Towards a Logic of Epistemic Theory of Measurement

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Abstract. We propose a logic to reason about data collected by a number of measurement systems. The semantic of this logic is grounded on the epistemic theory of measurement that gives a central role to measurement devices and calibration. In this perspective, the lack of evidences (in the available data) for the truth or falsehood of a proposition requires the introduction of a third truth-value (the undetermined). Moreover, the data collected by a given source are here represented by means of a possible world, which provide a contextual view on the objects in the domain. We approach (possibly) conflicting data coming from different sources in a social choice theoretic fashion: we investigate viable operators to aggregate data and we represent them in our logic by means of suitable (minimal) modal operators.

Keywords: measurement theory, social-choice theory, three-valued logic, logic of evidence, epistemic logic

1 Introduction

The need for grounding rational beliefs, for understanding what supports the epistemic states of agents, is vastly acknowledged. Grounding is particularly relevant for scientific claims that are usually justified in terms of observations and of empirical data. In this context, data cannot be private, they must be shared and trusted by different subjects at different times and in different places. According to [15], the *objectivity* and the *inter-subjectivity* of the scientific results distinguish *measurement* from *evaluation*. Objectivity concerns the independence of the measurements of a given property (of a measurand) of other properties, measuring devices, and environmental conditions. Inter-subjectivity regards the sharing of measurements and it is achieved by establishing measurement standards and calibration procedures for devices. Measurement theories play then a central role for the collection and the sharing of trusted data, a pre-requisite for grounding empirical science. Our aim is to develop a logic that explicitly represents how propositions connect to empirical data by exploiting the *epistemic measurement theory* (EMT) introduced in [9, 11, 15, 16].

Epistemic logics [26] and evidence logics [3] have already considered the grounding of epistemic states. Evidences for beliefs are encoded as sets of (or, in neighborhood semantics, as families of sets of) *possible worlds* that, in these logics, reduce in fact to plain and unstructured indexes.

By contrast, in our proposal, possible worlds are structured entities that explain the contextual nature of epistemic measurements. Possible worlds, called here *states*, are then characterised in terms of EMT. A world provides a group of *datasets*, i.e., a set of data (about the objects in the domain) collected by batteries of measurement systems. Intuitively, a world gathers the (possibly *partial*) information about the objects in the domain supplied by a given source. The truth of a proposition at a certain state is then *contextual* because it depends on the data collected by the source, i.e., on the measurement systems available at that source and on the performed measurements. As we shall see, the partiality of data forces the use of a three-valued semantics. A given dataset may not contain enough information to establish neither the truth nor the falsity of a proposition. A third truth-value, the *undetermined* (or unknown) is required. The interpretation of the undetermined value that we endorse here is close to the one introduced by Kleene [13] to represent the situation where an algorithm does not terminate yielding a 'true' or 'false' output.

Ideally, objectivity and inter-subjectivity guarantee that all the data collected by calibrated devices are consistent and sharable without errors. However, in a realistic scenario, measurement devices may be used in unsuitable environmental conditions or following wrong procedures. They may also malfunction or lose calibration during their life (typically, devices are not re-calibrated at every use). These issues may especially be present in (*i*) large-scale and distributed collaborative science, that often relies on (user-generated) data which are collected, for instance, through sensors in mobile and ubiquitous devices; and (*ii*) scientific endeavour that relies on tests, e.g. neuropsychological, clinical or behavioural, where the scores of the tests are the result of very complex procedures that aggregate heterogeneous measurements. Because of these complications, conflicting datasets may exist, i.e., the sources of data may disagree.

To approach conflicting sources, we extend our logic with modal operators inspired by social choice theory and judgment aggregation [14]. These operators represent different strategies to aggregate the heterogeneous data collected by several (possibly conflicting) sources. We shall see that a careful analysis of such procedures is required, as some procedures do not guarantee the consistency of the aggregated data. Moreover, in a scientific scenario, the data collected by the sources are often *sparse*, i.e., typically, only few sources have information about the same objects. This scenario is different from the one of judgement aggregation, where 'abstainers' are usually not the majority. The standard aggregation procedures need then to be adapted to take into account what are the sources that have relevant information about a given proposition.

By aggregating the information coming from heterogeneous epistemic contexts, these modal operators introduce a *de-contextualisation*, as intended in [19], of the truth of propositions. Aggregated statements are cross-contextual, they integrate (following a given strategy) the perspectives of several epistemic contexts (see Sect. 3.4 for more details).

The paper is organised as follows. Section 2 introduces EMT and precisely defines states as datasets collected by batteries of measurement systems. Section

3 introduces the logical framework for representing and reasoning about data collected by a single source and the modal aggregation operators. Moreover, it discusses the contextual nature of the measurement statements. Section 4 concludes the paper.

2 Measurement systems and datasets

According to the *Representational Measurement Theory* (RMT) [25], measurement consists in building a mapping from an *empirical* relational structure to a *numerical* relational structure such that the relations among numbers represent the empirical relations among objects. Despite the precise and deep mathematical treatment, RMT seems too abstract to be used in empirical contexts [9]. One first problem concerns the fact that RMT focuses on quantitative measurement (interval or ratio properties). Secondly, RMT considers the empirical relational structure (and the axioms governing it) as given. The problems of founding measurement on empirical methods and of data sharing are not addressed.

The perspective of *qualitative* classification (nominal or ordinal properties), that plays a fundamental role in disciplines like psychology, medicine, or sociology, has been addressed by the theory of *weak measurement* introduced by Finkelstein [8] and further elaborated by Mari [15]. In this weak perspective, measurement becomes "uncorrelated with quantification: the measurability of a property is a feature derived from experiment, not algebraic constraints" [15, p.2894]. Thus, not all the empirical structures are mapped into numerical structures, they can also be mapped into symbolic classification systems, where symbols may have a weak organization, not necessarily an algebraic one. Moreover, Frigerio and colleagues [9] follows this weak perspective by presenting a formal model that grounds measurement on *measurement systems* (MSs). Roughly speaking, an MS is a (physical) device that is able to interact with the system under measurement (SUM)[1] and that is characterized by a set of empirically discernible states and relations to which symbols are conventionally associated. The output of the interaction between an MS and a SUM is a piece of (symbolic) information. While weak measurement (as well as RMT) assumes SUMs to be states or individual properties (tropes) of objects, following [16], we do not commit to these kinds of entities and consider SUMs to be objects, i.e., MSs are mediators between (external) objects and measurements, sorts of physical embodiments of the classification systems ([17] provides additional details).

Definition 1. (Measurement system) *A measurement system is a tuple* $\mathcal{M} = \langle d, O, \mathcal{E}, \kappa, \mathcal{S}, \lambda \rangle$ *where:*

- d *is a device (usually a physical object);*
- O *is the set of objects the device* d *is able to interact with;*

[1] MSs are "provided with instructions specifying how such interaction must be performed and interpreted" [9].

- $\mathcal{E} = \langle U, R_1, \ldots, R_n \rangle$ *is an empirical structure, i.e., a structure where* U *is the set of empirically discernible states of all possible complex systems resulting from the interaction of any object* $o \in O$ *with* d *(noted by* $d \bullet o$*) and* R_i *are empirically discernible relations among the states in* U*;*[2]
- $\kappa : O \to U$ *is the interaction function that associates to an object* $o \in O$ *the state of the complex system* $d \bullet o$*;*
- $\mathcal{S} = \langle S, RS_1, \ldots, RS_n \rangle$ *is a symbolic structure, i.e., a structure where* S *is a set of symbols and* RS_i *are relations defined on* S*;*[3]
- $\lambda : U \to S$ *is the symbolization function, a one-to-one function between* U *and* S*, such that* $R_i(u_1, \ldots, u_n)$ *iff* $RS_i(\lambda(u_1), \ldots, \lambda(u_n))$*.*

The states U (through the interaction function κ) induce a partition on the set of objects O: $o \approx o'$ iff $\kappa(o) = \kappa(o')$ (U establishes the *resolution* of d). Similarly, each R_i (and each RS_i) induces a relation on objects: $\bar{R}_i(o_1, \ldots, o_n)$ iff $R_i(\kappa(o_1), \ldots, \kappa(o_n))$. The empirical structure is here determined by the MS that *induces* a structure on objects (by interacting with them), i.e., an MS (and the measurement procedure) provides an empirical access point to the world. The symbolization function and the symbolic structure allow to abstract from the empirical structure, they provide a symbolic encoding, i.e., $\mathcal{S}$ contains the whole information in $\mathcal{E}$ but in a *communicable* and *manipulable* form. Different measurement systems can then share the same symbolic structure allowing for alternative ways to measure the same kind of properties.

As we discussed in the introduction, the objectivity and inter-subjectivity of data is obtained via measurement standards and calibration. A measurement standard establishes a set of physical (or theoretical) objects that is isomorphic to the symbols in the classification system, i.e., they are the perfect realization of the properties represented by the symbols. Calibration determines a one-to-one correspondence between the (relations between the) positions of the pointers of an MS and the (relations between the) properties in the classification system, i.e., the positions of the pointers and the output symbols stand for properties, they have a meaning.[4] Thus, a measurement standard determines a classification system while calibration individuates all the MSs that can be (interchangeably) used to classify objects in this system.

As we anticipated in the introduction, we do not commit to perfect calibration. The MSs have been calibrated, but nothing guarantees that, at every time data are collected, the MSs are correctly used and still calibrated. The outputs of a single MS have a shared and precise meaning and are consistent, but conflicting data collected by different MSs may exist.

A *dataset* groups all the measurements collected by a single MS.

Definition 2. (Dataset) *A dataset is a couple* $\mathcal{D} = \langle \mathcal{M}, D \rangle$ *where:*

- $\mathcal{M} = \langle d, O, \mathcal{E}, \kappa, \mathcal{S}, \lambda \rangle$ *is a measurement system;*

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[2] Notice that $\mathcal{E}$ refers to potential interactions with objects, i.e., by abstracting from specific objects, it depends only on the (physical) structure of d.

[3] Differently from RMT, $\mathcal{S}$ is not necessarily a *numerical* structure.

[4] See [16] for the formal details.

– *D is the set of data collected by $\mathcal{M}$, i.e., the (possibly empty) set of pairs $\langle o, m \rangle$ such that $\lambda(\kappa(o)) = m$.*

Note that D is consistent by construction, it is not possible to have $\langle o, m_1 \rangle$, $\langle o, m_2 \rangle \in D$ with $m_1 \neq m_2$.

An MS is able to classify objects along a single classification system. However, one can have data that concern different properties of the same object.[5] The notion of *measurement battery* (MB) extends the one of MS by considering sets of MSs able to classify objects along several classification systems (e.g., a thermometer together with a scale, a ruler, etc.).[6]

Given a set $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of MSs, we denote by S_i the set of symbols of $\mathcal{M}_i$ and by $\mathbb{RS}_i^h$ the set of h-ary relations on S_i-symbols in $\mathcal{M}_i$.

Definition 3. (Measurement battery) *A measurement battery is a finite set of MSs $\mathfrak{M} = \{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ such that, for all $\mathcal{M}_i = \langle d_i, X_i, \mathcal{E}_i, \kappa_i, \mathcal{S}_i, \lambda_i \rangle$ and $\mathcal{M}_j = \langle d_j, X_j, \mathcal{E}_j, \kappa_j, \mathcal{S}_j, \lambda_j \rangle \in \mathfrak{M}$ with $i \neq j$, we have that:*

1. *$X_i = X_j = O$, i.e., every $\mathcal{M}_i \in \mathfrak{M}$ is about the same set of objects O; and*
2. *$S_i \cap S_j = \emptyset$, i.e., the symbols of the MSs in $\mathfrak{M}$ are disjoint.*

A *state* collects all the datasets provided by the MSs in an MB. MBs and states are the multidimensional counterparts of, respectively, MSs and datasets.

Definition 4. (State) *A state s is a set of datasets s.t. their respective MSs form a MB, i.e., $s = \{\langle \mathcal{M}_1, D_1 \rangle, \ldots, \langle \mathcal{M}_n, D_n \rangle\}$ where $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ is a MB.*

Note that, since each MS is consistent, the condition 2 in Definition 3 guarantees the consistency of states (i.e. it is not possible to have $\langle o, m_1 \rangle$, $\langle o, m_2 \rangle$ such that $m_1 \neq m_2$ and m_1, m_2 are in the same symbolic structure).

Finally, we introduce a finite set of states $\mathfrak{S}$ to model data coming from distinct measurement batteries. Single states do not contain any contradictory measurement, while different states can disagree. This disagreement is due to the use of different MSs that classify objects along the same system of properties.

3 A logic for measurement

We present a predicative modal logic to represent and reason about the data provided by a number of MBs. We shall see that a single state provides sufficient information to define the semantics of logical connective and quantifiers. We start by defining the predicative structure, then we shall discuss several modal operators that may be used to represent aggregations of MBs.

[5] In terms of the theory of *conceptual spaces* [10], single classification systems correspond to the *domains* of a conceptual space (e.g., color, taste, shape, temperature, etc.), while the whole space requires the composition of several systems.

[6] It is possible to extend the notion of MB to allow to have different MSs relative to the same classification system, e.g., different scales, different thermometers, etc.

3.1 Syntax and semantics

The vocabulary of our predicative language $\mathcal{L}$ contains: a set of individual constants $\mathbf{C} = \{c_1, c_2, \dots\}$, a set of individual variables $\mathbf{V} = \{x_1, x_2, \dots\}$, a set of n-ary ($n \geq 1$) predicates $\mathbf{R} = \{R_1^1, R_2^1, \dots, R_1^2, R_2^2, \dots, R_i^j, \dots\}$, the set $\{\neg, \wedge, \vee, \rightarrow\}$ of connectives, and the set $\{\forall, \exists\}$ of quantifiers . The set of atomic formula *Atom* of $\mathcal{L}$ is defined as follows: $Q_i^j(a_1, \dots, a_j) \in \mathit{Atom}$ iff $Q_i^j \in \mathbf{R}$ and $a_1, \dots, a_j \in \mathbf{C}$. This definition inductively extends to the full predicative language as usual.

Given a state $s = \{\langle \mathcal{M}_1, D_1\rangle, \dots, \langle \mathcal{M}_l, D_l\rangle\}$, we denote by $\delta(s)$ the set of all measurements that are present in some dataset of s, i.e., $\delta(s) = D_1 \cup \ldots \cup D_l$.

Definition 5. (Measurement model) *A* measurement model *for $\mathcal{L}$ is a tuple $M = \langle s, \varepsilon, \iota\rangle$ where:*

- *s is a state concerning the set of objects O, i.e. a set $\{\langle \mathcal{M}_1, D_1\rangle, \dots, \langle \mathcal{M}_l, D_l\rangle\}$;*
- *ε is a function that maps individual constants into objects, $\varepsilon : \mathbf{C} \rightarrow O$;*
- *ι is a function that maps:*
  - *unary predicates into symbols of the MSs in the MB in s:* $\iota : \mathbf{R^{(1)}} \rightarrow S_1 \cup \dots \cup S_l$;
  - *n-ary ($n \geq 2$) predicates into n-ary relations of the MSs in the MB in s:* $\iota : \mathbf{R^{(n)}} \rightarrow \mathbb{RS}_1^n \cup \dots \cup \mathbb{RS}_l^n$.

The domain of the interpretation is then given by the set of objects of the state s, i.e., by O, the interpretation of the individual constants is provided by ε, and the interpretation of predicates is provided by ι. We shall introduce the interpretation for the variables when discussing the semantics of quantifiers.

The valuation function $||\cdot||_M$ maps formulas to a suitable set of truth-values. Since unary predicate and n-ary ($n \geq 2$) relations have slightly different interpretations, we present their semantics separately. Moreover, to reflect a verificationist perspective on truth-making, we assume three truth values $\{t, f, u\}$. Intuitively, *true* means that there exists a verifier of ϕ in $\delta(s)$, *false* means that there exists a falsifier of ϕ in $\delta(s)$, and *undetermined* means that there is neither a verifier nor a falsifier of ϕ in $\delta(s)$.

The semantics for atomic formulas involving unary predicates is defined as:

- $||P(a)||_M = t$ iff $\langle \varepsilon(a), \iota(P)\rangle \in \delta(s)$;
- $||P(a)||_M = f$ iff there exists $\langle \varepsilon(a), m\rangle \in \delta(s)$ with m and $\iota(P) \in S_i$, for some i, and $m \neq \iota(P)$;
- $||P(a)||_M = u$ iff otherwise.

A falsifier of $P(a)$ is then a measurement of the object $\varepsilon(a)$ along the same system of properties of $\iota(P)$. For instance, to falsify $\mathit{1KG}(a)$, among the data available in s, one needs to find a *weight*-measurement of $\varepsilon(a)$ with a result different from $\iota(\mathit{1KG})$. We can follow this idea because the symbols in the S_i are considered as mutually exclusive, i.e., in principle, the measurements of a single object along a given classification system cannot result in different outputs.

The case of n-ary relations, for $n \geq 2$, is captured by the following definition:

– $||R(a_1, \dots, a_n)||_M = t$ iff for $1 \le i \le n$, there exist $\langle \varepsilon(a_i), m_i \rangle \in \delta(s)$ such that $\langle m_1, \dots, m_n \rangle \in \iota(R)$;
– $||R(a_1, \dots, a_n)||_M = f$ iff for $1 \le i \le n$ there exist $\langle \varepsilon(a_i), m_i \rangle \in \delta(s)$ such that $m_i \in S_l$, $\iota(R) \in \mathbb{RS}_l^n$, and $\langle m_1, \dots, m_n \rangle \notin \iota(R)$;
– $||R(a_1, \dots, a_n)||_M = u$ iff otherwise.

Negation, conjunction and disjunction are defined according to Kleene three-valued semantics, see Table 1.a-c. Intuitively, $\neg A$ is true (false) only when there exist data that support the falsity (truth) of A. When A is undetermined also $\neg A$ is undetermined, i.e., when we lack support for the falsity or truth of A, we also lack support for the falsity or truth of $\neg A$. The data that falsify one conjunct are enough to falsify the whole conjunction, while when one conjunct is undetermined. Dual considerations hold for the disjunction. Implication is more problematic. In Kleene logic, the implication is defined, as usual, by $\neg A \vee B$. In this case, when both A and B are undetermined, according to Table 1.a&c, $A \to B$ is also undetermined. This seems empirically plausible but it clashes with the idea that the logical principle $A \to A$ holds even when A is undetermined. Moreover, the refusal of $A \to A$ results in a very weak logic. Thus, to obtain a well-behaved logical implication, three-valued logics usually add the Lukasiewicz implication that has the truth-table in Table 1.d, cf.[2]. With respect to the classical definition of the implication, the only difference is that when both A and B are undetermined, $A \to B$ is true rather than undetermined.

Table 1. Truth-tables for connectives

(a)

$\neg$	t	u	f
	f	u	t

(b)

$\wedge$	t	u	f
t	t	u	f
u	u	u	f
f	f	f	f

(c)

$\vee$	t	u	f
t	t	t	t
u	t	u	u
f	t	u	f

(d)

$\to$	t	u	f
t	t	u	f
u	t	t	u
f	t	t	t

Let $A(x)$ be a formula with x among its free variables and let $\sigma : \mathbf{V} \to O$ an assignment of the variables to the elements of O.

– $||\forall x A||_{M,\sigma} = t$ iff for every $d \in O$, $||A||_{M,\sigma(x/d)} = t$;
– $||\forall x A||_{M,\sigma} = f$ iff there is a $d \in O$ such that $||A||_{M,\sigma(x/d)} = f$;
– $||\forall x A||_{M,\sigma} = u$ iff otherwise.

The existential quantifier is defined by $\exists x A(x) \leftrightarrow \neg \forall x \neg A(x)$. We say that a formula ϕ is satisfiable if there exists a model M such that $||\phi||_M = t$. A formula ϕ is *valid* iff for every model M, $||\phi||_M = t$.

The Hilbert system for propositional first-order Lukasiewicz three-valued logic is proposed in [2, 12].

3.2 Dataset aggregation and modal operators

A single state provides sufficient information to express and to reason about the formulas that are made true by a single MB. In this section, we aim at addressing possible disagreements about the data provided by distinct MBs by defining modalities that aggregate datasets. Each MB (its associated state) is then viewed as a source of data to be submitted to an aggregation procedure that has the task of integrating datasets and solving possible inconsistencies.

We assume a finite set of N states $\mathfrak{S}$ and we extend $\mathcal{L}$ by adding a number of modal operators $\Box_F$ that depend on a certain aggregation function F. An *aggregation function* is a function F that maps N-tuples of truth-values associated to formulas to a collective/aggregated assignment of truth-values to that formula, i.e., $F : \{t, u, f\}^N \to \{t, u, f\}$. By defining aggregators by means of F, we are assuming that the method for aggregation is the same for every statement (a property called *neutrality* in judgment aggregation) and that the method is the same for any tuple of truth values (*independence*), cf.[7]. Moreover, we are defining aggregators on three possible truth-values, thus the standard definitions of the theory of judgment aggregation have to be adapted, cf.[6, 23, 21].[7]

The $\Box_F$ operators aggregate the truth-values of the formulas that hold in the various states, thus no new formula can be introduced in the aggregated outcome. We follow here a *coarse*, rather than *fine-grained*, aggregation of formulas (cf. [23, 17]), where in fact each collectively accepted formula must be accepted by at least one state. Coarse aggregations often fail to elect an aggregated formula that is a good trade off between the individual sources. For instance, suppose that state 1 makes true $\mathit{1KG}(a)$ and state 2 makes true $\mathit{3KG}(a)$. A fine grained aggregation allows to introduce a formula that expresses the *mean* of the weights, i.e., $\mathit{2KG}(a)$, whereas a coarse aggregation cannot. A model of a fine-grained aggregation in the context of measurement is left for future work. We refer to [24] for an approach to fine-grained aggregation that can be applied to the logic of measurement.

For the sake of example, we introduce a few aggregation functions. The first example is the unanimous aggregator that associates a certain truth value only if every state (MB) in $\mathfrak{S}$ agrees on that truth value.

$$un(x_1, \dots, x_N) = \begin{cases} x_i, & \text{if for all } i,\ j \text{ we have } x_i = x_j; \\ u, & \text{otherwise.} \end{cases}$$

For the simple majority rule we assume that maj returns true (false) if the majority of states accept the truth (falsity), and it returns u in any other case.[8]

$$maj(x_1, \dots, x_N) = \begin{cases} t, & \text{if } |\{x_i \mid x_i = t\}| > N/2; \\ f, & \text{if } |\{x_i \mid x_i = f\}| > N/2; \\ u, & \text{otherwise.} \end{cases}$$

[7] A treatment for a larger class of aggregators in social choice is presented in [23]. The motivation for the present treatment is that it easily allows for viewing aggregators as modalities. An overview of functions used to aggregated data is discussed in [4].

[8] The majority rule is generalized by *quota rules* that specify a threshold for acceptance of a certain truth-value. In this case, to define F as a function, we have to separately define quota rules for true, false, and undetermined.

The majority rule can be adapted to select only informative votes, that is, MBs that return true or false. We label this aggregator *determined majority.*

$$dmaj(x_1, \ldots, x_N) = \begin{cases} t, \text{ if } |\{x_i \mid x_i = t\}| > (N - |\{x_i \mid x_i = u\}|)/2; \\ t, \text{ if } |\{x_i \mid x_i = f\}| > (N - |\{x_i \mid x_i = u\}|)/2; \\ u, \text{ otherwise.} \end{cases}$$

The previous aggregators are *anonymous*, namely any permutation of the MBs provides the same value, i.e., the reliability of the MBs is not considered. However, aggregators that use information about the reliability of MBs, when available, can be designed. Suppose to have a reliability partial order $\preceq$ defined on the states $\mathfrak{S}$. It is possible, for instance, to define a family of aggregators that associate truth-value x if the most reliable n sources wrt. $\preceq$ agree on x and u otherwise. Moreover, to handle disagreement among the most reliable source, one can use an auxiliary aggregation procedure, e.g. the majority rule. A detailed analysis of the properties of these aggregators is left for future work. However, we want to highlight that the reliability structure of the states, of the contextual information provided by the MBs, allows to define more refined aggregations. Additional meta-information could clearly be taken into account. For instance, one could consider the W3C PROV-ontology[9] to explicitly represent some characteristics of the MBs and of the measurement processes.

The language $\mathcal{L}$ can then be extended by adding a number of modal operators $\Box_F$ that depend on the aggregator F:

$$\mathcal{L}_{\Box_F} ::= \phi \in \mathcal{L} \mid \Box_F \phi$$

where the possible nesting of modalities is excluded, cf.[18].

A *modal structure* is a couple $\langle \mathfrak{S}, F \rangle$, where $\mathfrak{S}$ is a set (with cardinality N) of states all about the same set of objects O and F is an aggregation function. A *model* M for our modal logic is then obtained by adding for each state $s \in \mathfrak{S}$, the interpretation ε_s for the individual constants and the interpretation ι_s for the predicates.

The semantics of the non-modal formulas of $\mathcal{L}$ is the one provided in Section 3.1, now relative to a state $s \in \mathfrak{S}$, i.e., $||\phi||_{M,s} = ||\phi||_{\langle s, \varepsilon_s, \iota_s \rangle}$.

The semantics of modal formulas relies on the function F:

$$||\Box_F \phi||_{M,s} = F(||\phi||_{\langle s_1, \varepsilon_{s_1}, \iota_{s_1} \rangle}, \ldots, ||\phi||_{\langle s_N, \varepsilon_{s_N}, \iota_{s_N} \rangle}).$$

Note that the truth-value of any modal formula is the same in all the states in $\mathfrak{S}$. We can construe the modal formulas as assessed wrt. the whole set of states $\mathfrak{S}$, rather than wrt. a single state. Moreover, by our definition of aggregators, every F is *systematic* [18], i.e., if $\models \phi \leftrightarrow \psi$, then $\models \Box_F \phi \leftrightarrow \Box_F \psi$. The modalities $\Box_F$ are then well-defined and they validate the *rule of equivalents* (RE) of minimal modal logic [5, 18]. An axiomatisation of the minimal modal extension of three-valued logic can then be given by adding (RE) to the propositional axioms.

$$(\textbf{RE}) \quad \vdash \phi \leftrightarrow \psi, \text{ then } \vdash \Box_F \phi \leftrightarrow \Box_F \psi$$

[9] See https://www.w3.org/TR/prov-overview

To characterise the aggregators, even in the bivalent case, a number of additional axioms are required, see for instance [18] for the case of the majority aggregator. We leave this aspect to a future work.

The condition (RE) does not constrain the way ε_s and ι_s can vary across the different states $s \in \mathfrak{S}$. In empirical terms, it is plausible to assume that the interpretation of the individual constants is fixed for every $s \in \mathfrak{S}$, i.e., for every $s, s' \in \mathfrak{S}$ and $c \in \mathbf{C}$ we have $\varepsilon_s(c) = \varepsilon_{s'}(c)$. The Barcan formula and its converse (BC), cf.[1], allows to axiomatise this property (i.e., a fixed domain assumption).

$$(\mathbf{BC}) \quad \Box_F \forall x A(x) \leftrightarrow \forall x \Box_F A(x)$$

We also assume that the interpretation of predicates is stable across states, i.e., for every $s, s' \in \mathfrak{S}$ and $P \in \mathbf{R}$ we have $\iota_s(P) = \iota_{s'}(P)$. On the one hand this is empirically plausible: by means of measurement standards and calibration, measurement theories aim at guaranteeing the sharing of data collected by different MBs, i.e., they individuate a set of reference systems. A predicate needs then to have a stable intension, to always refer to the same symbol. On the other hand, this is in line with standard modal logic where the intension of a predicate P is represented by a unique function that provides, for each world, the extension of P in such world. In our framework, the extension of a predicate P in a state s may be defined as the set of objects $o \in O$ such that $\langle o, \iota(P) \rangle \in \delta(s)$, which of course can vary in different states.

3.3 Reasoning about aggregated data: possible inconsistency

We informally discuss a few issues in reasoning about aggregated data beyond the minimal principle assumed by (RE). Consider the following example.

Example 1. Suppose $\mathfrak{S} = \{s_1, s_2, s_3\}$, where the datasets provide information about weights, lengths, and colours. Suppose we assess the atomic propositions $\mathit{1KG}(a)$, $\mathit{1MT}(a)$, $\mathit{RED}(a)$, and $\neg \mathit{RED}(a)$, which are grounded on their respective datasets. The profile of truth-values for each state is reported in Table 2. Consider now the formula $\lambda = \forall x((\mathit{1KG}(x) \wedge \mathit{1MT}(x)) \leftrightarrow \mathit{RED}(x))$ representing a law that relates weights and lengths with colours. According to the semantics of the connectives previously introduced, each state in $\mathfrak{S}$ validates such law.

In this scenario, the aggregation by majority of the data exhibits a case of *discursive dilemma*, [14]. In empirical terms, the law is consistent with all the single sources, but not with the aggregated data, preventing, in this case, an inductive generalisation. This means that aggregators may in principle provide inconsistent information even if every input is consistent.

In order to infer the inconsistent outcome in our modal setting, three principles of reasoning are required.[10]

$$(\mathbf{RM}) \quad \text{if} \vdash \phi \to \psi, \text{ then} \vdash \Box_F \phi \to \Box_F \psi$$

[10] Note that an analogous argument applies to the determined majority rule.

Table 2. Truth-values profile of the example 1

	$1KG(a)$	$1MT(a)$	$RED(a)$	$\neg RED(a)$	$\forall x((1KG(x) \wedge 1MT(x)) \leftrightarrow RED(x))$
s_1	t	t	t	f	t
s_2	t	f	f	t	t
s_3	f	t	f	t	t
maj	t	t	f	t	t

$$(\mathbf{C}) \quad \Box_F \phi \wedge \Box_F \psi \to \Box_F(\phi \wedge \psi)$$

$$(\bot) \quad \neg \Box_F \bot$$

(RM) is the monotonicity principle, the principle (C) allows for combining aggregated information, and ($\bot$) excludes possibly inconsistent aggregated data By assuming (RM), (C) and ($\bot$), together with the axioms for the propositional logic, the calculus becomes inconsistent for the majoritarian aggregation.

In Example 1, $\Box_{maj} 1KG(a)$ and $\Box_{maj} 1MT(a)$ are true, therefore by (C) we infer $\Box_{maj}(1KG(a) \wedge 1MT(a))$. Since the law λ is true in every state, $\Box_{maj}\lambda$ is true. From $\Box_{maj}(1KG(a) \wedge 1MT(a))$ and $\Box_{maj}\lambda$, by (C) and (RM), we infer $\Box_{maj} RED(a)$. However, we also have $\Box_{maj}\neg RED(a)$, since a majority of states makes $RED(a)$ false. By (C) we obtain $\Box_{maj}(RED(a) \wedge \neg RED(a))$ and by (RE), since every contradiction is logically equivalent, we obtain $\Box_{maj}\bot$, against ($\bot$).

The principle (RM) legitimates the use of a logical inference at the level of aggregated data. E.g., it justifies to infer $\Box_{maj} RED(a)$ from $\Box_{maj} 1KG(a)$ and $\Box_{maj} 1MT(a)$ via $\Box_{maj}\lambda$. Notice that (RM) applies regardless of the majority that supports those data, the actual set of states that produces them.

While the principle (C) appears a reasonable principle for combining aggregated data, in fact it is also insensitive to the fact that possibly distinct, although overlapping, sets of MBs can be the source of the data. In the example, s_1 and s_2 agree on $1KG(a)$, whereas s_1 and s_3 agree on $1MT(a)$.

(RM) and (C) seem to identify two types of reasoning: an *intra-state* reasoning, where each state reasons about the data by means of the law, and an *inter-state* reasoning, where reasoning is performed at the level of aggregated data, by means of the law. It is in fact possible to separate the two forms of reasoning; for instance, by distinguishing two types of combinations of data (i.e., conjunction), one that applies to the case where the same states support a number of data, the other that combines data produced by distinct sets of states. This move is capable of restoring consistency, although it requires to enter the realm of substructural logics for modelling reasoning about aggregated data [20–22]. In fact, the possible inconsistency of the aggregated sets depends only on the meaning of logical connectives, not on the atomic formula produced by the MBs. If our language only contains atomic proposition, e.g., we prevent talking about $1KG(a) \wedge 1MT(a)$ and we content with $1KG(a)$ and $1MT(a)$ or we exclude laws to connect data, the majority is indeed consistent. Hence, it is worthy to investigate logical operators that preserve consistency under the majority rule and suitably represent the rules of reasoning about aggregated data.

By contrast, a simple solution, is to accept that there might be cases of inconsistent data aggregation and give up the axiom ($\perp$). Note that the non-anonymous procedures that we defined do indeed preserve consistency, however they rest on the demanding assumption of knowing in advance the most reliable MBs. More sophisticated aggregators require dropping the systematicity assumption that we embraced here and they are therefore left for future work.

3.4 The contextual nature of measurement

We discuss now the contextual nature of measurement statements in the model that we proposed. We have introduced two types of statements expressing measurement: non-modal statements, that are assessed with respect to a single state, and modal statements that are assessed with respect to a plurality of states, by aggregating the information therein.

It useful here to apply the distinction between two interpretations of context proposed in [19]. A context can be intended in an *objective* (or *ontological*) way, i.e., basically as a metaphysical state of affairs, or in a *subjective* (or *epistemic*) way, i.e., basically as a cognitive representation of the world. According to this dichotomy, the non-modal statements of our logic are close to the epistemic view of context. The holding of a formula (e.g., $1KG(a)$) at a given state depends on the considered measurement battery, on its representation systems (the symbolic classification systems), on the resolution of the devices in the battery, and on the actual measurement processes performed. As observed in the introduction, it is however important to note that the degree of subjectivity of measurements is lower than the one of personal evaluations, opinions, perceptions, etc.

By contrast, the modal statements of our logic aggregate the information coming from several epistemic contexts. Thus, on the one hand, the aggregated statements are not immediately objective, in the above sense, as they are always mediated by the measurement systems, they are not directly reducible to real states of affairs, to sets of features of the world. On the other hand they are not merely subjective, as they balance between the viewpoints of different epistemic contexts. The aggregated statements seem then to constitute a further type of context, which we may term an *inter-subjective* context, which results from the aggregation of a number of subjective (epistemic) contexts.

We suggest an analogy between the aggregation of different epistemic context with the operation of *de-contextualisation* used in [19] to dismiss the demanding idea of an ontological context, while preserving the possiblity of an objective context, as resulting from intersubjective agreement (viz. "Objectivity is always a result of our interaction, not a datum", [19], p. 283.)

In this sense, a theory of the aggregation of heterogeneous (epistemic) contexts may serve as the formal backbone of a theory of de-contextualisation, viewed as a theory of multiagent interaction. We leave the development of this suggestion to a dedicated work.

4 Conclusion

This work has three main contributions. Firstly, we introduced an explicit definition of states in terms of the epistemic theory of measurement. States are not simple indexes for possible worlds, they are sets of measurements collected by MBs. The datasets associated to each MB depend on the nature of the MB, therefore the information that we may assess at a state has indeed a contextual nature explained and justified in terms of the epistemic measurement theory.

The second contribution concerns the characterisation of the meaning, or more precisely the intension, of the properties represented by the predicates in $\mathcal{L}$. The theory of measurement allows us to interpret (unary) predicates into symbols that, by means of measurement standards, are conventionally assigned to perfect realisations of properties. While standard modal logic encapsulates the intension of a predicate into a function from worlds to sets of objects, our approach is more descriptive and operative, it associates a computational 'recipe' to a predicate: to calculate the extension of a predicate in a given state, one needs to look for the measurements (in such state) that have as output the symbol associated to the predicate.

Thirdly, we introduced modal operators to model aggregators of (possibly conflicting) data and we discussed the contextual nature of measurement statements distinguishing the device-based measurement and the aggregated measurement.

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