

1 **Title:** Buddhist Thought on Emptiness and Category Theory

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18 Abstract

19 Notions such as Sunyata, Catuskoti, and Indra's Net, which figure prominently in Buddhist
20 philosophy, are difficult to readily accommodate within our ordinary thinking about everyday
21 objects. Famous Buddhist scholar Nagarjuna considered two levels of reality: one called
22 conventional reality and the other ultimate reality. Within this framework, Sunyata refers to the
23 claim that at the ultimate level objects are devoid of essence or "intrinsic properties", but are
24 interdependent by virtue of their relations to other objects. Catuskoti refers to the claim that four
25 truth values, along with contradiction, are admissible in reasoning. Indra's Net refers to the
26 claim that every part of a whole is reflective of the whole. Here we present category theoretic
27 constructions which are reminiscent of these Buddhist concepts. The universal mapping
28 property definition of mathematical objects, wherein objects of a universe of discourse are
29 defined not in terms of their content, but in terms of their relations to all objects of the universe is
30 reminiscent of Sunyata. The objective logic of perception, with perception modeled as [a
31 category of] two sequential processes (sensation followed by interpretation), and with its truth
32 value object of four truth values, is reminiscent of the Buddhist logic of Catuskoti. The category
33 of categories, wherein every category has a subcategory of sets with zero structure within which
34 every category can be modeled, is reminiscent of Indra's Net. Our thorough elaboration of the
35 parallels between Buddhist philosophy and category theory can facilitate better understanding of
36 Buddhist philosophy, and bring out the broader philosophical import of category theory beyond
37 mathematics.

38 **Introduction**

39 Buddhist philosophy, especially Nagarjuna's Middle Way (Garfield, 1995; Siderits and Katsura,
40 2013), is intellectually demanding (Priest, 2013). The sources of the difficulties are many. First
41 it argues for two realities: conventional and ultimate (Priest, 2010). Next, ultimate reality is
42 characterized by Sunyata or emptiness, which is understood as the absence of a fundamental
43 essence underlying reality (Priest, 2009). Equally importantly, contradictions are readily
44 deployed, especially in Catuskoti, as part of the characterization of reality (Deguchi, Garfield,
45 and Priest, 2008; Priest, 2014). Lastly, reality is depicted as Indra's Net—a whole, whose parts
46 are reflective of the whole (Priest, 2015). The ideas of relational existence, admission of
47 contradictions, and parts reflecting the whole are seemingly incompatible with our everyday
48 experiences and the attendant conceptual reasoning used to make sense of reality. However,
49 notions analogous to these ancient Buddhist ideas are also encountered in the course of the
50 modern mathematical conceptualization of reality. These parallels may be, in large part, due to
51 'experience' and 'reason' that are treated as the final authority in both mathematical sciences and
52 Buddhist philosophy. Here, we highlight the similarities between Buddhist philosophy and
53 mathematical philosophy, especially category theory (Lawvere and Schanuel, 2009). The
54 resultant cross-cultural philosophy can facilitate a proper understanding of reality—a noble goal
55 to which both Buddhist philosophy and mathematical practice are unequivocally committed.

56

57 **Two Realities**

58 There are, according to Buddhist thought, two realities: the conventional reality of our everyday
59 experiences and the ultimate reality (Priest, 2010; Priest and Garfield, 2003). In our

60 conventional reality, things appear to have intrinsic essences. It is sensible, at the level of
61 conventional reality, to speak of essences of objects, but at the level of ultimate reality there are
62 no essences, and everything exists but only relationally. There is an analogous situation in
63 mathematics. On one hand, mathematical objects can be characterized in terms of their relations
64 to all objects, in which case the nature of an object is determined by the nature of its relationship
65 to all objects. In a sense, there is nothing inside the object; an object is what it is by virtue of its
66 relations to all objects. The objects of mathematics are, as Resnik (1981, p. 530) notes,
67 “positions in structures”, which is in accord with the Buddhist understanding of things as “loci in
68 a field of relations” (Priest, 2009, p. 468). However, there is another level of mathematical
69 reality, wherein we can speak of essences of objects (e.g. theories of objects; Lawvere and
70 Rosebrugh, 2003, pp. 154-155). For example, one can characterize a set as a collection of
71 elements or “sum” of basic-shaped figures (**1**-shaped figures, where $\mathbf{1} = \{\bullet\}$), with basic shapes
72 understood as essences (Lawvere, 1972, p. 135; Lawvere and Schanuel, 2009, p. 245; Reyes,
73 Reyes, and Zolfaghari, 2004, p. 30). Similarly, every graph is made up of figures of two basic-
74 shapes (arrow- and dot-shaped figures; Lawvere and Schanuel, 2009, p. 150, 215). This
75 characterization of an object in terms of its contents i.e. basic shapes or essences (Lawvere,
76 2003, pp. 217-219; Lawvere, 2004, pp. 11-13) can be contrasted with the relational
77 characterization, wherein each and every object of a universe of discourse (a mathematical
78 category; Lawvere and Schanuel, 2009, p. 17) is characterized in terms of its relationship to all
79 objects of the universe or category (see Appendix A1). The relational nature of mathematical
80 objects, as elaborated below, is reminiscent of the Buddhist notion of emptiness—an assertion
81 that objects are what they are not by virtue of some intrinsic essences but by virtue of their
82 mutual relationships.

83

84 **Emptiness**

85 According to Buddhist philosophy, everything is empty and the totality of empty things is empty.
 86 Here, emptiness is understood as the absence of essences. Things, in the ultimate analysis, are
 87 what they are and behave the way they do not because of [some] essences inherent in them, but
 88 by virtue of their mutual relationships (Priest, 2009). This idea of relational existence has
 89 parallels in mathematical practice. Mathematical objects of a given mathematical category (e.g.
 90 category of sets) are what they are not by virtue of their intrinsic essences but by virtue of their
 91 relations to all objects of the category. For example, a single-element set is a set to which there
 92 exists exactly one function from every set (Lawvere and Schanuel, 2009, p. 213, 225). Note that
 93 the singleton set is characterized not in terms of what it contains (a single element), but in terms
 94 of how it relates to all sets of the category of sets. In a similar vein, the truth value set $\Omega =$
 95 $\{\text{false}, \text{true}\}$ is defined in terms of its relation to all sets of the category of sets. The truth value
 96 set, instead of being defined as a set of two elements ‘false’ and ‘true’, is defined as a set Ω such
 97 that functions from any set X to the set Ω are in one-to-one correspondence with the parts of X
 98 (ibid, pp. 339-344). To give one more example, product of two sets is defined not by specifying
 99 the contents of the product set (pairs of elements), but by characterizing its relationship to all
 100 sets. More explicitly, the product of two sets A and B is a set $A \times B$ along with two functions
 101 (projections to the factors) $p_A: A \times B \rightarrow A$, $p_B: A \times B \rightarrow B$ such that for every set Q and any pair
 102 of functions $q_A: Q \rightarrow A$, $q_B: Q \rightarrow B$, there is exactly one function $q: Q \rightarrow A \times B$ satisfying both
 103 the equations: $q_A = p_A \circ q$ and $q_B = p_B \circ q$, where ‘ \circ ’ denotes composition of functions (ibid, pp.
 104 339-344). The universal mapping property definition of mathematical constructions brought to
 105 sharp focus the relational nature of mathematical objects. It conclusively established that “the

106 substance of mathematics resides not in Substance (as it is made to seem when \in [membership]
107 is the irreducible predicate, with the accompanying necessity of defining all concepts in terms of
108 a rigid elementhood relation) but in Form (as is clear when the guiding notion is isomorphism-
109 invariant structure, as defined, for example, by universal mapping properties)” (Lawvere, 2005,
110 p. 7). More broadly, Yoneda lemma (Lawvere and Rosebrugh, 2003, pp. 249-250; Appendix
111 A1), according to which a mathematical object of a given universe of discourse (i.e. category) is
112 completely characterized by the totality of its relations to all objects of the universe (category), is
113 an unequivocal assertion of the relational nature of mathematical objects. Yoneda lemma, as
114 pointed out by Barry Mazur, establishes that “an object X of a category \mathcal{C} is determined by the
115 network of relationships that the object X has with all the other objects in \mathcal{C} ” (Mazur, 2008).
116 Thus the Buddhist idea of emptiness or relational existence finds resonance in mathematical
117 practice, especially in terms of universal mapping properties and the Yoneda lemma.

118 However, note that according to the Buddhist doctrine of emptiness, not only is
119 everything empty, but the totality of empty things is also empty (Priest, 2009). In other words,
120 even the notion of relational existence is empty i.e., emptiness is not the essence of existence;
121 emptiness is also empty. This idea of emptiness being empty is much more challenging to
122 comprehend. When we say that objects are empty, we are saying that objects are mere locations
123 in a network of relations. But when we say that the totality of empty things is empty, we are
124 asserting that the existence of totality is also relational just like that of the objects in the totality.
125 What is not immediately clear is how are we to think of relations especially when all we have is
126 the totality i.e., one object. Within mathematics, note that the totality of all objects (along with
127 their mutual relations) forms a category. More importantly, categories are objects in the category
128 of categories (Lawvere, 1966), and hence the totality of objects i.e. category is also empty or

129 relational as much as the objects of a category. Thus the idea of Sunyata (everything is empty)
130 resonates with the relational nature of objects and of the totality of objects (within the
131 mathematical framework of the category of categories).

132 Equally importantly, Nagarjuna's Middle Way, having gone to great lengths to
133 distinguish two realities (conventional essences vs. ultimate emptiness) identifies the two: "there
134 is no distinction between conventional reality and ultimate reality" (Deguchi, Garfield, and
135 Priest, 2008, p. 399). Contradictions (such as these) within Buddhist philosophy, on a superficial
136 reading, are diagnostic of irrational mysticism. However, as we point out in the following,
137 contradictions also figure prominently in the foundations of mathematical modeling of reality. In
138 light of these parallels, 'contradiction' may be intrinsic to the nature of reality, which is the
139 common subject of both Buddhist and mathematical investigations, and not a sign of faulty
140 Buddhist reasoning.

141

142 **Contradiction**

143 Within the Buddhist philosophical discourse, one often encounters contradictions and these
144 contradictions are treated as meaningful (Deguchi, Garfield, and Priest, 2008; Priest, 2014).
145 There is an analogous situation in mathematics. Though not every contradiction is sensible,
146 there are sensible contradictions such as the boundary of an object A formalized as 'A and not A'
147 (Lawvere, 1991, 1994a, p. 48; Lawvere and Rosebrugh, 2003, p. 201). More importantly, within
148 mathematical practice, it is now recognized that contradictions do not necessarily lead to
149 inconsistency (an inconsistent system, according to Tarski, is where everything can be proved;
150 Lawvere, 2003, p. 214). Of course, admitting a contradiction invariably leads to inconsistency in

151 classical Boolean logic. In logics more refined than Boolean logic contradiction does not
152 necessarily lead to inconsistency. This recognition is very important, especially since
153 contradiction plays a foundational role in mathematical practice. Briefly, Cantor's definition of
154 SET is, as pointed out by F. William Lawvere, "a strong contradiction: its points are completely
155 distinct and yet indistinguishable" (ibid, p. 215; Lawvere, 1994a, pp. 50-51). Zermelo, and most
156 mathematicians following him, concluded that Cantor's account of sets is "incorrigibly
157 inconsistent" (Lawvere, 1994b, p. 6). Lawvere, using adjoint functors, showed that Cantor's
158 definition is "not a conceptual inconsistency but a productive dialectical contradiction" (Lawvere
159 and Rosebrugh, 2003, pp. 245-246), which is summed up as the unity and identity of adjoint
160 opposites (Lawvere, 1992, pp. 28-30; Lawvere, 1996).

161 A related notion is *catuskoti*, which is routinely employed in Buddhist reasoning (Priest,
162 2014; Westerhoff, 2006). To place it in perspective, in the familiar Boolean logic, any
163 proposition is either true or false. Put differently, there are only two possible truth values, and
164 they are mutually exclusive and jointly exhaustive. Unlike Boolean logic, in Buddhist reasoning
165 more than two truth values are admissible. In the Buddhist logic of *Catuskoti*, a proposition can
166 possibly take, in addition to the familiar truth values of 'true' or 'false', the truth values of 'true
167 and false', or 'not true and not false'. Given a proposition *A*, there are four possibilities: 1. *A*, 2.
168 not *A*, 3. *A* and not *A*, 4. not *A* and not not *A*. Here contradiction is admissible, i.e. '*A* and not
169 *A*' is a possible state of affairs, which is reminiscent of the boundary operation and the unity and
170 identity of adjoint opposites in mathematics, alluded to earlier. Moreover, double negation is not
171 same as identity operation as in the case of, to give one example, the non-Boolean logic of
172 graphs (Lawvere and Schanuel, 2009, p. 355). Note that if not not $A = A$, then the fourth truth
173 value of *catuskoti* is equal to the third.

174 As an illustration of how the four truth values of catuskoti could be a reflection [of an
175 aspect] of reality, we consider the category of percepts. Perception involves two sequential
176 processes of sensation followed by interpretation (Albright, 2015; Croner and Albright, 1999).
177 So, we define the category of percepts as a category of two sequential functions of decoding after
178 coding. The truth value object of the category of percepts has four truth values (Appendix A2).
179 Thus the objective logic of perception, with its truth value object of four truth values, is
180 reminiscent of the Buddhist logic of catuskoti (see Linton, 2005).

181

182 **Indra's Net and Zero Structure**

183 Another important concept in Buddhist philosophy is the idea of Indra's Net, wherein reality is
184 compared to a vast network of jewels such that every jewel is reflective of the entire net (Priest,
185 2015). In abstract terms, reality is characterized as a whole wherein every part is reflective of
186 the whole. Admittedly, this Buddhist characterization of reality sounds mystifying, but there is
187 an analogous situation, involving part-whole relations, in mathematics.

188 How can a part of a whole reflect the whole? First, note that mathematical structures of
189 all sorts can be modeled in the category of sets (Lawvere and Schanuel, 2009, pp. 133-151). Sets
190 have zero structure (Lawvere, 1972, p. 1; Lawvere and Rosebrugh, 2003, p. 1, 57; Lawvere and
191 Schanuel, 2009, p. 146). Negating the structure (cohesion, variation) inherent in mathematical
192 objects, Cantor created sets: mathematical structures with zero structure (Lawvere, 2003, 2016;
193 Lawvere and Rosebrugh, 2003, pp. 245-246). In comparing his abstraction of sets with zero
194 structure to the invention of number zero, Cantor considered sets as his most profound
195 contribution to mathematics (Lawvere, 2006). Sets, by virtue of having zero structure, serve as a

196 blank page—an ideal background to model any category of mathematical objects (Lawvere,
197 1994b; Lawvere and Rosebrugh, 2003, pp. 154-155). However, structureless sets are a small
198 part—the only part—of the mathematical universe which reflects all of mathematics. It seemed
199 so until Lawvere axiomatized the category of categories (Lawvere, 1966; Lawvere and Schanuel,
200 2009, pp. 369-370). Along the lines of Cantor’s invention of structureless sets, Lawvere defined
201 a subcategory of structureless (discrete, constant) objects within a category by negating its
202 structure (cohesion, variation; Lawvere, 2004, p. 12; Lawvere and Schanuel, 2009, pp. 358-360,
203 372-377). Thus, within any category of mathematical objects, there is a part, a structureless
204 subcategory, which is like the category of sets in having zero structure, and hence serves as a
205 background to model all categories of mathematical objects (Lawvere, 2003; Lawvere and
206 Menni, 2015; Picado, 2008, p. 21). Modeling a category of mathematical objects requires, in
207 addition to the subcategory with zero structure, another subcategory objectifying the structural
208 essence(s) of the objects of the category, i.e. the theory of the given category of mathematical
209 objects. Finding the theory subcategory also depends on the structureless subcategory, by way of
210 contrasting or negating the structureless subcategory (Lawvere, 2007). Once we have the
211 subcategory with zero structure and the subcategory objectifying the essence (theory) of a given
212 category, interpreting the theory subcategory into the structureless subcategory gives us models
213 of the given category of mathematical objects. Thus, thanks to the recognition of significance of
214 Cantor’s zero structure, every mathematical category can be modelled in any category of the
215 category of categories.

216 If we compare the category of categories to Indra’s net, then categories within the
217 category of categories would correspond to jewels in Indra’s net. Just as in the case of Indra’s
218 net, wherein every jewel in the network of jewels is reflective of the entire network, in the

219 category of categories every category (part) of the category of categories (whole) reflects the
220 whole. For example, the category of dynamical systems is a part of the category of categories.
221 Within the category of dynamical systems, we have the constant subcategory (obtained by
222 negating the variation) of dynamical systems (wherein every state is a fixed point), which is like
223 the category of sets, and within which any category can be modeled. Similarly, the category of
224 graphs is another part of the category of categories. Within the category of graphs there is the
225 discrete subcategory (obtained by negating the cohesion) of graphs (with one loop on each dot),
226 which is also like the category of sets, and hence can model every category. Thus, we find that
227 within the category of categories, every part is reflective of the whole, which is reminiscent of
228 the Buddhist depiction of reality as Indra's Net: a whole with parts reflective of the whole.

229

230 **Conclusion**

231 There are similarities between Buddhist philosophy and mathematical practice, especially with
232 regard to essence vs. emptiness, contradictions, and part-whole relations. These similarities
233 might be a natural consequence of identical objectives—understanding reality and commitment
234 to truth—and identical means—experience and reason—employed towards those ends. It is in
235 this respect that the practices of the two—mathematicians and Buddhists—can be compared. Our
236 exercise, on one hand, can help better appreciate the rationality of Buddhist reasoning.

237 Oftentimes, admission of contradiction (as in *catuskoti*) tends to be equated with irrational
238 mysticism. However, as we have seen, contradictions are also an integral and indispensable part
239 of the mathematical understanding of reality. On the other hand, in drawing parallels between

- 240 Buddhist thought and mathematical practice, we hope to have brought out the broad
- 241 philosophical import of category theory beyond mathematics.

242 **Appendices**

243

244 **A1. Yoneda lemma**

245 We begin with an intuitive introduction to the mathematical content of Yoneda lemma (Lawvere
 246 and Rosebrugh, 2003, pp. 175-176, 249). With simple illustrations of figures-and-incidences
 247 (along with [its dual] properties-and-determinations) interpretations of mathematical objects, we
 248 prove the Yoneda lemma (Lawvere and Schanuel, 2009, pp. 361, 370-371). Broadly speaking,
 249 Yoneda lemma is about [properties of] objects [of categories] and their mutual determination.

250 First, let us consider a function

251
$$f: A \rightarrow B$$

252 We can think of the function f as (i) a figure of shape A in B , i.e., an A -shaped figure in B . For
 253 example, in the category of graphs, a map

254
$$d: D \rightarrow G$$

255 from a graph D (consisting of one dot) to any graph G is a D -shaped figure in G , i.e., a dot in the
 256 graph G . We can also think of the same function f as (ii) a property of A with values in B , i.e., a
 257 B -valued property of A (Lawvere and Schanuel, 2009, pp. 81-85). For example, with sets, say,
 258 $\text{Fruits} = \{\text{apple, grape}\}$ and $\text{Color} = \{\text{red, green}\}$, a function

259
$$c: \text{Fruits} \rightarrow \text{Color}$$

260 (with $c(\text{apple}) = \text{red}$ and $c(\text{grape}) = \text{green}$) can be viewed as Color -valued property of Fruits .

261 Now let us consider two figures: an X-shaped figure in A

$$262 \quad x_A: X \rightarrow A$$

263 and a Y-shaped figure in A

$$264 \quad y_A: Y \rightarrow A$$

265 Given a transformation from the shape X to the shape Y, i.e. an X-shaped figure in Y

$$266 \quad x_Y: X \rightarrow Y$$

267 we find that the X-shaped figure in Y (x_Y) induces a transformation of a Y-shaped figure in A
 268 into an X-shaped figure in A via composition of maps

$$269 \quad y_A \circ x_Y = x_A$$

270 (where ‘ \circ ’ denotes composition) displayed as a commutative diagram



274 showing the transformation of a Y-shaped figure in A (y_A) into an X-shaped figure in A (x_A) by
 275 an X-shaped figure in Y (x_Y) via composition of maps.

276 As an illustration, consider an object (of the category of graphs) i.e., a graph G (shown
 277 below):

278

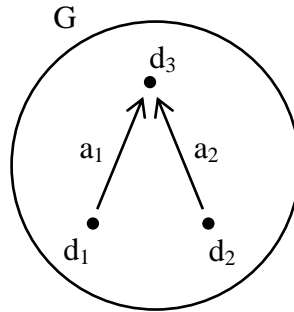
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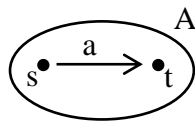
283



284 and a shape graph [arrow] A with exactly one arrow 'a', along with its source 's' and target 't',

285 as shown:

286



287 along with an A-shaped figure in G

288

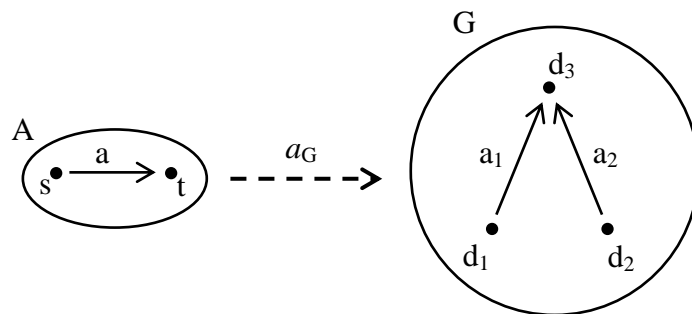
$$a_G: A \rightarrow G$$

289 displayed as:

290

291

292



293 with, say,

294

$$a_G(a) = a_1$$

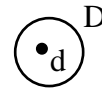
295 This A-shaped figure in G , i.e. the graph map a_G maps the [only] arrow 'a' in the shape graph A

296 to the arrow ' a_1 ' in the graph G , while respecting the source (s) and target (t) structure of the

297 arrow 'a', i.e., with arrow 'a' in shape A mapped to arrow 'a₁' in the graph G, the source 's' and
 298 target 't' of the arrow 'a' are mapped to the source 'd₁' and target 'd₃' of arrow 'a₁', respectively.

299 Next, consider another shape graph [dot] D with exactly one dot 'd'

300



301 along with a D-shaped figure in A

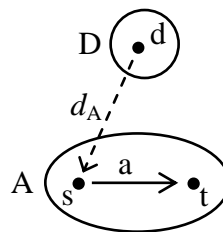
302 $d_A: D \rightarrow A$

303 with

304 $d_A(d) = s$

305 i.e., the graph map d_A maps the dot 'd' in the graph D to the dot 's' in the graph A, i.e. the source
 306 dot 's' of the arrow 'a', as shown below:

307



308

309 This graph map d_A from shape D to shape A induces a transformation of the (above) A-shaped
 310 figure in graph G

311 $a_G: A \rightarrow G$

312 into a D-shaped figure in G

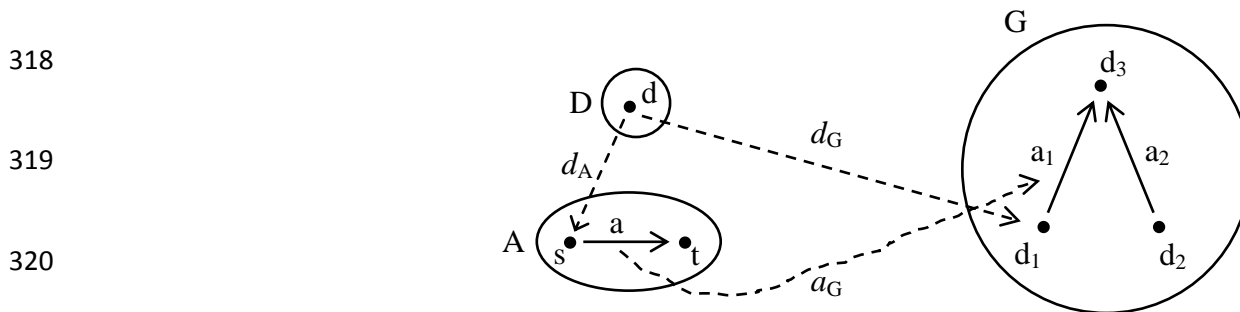
313 $d_G: D \rightarrow G$

314 via composition of graph maps

315
$$d_G = a_G \circ d_A$$

316 i.e., $d_G(d) = a_G \circ d_A(d) = a_G(s) = d_1$

317 as depicted below (Lawvere and Schanuel, 2009, pp. 149-150):



322 In general, every X-shaped figure in Y transforms a Y-shaped figure in A into an X-

323 shaped figure in A i.e., every map

324
$$x_Y: X \rightarrow Y$$

325 induces a map in the opposite direction (contravariant; Lawvere, 2017; Lawvere and Rosebrugh,

326 2003, p. 103; Lawvere and Schanuel, 2009, p. 338)

327
$$A^{x_Y}: A^Y \rightarrow A^X$$

328 where A^Y is the map object of the totality of all Y-shaped figures in A, A^X is the map object of

329 the totality of all X-shaped figures in A, and with the map A^{x_Y} of map objects defined as

330
$$A^{x_Y}(y_A: Y \rightarrow A) = y_A \circ x_Y = x_A: X \rightarrow A$$

331 assigning a map x_A in the map object A^X to each map y_A in the map object A^Y . Thus, the figures
 332 in an object A of all shapes (all X -shaped figures in A for every object X of a category) along
 333 with their incidences

$$334 \quad A^{x_Y}: A^Y \rightarrow A^X$$

335 induced by all changes of figure shapes

$$336 \quad x_Y: X \rightarrow Y$$

337 (i.e. every map in the category) together constitute the geometry of figures in A , i.e., a complete
 338 picture of the object A . Summing up, we have the complete characterization of the geometry of
 339 every object A of a category in terms of the figures of all shapes (objects of the category) and
 340 their incidences (induced by the maps of the category) in the object A (Lawvere and Schanuel,
 341 2009, pp. 370-371).

342 Let us now examine how figures of a shape X in an object A are transformed into figures
 343 of the [same] shape X in an object B . We find that an A -shaped figure in B

$$344 \quad a_B: A \rightarrow B$$

345 induces a transformation of an X -shaped figure in A

$$346 \quad x_A: X \rightarrow A$$

347 into an X -shaped figure in B

$$348 \quad x_B: X \rightarrow B$$

349 via composition of maps

350
$$a_B \circ x_A = x_B$$

351 displayed as a commutative diagram

352
$$\begin{array}{ccc} X & & B \\ & \searrow^{x_A} & \nearrow^{x_B = a_B \circ x_A} \\ & A & \xrightarrow{a_B} B \end{array}$$

353

354 showing the transformation of an X-shaped figure in A (x_A) into an X-shaped figure in B (x_B) by
355 an A-shaped figure in B (a_B) via composition of maps. Thus, every map

356
$$a_B: A \rightarrow B$$

357 induces a map in the same direction (covariant; Lawvere and Rosebrugh, 2003, pp. 102-103,
358 109; Lawvere and Schanuel, 2009, p. 319)

359
$$a_B^X: A^X \rightarrow B^X$$

360 where A^X is the map object of all X-shaped figures in A, B^X is the map object of all X-shaped
361 figures in B, and with the map a_B^X defined as

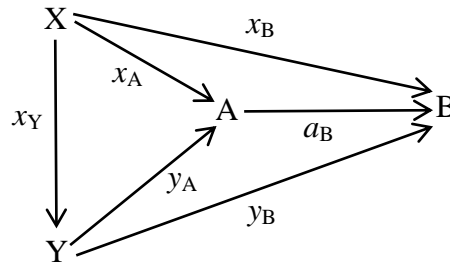
362
$$a_B^X(x_A: X \rightarrow A) = a_B \circ x_A = x_B: X \rightarrow B$$

363 assigning a map x_B in the map object B^X to each map x_A in the map object A^X . Thus, the totality
364 of maps a_B^X of map objects (for all objects and maps of the category) induced by a map a_B from
365 A to B constitutes a covariant transformation of the figure geometry of object A into that of B,
366 i.e., specifies how figures-and-incidences in A are transformed into figures-and-incidences in B.

367 Putting together these two transformations: (i) the covariant transformation of X-shaped
368 figures in A into X-shaped figures in B induced by an A-shaped figure in B, and (ii) the

369 contravariant transformation of Y-shaped figures in A into X-shaped figures in A induced by an
 370 X-shaped figure in Y, we obtain the diagram (Lawvere and Schanuel, 2009, p. 370):

371



374

375 from which we notice that there are two paths to go from a Y-shaped figure in A (y_A) to an X-
 376 shaped figure in B (x_B):

377 Path 1. First we map the Y-shaped figure in A (y_A) into an X-shaped figure in A (x_A) along the
 378 X-shaped figure in Y (x_Y) via composition of the maps

379

$$y_A \circ x_Y$$

380 and then map the composite X-shaped figure in A ($y_A \circ x_Y$) into an X-shaped figure in B along
 381 the A-shaped figure in B (a_B) via composition

382

$$a_B \circ (y_A \circ x_Y)$$

383 Path 2. First we map the Y-shaped figure in A (y_A) into a Y-shaped figure in B (y_B) along the A-
 384 shaped figure in B (a_B) via composition of the maps

385

$$a_B \circ y_A$$

386 and then map the composite Y-shaped figure in B ($a_B \circ y_A$) into an X-shaped figure in B along
 387 the X-shaped figure in Y (x_Y) via composition

388
$$(a_B \circ y_A) \circ x_Y$$

389 Based on the associativity of composition of maps (Lawvere and Schanuel, 2009, pp. 370-371),

390 we find that

391
$$a_B \circ (y_A \circ x_Y) = (a_B \circ y_A) \circ x_Y$$

392 i.e., the two paths of transforming a Y-shaped figure in A

393
$$y_A: Y \rightarrow A$$

394 into an X-shaped figure in B give the same map

395
$$a_B \circ y_A \circ x_Y = x_B: X \rightarrow B$$

396 Since the associativity of composition of maps hold for all maps of any category (Lawvere and

397 Schanuel, 2009, p. 17), we find that every A-shaped figure in B induces a covariant

398 transformation of the figure geometry of A into the figure geometry of B. More explicitly, each

399 A-shaped figure in B

400
$$a_B: A \rightarrow B$$

401 induces a commutative diagram (of maps of map objects)

402
$$\begin{array}{ccc} A^X & \xrightarrow{a_B^X} & B^X \\ \uparrow A^{x_Y} & & \uparrow B^{x_Y} \\ A^Y & \xrightarrow{a_B^Y} & B^Y \end{array}$$

405 satisfying

$$406 \quad a_B^X \circ A^{xY} = B^{xY} \circ a_B^Y$$

407 for every map in the category, and hence a natural transformation (compatible with the
 408 composition of maps) of the figure geometry of A into the figure geometry of B. To see the
 409 commutativity, consider a Y-shaped figure in A, i.e. a map y_A of the map object A^Y and evaluate
 410 the above two composites:

$$411 \quad a_B^X \circ A^{xY}(y_A) = a_B^X(y_A \circ x_Y) = a_B \circ (y_A \circ x_Y)$$

$$412 \quad B^{xY} \circ a_B^Y(y_A) = B^{xY}(a_B \circ y_A) = (a_B \circ y_A) \circ x_Y$$

413 Again, according to the associativity of the composition of maps

$$414 \quad a_B \circ (y_A \circ x_Y) = (a_B \circ y_A) \circ x_Y = a_B \circ y_A \circ x_Y$$

415 and hence both composites map each Y-shaped figure in A (a map in the map object A^Y)

$$416 \quad y_A: Y \rightarrow A$$

417 to the X-shaped figure in B (a map in the map object B^X)

$$418 \quad a_B \circ y_A \circ x_Y = x_B: X \rightarrow B$$

419 Since we have the above commutativity for every shape (object) and figure (map), i.e. for all
 420 objects and maps of the category, we conclude that an A-shaped figure in B corresponds to a
 421 natural transformation (respectful of figures-and-incidences) of the figure geometry of A into the
 422 figure geometry of B.

423 Now we formally show that every A-shaped figure in B

$$424 \quad a_B: A \rightarrow B$$

425 of a category \mathcal{C} can be represented as a natural transformation

$$426 \quad n^{aB}: \mathcal{C}(-, A) \rightarrow \mathcal{C}(-, B)$$

427 from the domain functor $\mathcal{C}(-, A)$ constituting the figure geometry of the object A to the
 428 codomain functor $\mathcal{C}(-, B)$ constituting the figure geometry of the object B, which is the core
 429 mathematical content of the Yoneda lemma (Lawvere and Rosebrugh, 2003, p. 249): “maps in
 430 any category can be represented as natural transformations” (Lawvere and Schanuel, 2009, p.
 431 378). Since natural transformations represent structure-preserving maps between objects, the
 432 domain (codomain) functor of a natural transformation represents the domain (codomain) object
 433 of the structure-preserving map.

434 Let us define the (domain) functor

$$435 \quad \mathcal{C}(-, A): \mathcal{C} \rightarrow \mathcal{C}$$

436 as: for each object X of the category \mathcal{C}

$$437 \quad \mathcal{C}(-, A)(X) = A^X$$

438 where A^X is the map object of all X-shaped figures in A

$$439 \quad x_A: X \rightarrow A$$

440 and, for each map

$$441 \quad x_Y: X \rightarrow Y$$

442 of the category \mathcal{C}

$$443 \quad \mathcal{C}(-, A)(x_Y: X \rightarrow Y) = A^{x_Y}: A^Y \rightarrow A^X$$

444 where A^Y is the map object of all Y-shaped figures in A , and with the map A^{x_Y} of map objects
 445 defined as

$$446 \quad A^{x_Y} (y_A: Y \rightarrow A) = y_A \circ x_Y = x_A: X \rightarrow A$$

447 assigning a map x_A in the map object A^X to each map y_A in the map object A^Y . Thus the functor

$$448 \quad C(-, A): C \rightarrow C$$

449 in assigning to each map

$$450 \quad x_Y: X \rightarrow Y$$

451 (of the domain category C) its [induced] map [of map objects]

$$452 \quad C(-, A) (x_Y: X \rightarrow Y) = C(-, A) (Y) \rightarrow C(-, A) (X) = A^{x_Y}: A^Y \rightarrow A^X$$

453 (of the codomain category C) is contravariant, i.e. a transformation of a shape X into a shape Y

454 induces a transformation (in the opposite direction) of Y-shaped figures in A into X-shaped

455 figures in A (Lawvere and Rosebrugh, 2003, pp. 236-237).

456 Now, we check to see if $C(-, A)$ preserves identities, i.e. whether

$$457 \quad C(-, A) (1_X: X \rightarrow X) = 1_{C(-, A)(X)}$$

458 for every object X . Evaluating

$$459 \quad C(-, A) (1_X: X \rightarrow X) = A^{1_X}: A^X \rightarrow A^X$$

460 at a map

$$461 \quad x_A: X \rightarrow A$$

462 we find that

$$463 \quad A^{1_X} (x_A: X \rightarrow A) = (x_A \circ 1_X) = x_A: X \rightarrow A$$

464 (for every map x_A in the map object A^X). Next, evaluating

$$465 \quad 1_{C(-, A)(X)} = 1_{A^X}: A^X \rightarrow A^X$$

466 at the map

$$467 \quad x_A: X \rightarrow A$$

468 we find that

$$469 \quad 1_{A^X} (x_A: X \rightarrow A) = (x_A \circ 1_X) = x_A: X \rightarrow A$$

470 (for every map x_A in the map object A^X). Since

$$471 \quad A^{1_X} = 1_{A^X}$$

472 i.e.

$$473 \quad C(-, A)(1_X: X \rightarrow X) = 1_{C(-, A)(X)}$$

474 for every object X of the category \mathcal{C} , we say $C(-, A)$ preserves identities.

475 Next, we check to see if $C(-, A)$ preserves composition. Since $C(-, A)$ is contravariant,

476 we check whether

$$477 \quad C(-, A)(y_Z \circ x_Y) = C(-, A)(x_Y) \circ C(-, A)(y_Z)$$

478 where $y_Z: Y \rightarrow Z$. Evaluating

479
$$\mathbf{C}(-, \mathbf{A})(y_Z \circ x_Y) = \mathbf{A}^{(y_Z \circ x_Y)}$$

480 at any map z_A in the map object \mathbf{A}^Z , we find that

481
$$\mathbf{A}^{(y_Z \circ x_Y)}(z_A) = z_A \circ (y_Z \circ x_Y)$$

482 Next, we evaluate

483
$$\mathbf{C}(-, \mathbf{A})(x_Y) \circ \mathbf{C}(-, \mathbf{A})(y_Z) = (\mathbf{A}^{x_Y} \circ \mathbf{A}^{y_Z})$$

484 also at the map z_A

485
$$(\mathbf{A}^{x_Y} \circ \mathbf{A}^{y_Z})(z_A) = \mathbf{A}^{x_Y}(z_A \circ y_Z) = (z_A \circ y_Z) \circ x_Y$$

486 Since

487
$$z_A \circ (y_Z \circ x_Y) = (z_A \circ y_Z) \circ x_Y$$

488 by the associativity of the composition of maps, we have composition preserved

489
$$\mathbf{C}(-, \mathbf{A})(y_Z \circ x_Y) = \mathbf{C}(-, \mathbf{A})(x_Y) \circ \mathbf{C}(-, \mathbf{A})(y_Z)$$

490 Having checked that

491
$$\mathbf{C}(-, \mathbf{A}): \mathbf{C} \rightarrow \mathbf{C}$$

492 with

493
$$\mathbf{C}(-, \mathbf{A})(X) = \mathbf{A}^X$$

494
$$\mathbf{C}(-, \mathbf{A})(x_Y: X \rightarrow Y) = \mathbf{A}^{x_Y}: \mathbf{A}^Y \rightarrow \mathbf{A}^X$$

495 where $\mathbf{A}^{x_Y}(y_A) = y_A \circ x_Y$, is a contravariant functor, we consider another contravariant functor

496 $C(-, B): C \rightarrow C$

497 with

498 $C(-, B)(X) = B^X$

499 $C(-, B)(x_Y: X \rightarrow Y) = B^{x_Y}: B^Y \rightarrow B^X$

500 where $B^{x_Y}(y_B) = y_B \circ x_Y$.

501 With the two functors $C(-, A)$ and $C(-, B)$ representing the [figure geometry of] objects

502 A and B , respectively, we now show that every structure-preserving map

503 $a_B: A \rightarrow B$

504 is represented by a natural transformation

505 $n^{a_B}: C(-, A) \rightarrow C(-, B)$

506 More explicitly, given a map a_B , we can construct a natural transformation n^{a_B} . A natural

507 transformation n^{a_B} from the functor $C(-, A): C \rightarrow C$ to the functor $C(-, B): C \rightarrow C$ assigns to

508 each object X of the domain category C (of both domain and codomain functors) a map

509 $a_B^X: A^X \rightarrow B^X$

510 (in the common codomain category C) from the value of the domain functor at the object X , i.e.

511 $C(-, A)(X) = A^X$ to the value of the codomain functor at X , i.e. $C(-, B)(X) = B^X$; and to each

512 map $x_Y: X \rightarrow Y$ (in the common domain category C), a commutative square (in the common

513 codomain category C) shown below:

514

515

516

517

518

519

$$\begin{array}{ccc}
 A^X & \xrightarrow{a_B^X} & B^X \\
 \uparrow A^{x_Y} & & \uparrow B^{x_Y} \\
 A^Y & \xrightarrow{a_B^Y} & B^Y
 \end{array}$$

520 satisfying

521

$$a_B^X \circ A^{x_Y} = B^{x_Y} \circ a_B^Y$$

522 (Lawvere and Rosebrugh, 2003, p. 241; Lawvere and Schanuel, 2009, pp. 369-370). We have

523 already seen that with the composition-induced maps (of map objects):

524

$$A^{x_Y}(y_A) = y_A \circ x_Y$$

525

$$a_B^X(x_A) = a_B \circ x_A$$

526

$$a_B^Y(y_A) = a_B \circ y_A$$

527

$$B^{x_Y}(y_B) = y_B \circ x_Y$$

528 the required commutativity:

529

$$a_B^X \circ A^{x_Y}(y_A) = a_B^X(y_A \circ x_Y) = a_B \circ (y_A \circ x_Y)$$

530

$$B^{x_Y} \circ a_B^Y(y_A) = B^{x_Y}(a_B \circ y_A) = (a_B \circ y_A) \circ x_Y$$

531 is given by the associativity of the composition of maps

$$532 \quad a_B \circ (y_A \circ x_Y) = (a_B \circ y_A) \circ x_Y = a_B \circ y_A \circ x_Y$$

533 Thus, each A-shaped figure in B (a_B) is a natural transformation (n^{a_B} ; homogenous with respect
534 to composition of maps) of the figure geometry $\mathcal{C}(-, A)$ of A into the figure geometry $\mathcal{C}(-, B)$
535 of B.

536 Furthermore, we can obtain the set $|B^A|$ of all A-shaped figures in B based on the 1-1
537 correspondence between A-shaped figures in B and the points (i.e. maps with terminal object T
538 of the category \mathcal{C} as domain; Lawvere and Schanuel, 2009, pp. 232-234) of the map object B^A .
539 This 1-1 correspondence, which follows from the universal mapping property defining
540 exponentiation, along with the fact that the terminal object T is a multiplicative identity
541 (Lawvere and Schanuel, 2009, pp. 261-263, 313-314, 322-323), involves the following two 1-1
542 correspondences between three maps:

$$543 \quad \begin{array}{c} T \rightarrow B^A \\ \hline T \times A \rightarrow B \\ \hline A \rightarrow B \end{array}$$

546 Yoneda lemma says, in terms of our figures-and-incidences characterization of objects,
547 that the set $|B^A|$ of A-shaped figures in B

$$548 \quad a_B: A \rightarrow B$$

549 is isomorphic to the set $|\mathcal{C}(-, B)^{\mathcal{C}(-, A)}|$ of natural transformations

$$550 \quad n^{a_B}: \mathcal{C}(-, A) \rightarrow \mathcal{C}(-, B)$$

551 of the figure geometry of A into that of B. The required isomorphism of sets

$$552 \quad |B^A| = |C(-, B)^{C(-, A)}|$$

553 follows from the 1-1 correspondence between A-shaped figures in B and the natural
 554 transformations (compatible with all figures and their incidences) of the figure geometry of A
 555 into that of B, which we have already shown (see also Lawvere and Rosebrugh, 2003, p. 104,
 556 174).

557 Dually, a map

$$558 \quad A \rightarrow B$$

559 viewed as a B-valued property on A induces a natural transformation

$$560 \quad C(B, -) \rightarrow C(A, -)$$

561 of the function algebra of B into that of A (Lawvere and Rosebrugh, 2003, p. 249). Here also the
 562 proof of Yoneda lemma involves two transformations: (i) Contravariant: a map from an object A
 563 to an object B induces a transformation of properties of B into properties of A, for each type
 564 (object) of the category, and (ii) Covariant: a map from a type T to a type R (of properties)
 565 induces a transformation of T-valued properties into R-valued properties, for every object of the
 566 category. The calculations involved in proving Yoneda lemma in this case of function algebras
 567 are same as in the case of figure geometries, except for the reversal of arrows due to the duality
 568 between function algebra and figure geometry (Lawvere and Rosebrugh, 2003, p. 174; Lawvere
 569 and Schanuel, 2009, pp. 370-371). More specifically, function algebras and figure geometries
 570 are related by adjoint functors (Lawvere, 2016).

571 **A2. Four Truth Values of the Logic of Perception**

572 Conscious perception involves two sequential processes of sensation followed by interpretation:

573
$$\text{Physical stimuli} \rightarrow \text{Brain} \rightarrow \text{Conscious Percepts}$$

574 (Albright, 2015; Croner and Albright, 1999), which can be thought of as

575
$$X - \text{coding} \rightarrow Y - \text{decoding} \rightarrow Z$$

576 and objectified as two sequential processes:

577
$$A - f \rightarrow B - g \rightarrow C$$

578 Without discounting that the processes of sensation and interpretation are much more structured

579 than mere functions, and with the objective of simplifying the calculation of truth value object,

580 we model percept as an object made up of three [component] sets C, B, and A, which are sets of

581 physical stimuli, their neural codes, and interpretations, respectively, and two [structural]

582 functions f and g specifying for each interpretation in A the neural code in B (of which it is an

583 interpretation) and for each neural code in B the physical stimulus in C (of which it is a

584 measurement), respectively (see Lawvere and Rosebrugh, 2003, pp. 114-117). The logic of [the

585 category of] perception, whose objects are two sequential functions is determined by its truth

586 value object (Lawvere and Rosebrugh, 2003, pp. 193-212; Lawvere and Schanuel, 2009, pp.

587 335-357; Reyes, Reyes, and Zolfaghari, 2004, pp. 93-107). The truth value object of a category

588 is an object Ω of the category such that parts of any object X are in 1-1 correspondence with

589 maps from the object X to the truth value object Ω . Since parts of an object are monomorphisms

590 with the object X as codomain, for each monomorphism with X as codomain there is a

591 corresponding X-shaped figure in Ω .

592 In order to calculate the truth value object, first we need to define maps between objects
 593 of the category of percepts. A map from an object

594
$$A - f \rightarrow B - g \rightarrow C$$

595 to an object

596
$$A' - f' \rightarrow B' - g' \rightarrow C'$$

597 is a triple of functions

598
$$p: A \rightarrow A', q: B \rightarrow B', r: C \rightarrow C'$$

599 satisfying two equations

600
$$q \circ f = f' \circ p, r \circ g = g' \circ q$$

601 which make the two squares in the diagram

602

603

604

605

606

$$\begin{array}{ccc}
 A & \xrightarrow{p} & A' \\
 f \downarrow & & \downarrow f' \\
 B & \xrightarrow{q} & B' \\
 g \downarrow & & \downarrow g' \\
 C & \xrightarrow{r} & C'
 \end{array}$$

607 commute, i.e. ensure that maps between objects preserve the structural essence of the category

608 (Lawvere and Schanuel, 2009, pp. 149-150).

609 Now that we have maps of the category of percepts defined, we can calculate its truth
 610 value object. The truth value object of a category is calculated based on the parts of the basic
 611 shapes (essence) constituting the objects of the category. In the category of sets, one-element set
 612 $\mathbf{1}$ ($= \{\bullet\}$) is the basic shape in the sense any set is made up of elements (see Posina, Ghista, and
 613 Roy, 2017 for the details of the calculation of basic shapes, i.e. theory subcategories of various
 614 categories). Since the set $\mathbf{1}$ is the also the terminal object (i.e. an object to which there is exactly
 615 one map from every object; Lawvere and Schanuel, 2009, pp. 213-214) of the category of sets,
 616 and since every set is completely determined by its points (terminal object-shaped figures), we
 617 can determine the truth value object of the category of sets by determining its points, i.e. maps
 618 from $\mathbf{1}$ to the (yet to be determined) truth value object. According to the definition of truth value
 619 object, $\mathbf{1}$ -shaped figures in the truth value object are in 1-1 correspondence with parts of $\mathbf{1}$. Since
 620 the terminal set $\mathbf{1}$ has two parts: $\mathbf{0}$ ($= \{\}$) and $\mathbf{1}$, the truth value set has two points (elements).
 621 Thus, the truth value object of the category of sets is $\mathbf{2}$ ($= \{\text{false}, \text{true}\}$).

622 Along similar lines, let us calculate the terminal object of the category of percepts. Since
 623 there is only one map from any object (two sequential functions) to the object T (two sequential
 624 functions from one-element set to one-element set):

$$625 \quad \mathbf{1} \rightarrow \mathbf{1} \rightarrow \mathbf{1}$$

626 the terminal object of the category of percepts is T. Since parts of the terminal object T
 627 correspond to the points of the truth value object, let's look at the parts of the terminal object.

628 The terminal object T

$$629 \quad \mathbf{1} \rightarrow \mathbf{1} \rightarrow \mathbf{1}$$

630 has four parts:

631 Part 1 ($O: 0 \rightarrow T$)

0 **1**

632 ↓

0 **1**

633 ↓

0 **1**

634

635 Part 2 ($O_I: 0_1 \rightarrow T$)

0 **1**

636 ↓

0 **1**

637 ↓

1 → **1**

638

639 Part 3 ($O_2: 0_2 \rightarrow T$)

0 **1**

640 ↓

1 → **1**

641 ↓ ↓

1 → **1**

642

643 Part 4 ($I: T \rightarrow T$)

1 → **1**

644 ↓ ↓

1 → **1**

645 ↓ ↓

1 → **1**

646

647 These four parts correspond to the four points (global truth values) of the truth value object,

648 which means that the component set (of the truth value object) corresponding to the stage of

649 interpretations is a four-element set $\mathbf{4} = \{0, 0_1, 0_2, I\}$. Since objects in the category of perception
 650 (two sequential functions) are not completely determined by points, we look for all other basic
 651 shapes that are needed to completely characterize any object of two sequential functions. The
 652 other basic shapes, besides the terminal object T , are: domains of the parts 0_2 and 0_1 of the
 653 terminal object T , i.e. shape 0_2

$$654 \quad \mathbf{0} \quad \mathbf{1} \rightarrow \mathbf{1}$$

655 and shape 0_1

$$656 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{1}$$

657 Since the basic shape object 0_2 has three parts (0 , 0_1 , and I), there are three 0_2 -shaped figures in
 658 the truth value object, and since the object 0_1 has two parts (0 and I), there are two 0_1 -shaped
 659 figures in the truth value object, which means that the component set (of the truth value object)
 660 corresponding to the stage of neural coding is a three-element set $\mathbf{3} = \{0, 0_1, I\}$, while the
 661 component set (of the truth value object) corresponding to the stage of physical stimuli is a two-
 662 element set $\mathbf{2} = \{0, I\}$. Putting it all together we find that the truth value object of the category
 663 of percepts is:

$$664 \quad \mathbf{4} - j \rightarrow \mathbf{3} - k \rightarrow \mathbf{2}$$

665 We still have to determine the functions j and k , which can be done by examining the structural
 666 maps between the basic shapes

$$667 \quad 0_1 - c \rightarrow 0_2 - d \rightarrow T$$

668 which as a subcategory constitutes the theory (abstract essence) of the category of two sequential
 669 functions. More explicitly, the incidence relations between the three basic-shaped figures in the

670 truth value object are calculated from the inverse images of the parts of the basic shapes (0_1 , 0_2 ,
 671 and T) along the structural maps (d and c). The inverse images of each one of the four points (0 ,
 672 0_1 , 0_2 , and 1 corresponding to the four parts of the terminal object T) along the structural maps
 673 decoding d and coding c give for each one of the four global truth values $\mathbf{4} = \{0, 0_1, 0_2, 1\}$ its
 674 value in the truth value sets $\mathbf{3} = \{0, 0_1, 1\}$ and $\mathbf{2} = \{0, 1\}$ of the previous stages of neural codes
 675 and physical stimuli. For example, the global truth value 0_2 corresponds to the part 0_2 of the
 676 basic shape T, and its inverse image along the structural map $d: 0_2 \rightarrow T$ is the entire basic shape
 677 0_2 , which corresponds to the truth value 1 (of stage $\mathbf{3}$); and the inverse image of the entire object
 678 0_2 along the structural map $c: 0_1 \rightarrow 0_2$ is the entire basic shape 0_1 , which corresponds to the truth
 679 value 1 (of stage $\mathbf{2}$). Along these lines we find that

$$680 \quad j(0) = 0, j(0_1) = 0_1, j(0_2) = 1, j(1) = 1$$

$$681 \quad k(0) = 0, k(0_1) = 1, k(1) = 1$$

682 which completely characterizes the truth value object

$$683 \quad \mathbf{4} - j \rightarrow \mathbf{3} - k \rightarrow \mathbf{2}$$

684 of the category of percepts.

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