. Thus, on the explicative interpretation, more is required for a successful "reduction" than what Blanchette claims.

The more fundamental problem is that Blanchette criticizes a strawman. For Blanchette takes what, on the explicative interpretation, is a *constraint on definitions*—namely, that they preserve apparent truth-values and derivability relations—and represents it as a *sufficient condition for the success of Frege's project*. But this is a mistake. On the explicative interpretation, the success of Frege's project does not merely consist in constructing Begriffsschrift proofs, but Begriffsschrift proofs *from certain axioms*. These axioms are to be logical truths. Thus, the empiricist account that Blanchette provides would fail to vindicate Frege's logicism not necessarily because of defective definitions but because the axioms with which it begun, being about pebbles, are clearly not logical truths.[23]

VI. Conclusion

Frege's writings present his readers with an exegetical puzzle. How are we to understand his logicism, given the apparent strangeness of his definitions and his later claims about definitions? On the analysis interpretation, the puzzle is resolved by claiming that Frege's definitions in the *Foundations* and *Basic Laws* were either analyses of pre-proof terms or allowed him to prove

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[22] Blanchette gives another counterexample involving geometry: "it is unproblematic to reinterpret the terms "point," "line," etc. in such a way that Euclid's axiom-sentences express analytic truths about collections of extensions and relations thereon" (2012: 85). Yet Frege would not have thought this shows the truths of geometry to be grounded in logic. Now this counterexample fails because Frege would not have regarded the axioms of geometry as logical truths. But it fails for a second reason: Frege would have objected to reinterpreting the axioms of geometry, a point Blanchette recognizes (2012: chp. 5). It is thus unclear why Blanchette includes this additional counterexample.

[23] There is some dispute about the status of the axioms. Weiner (2004b: 120ff.) argues that the axioms are not only to be logical truths but *evidently* logical truths. Jeshion (2001, 2004) argues for a weaker constraint. Either interpretation is consistent with my point here.

claims that were analyses of pre-proof claims. I've argued against the analysis interpretation, specifically, that it cannot make sense of Frege's writings, and the exegetical support mustered for it either does not support it or does not support it over the explicative interpretation. While I've not provided the explicative interpretation's resolution to this puzzle, I've shown that its main competitor is seriously problematic.[24]

Works Cited

Beaney, Michael. (1997). Eds. *The Frege Reader*. Oxford: Blackwell.

Benacerraf, Paul (1981), "Frege: The Last Logicist," in Peter French, Teodore Uehling,and Howard Wettstein, eds. *The Foundations of Analytic Philosophy*, volume VI of *Midwest Studies in Philosophy*, pp. 17-35.

Blanchette, Patricia. (1994). "Frege's Reduction." *History and Philosophy of Logic*. 15.1: 85-103.

Blanchette, Patricia. (2012). *Frege's Conception of Logic*. New York: Oxford University Press.

Dummett, Michael. (1981). *The Interpretation of Frege's Philosophy*. Cambridge, Mass.: Harvard University Press.

Dummett, Michael. (1991a). "Frege and the Paradox of Analysis." In Michael Dummett, *Frege and Other Philosophers*. Oxford: Clarendon Press.

Dummett, Michael. (1991b). *Frege: Philosophy of Mathematics*. Cambridge, Mass.: Harvard University Press.

Ebert, Philip A. & Marcus Rossberg with Crispin Wright. (2013). Eds. and Trans. *Basic Laws of Arithmetic,* volumes 1&2 by Gottlob Frege. Oxford: Oxford University Press.

Frege, Gottlob. (1879). *Begriffsschrift*. In van Heijenoort (1967).

Frege, Gottlob. (1881). "Boole's Logical Calculus and the Concept Script." In Hermes, Kambartel, an Kaulbach (1979).

Frege, Gottlob. (1882). "Letter to Marty." In Beany (1997).

Frege, Gottlob. (1884 [1980]). *The Foundations of Arithmetic.* Trans. J. L. Austin. Evanston, Illinois. Northwestern University Press.

Frege, Gottlob. (1885). "On Formal Theories of Arithmetic." In In McGuiness (1984).

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Frege, Gottlob. (1890-2). "Draft Towards a Review of Cantor's *Gesammelte Abhandlungen zur Lehre vom Transfiniten*." In Hermes, Kambartel, an Kaulbach (1979).

Frege, Gottlob. (1891). "Function and Concept." In Beany (1997).

Frege, Gottlob. (1893). *Basic Laws of Arithmetic, volume 1*. In Ebert & Rossberg with Wright (2013).

Frege, Gottlob. (1894). "Review of E. G. Husserl, *Philosophy of Arithmetic*." In McGuiness (1984).

Frege, Gottlob. (1895). "A Critical Elucidation of Some Points in E Schröder, *Lectures on the Algebra of Logic*." In McGuiness (1984).

Frege, Gottlob. (1899). "On Mr. H. Schubert's Numbers." In McGuiness (1984).

Frege, Gottlob. (1903a). "On the Foundations of Geometry: First Series." In McGuiness (1984).

Frege, Gottlob. (1903b). *Basic Laws of Arithmetic, volume 2*. In Ebert & Rossberg with Wright (2013).

Frege, Gottlob. (1906). "On the Foundations of Geometry: Second Series." In McGuiness (1984).

Frege, Gottlob. (1914). "Logic in Mathematics." In Hermes, Kambartel, and Kaulbach (1979).

Hermes, Hans, Friedrich Kambartel, and Friedrich Kaulbach. (1979). Eds. *Gottlob Frege: Posthumous Writings*. Oxford: Basil Blackwell.

Horty, John. (2007). *Frege on Definitions*. New York: Oxford University Press.

Jeshion, Robin. (2001). "Frege's Notion of Self-Evidence." *Mind*. 110.440: 937-976.

Jeshion, Robin. (2004). "Frege: Evidence for Self-Evidence." *Mind*. 113.449: 131-8.

Kant, Immanuel (1781/7), *The Critique of Pure Reason*, trans. Norman Kemp Smith, (New York: St. Martin's Press).

Kneale, William and Martha Kneale. (1962). *The Development of Logic*. Oxford: Clarendon Press.

McGuiness, Brian. (1984). Eds. *Gottlob Frege: Collected Papers on Mathematics, Logic, and Philosophy*. Basil Blackwell.

Nelson, Michael. (2008). "Frege and the Paradox of Analysis." *Philosophical Studies*. 137: 159-181.

Noonan, Harold W. (2001). *Frege: A Critical Introduction*. Malden, MA: Polity.

Reck, Erich H. (2007). "Frege-Russell Numbers: Analysis or Explication?" In *The Analytic Turn*, ed. Michael Beaney. London: Routledge, 33-50.

Shieh, Sanford. (2008). "Frege on Definitions." *Philosophy Compass*. 3/5: 992-1012.

Tappenden, Jamie. (1995). "Extending Knowledge and "Fruitful Concepts": Fregean Themes in the Foundations of Mathematics." *Nous*. 29.4: 427-467.

Van Heijenoort, Jean. (1967). Eds. *From Frege to Gödel.* Cambridge, Mass.: Harvard University Press.

Weiner, Joan. (1990). *Frege in Perspective*. Ithaca: Cornell University Press.

Weiner, Joan. (2004a). *Frege Explained*. La Salle, Illinois: Open Court.

Weiner, Joan. (2004b). "What Was Frege Trying to Prove? A Response to Jeshion." *Mind*. 113.449: 115-130.

Weiner, Joan. (2007). "What's In a Numeral? Frege's Answer." *Mind*. 116.463: 677-716. doi:10.1093/mind/fzm677

Weiner, Joan. (2010). "Frege's Project." In *The Cambridge Companion to Frege*, eds. Michael Potter and Tom Ricketts. Cambridge: Cambridge University Press, 32-62.