Steps towards an axiomatic pregeometry of space-time

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Abstract

We present a deductive theory of space-time which is realistic, objective, and relational. It is realistic because it assumes the existence of physical things endowed with concrete properties. It is objective because it can be formulated without any reference to cognoscent subjects or sensorial fields. Finally, it is relational because it assumes that space-time is not a thing but a complex of relations among things. In this way, the original program of Leibniz is consummated, in the sense that space is ultimately an order of coexistents, and time is an order of succesives. In this context, we show that the metric and topological properties of Minkowskian space-time are reduced to relational properties of concrete things. We also sketch how our theory can be extended to encompass a Riemannian space-time.

1 Introduction

Space-time is a primitive (*i.e.* non-derivable) concept in every physical theory. Even the so-called space-time theories, like General Relativity, do not deal with the nature of space-time but with its geometrical structure. The question "what is space-time?" precedes the formulation of any specific physical theory, and belongs to what is usually known as Protophysics (*i.e.* the branch of scientific ontology concerned with the basic assumptions of Physics).

We must recall that the ontological status of space-time has been a subject of debate for physicists and philosophers during the last 300 years. The kernel of this debate has been the confrontation of two antagonic positions: absolutism and relationalism. The former considers space-time as much a thing as planets and electrons are, *i.e.* space-time would be a physical entity endowed with concrete properties. This is the position held by Newton in his renowned discussion with Leibniz (mediated by S. Clarke: Alexander, 1983), and also by J. Wheeler in the geometrodynamical approach to physics (*e.g.* Misner *et al.*, 1973). The relationalism instead asserts that space-time is not a thing but a complex of relations among physical things. In Leibniz's words: "I have said more than once, that I hold space to be something merely relative, as time is; that I hold it to be an order of coexistents, as time is an order of successions" (see Alexander 1983).

An important consequence of Leibniz's ideas is that if space-time is not an ontological primitive, then it should be possible to construct it starting from a deeper ontological level. That is to say, the spatiotemporal relations should be definable from more fundamental relations. There have been several attempts to demonstrate the relational nature of space-time, both subjectivistic and phenomelogical (*e.g.* Carnap 1928 and Basri 1966) and objective and realistic (Bunge and García Maynez 1977, and Bunge 1977). We think that a deductive theory of space-time cannot be built with blocks that are alien to the physical discourse (such as cognoscent subjects or sensorial fields) in order to be compatible with contemporary physical theories. In this sense, we coincide with Bunge's approach, which only assumes the presuppositions common to the entire physical science (see Bunge 1977).

We present here a new formulation, realistic and objective, of the relational theory of space-time, based on the scientific ontology of Bunge (1977, 1979). The theory will be displayed as an axiomatic system, in such a way that its structure will turn out to be easily analyzable [1]. The construction of the theory lays on the notion of interaction among basic things, and on the notion of simultaneity.

At this point, we should mention that the search for a Quantum Theory of Gravity has triggered an intense research on the nature of space-time (for an exhaustive review, see Gibbs 1996). The aim of this research is to build a theory ("pregeometry") from which all the "properties" of space-time (like continuity and dimensionality) can be explained [2]. This kind of pregeometry should be the consequence of the unavoidable merging of Quantum Mechanics and General Relativity at very small distances. We emphasize that the pregeometry we propose here is valid only for lengths above a minimum length, which is suggested to be the Planck length by arguments based on the (yet unknown) theory of Quantum Gravity (Garay 1995).

The structure of the paper is as follows: in Section 2 we offer a brief account of the main ontological assumptions of the theory. Section 3 contains some formal tools, such as uniform spaces, to be used later. The axiomatic core is presented in Section 4. Finally, in Section 5 we give a short sketch of an extension of these ideas to Riemannian space-times, we compare our theory with the theory of Bunge (1977) and, to close, some observations on the nature of space-time are pointed out.

2 Ontological Background

In this section we give a brief synopsis of the ontological presuppositions that we take for granted in our theory. For greater detail see Bunge (1977, 1979) and Perez-Bergliaffa *et al.* (1996). The basic statements of the ontology can be formulated as follows:

1. There exist concrete objects x, named *things.* The set of all the things is denoted by Θ.
2. Things can juxtapose ($\dot{+}$) and superimpose ($\dot{\times}$) to give new things:
$$x \dot{+} y = \{x, y\} \in \Theta$$
$$x \dot{\times} y \in \Theta$$
3. The null thing is a fiction introduced in order to give a structure of Boolean algebra to the laws of composition of things:
$$\begin{aligned} x \dot{+} \diamond &= x \\ x \dot{\times} \diamond &= \diamond \end{aligned}$$
4. Two things are separated if they do not superimpose:
$$x \wr y \Leftrightarrow x \dot{\times} y = \diamond$$
5. Let T a set of things. The *aggregation* of T (denoted $[T]$) is the supremum of T with respect to the operation $\dot{+}$.

[1]On the advantadges of the axiomatic method see Perez Bergliaffa *et al.* (1993) and references therein.
[2]Let us recall that, according to our view, space-time is not a thing. Consequently, it cannot have properties.

6. The world ($\Box$) is the aggregation of all things:

$$\Box = [\Theta] \Leftrightarrow (x \sqsubset \Box \Leftrightarrow x \in \Theta)$$

where the symbol '$\sqsubset$' means 'to be part of'. It stands for a relation between concrete things and should be not mistaken with '$\in$', which is a relation between elements and sets (*i.e.* abstract entities).

7. All things are made out of basic things $x \in \Xi \subset \Theta$ by means of juxtaposition or superimposition. The basic things are elementary or primitive:

$$(x, y \in \Xi) \wedge (x \sqsubset y) \Rightarrow x = y$$

8. Things x have *properties* $P(x)$. These properties can be intrinsic or relational.

9. The *state* of a thing x is a set of functions from a domain of reference M to the set of properties $\mathcal{P}$. The set of the accesible states of a thing x is the *lawful state space* of x: $S_{\rm L}(x)$. The state of a thing is represented by a point in $S_{\rm L}(x)$.

10. A *legal statement* is a restriction upon the state functions of a given class of things. A *natural law* is a property represented by an empirically corroborated legal statement.

11. The *ontological history* $h(x)$ of a thing x is a part of $S_{\rm L}(x)$ defined by

$$h(x) = \{\langle t, F(t)\rangle | t \in M\}$$

where t is an element of some auxiliary set M, and F are the functions that represent the properties of x.

12. Two things *interact* if each of them modifies the history of the other:

$$x \bowtie y \Leftrightarrow h(x \dot{+} y) \neq h(x) \cup h(y)$$

13. A thing $x_{\rm f}$ is a *reference frame* for x iff
(i) M equals the state space of $x_{\rm f}$, and
(ii) $h(x \dot{+} f) = h(x) \cup h(f)$

14. A *change* of a thing x is an ordered pair of states:

$$(s_1, s_2) \in E_{\rm L}(x) = S_{\rm L}(x) \times S_{\rm L}(x)$$

A change is called an *event*, and the space $E_{\rm L}(x)$ is called the *event space* of x.

15. An event e_1 *precedes* another event e_2 if they compose to give $e_3 \in E_{\rm L}(x)$:

$$e_1 = (s_1, s_2) \ \wedge \ e_2 = (s_2, s_3) \Rightarrow e_3 = (s_1, s_3)$$

The ontology sketched here (due mainly to M. Bunge) is realistic, because it assumes the existence of things endowed with properties, and objective, because it is free of any reference to cognoscent subjects.

We will base the axiomatic formulation of the pregeometry of space-time on this ontology and on the formal tools that will be described in the following.

3 Formal Tools

3.1 Topological Spaces

We give here just a brief review; for details the reader is referred to Thron (1966) and references therein.

D 1 $\mathcal{P}(A) =_{\mathrm{Df}} \{X/X \subset A\}$ *is the* power set *of the set* A.

D 2 *Let* A *be a set. A subset* $\mathcal{Z}$ *of* $\mathcal{P}(A)$ *is a* topology *on* A *if*

1. $\emptyset \in \mathcal{Z}$, $A \in \mathcal{Z}$.
2. *if* $A_i \in \mathcal{Z}$, $i \in [i_1, ..., i_n]$, *then* $\bigcup_{i=1}^{n} A_i \in \mathcal{Z}$
3. *if* $A_i \in \mathcal{Z}$, $i \in [i_1, ..., i_n]$, *then* $\bigcap_{i=1}^{n} A_i \in \mathcal{Z}$

The elements of $\mathcal{Z}$ *are usually known as the* open sets *of* A. *The pair* $(A, \mathcal{Z})$ *is called a* topological space. *The elements of* A *on which a topology* $\mathcal{Z}$ *is defined are the* points *of the space* $(A, \mathcal{Z})$

D 3 *A family* $\mathcal{B} \in \mathcal{P}(A)$ *is a* base *iff the family* $\mathcal{Z}$ *of all unions of elements of* $\mathcal{B}$ *is a topology on* $\bigcup\{B/B \in \mathcal{B}\}$. *It is said then that* $\mathcal{Z}$ *is the topology* generated *by* $\mathcal{B}$.

3.2 Filters

D 4 *A nonempty family* $\mathcal{B}$ *of subsets of a set* A *is a* filter *on* A *iff:*

1. $A_1 \in \mathcal{F} \wedge A_2 \in \mathcal{F} \Rightarrow A_1 \cap A_2 \in \mathcal{F}$
2. $B \supset A \in \mathcal{F} \Rightarrow B \in \mathcal{F}$
3. $\emptyset \notin \mathcal{F}$

D 5 *A nonempty family* B *of subsets of a set* A *is called a* filter base *on* A *provided* $\mathcal{B}$ *does not contain the empty set and provided the intersection of any two elements of* $\mathcal{B}$ *contains an element of* $\mathcal{B}$

3.3 Uniform spaces

D 6 *A nonvoid family* Λ *of subsets of* $A \times A$ *is a* uniformity *on* A *iff*

1. $L \supset \Delta = \{(x, x)/x \in A\}$ *for all* $L \in \Lambda$
2. $C \supset L \in \Lambda$ *implies* $C \in \Lambda$
3. *If* $L_1, L_2 \in \Lambda \Rightarrow L_1 \cap L_2 \in \Lambda$
4. *If* $L \in \Lambda \Rightarrow L^{-1} = \{(x, y)/(y, x) \in L\} \in \Lambda$
5. *For all* $L \in \Lambda$ *there exists a* $K \in \Lambda$ *such that* $K \circ K \subset L$, *where* $K \circ K = \{(x, y)/\exists z/(x, z) \in K, (z, y) \in K\}$

D 7 *The pair formed by* (A, Λ) *is called a* uniform space.

D 8 *If* Λ *satisfies that* $\bigcap L \in \Lambda = \Delta \Rightarrow \Lambda$ *is a* separated (or Hausdorff) uniformity.

Remark:

Notice that a uniformity is a filter on $A \times A$ each element of which contains Δ. Property 4 is a symmetry property, whereas property 5 is an abstract version of the triangle inequality.

D 9 *A set B is called* everywhere dense *in a set A iff $\bar{B}$ (the closure of B) $\supset A$.*

D 10 *A topological space (X, τ) is* separable *iff there exists an everywhere-dense subset of X which is denumerable.*

D 11 *A filter $\mathcal{F}$ in a uniform space (A, Λ) is called a* Cauchy filter *iff, given $L \in \Lambda$ there exist $M \in \mathcal{F}$ such that $M \times M \subset L$. A uniform space is* complete *iff every Cauchy filter has a limiting point.*

T 1 *Every separated uniform space has a* completion*. That is to say, one can always add "ideal elements" to complete the space.*

Proof:

see Thron (1966), pp. 184-185.

3.4 Metric spaces

D 12 *Let X be a set. A function $d : X \times X \mapsto \Re^+$ is a* metric *on X iff:*

1. *$d(x, y) = 0 \Rightarrow x = y$ for all $x, y \in X$*
2. *If $x = y \Rightarrow d(x, y) = 0$ for all $x, y \in X$*
3. *$d(x, y) = d(y, x)$ for all $x, y \in X$*
4. *$d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$*

D 13 *The pair (X, d) is a* metric space*.*

T 2 Theorem of metrization: *A uniform space is* metrizable *if and only if it is separable and its uniformity has a numerable base (Kelley 1962).*

T 3 Theorem of isometric completion: *Any metric space is isometric to a subspace dense in a complete metric space (Kelley 1962).*

T 4 *Let S be a subset of X. Let $(X, \mathcal{H})$ a uniform space. Then the family $\mathcal{H}_S = \{H \bigcup (S \times S)/H \in \mathcal{H}$ is a uniformity on S (called the relativized uniformity), and $\tau_{\mathcal{H}_S} = (\tau_{\mathcal{H}})_S$.*

4 Axiomatics

We present now the axiomatic core of our formulation. The generating basis of primitive concepts is:

$$B = \{\Xi, \mathcal{P}, S_{\rm L}, E_{\rm o}, E_{\rm G}, T_{\rm u}, \dot{+}, \dot{\times}, \leq, c\}.$$

The different symbols are characterized by the ontological background (Sect. 2) and a set of specific axioms. We shall classify these axioms in ontological (o), formal (f), and semantical (s), according to their status in the theory.

A 1 (o) *For each $x \in \Xi$ there exists a single ordering relation:*

$$s_1 \leq s_2 \iff s_2 = g(s_1)$$

where $g : S_{\rm L} \to S_{\rm L}$ is a legal statement.

A 2 (s) *The set of legal states of x, $S_{\rm L}(x)$, is* temporally ordered *by the relation $\leq$.*

D 14 $s_1 \leq s_2 \iff s_1$ precedes temporally s_2.

Remark:

The relation $\leq$ is a partially ordering relation: there are states that are not ordered by $\leq$ (*e.g.* given the initial conditions x_0, v_0, there are states, which are characterized by the values of x and v, that can not be reached by a classical particle).

D 15 *A subset of $S_{\rm L}(x)$ totally ordered by the relation $\leq$ is called* proper history *of x.*

A 3 (o) *For each thing x, there exists one and only one proper history.*

A 4 (o) *If the entire set of states of an ontological history is divided in two subsets $h_{\rm p}$ and $h_{\rm f}$ such that every state in $h_{\rm p}$ temporally precedes any state in $h_{\rm f}$, then there exists one and only one state s_0 such that $s_1 \leq s_0 \leq s_2$, where $s_1 \in h_{\rm p}$ and $s_2 \in h_{\rm f}$. In symbols:*

$$(\forall s_1)_{h_{\rm p}} (\forall s_2)_{h_{\rm f}} (s_1 \leq s_2)(\exists s_0)(s_1 \leq s_0 \leq s_2)$$

Remark:

This axiom expresses the notion of *ontological continuity.*

D 16 *$h_{\rm p}$ is called the* past *of s_0, and $h_{\rm f}$ is called the* future *of s_0.*

Remark:

Notice that past and future are meaningful concepts just when they are referred to a given state s_0.

A 5 (o) *For every thing x, there exists another thing $x_{\rm t}$ called* clock*, and an injective application ψ such that:*

1. $\psi_t : S_{\mathrm{L}}(x_t) \to S_{\mathrm{L}}(x)$
2. *Given* t, $t' \in S_{\mathrm{L}}(x_t)$*:* $t \leq t' \Rightarrow \psi(t) \leq \psi(t')$

T 5 *Given a thing* x *with ontological history* $h(x)$ *and an arbitrary system of units* U_τ *there exists a bijection:*

$$\mathcal{T} : h \times U_\tau \leftrightarrow \Re$$

that gives a parametrization $s_x = s_x(\tau)$.

Proof:

From **A3**, **A4**, and Rey Pastor *et al.* (1952).

D 17 *The variable* τ *is called* proper time *of* x.

T 6 *Let* x_t *be a clock for* x*, with event space* $E_{\mathrm{L}}(x_t)$*, and* U_t *an arbitrary system of units. There exists a bijection*

$$T : E_{\mathrm{L}}(x_t) \times U_t \leftrightarrow \Re$$

that provides a parametrization $s = s(\tau)$.

Proof:

Generalization of **T5**.

D 18 t *is the* duration *of an event of* x *respect to the clock* x_t.

This is what we need to say about time. For more details, see Bunge (1977).

A 6

$$(\forall x)(x \sqsubset \Box)(\exists y)(y \sqsubset \Box \wedge y \bowtie x)$$

Remark:

This latter axiom states that there exist no completely isolated things.

We shall show now that the relation of interaction $\bowtie$ (see Sect. 2), generalized in a convenient way, induces a uniform structure (see Sect. 3) on the set of basic things. It is important to note that the relation $\bowtie$ is symmetric but neither reflexive nor transitive. However, it is always possible to define a reflexive-transitive closure of a given relation (see Salomaa 1973). The closure $\bowtie^*$ of the relation of interaction is the set of pairs of basic things that interact either directly or by means of a chain (finite or infinte) of basic things.

Now the following theorem can be proved:

T 7 *The relation* $\bowtie^*$ *defines a uniform structure on* Ξ.

Proof:

Every equivalence relation defines a uniform structure on a set (Thron 1966).

Remark:

Armed with this theorem, we will be able to give space a uniform structure.

In order to introduce the concept of space we shall use the notion of *reflex action* between two things. Intuitively, a thing x acts on another thing y if the presence of x disturbs the history of y. Events in the real world seem to happen in such a way that it takes some time for the action of x to propagate up to y. This fact can be used to construct a relational theory of space *á la* Leibniz, that is, by taking space as a set of equitemporal things. It is necessary then to define the relation of simultaneity between states of things.

Let x and y be two things with histories $h(x_\tau)$ and $h(y_\tau)$, respectively, and let us suppose that the action of x on y starts at τ_x^0. The history of y will be modified starting from τ_y^0. The proper times are still not related but we can introduce the reflex action to define the notion of simultaneity. The action of y on x, started at τ_y^0, will modify x from τ_x^1 on. The relation "the action of x is reflected on y and goes back to x" is the reflex action. Historically, G. Galilei (1945) introduced the reflection of a light pulse on a mirror to measure the speed of light. With this relation we will define the concept of simultaneity of events that happen on different basic things (see also Landau & Lifshitz 1967).

We have already seen in Sect. 2 that a thing x acts upon a thing y if the presence of x modifies the history of y:

$$x \rhd y =_{\mathrm{Df}} h(y|x) \neq h(y)$$

where $h(y|x)$ represents the history of y in the presence of x.

The *total action* of x upon y is

$$\mathcal{A}(x,y) = h(y|x) \cap \overline{h(y)},$$

where the overline denotes the complement.

Let us define now the history of x after τ_x^0 as

$$h(x,\tau_x^0) = h(x)|_{\tau_x > \tau_x^0}$$

and similar definitions for $h(y,\tau_y^0)$ and for the history of y after τ_y^0 in the presence of x after τ_x^0, denoted here as $h(\langle y,\tau_y^0\rangle,\langle x,\tau_x^0\rangle)$.

The total action of x after τ_x^0 on y after τ_x^0 is

$$\mathcal{A}(y,x^0) = h(y|x) \cap \overline{h(\langle y,\tau_y^0\rangle,\langle x,\tau_x^0\rangle)}$$

In a similar way we define the action of y on x posterior to τ_y^1.

τ_y^0 is the minimum value of the proper time of y for which the action of x posterior a τ_x^0 is felt.

$$\tau_y^0 = \inf\{\tau_y|\mathcal{A}(y,x^0)\}$$

This quantity always exists, because of ontological continuity, enforced in **A4**.

Similarly, we define τ_x^1:

$$\tau_x^1 = \inf\{\tau_x|\mathcal{A}(x,y^0)\}$$

Finally we can introduce a relation between the three instants involved in the reflex action. We will call $\mathcal{R} < \tau_x^0, \tau_y^0, \tau_x^1 >$ the relation given by the set of ordered 3-tuples and established by the previous equations.

Let us go back to the axiomatics.

A 7 (o) *Given two different and separated basic things x and y, there exists a minimum positive bound for the interval $(\tau_x^1 - \tau_x^0)$, defined by $\mathcal{R}$.*

Remark:

Hereafter we shall deal only with 3-tuples $< \tau_x^0, \tau_y^0, \tau_x^1 >$ that satisfy the minimum condition.

D 19 τ_y^0 *is simultaneous with* $\tau_x^{1/2} =_{\mathrm{Df}} 1/2(\tau_x^0 + \tau_x^1)$.

T 8 τ_x *and* τ_y *can be* synchronized *by the simultaneity relation.*

Proof:

There exists a bijection between τ_x and τ_y because $\mathcal{R}^{-1}$, the inverse of $\mathcal{R}$, is well-defined.

Comment: As we know from General Relativity, the simultaneity relation is transitive only in special reference frames called *synchronous* (Landau and Lifshitz 1967). We then include the following axiom:

A 8 (f) *Given a set of basic things* $\{x_1, x_2, ...\}$*, there exists an assignation of proper times* $\tau_1, \tau_2, ...$ *such that the relation of simultaneity is transitive.*

T 9 *The relation of simultaneity is an equivalence relation.*

Proof:

From **T8** and **A7**.

Remark:

We should mention that, because of **T5** and **D17**, the history of a given thing is parametrized by its proper time τ. Then, the relation of simultaneity is defined not over things but over states of things.

D 20 *The equivalence class of states defined by the relation of simultaneity on the set of all basic things is the* ontic space E_{o}.

T 10 *The ontic space* E_{o} *has a uniform structure.*

Proof:

Let S be a set of states of things related by the simultaneity relation. Because of the uniqueness of the ontological history postulated in **A3**, there is a one-to-one relation between a state in S and a given thing, and then S is isomorphic to a subset of Ξ. Then, by **T4**, S is a uniform space.

A 9 *There exists a subset* D *in the set of simultaneous states of interacting things* S *that is denumerable and dense in* S.

Remark:

This axiom requires space to be a *plenum*. Indeed, this hypothesis (introduced by Aristotle and later supported by Leibniz) is central to Quantum Physics, and it permits the prediction of a plethora of vacuum phenomena (like the Casimir effect), in good agreement with observation.

A 10 *Each* $x \in \Xi$ *interacts with a denumerable set of basic things.*

T 11 *The power set of Ξ reduced to the equivalence class is a basis for the uniformity (Bourbaki 1964).*

So now we are in the conditions of the th. of metrization:

T 12 *The ontic space is metrizable.*

Proof:

Immediate, from **T2**.

D 21 *τ_u is the proper time of a reference thing x_{f} .*

T 13 $(\forall x)_{\Theta}(\exists f_x)(\tau_x = f_x^{-1}(\tau_u))$

Proof:

Immediate, from **A5**.

Remark:

We shall call τ_u the *universal time.*

The ontic space E_{o} is still devoid of any geometric properties and consequently cannot represent the physical space. We postulate then:

A 11 (f) *The metrization of the ontic space is given by*

$$d(x, y) = \frac{1}{2} c |\tau_{\mathrm{x}}^1 - \tau_{\mathrm{x}}^0|$$

where c is a constant with appropriate dimensions, and the distance is evaluated at τ_y^0, which is simultaneous with $\tau_x^{1/2}$.

T 14 *The ontic space is isometric to a subspace dense in a complete space.*

Proof:

The proof follows immediately form the theorem of isometric completion (**T3**).

D 22 *The complete space mentioned in* **T14** *is called* geometric space E_{G}.

Remark:

Because of the isometry mentioned in **T14**, E_{G} inherits the metric of E_{o}. Besides, note that every filter of Cauchy has a limiting point in E_{G}, because this space is complete.

D 23 *The elements of the completion are called* ideal things.

Remark:

It should be noted that the ideal things (which are abstract objects) do not belong to the ontic space but to the geometric space.

A 12 (f) *The points in E_{G} satisfy the following conditions:*

1. *Given two points x and y there exists a third point y aligned with x and z.*
2. *There exist three nonaligned points*
3. *There exist four noncoplanar points.*
4. *There exist only three spatial dimensions* [3].

Remark:

All the conditions in **A12** can be expressed in terms of the distance $d(a,b)$. For instance, (1) can be written in the form:

$$\begin{aligned} (\forall(a))_G(\forall(b))_G(\exists c)_G\,(d(a,b)+d(b,c) &= d(a,c)\vee && (1)\\ d(a,c)+d(c,b) &= d(a,b)\vee && (2)\\ d(c,a)+d(a,b) &= d(a,c)) && (3) \end{aligned}$$

For details, see Blumenthal (1965).

T 15 *The geometric space E_{G} is globally Euclidean.*

Proof:

From **A12**, see Blumenthal (1965).

Remark:

The ontic space E_{o} is not Euclidean but dense on an Euclidean space (*i.e.* E_{G}). This is a consequence of the fact that a *"sequence of Cauchy of things"* does not have in general a thing as a limit.

D 24 T_{u} *is the the bijection alluded to in* **T6** *for the reference thing x_{f} with proper time τ_{u}.*

T 16 *There exists a nontrivial geometric structure on $E_{\mathrm{G}}\times T_{\mathrm{u}}$.*

Proof:

Let us introduce a Cartesian coordinate system in E_{G}, with origin located in the reference thing x_{f}. From **T6**, **D19** and **A11**,

$$(t_{\mathrm{y}}-t_{\mathrm{x}})^2=\left[\frac{d(y,x)}{c}\right]^2 \qquad (4)$$

This equation describes a sphere of radius $c\delta t$ centered at x. Then the family of spheres $S(x,t_{\mathrm{x}})\subset E_{\mathrm{G}}\times T_{\mathrm{u}}$ defines a geometric structure on $E_{\mathrm{G}}\times T_{\mathrm{u}}$.

A 13 (o) *The* cones of action *determined by (4) are independent of the reference thing x_{f}.*

[3]This may not be true at the Planck scale. See for instance Tegmark 1997.

T 17 *The quadratic form*

$$\delta s^2 = (c\delta t)^2 - (\delta \vec{r})^2 \tag{5}$$

is invariant under changes of the reference thing.

Cor. 1 $E_{\mathrm{G}} \times T_{\mathrm{u}}$ *has a Minkowskian structure.*

T 18 *The only coordinate transformations that leave invariant the quadratic form (5) are the Lorentz transformations.*

A 14 (s) $E_{\mathrm{G}} \times T_{\mathrm{u}}$ *represents the physical space-time.*

This axiom completes the formulation of the theory. It should be noted that the spatial relations that we perceive are defined between macroscopic (*i.e.* composed) things. Our system of axioms can handle also this situation (which strictly does not belong to Protophysics but to Physics), if we incorporate a specific model for a given thing, which should be based on an explicit form of the interaction between basic things.

5 Further comments

5.1 Extension to quantum basic things

The relational theory of space-time that we developed in the precedent section is based on the concept of basic thing. We should remark that this is a theory-dependent concept, *i.e.* two different theories may take different sets of things as basic. In this sense, our theory is classical (as opposed to quantum), because **T2** enforces the separability of basic things. However, it can be conveniently modified to serve as a part of the protophysics of Quantum Theories, and this we shall do in the following.

The main problem is the fact that quantum particles can superimpose (*i.e.* the distance between two of them can be zero while the particles are still distinguishable). To incorporate this fact, we shall relax **D12**, keeping only items (2), (3), and (4). With this modification, **D12** defines a pseudometric instead of a metric (Kelley 1962).

Now we replace **T2** with

T 19 Theorem of pseudometrization: *a uniform space is pseudometrizable if its uniformity has a denumerable base.*

So, due to **A10**, E_{o} is pseudometrizable. Morover, it can be completed:

T 20 *Every pseudometric space is isometric to a subspace dense in a complete pseudometric space.*

We shall call this last space E_p (*pregeometric space*). But we know that in Quantum Mechanics the Euclidean space is included in the corresponding protophysics (Perez Bergliaffa *et al.* 1993). To recover Euclidean space, we begin by introducing the concept of ontic point:

D 25 *Let* $X \sqsubset \Xi$ *be a family of basic things. We say that* X *is a* complete family of partially superimposed things *if*

1. $(\forall x)_X (\forall y)_X (x \dot{\times} y \neq \diamond)$
2. $(\forall x)_{\bar{X}} (\exists y)_X (x \wr y)$

We shall call this kind of family an *ontic point*, because the (pseudometric) distance between any two components of the family is zero.

A 15 *Let ξ and η two ontic points. Then* $(\forall x_i)_\xi(\forall y_j)_\eta(\exists\, C(\xi,\eta))(d_p(x_i,y_j) < C(\xi,\eta))$.

D 26 *Let ξ and η be two ontic points. The distance between them is given by*

$$d_G(\xi,\eta) =_{\mathrm{Df}} \sup_{(i,j)} d_p(x_i,y_j) \tag{6}$$

with $x_i \in \xi$ and $y_j \in \eta$.

Remark:

The axiom **A15** guarantees that this distance is well-defined.

T 21 *The set of ontic points, toghether with the distance function (6) is a metric space.*

Proof: *the items (2), (3), and (4) of* **D12** *are trivially satisfied because d_p is a pseudometric. Regarding the first item, if ξ and η are two ontic points, there exist $x_i \in \xi$ and $y_j \in \eta$ such that $x_i \wr y_j$. The condition $d_p(x_i,y_j) > 0$ is satisfied because d_p is a pseudometric. Then, $\xi \neq \eta \Rightarrow d_G(\xi,\eta) > 0$, which can be written as $d_G(\xi,\eta) = 0 \Rightarrow \xi = \eta$.*

T 22 Theorem of isometric completion: *any metric space is isometric to a subspace dense in a complete metric space (Kelley 1962).*

D 27 *The completation of the space of ontic points is the geometric space E_{G}.*

From this point, the construction goes on as in the previous case.

5.2 Extension to Riemannian spaces

Our theory is a pregeometry for a Minkowskian space-time. Gravitational physics, however, requieres more complex structures. In this section we shall sketch the necessary steps that lead to a Riemanian space, which is used in General Relativity (Covarrubias, 1993), in an informal way, avoiding the technicalities.

A Riemannian space can be obtained from our theory using a tetrad formulation. For this we need at least an axiom elucidating the connection between ontic and geometric spaces:

A 16 (f) *The geometric space E_{G} is the tangent space to the ontic space at the reference thing x_{f}.*

With this axiom, the connection between ontic and geometrical spaces will be purely local. The full space will be constructed by pasting together patches of quasieclidean pieces. The following axiom sketches the way this can be done:

A 17 (f) *There exists a parallel displacement operator ω connecting the components of vectors (*i.e. *elements of the tangent space) tangent to E_{o} on neighbouring things.*

With the parallel displacement, a covariant derivative can be defined in the usual way. The usual property of the Riemannian conection (Ricci coefficients) must be posited. The following axiom will do the job:

A 18 (f) *The covariant derivative* ∇ *annihilates the metric.*

Since we are working with a transitive simultaneity relation (which is equivalent to use a synchronous reference frame in any metric theory of gravitation (Landau & Lifchitz 1967) a Riemannian space will define a unique pseudoriemannian spacetime. In this way a protophysics for a rigurous formulation of General Relativity (such as Covarrubias (1993)) and more general theories of gravitation will be obtained.

The above scheme must be completed in several ways. Accurate definitions should be given of the different constructs defined. Also several axioms should be introduced to ensure a differential manifold structure on a suitable completition of the ontic space. We shall not pursue further this matter here, but in a future communication.

5.3 Comparison with the theory of Bunge

As we have mentioned in the Introduction, protophysical theories can be classified as subjective and objective, according to whether cognoscent subjects and/or sensorial fields are considered as basic objects or not. Bunge (1977) developed an objective and realistic relational theory of space time and made a clear cut comparison with other subjective and objective theories. In this section, we shall limit ourselves to a comparison of the objective and realistic theory of Bunge (1977) with the one developed in the present paper.

Bunge's theory of space is based on the *interposition relation* $(x|y|z)$ that can be read "y interposes between x and z". The properties of this relation are posited in Axioms 6.1 to 6.6 of Bunge (1977). In this section we shall show how the corresponding relation can be constructed in the present theory.

We shall first define a similar relation between basic things.

D 28 *Let* $x, y, z \in \Xi$*. We shall say that* $[x|y|z]_\Xi$ *if*

$$(d(x,y) + d(y,z) = d(x,z)) \wedge (x \wr y \wr z \wr x) \vee (x = y = z))$$

The next theorem proves that in our theory, the interposition relation holds between basic things if and only if it is valid in Bunge's theory:

T 23 *Let* $x, y, z \in \Xi$*. Then* $[x|y|z]_\Xi$ *iff* $(x|y|z)$*.*

Proof:

The proof consists in showing that $[x|y|z]_\Xi$ satisfies each of the seven conditions *(i)–(vii)* of Axiom 6.1 in Bunge (1977).

In order to define an interposition relation for general things we shall use our interposition relation for basic things:

D 29 *Let* $\xi, \eta, \zeta \in \Theta$*. Then* $[\xi|\eta|\zeta]_\Theta$ *either if they are equal* ($\xi = \eta = \zeta$) *or if there exists three separate basic things* $x, y, z \in \Xi$ *that are parts of one thing but not of the others and that interpose.*

$$\begin{aligned}
[\xi|\eta|\zeta]_\Theta \quad =_{\mathrm{Df}} \quad & (\xi = \eta = \zeta) \vee \\
& \exists (x, y, z \in \Xi) \{[(x \sqsubset \xi) \wedge (x \wr \eta) \wedge (x \wr \zeta)] \wedge \\
& [(y \sqsubset \eta) \wedge (y \wr \xi) \wedge (y \wr \zeta)] \wedge \\
& [(z \sqsubset \zeta) \wedge (z \wr \eta) \wedge (z \wr \xi)] \wedge \\
& [x|y|z]_\Xi\}
\end{aligned} \tag{7}$$

T 24 *Let $\xi, \eta, \zeta \in \Theta$. Then $[\xi|\eta|\zeta]_\Theta$ if $(\xi|\eta|\zeta)$.*

In the same way, with appropriate definitions, it is possible to show that the remaining postulates of Bunge's theory can be recovered as theorems in our formulation.

The time theory exposed in the first part of our axiomatics is essentially the same theory exposed in Bunge (1977), although our axioms are somewhat different. The main differences lies in Axms. 3 and 4. The first one, not explicitly stated in Bunge (1977), forbids "gardens of bifurcating paths" (Borges, 1967) or, in general, more than one time-like direction. The second axiom may be taken as a reformulation of the heraclitean principle: "Panta rhei".

From a formal point of view, the present theory of space-time is very different from the theory in Bunge (1977). This is because our fundamental relation of "reciprocal action" is very limitative and the related axioms are extremely strong: we are led almost without ambiguity to a Minkowskian structure of space-time.

Finally, it is important to remark that, since both theories have the same referents (namely, things and their properties), they are referentially equivalent, realistic and objective relational theories of space and time.

5.4 The nature of space-time

In the present theory, space-time is not a thing but a substantial property of the largest system of things, the world $\square$, emerging from the set of the relational properties of basic things. Thus, any existential quantification over space-time can be translated into quantification over basic things. This shows that space-time has no ontological independence, but it is the product of the interrelation between basic ontological building blocks. For instance, rather than stating "space-time possesses a metric", it should be said: "the evolution of interacting things can be described attributing a metric tensor to their spatio-temporal relationships". In the present theory, however, space-time is intrepreted in an strictly materialistic and Leibnitian sense: it is an order of succesive material coexistents.

6 Summary

We have developed a materialistic relational theory of space-time, that carries out the program initiated by Leibniz, and provides a protophysical basis consistent with any rigorous formulation of General Relativity. Space-time is constructed from general concepts which are common to any consistent scientific theory. It is shown, consequently, that there is no need for positing the independent existence of space-time over the set of individual things.

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