Rational Number Representation by the Approximate Number System

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Abstract

The approximate number system (ANS) enables organisms to represent the approximate number of items in an observed collection, quickly and independently of natural language. Recently, it has been proposed that the ANS goes beyond representing natural numbers by extracting and representing rational numbers (Clarke & Beck, 2021a). Prior work demonstrates that adults and children discriminate ratios in an approximate and ratio-dependent manner, consistent with the hallmarks of the ANS. Here, we use a well-known "connectedness illusion" to provide evidence that these ratio-dependent ratio discriminations are (a) based on the perceived number of items in seen displays (and not just non-numerical confounds), (b) are not dependent on verbal working memory, or explicit counting routines, and (c) involve representations with a part-whole (or subset-superset) format, like a fraction, rather than a part-part format, like a ratio. These results vindicate key predictions of the hypothesis that the ANS represents rational numbers.

Keywords: numerosity perception; approximate number system; rational number; ratio; connectedness illusion

1. Introduction

Humans, and many non-human animals, possess a *number sense* (Dehaene, 1997) or *approximate number system* (ANS), that represents number. Evidence for this system is found in many creatures, including fish (e.g., Agrillo et al., 2008), rats (e.g., Platt & Johnson, 1971), pigeons (e.g., Emmerton et al., 1997), monkeys (e.g., Brannon & Terrace, 1998), human infants (e.g., Xu & Spelke, 2000), pre-numerate human children (e.g., Mix et al., 2002), human adults whose language lacks precise number words (Pica et al., 2004), as well as human adults with a formal math education (e.g., Barth et al., 2003). In addition, the system is found to support a diverse range of numerical computations – for instance, ordinal comparisons (Temple and Posner, 1998), the ability to identify two collections as equinumerous (Barth et al., 2003), number estimations (Cordes et al., 2001), as well as addition (McCrink & Wynn, 2004), subtraction (Barth et al., 2006), multiplication (Qu, Szkudlarek, & Brannon, 2021; McCrink & Spelke, 2010) and division operations (McCrink & Spelke, 2016; Szkudlarek et al., 2022). Consequently, an orthodox view in cognitive science is that the ANS is widespread in nature, operates independently of natural language, and enables humans and other organisms to process number throughout the lifespan, albeit approximately and in accord with Weber's Law.

Despite this prevailing orthodoxy, relatively little attention has been paid to the kinds of number that the ANS represents. The vast majority of ANS research focusses exclusively on the system's representation and discrimination of natural numbers; for instance, the system's capacity to discriminate *8* items from *16* items, or to add the number of whole items in two collections together. While some have proposed that the ANS goes beyond representing natural number by representing real numbers quite generally (e.g., Gallistel & Gelman, 2000; Dramkin & Odic, 2021), arguments in favour of this conjecture have been roundly criticized (Laurence & Margolis, 2005; Carey 2009; Clarke & Beck 2021a; Clarke & Beck 2021b; Samuels & Snyder 2024; see also Gallistel 2021 who clarifies that he no longer endorses his original argument). Consequently, it is tempting to suppose that the ANS is exclusively in the business of representing whole or natural number.

A recent article by Clarke and Beck (2021a) questions this temptation. The authors of this paper agree that there is little reason to think that the ANS represents irrational numbers, and hence the real numbers in general. Nevertheless, they propose that there is provisional reason to think the ANS goes beyond representing natural numbers (like 1, 2, 3, 4, 5…), by representing rational numbers, which can be expressed as a ratio or fraction among natural numbers (e.g., 1:3 or ¼). To this end, they lean on research that demonstrates sensitivity to proportional relations in non-human animals, human infants, and young children. For example, six-month-old infants, who have been habituated to a specific ratio of blue to yellow elements, look significantly longer at arrays with a novel ratio (McCrink & Wynn, 2006). Slightly older infants rely on the ratio of preferred to non-preferred lollipops in two collections to decide which of two samples is more likely to contain a desired lollipop (Denison & Xu, 2014; see also: Xu & Garcia, 2008; Xu & Denison, 2009; Denison & Xu, 2010; Fontanari et al., 2014; and Kayhan et al., 2018). Similarly, non-human apes (Rakoczy et al., 2014; Eckert et al., 2018) and monkeys (Drucker et al., 2016; Tecwyn et al., 2017) can reliably distinguish arrays based on the ratio of preferred to non-preferred food items.

Sensitivity to ratios or probabilities so early in development is surprising given that learning symbolic notations for rational numbers (e.g., fractions, ratios, and percentages) is notoriously difficult for children (e.g., Lortie-Forgues, Tian, & Siegler, 2015). In fact, Piaget (Inhelder & Piaget, 1958; Piaget & Inhelder, 1951/1975) argued that proportional reasoning is only possible once children understand formal operations. Subsequent studies, however, provide evidence for an intuitive understanding of non-symbolic proportional relations with discrete quantities that emerges long before the capacity to grasp formal operations obtains (e.g., Siegler et al., 2013; Matthews & Chesney, 2015; Lewis et al., 2016; Matthews et al., 2016; Bhatia et al., 2020; Binzak & Hubbard, 2020; O'Grady & Xu, 2020). To give a concrete example, Szkudlarek and Brannon (2021) presented 6–8-year-old children with pairs of gumball machines, each containing blue and white 'gumballs' (dots) or blue and white Arabic numerals specifying the number of blue and white gumballs that each machine contained. The children were tasked with reporting which gumball machine was most likely to randomly produce a single gumball of a preferred color. They were significantly above chance in both symbolic and non-symbolic versions of the task (i.e., whether the numbers of gumballs were represented using Arabic numerals or depicted as a collection of dots), even though they had not begun to study fractions in school and still could not make or understand precise fraction comparisons. Since subsequent analyses revealed that they could not have relied exclusively on simpler heuristics (e.g., choosing machines which contained the larger absolute number of desired gumballs, or the smaller absolute number of undesired gumballs), the results of this study indicate that children were discriminating the ratio of blue to white gumballs in each machine and subsequently comparing those ratios, with accuracy predicted by Weber's Law (i.e., the ratio between the ratios of gumballs in each machine).

In this latter respect, children's ratio discriminations mirrored the performance profile of the numerical discriminations facilitated by the ANS. But is ratio-dependent performance in a ratio-comparison task, such as that used by Szkudkarek and Brannon (2021), evidence that the ANS represents rational number? Clarke and Beck (2021a) conjecture that it is. On their view, results of this sort recommend a two-stage model of ANS processing. When participants are presented with two arrays of discrete items, an initial stage of ANS processing extracts the natural number of elements in its relevant subsets (e.g., the number of blue gumballs, white gumballs, and/or blue and white gumballs). A second stage of ANS processing is then able to compare these natural numbers, to extract the fraction or ratios that hold between them. One possibility is that this second stage of processing is an independent system from the first system that extracted natural numbers, in the sense of being a distinct module or mechanism (see: Mandelbaum 2013), operating according to its own proprietary algorithms (Clarke 2021). However, Clarke and Beck propose that ratio processing of this sort still qualifies as a *bona fide* constituent of the ANS since the ANS is best understood in terms of its computational-level description (Marr 1982). While ANS processing, may or may not comprise a myriad of sub-systems, this second stage of ratio processing belongs to a broader system which is *demarcated by its functioning to represent numbers* (*rational* and *natural numbers*) *independently of flexible thought and in a manner that is imprecise and conforms to Weber's Law*. In this way, Clarke and Beck propose that 'the ANS' or 'number sense' is to be understood and demarcated in something like the way that 'the visual system' has been understood or demarcated by appeal to its function of detecting *what is where* from light on the retina, irrespective of the underlying architecture and algorithms the system employs (*ibid.*).

In the present study, we sought to identify and adjudicate three untested assumptions on which Clarke and Beck's two-stage model of ANS processing relies. Firstly, we note that their two-stage model would be put under pressure if it turned out that participants in the abovementioned experiments were relying on continuous properties of the observed displays when comparing ratios. Prior work demonstrates that children and adults are adept at making ratio comparisons with continuous quantities (e.g., Boyer, Bradley, & Greer, 2024). Furthermore, in Szkudlarek and Brannon's (2021) study, the size and spacing of the dots was not controlled for. Thus, it is possible that children's discriminations were based on an appreciation for the ratios/proportions among continuous (non-numerical) properties of the stimuli – for instance, between the total surface area of blue dots on the screen and the total surface area of white dots on the screen, thereby undermining Clarke and Beck's suggestion that ratio comparisons depended on an initial stage of *natural number extraction by the ANS*. Indeed, similar worries could be raised in relation to many of the abovementioned ratio experiments purporting to demonstrate numerically based ratio discriminations, which is problematic given existing critiques of the ANS which assert that its comparisons pertain to non-numerical values (Gebuis et al., 2016; Leibovich et al., 2017; though see Park et al., 2021). While it might be replied that children from Szkudlarek and Brannon's (2021) study performed equally well in a symbolic version of the task, which effectively eliminates these confounds, symbolic number comparisons may rely on orthogonal processes (Gomez, 2021; Krajsci et al., 2022). Thus, Experiment 1 of the present study sought to adjudicate these concerns directly by employing the 'connectedness illusion' (Franconerri et al., 2009; He et al., 2009). Connectedness systematically alters the ANS's discrimination of natural number without altering continuous attributes of the collections. In this way we provide evidence that non-symbolic ratio comparisons are based on natural number, as extracted by an ANS, and not just non-numerical properties of seen collections (e.g., their total surface area, density, or average brightness), consistent with Clarke and Beck's hypothesis.

A second argument that ratio comparisons (of the sort investigated by Szkudlarek and Brannon [2021]) need not imply that the ANS represents rational numbers, concerns the observation that performance might simply reflect domain general analogical reasoning. For example, Hecht et al. (2021) argue that "The ANS represents natural numbers, and there are separate, non-numeric processes that can be used to represent ratios across a wide range of domains, including natural numbers." Such claims admit various interpretations. But a strong reading, which would be particularly problematic for Clarke and Beck's (2021a) computational-level hypothesis, will maintain that while natural numbers are extracted by the ANS, ratios or fractions among these natural numbers are simply derived by domain general resources involved in flexible thought. For instance, having represented that there are 8 or 8ish blue items in one subset or collection and 10 or 10ish red items in another subset or collection, participants might appreciate that these numbers stand in an 8 to 10 ratio to one another, by employing post-perceptual cognitive resources involved in domain general reasoning. Indeed, Dramkin and Odic (2021) note that, if this were so, the resulting ratio discriminations (which conform to Weber's Law) might inherit their characteristic imprecision from the imprecision of the ANS's representations of natural number which serve as input to this central cognitive process of ratio discrimination. To adjudicate this concern, we conducted a second experiment in which participants engaged in a secondary verbal shadowing task, while performing the ratio comparison task tested in Experiment 1. While this secondary task taxes verbal working memory, in ways that are known to hinder verbal arithmetic (Hecht, 2002; Seitz & Schumann-Hengsteler, 2002; Imbo & Vandierendonck, 2007) and analogical reasoning (Waltz et al., 2000), we find that participants' performance in this second experiment is largely unaffected as compared to those tested in Experiment 1. These findings thereby vindicate a further bold prediction of Clarke and Beck's model.

Finally, some researchers have objected that even if extant evidence supports the view that the ANS represents *ratios*, ratios are not rational numbers. Critiques of this type take different forms (see: Gomez, 2021; Lyons, 2021). For instance, Ball (2021) objects that, unlike genuine rational numbers (e.g., numbers expressed as fractions or decimals), ratios lack an additive structure. So, while fractions and decimals are apt to be added and subtracted (e.g., ¼ + ¼ = ½), ratios are not like this: if the blue and red balls in one box stand in a 1:3 ratio, while the blue and red balls in a second box stand in a 2:3 ratio, we have no way of knowing what ratio the blue and red balls would stand in if they were both poured into a single box without first establishing the absolute number of blue and red balls in either box. Somewhat distinctly, Peacocke (2021) objects that genuine rational number representation by the ANS requires that values be placed on a single "rational number line". Clarke and Beck (2021b) have responded to these concerns on broadly philosophical grounds. But while the current study was not designed to address these issues directly, we conjecture that there are *empirically detectable* differences between representations of ratios and fractions in the abovementioned experiments. Perhaps most notably: Where ratio representations specify a part-part relation (i.e., to say that there is a 1:3 ratio of blue to red balls in a bucket is to say that for every ball in the blue subset of the collection there are three balls in the red subset of the collection), fractions encode a part-whole relation (i.e., to say that ¼ of the balls are blue is to say that one out of every four balls is blue). We present posthoc analyses, showing that participants' pattern of performance in Experiments 1 and 2 is straightforwardly predicted by their employment of representations with part-whole structure, which is thereby akin to the format of a fraction rather than a literal ratio. We consider alternative explanations but conjecture that our findings serve as a proof of principle, showing how progress can be made investigating the format and content of sub-linguistic proportional representations.

2. Experiment 1

Previous research shows that humans can discriminate non-symbolic ratios when presented with arrays of discrete items, quickly, efficiently, and without recourse to explicit counting, albeit imprecisely and in accord with Weber's Law. However, it remains unclear whether these ratio comparisons pertain to ratios among the *number* of perceived elements in a display, or continuous properties of the collections compared. For instance, if all elements in a collection are of a single size, discriminating the ratios among relevant elements could be based on an appreciation for their non-numerical properties, like their total surface areas (but see Park et al., 2021), or perhaps their density, brightness, and convex hull. Indeed, this concern is pressing, given prior claims that children find continuous proportions easier to compare than proportions based on discrete items (e.g., Boyer, Levine, & Huttenlocher, 2008; Jeong et al., 2007; Singer-Freeman & Goswami, 2001; Spinillo & Bryant, 1999; Park, Viegut & Matthews, 2020).

To adjudicate this concern, our first experiment takes advantage of the 'connectedness illusion' (Franconerri et al., 2009). In the connectedness illusion, items (e.g., circles) are connected with thin lines, effectively turning pairs of items into single dumbbell-shaped objects. Under these conditions, observers enjoy a significant reduction in the perceived number of items in the display, as compared with an unconnected condition, in which the same lines are present but do not connect circles (He et al., 2009) or contain small breaks (Franconerri et al., 2009; see also Kirjakovski &

Matsumoto, 2016). Crucially, these results persist, even when participants are instructed to ignore the lines and to focus only on the circles in the displays. This is striking, since arrays containing thin lines which *connect* circles barely differ with respect to their total surface area, density, convex hull, or average brightness, when compared with arrays containing an identical number of circles and free-floating lines. Thus, the reduction in perceived number appears to reflect the fact that the ANS functions to track and enumerate discrete and bounded whole objects, rather than their continuous properties (Clarke & Beck, 2021a).[1]

We reasoned that if approximate ratio comparisons are based on prior representations of whole numbers (i.e., the natural number of items in relevant subsets and/or supersets), produced by the ANS, as Clarke and Beck's (2021a) two-stage model recommends, these comparisons should be systematically affected by connectedness. After all, it is widely accepted that the ANS's extraction of natural number is influenced in this way (see above). By contrast, rival accounts have no such commitment. Since recent studies suggest that non-ANS-based representations of natural number are not influenced by the connectedness illusion, even in visual discrimination tasks (Fornaciai & Park 2021), rival architectures which propose that ratio processing bypasses the ANS entirely, or propose that its whole number representations might be "an emergent property of ratio perception (i.e., when the denominator is 1)" (Hubbard & Matthews 2021: 35), could find independent support if ratio processing proved immune to the effects of connectedness.

Accordingly, we replicated Szkudlarek and Brannon's (2021) gumball task (described previously) but with an added manipulation: We connected pairs of gumballs to systematically manipulate their perceived number. Our question was whether perceived ratios were systematically altered as compared to a baseline condition in which gumballs were left unconnected.

To illustrate, imagine that you are presented with an array containing 16 blue items and 10 red items wherein blue items are the favorable (target) color (i.e., instructions require choosing the machine with the best odds of randomly returning a single blue gumball in a blind draw). In a baseline condition, with no connections, 16:10 would be a highly favorable ratio since 16 is almost double 10. But if the 16 blue items were connected into 8 pairs this could reduce the perceived number of bounded blue items, extracted by the ANS, such that the ratio would no longer be as favorable. By contrast, if the red items were connected this could increase the perceived favorability of the ratio of blue to red items since it would reduce the perceived number of reds. In either case, if participants are simply relying on the ratio among continuous properties, connectedness should have a negligible effect because this manipulation does not change the total surface area, perimeter, convex hull, or average brightness of red and/or blue items.

Replicating Szkudlarek and Brannon (2021), we predicted that the ratio among the ratios in the two arrays would modulate performance, as measured in a baseline condition where neither stimulus contained connected items. In addition, we hypothesized that, if ratio comparisons were based on the prior extraction of the natural number of blue and red items in the display by the

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[1] This point is tacitly conceded, even by those who are critical of a *number* sense. For instance, Durgin (2008) argues that number adaptation effects simply reflect the visual system's tendency to adapt to *density*. While our study takes no stand on the existence or interpretation of number adaptation effects, Durgin nevertheless seems to accept that connectedness controls for density and other confounds; as such, he focusses on questioning whether connectedness affects apparent cases of number adaptation (Shepherd & Durgin 2016; see also Yousif et al. 2023).

ANS, performance would be systematically influenced by the connectedness of the items (even when subjects tried to ignore these connections and focus only on the dots). Thus, when favorable items in a ratio are connected, this will reduce the perceived number of favorable items, and in turn, reduce the probability with which that ratio is chosen compared to a baseline condition with no connections. In contrast when connections are applied to unfavorable items in a ratio, this should reduce the perceived number of those items and, thus, increase the perceived favorability of the ratio compared to a baseline condition. In sum, we predicted that perceived number would be influenced by the number of bounded objects in seen collections (Spelke, 1990; c.f. Green, 2018), and that connectedness would systematically influence the perceived ratio of the favorable to unfavorable items in the array, as Clarke and Beck's two-stage model predicts.

2.1 Method

In this and subsequent experiments, the sample size, primary dependent variables, and key statistical tests were determined prior to data collection and were pre-registered. Those pre-registrations can be accessed here:
https://osf.io/6gbcz/?view_only=3fa268d3afdc47438d70d967221970cb

2.1.1 Participants

Thirty-nine adults participated in this experiment (Mean age = 22.57 years, SD = 5.76; 24 male participants, 15 female participants, and 0 non-binary participants). Participants were recruited online, using Prolific, tested online, and paid a modest sum for their participation. Participants provided informed consent to a protocol approved by the Institutional Review Board in a large research university. Each participant was required to confirm that they had normal or corrected-to-normal vision, were completing the task on a computer with a screen-size of 13-16 inches diagonal length and did not suffer from color blindness (color sightedness was also independently verified in an online test). Additional preregistered exclusion criteria were 1) a side bias with greater than 65% of responses to the left or right side, 2) failure to respond to more than 10% of trials, 3) at or below chance performance on baseline trials, 4) mean accuracy more than 1.5 interquartile ranges below Q1 of the whole sample's distribution. These criteria were chosen to ensure that participants were engaging with the task, and not answering randomly, or employing some task-irrelevant strategy. Following the pre-registered criteria, 3 participants were excluded because they did not complete all the experimental blocks; 10 participants were removed because they did not perform above chance expectations on baseline trials where all elements were unconnected. The final sample size was 26 (Mean age = 22.47, SD = 1.89; 18 male participants, 8 female participants, 0 non-binary participants).

2.1.2 Stimuli

Stimuli were 105 different spatially intermixed red and blue dot arrays with thin lines either connecting two adjacent dots or free floating within white squared panels against a neutral grey background. As shown in Figure 1, in each stimulus array, there were both blue dots (RGB: 0 0 255) and red dots (RGB: 255 0 0) that were spatially intermixed. Within a single array, the absolute number of blue and red dots varied near-uniformly from 5 - 36 and the ratio of blue to red dots varied from 0.667 – 1.5.

Lines were generated with random orientations and spatial positions and matched the colors of the dots that they connected. Stimuli were constructed such that they contained the minimum number of lines that would be needed to ensure that each dot was connected with one other of the same color. However, if a stimulus contained an uneven number of to-be-connected dots, a triplet was connected. Thus, a connected stimulus with 8 blue dots, would warrant 4 thin blue lines to connect the 8 dots into 4 dumbbell-shaped objects and a connected stimulus with 9 red dots would warrant 5 red lines to connect the 9 dots into 3 pairs and one triplet. Neither connecting nor free floating lines were permitted to cross each other or touch.

There were two kinds of stimuli in this experiment. In c*onnected* stimuli the blue or red dots were connected by blue or red lines into pairs (and a single triplet if there were an odd number of dots). *Unconnected* stimuli contained an identical number of blue and red lines however the lines were left free-floating and thus did not connect any of the dots. This enabled us to manipulate the perceived number of blue or red items, independently of their continuous confounds (e.g., the total red/blue surface area presented on the screen), thereby altering the ratio among numbers of blue to red bounded items without altering the ratios among their continuous confounds.

As shown in Figure 1, there were 5 different trial types: 1) trials in which both red and blue items were left unconnected in both arrays (the baseline condition); 2) trials in which favorable items in the less-favorable ratio were connected; 3) trials in which un-favorable items in the more-favorable ratio were connected; 4) trials in which un-favorable items in the less-favorable ratio were connected; 5) trials in which favorable items in the more-favorable ratio were connected. Each participant completed 280 trials in total, with 56 trials at each of the five trial types.

2.1.3 Design and Procedure

Participants completed consent and demographic forms before the start of the experimental tasks. On each trial, participants first saw a central fixation point for 500-ms. Then two stimuli were presented simultaneously on the screen, one to the left of the fixation point and one to the right. Stimuli remained on the screen for 1500ms, after which the stimuli and fixation point disappeared, and participants were instructed to press ‘f’ if the left-hand array was deemed to have a higher probability of randomly producing a favorable item, and ‘j’ if the right-hand array was deemed to have a higher probability of randomly producing a favorable item.

Half of the participants were randomly assigned blue as the favorable target, with the other half assigned red as the favorable target. In all cases, participants were explicitly told to ignore the lines, to attend only to the number of dots, and to indicate “which box has a higher ratio of [favorable dots] to [unfavorable dots]”. Prior to completing the task, participants completed an instruction block with multiple example trials, where the instruction to ignore lines was emphasized repeatedly and it was explained that whether lines happened to connect dots was irrelevant to the number of dots and, thus, irrelevant to the ratio of blue to red dots.

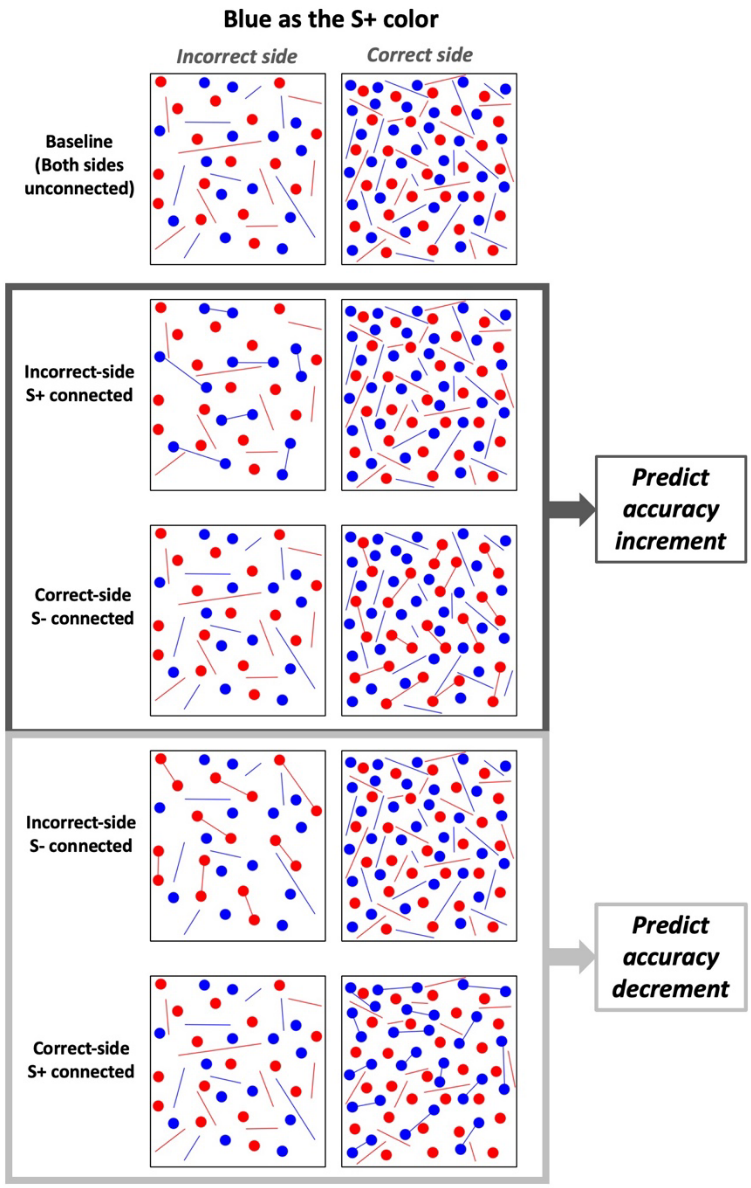

Fig. 1. Examples of standard stimuli for different connectedness conditions. When blue was assigned as the favorable color (S+), the side that contained a more favorable ratio of blue dots was referred to as the correct side, whereas the side that contained a less favorable ratio of blue dots was referred to as the incorrect side. There were three within-subject conditions: 1) Baseline condition where neither stimulus contained connected dots; 2) Predict-increment condition where the favorable dots (S+) on the incorrect side were connected or the undesirable dots (S-) on the correct side were connected; 3) Predict-decrement condition where the undesirable dots (S-) on the incorrect side were connected or the favorable dots (S+) on the correct side were connected.

2.2 Results

Overall participants performed significantly above chance expectations (Accuracy M = 76.63%, SD = 0.01, $t(25) = 53.01$, $p < .0001$, 95% CI [0.74, 0.80]; one-sample t-test). To test whether the ratio of ratios among the two stimuli affected accuracy on baseline trials, we ran a generalized linear mixed-effects model (GLMM). This model followed a binomial distribution predicting participants' item-level accuracy with the ratio of ratios as a fixed effect and a random effect of individuals. As shown in Fig 2a, we found a significant fixed effect of the ratio of ratios (β_{RR}= 0.79, SE = 0.10, Z = 8.19, p < .0001), indicating that accuracy parametrically varied based on the ratio of ratios between the two arrays, replicating Szkudlarek and Brannon's (2021) findings in elementary school children.

Our main pre-registered prediction was that accuracy would be lower on two of the trial types and higher on the other two trial types relative to baseline depending on whether the favorable (S+) dots were connected or the unfavorable (S-) dots were connected on the correct or incorrect side. We thus segregated the data into three within-subject conditions: 1) a Baseline Condition where there were no connections between dots; 2) a Predict Increment Condition where the favorable dots (S+) on the incorrect side were connected or the unfavorable dots (S-) on the correct side were connected; 3) a Predict Decrement Condition where the unfavorable dots (S-) on the incorrect side were connected or the favorable dots (S+) on the correct side were connected (See Figure 1). As shown in Fig. 2b, an ANOVA revealed a main effect of condition (F(1.4,34.5) = 38.33, p < .0001, η^2_p = .251). We conducted post-hoc pairwise comparisons between the Baseline Condition and each of the other two conditions, averaging across other factors (e.g., the ratio of ratios of the two stimuli, the assigned favorable colors). As predicted, when compared to the Baseline Condition which had no connecting lines (Accuracy M = 78.3%), the accuracy of ratio comparisons increased in the Predict Increment condition (Accuracy M = 81.9%, t(25) = 4.04, p = .001, Cohen's d = 0.79). Conversely, when compared to the Baseline Condition, participants were significantly less accurate in the Predict Decrement condition (Accuracy M = 70.6%, t(25) = 6.06, p < .0001, Cohen's d = 1.19).

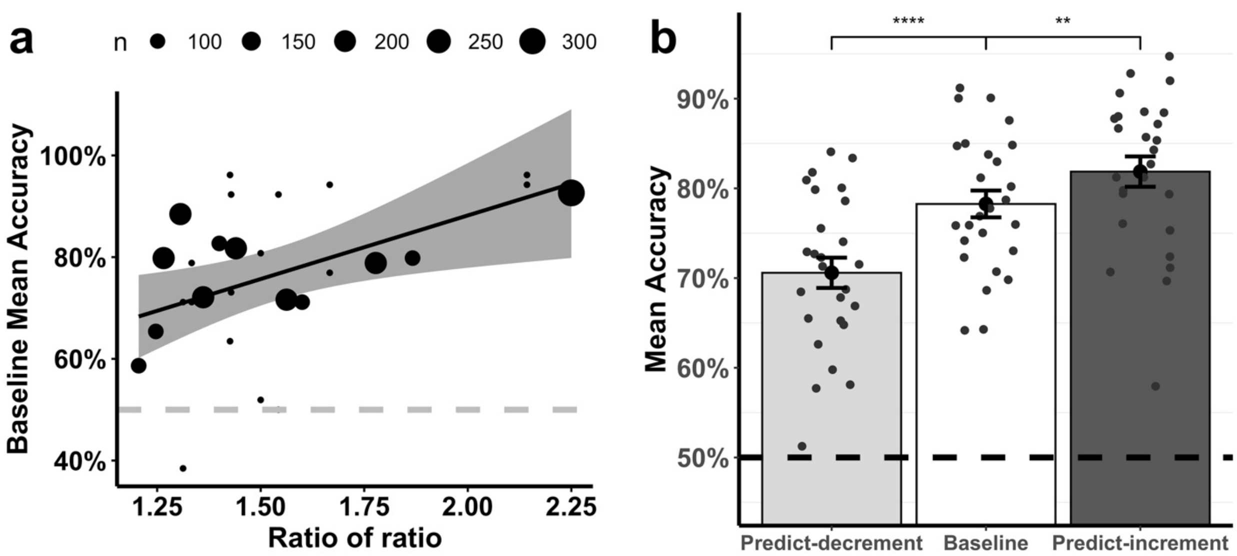

Fig. 2. Results of Experiment 1. (a) Scatter plot of mean accuracy in the baseline condition as a function of the ratio of ratios of the two stimuli. Performance on the ratio comparison task was dependent on the ratio of ratios. The grey dashed line indicates chance accuracy. The size of points denotes the number of trials in a visually proportional way. **(b)** Bar plot of mean accuracy on ratio comparisons in the different connectedness conditions. All error bars are *SEM*. Dashed line denotes chance accuracy. **p < .01, ***p < .001, ****p < .0001.

2.3 Discussion

Experiment 1 replicates prior work showing that humans can perform fast and intuitive visual ratio comparisons. In line with previous results from young children (Szkudlarek & Brannon, 2021), these ratio comparisons were imprecise, or approximate, with accuracy predicted by Weber's Law (i.e., the ratio among the ratios compared). In this respect, performance in our task mirrored the distinctive performance profile of the natural number discriminations facilitated by the ANS.

Experiment 1 extended this existing work by showing that our ability to accurately discriminate ratios is susceptible to the numerical "connectedness illusion" (He et al., 2009; Franconerri et al., 2009). When favorable items on *the correct side* (defined by the true number of red and blue dots regardless of connections), or non-favorable items on *the incorrect side*, were connected with thin lines, participants performed worse than in an otherwise identical baseline condition where all elements were left unconnected. Conversely, when non-favorable items on the correct side, or favorable items on the incorrect side were connected with thin lines, participants performed better than in an otherwise identical baseline condition where all elements were left unconnected. In both cases, this is what we would expect if connections among items leads to a reduction in their perceived number, with this (in turn) influencing the represented ratios among colored items.

Such results are predicted by Clarke and Beck's (2021a) two-stage model, according to which ratios or rational numbers are extracted via the ANS's prior extraction of natural number (i.e., the natural number of items in the subsets/supersets whose ratio is then compared). It is well established that the ANS's extraction of natural numbers is affected by the connectedness illusion, such that connecting two dots with a thin line (effectively turning two dots into a single dumbbell-shaped object) leads to a reduction in their perceived number (e.g., Franconerri et al. 2009). As such, Clarke and Beck's conjecture that ratios are based upon these natural number representations implies that connections among dots should affect perceived ratios/fractions in the above respects. Meanwhile, alternative architectures which deny that ratio processing is based on the ANS's prior extraction of natural numbers, and instead posit multiple routes to number extraction, or propose that "whole number representation might be an emergent property of ratio perception" (Hubbard & Matthews 2021: 35), share no such commitment. Indeed, while these alternative hypotheses may ultimately prove consistent with the observed effects of connectedness, this is non-obvious since independent representations of natural number in early visual cortex have proven immune to the connectedness illusion (Fornaciai & Park 2021). So, while our results are not definitive evidence for the proposal that ratio comparisons are based on the ANS's prior extraction of natural numbers, they support a bold prediction of the hypothesis, and place an explanatory burden on rival hypotheses.

Irrespective of these controversies, our results provide evidence that the ratios/rational numbers on which comparisons are based pertain to the ratios/fractions among *numbers* of items in observed arrays, rather than those items' continuous (or non-numerical) properties. Connectedness has virtually no effect on the non-numerical quantities of a visual array. This is because we included free-floating lines in all arrays (free-floating blue and red lines in baseline trials and free-floating lines of the unconnected color in connected arrays). As a result, connected arrays did not differ in their total surface area as compared with unconnected arrays (i.e., the total number of blue and/or red pixels on the screen was equivalent), and consequently did not differ in total brightness or density. It is thus hard to see how our results could be explained by mechanisms that simply track and represent ratios/fractions among continuous dimensions. By contrast, connecting two dots with a thin line *does* significantly affect the number of bounded items in a display, since it effectively turns two items into a single dumbbell-shaped object. Since our results show that connecting preferred or non-preferred items in a more/less favorable ratio systematically affects the perceived favorability of these ratios, our results indicate that approximate ratios/fractions are extracted via mechanisms which function to extract the natural number of bounded objects in observed displays. Our results thereby suggest that ratio/fraction representations are robustly numerical and concern *ratios/fractions among natural numbers of items*.

3. Experiment 2

Experiment 1 replicated previous work showing that humans can perform fast and intuitive ratio discriminations, albeit imprecisely and in accord with Weber's Law. Moreover, it extends this previous work by showing that ratio discriminations concern ratios of preferred to non-preferred *numbers* of items in displays, as extracted by mechanisms with the distinctive signature limits of the ANS. But while these findings are consistent with the suggestion that rational numbers are then extracted by a secondary stage of ANS processing, which extracts and represents ratios/fractions based on that system's prior extraction of natural number, it is an open question whether ratio comparisons of this sort are driven by the ANS itself (*pace* Clarke & Beck, 2021a). Perhaps most obviously, it could instead be that ratio comparisons result from participants' domain general reflection on the relationship between the natural numbers of items that their ANS has extracted – i.e., the number of blue items, red items, or red and blue items in each ratio – not least, because the ANS's characteristic imprecision in representing natural numbers could potentially explain why subsequent ratio comparisons conform to Weber's Law in the way we observed (Dramkin & Odic, 2021).

To address this concern, Experiment 2 sought to replicate Experiment 1 while requiring participants to engage in an additional verbal shadowing task, which was designed to consume verbal working memory and prevent explicit counting/arithmetic. We reasoned that, if ratios are being represented and processed by domain general cognitive processes and flexible thought, a distractor task that taxes verbal working memory would impair performance, as it is known to do in other cases of verbally mediated arithmetic (Hecht, 2002; Seitz & Schumann-Hengsteler, 2002; Imbo & Vandierendonck, 2007) and analogical reasoning (Waltz et al., 2000). But, if ratio comparisons are performed by independent processes, such as a relatively autonomous ANS that operates independently of flexible thought and verbal reasoning, we should expect the addition of this secondary task to have little effect on performance (see Cordes et al. 2001). Thus, our second experiment sought to test a further bold prediction of Clarke and Beck's two-stage model.

3.1 Method

3.1.1 Participants:

Thirty-one undergraduate students (Mean age = 19.78 years, SD = 1.18 years, 18 female participants, 13 male participants, 0 non-binary participants) participated in this experiment in exchange for course credit and were tested in a university laboratory. Participants provided informed consent to a protocol approved by the Institutional Review Board before starting the experiment. All participants reported normal or corrected-to-normal visual acuity and color vision (as in Experiment 1, color sightedness was also independently verified in an online test). Following the pre-registered exclusion criteria, 8 participants were excluded. (One participant was removed due to incomplete data. One participant was excluded because they exhibited a side bias where the right-side arrays were chosen on greater than 65% trials. Five participants were removed because they did not perform above chance expectations on baseline trials. One participant whose mean accuracy was more than 1.5 interquartile ranges below Q1 of the whole sample's distribution was removed.) The final sample size was 24 (Mean age = 19.79, SD = 1.22; 14 female participants, 10 male participants, 0 non-binary participant).

3.1.2 Materials and design:

Experiment 2 was designed to be as similar as possible to Experiment 1 except for the following deviations. Unlike Experiment 1, participants were required to perform an additional verbal shadowing task throughout the entire experiment. This involved repeating the words "Mary had a little lamb, her fleece was white as snow" over and over, whilst performing the ratio discrimination task from Experiment 1. To ensure that this additional verbal shadowing task was performed, participants were tested in a laboratory testing room with an experimenter monitoring their verbal recital. Before starting the experiment, participants practiced repeating the assigned sentence until they felt comfortable performing the verbal shadowing task. An experimenter monitored each participant's recitation throughout the experiment and was prepared to tap their shoulder if they paused the recitation. This only happened on one occasion and was quickly rectified. The same experimental program used in Experiment 1 was run on a Dell personal computer (Peirce, 2007). The monitor was 51 × 28.5 cm with a resolution of 1,920 ×1,080 pixels. Participants were at an average viewing distance of 56 cm from the screen. Otherwise, the design and stimuli used in the experiment were identical to that of Experiment 1.

3.2 Results

As in Experiment 1, accuracy was significantly above chance despite the requirement to perform an additional verbal shadowing task throughout (Accuracy M = 75.34%, SD = 0.01, t(23) = 58.38, $p < .0001$, 95% CI [0.73, 0.78]; one-sample t-test). As in Experiment 1, we found a significant effect of connectedness on task accuracy (F(2,46) = 35.32, $p < .0001$, $\eta^2_p = .249$). As shown in Fig 3b, accuracy was higher in the Predict Increment Condition compared to the Baseline Condition (t(23) = 4.63, $p < .001$, Cohen's d = 0.95) and lower in the Predict Decrement Condition compared to the Baseline Condition (t(23) = 4.81, $p < .001$, Cohen's d = 0.98). As shown in Fig 3a and consistent with Experiment 1, the ratio of ratios of the two arrays parametrically affected the accuracy of ratio comparisons. A GLMM controlling for the random effect of individuals revealed a significant fixed effect of the ratio of ratios on participants' item-level accuracy in the baseline condition (β_{RR}= 0.86, SE = 0.10, Z = 8.86, $p < .0001$), indicating that the approximate ratio discriminations conformed to Weber's Law.

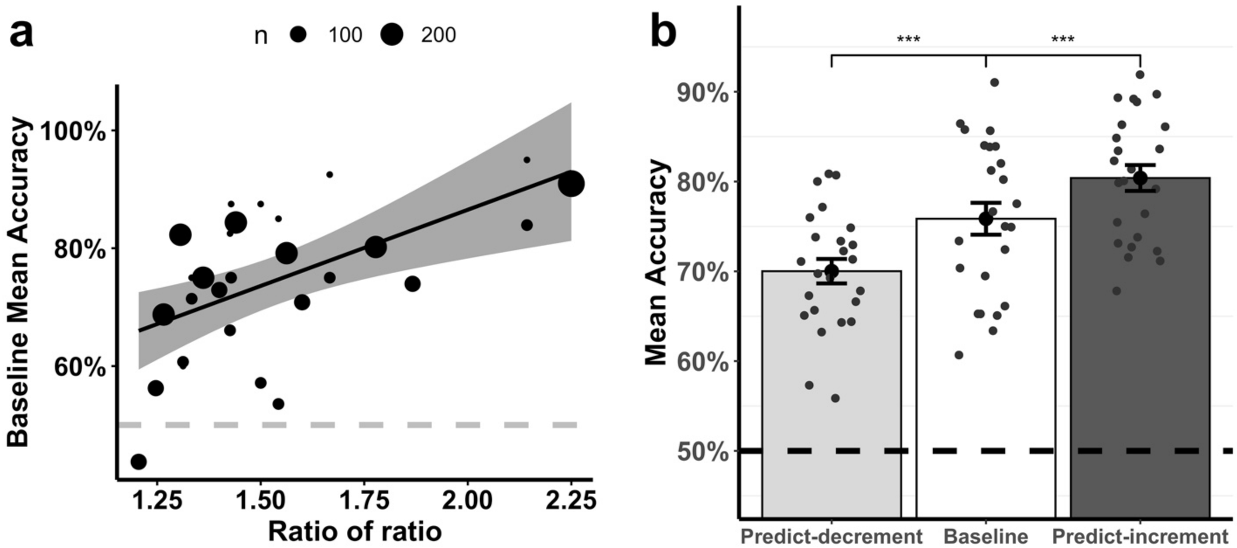

Fig. 3. Results of Experiment 2. Adding a verbal shadowing task did not significantly affect participants' performance on ratio discriminations compared to Experiment 1. The conventions of plots in (a) and (b) are the same as in Fig. 2.

To test whether the addition of our verbal shadowing task impaired ratio discriminations, we analyzed the data from participants tested in the dual-task paradigm used in Experiment 2 together with the data from participants tested in the single task paradigm from Experiment 1. We conducted a GLMM adding task format (with and without verbal shadowing) entered as a fixed effect while controlling for the fixed effect of the three connectedness conditions as well as the random effect of individuals. This GLMM followed a binomial error distribution with a logit link function. Consistent with our pre-registered prediction, we found no significant fixed effect of task format on participants' accuracy (β_{task}= -0.13, SE = 0.11, Z = -1.12, p = .26). This indicates that the addition of a secondary verbal shadowing task did not significantly disrupt approximate ratio comparisons.

3.3 Discussion

Experiment 2 replicated the results of Experiment 1 while requiring that participants simultaneously perform an additional verbal shadowing task, designed to consume verbal working memory, and prevent the use of explicit counting routines, mental arithmetic, or analogical reasoning (e.g., Waltz et al., 2000). Under these conditions, participants' performance continued to be predicted by the ratio among the ratios of items, again highlighting that the accuracy of ratio comparisons is predicted by Weber's Law. As in Experiment 1, we found that connecting target vs non-target items in a ratio systematically altered their perceived number and, thus, ratio discriminations accordingly. Finally, and most importantly, performance in Experiment 2 was not significantly impaired relative to performance in Experiment 1, despite the addition of the secondary verbal shadowing task.

These results are consistent with Clarke and Beck's (2021a) proposal that ratio comparisons rely on the ANS, defined in terms of its computational-level function of extracting numbers in accord with Weber's Law, *independently of flexible thought*. If the representations used in our task were constructed by domain general resources, involving in flexible thinking and verbal working memory, we would expect to find a reduction in task performance in Experiment 2 as compared with Experiment 1, due to the secondary verbal shadowing task. In contrast, our results are what we would expect if numerical ratios/fractions were extracted by mechanisms with the computational profile of an ANS, operating somewhat independently of flexible thought/central cognition and in a relatively encapsulated manner. In the general discussion we discuss alternative hypotheses that propose ratio processing is accomplished by something other than the ANS that nevertheless does not rely on verbal working memory.

4. Combined Analyses on Experiment 1 and 2:

Since the addition of a secondary verbal shadowing task did not significantly impact performance in Experiment 2 compared to Experiment 1, we next combined data from the two experiments (n=74) to increase power and conduct additional exploratory analyses. These subsequent analyses enabled us to confirm that performance in our task was driven by the ratios or proportions in target arrays, rather than simpler heuristics, and may also provide a window onto the underlying format of these proportional representations.

4.1 Heuristic analyses

Previous work with our experimental task and similar tasks suggests that performance is largely driven by the ratio of ratios, but that choices are biased by three heuristics that depend on the absolute number of items in the observed collections (Szkudlarek & Brannon, 2021; Falk et al., 2012; O'Grady & Xu, 2020). For instance, in Szkudlarek and Brannon (2021), children performed significantly above chance levels in both symbolic (using Arabic numerals) and non-symbolic (depicting quantities with dots) versions of the ratio-discrimination task, even before receiving any formal instruction dealing with fractions. Their subsequent analyses explored three incorrect heuristics that children could have relied on to perform the ratio tasks. These heuristics included the "More Good" strategy (choosing the machine with more of the preferred color gumballs), the "More Items" strategy (choosing the machine with a greater total number of items), and the "Less Bad" strategy (choosing the machine with fewer nonpreferred color gumballs) – strategies which need not imply anything more than natural number processing. Children's choices were significantly driven by a comparison of ratios in both the symbolic and nonsymbolic ratio comparison tasks, even when accounting for the possible use of each incorrect heuristic. Furthermore, children's errors were consistent with the "More Good" and "More Items" strategies but contradicted the "Less Bad" strategy.

Following Szkudlarek and Brannon (2021) we assessed the contribution of these heuristics to participants' performance in the collapsed data set that included Experiment 1 and 2. We first created a Ratio Model in which we identified for each trial whether using a true ratio strategy would lead to a left-side response (coded as 0) or a right-side response (coded as 1). We then constructed a model for each of the incorrect heuristic strategies. Each heuristic model indicated the left or right responses people would make if they used that heuristic (1 for choosing the right side and 0 for choosing the left side). To identify whether participants' responses deviated from each heuristic, we subtracted each heuristic model from participants' actual left and right choices and took the absolute value. This was referred to as the Actual Deviation from Heuristic Model (0 for responses aligned with each heuristic, 1 for responses different from that predicted by a given heuristic). Next, we identified whether responses predicted by the true Ratio Model deviated from that predicted by each Heuristic Model for which we created the Ratio Deviation from Heuristic

Model. This was calculated as the absolute difference values between each Heuristic Model and the Ratio Model. For a single trial, a 0 indicated that the Ratio Model and a given Heuristic Model predicted the same response whereas a 1 indicated that the Ratio Model predicted an opposite response from a given Heuristic Model.

To examine whether participants performed true ratio discriminations independently of the abovementioned heuristics, we ran a GLMM separately for each heuristic following a binomial distribution predicting participants' Actual Deviation from Heuristic Model as a function of the fixed effect of Ratio Deviation from Heuristic Model and the random effect of participants. If a participant was exclusively relying on a given heuristic to perform the task, then the Actual Deviations from Heuristic Model would indicate 0s and so the β value of the fixed effect should be close to zero. In contrast, a significantly positive β value would indicate that the Ratio Deviation from Heuristic Model better explained participants' performance. The three GLMMs for each of the three heuristics all revealed significant fixed effects of the Ratio Deviation from Heuristic Model, indicating choices predicted by the true Ratio Model when controlling for the use of each heuristic ("More Good": β_{RD}= 2.38, SE = 0.07, Z = 35.05, p < .0001; "Less Bad": β_{RD}= 2.69, SE = 0.07, Z = 36.44, p < .0001; "More Item": β_{RD}= 2.71, SE = 0.07, Z = 37.93, p < .0001). We thus found support for the ratio strategy controlling for each heuristic.

Although the above analysis demonstrates that adults did not exclusively rely on any of the heuristic strategies to perform ratio discriminations, it remains possible that the errors adults made were predicted by one or more heuristic. To test this, we constructed a GLMM for each heuristic predicting participants' actual errors on each trial with the Ratio Deviation from Heuristic Model entered as the fixed effect while controlling for the random effect of participants. A positive β value would indicate that people's errors were systematically predicted by the use of a given heuristic. We obtained significantly positive β values for the "More Good" strategy (β_{RD}= 0.87, SE = 0.07, Z = 12.80, p < .0001) and the "More Item" heuristic (β_{RD}= 0.83, SE = 0.07, Z = 12.38, p < .0001). We obtained a negative β value for the "Less Bad" heuristic (β_{RD}= -0.64, SE = 0.07, Z = -9.11, p < .0001) indicating that people's actual errors were the opposite of the responses predicted by choosing the array with fewer unfavorable items. In sum, our strategy analysis replicates prior work in children, providing evidence that participants use a true ratio comparison strategy when successfully discriminating seen ratios, and that errors in these tasks are more often explained by the "More Good" and "More Item" heuristics (Szkudlarek & Brannon, 2021).

4.2 The relative salience of favorable vs unfavorable items and implications for representational format

This exploratory analysis was inspired by the observation that connecting favorable items had a greater impact on performance than connecting unfavorable items.

As shown in Figure 4 when we examined all 5 trial types without collapsing the trial-types into three conditions we observed that accuracy increased for both predict increment trial types ('Correct-U' and 'Incorrect-F') and decreased for both predict decrement trial types ('Incorrect-U' and 'Correct-F') relative to baseline. When favorable items on the incorrect side were connected, accuracy increased to 83.1% compared to 76% on baseline trials (t(49) = 5.91, p < .0001, Cohen's d = 0.84). Meanwhile, when unfavorable items on the correct side were connected accuracy increased to 79.2% (t(49) = 2.80, p < .01, Cohen's d = 0.40). Conversely, accuracy decreased to 73.5% when unfavorable items on the incorrect side were connected again compared to 76% on baseline trials (t(49) = 3.53, p < .01, Cohen's d = 0.50). Meanwhile, when favorable items on the correct side were connected accuracy decreased to 67.1% (t(49) = 8.54, p < .0001, Cohen's d = 1.21). Thus, our main prediction that connectedness would systematically affect perceived number and, in turn, ratio comparison was supported even when analyzing at the level of trial-type.

Interestingly, however, connecting favorable items, had a larger impact on perceived ratios than connecting unfavorable items. Although both Predict-Increment conditions were greater than baseline, accuracy was higher when favorable items on the incorrect side were connected (83.1%) compared to when unfavorable items on the correct side (79.2%) were connected (t(49) = 3.18, p < .01, Cohen's d = 0.45). Similarly, although both trial types within the Predict-Decrement Condition were lower than baseline, accuracy when favorable items on the correct side (67.1%) were connected was significantly lower compared to when unfavorable items on the incorrect side (73.5%) were connected (t(49) = 4.89, p < .0001, Cohen's d = 0.69).

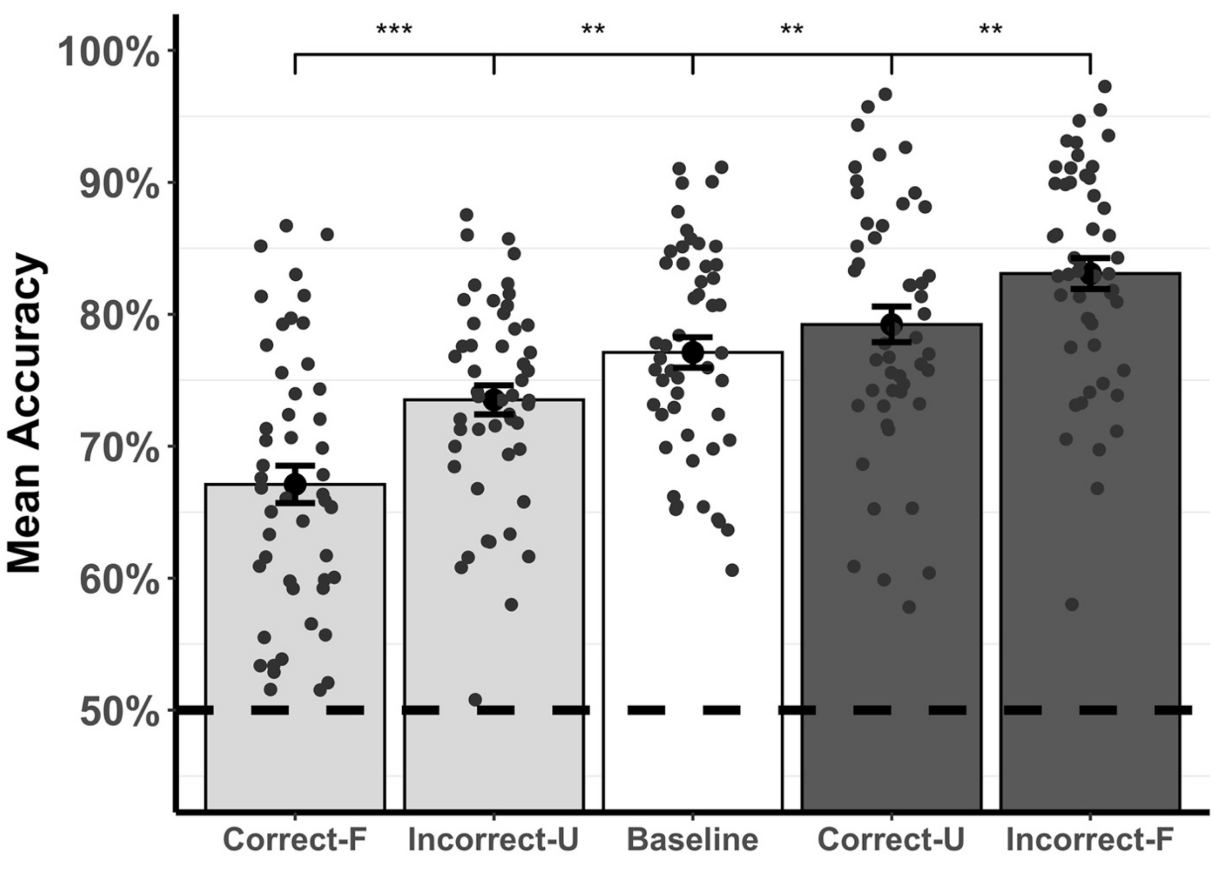

Fig. 4. Bar plot of mean accuracy on ratio comparisons across different connectedness trial types. All error bars are *SEM*. Dashed line denotes chance accuracy. Asterisks indicate the level of significance between two adjacent bars. **p < .01, ***p < .001.

A possible explanation for this asymmetry is that participants were employing proportional representations couched in a part-whole representational format (akin to a fraction, like 1/4) rather than a part-part format (akin to a genuine ratio, like 1:3).

To illustrate, consider an array containing 10 blue circles and 10 red circles where blue is the favorable color. Suppose that the proportion of blues is represented using a part-whole format (like the fraction, 10 blues/20 blues and reds). In this scenario, connecting blue items would reduce their perceived number, affecting the fraction of whole bounded blue items in the collection accordingly. And this should come as no surprise, since it would reduce the number of bounded blue items (referred to by the fraction's numerator) to 5, and the actual number of bounded blue and red items (referred to by the fraction's denominator) to 15. Thus, it would change the actual fraction of bounded blue items from 1/2 (10blues/20 blue and reds) to 1/3 (5blues/15 blue and reds). Meanwhile, connecting the red items would leave the number of bounded blue items referred to by the numerator unaffected (there would still be 10 bounded blue items in the collection). It would, however, lead to a reduction in the number referred to by the fraction's denominator, specifying the value of the superset (reducing this from 20 to 15). Hence, connections among red items would change the actual fraction of bounded blue items in the collection from 1/2 (10blues/20 blue and reds) to 10/15, or 2/3. We can summarize these observations by noting that connections among blue items would change the actual fraction of bounded favorable/blue items in the array from 10/20 (or ½) to 5/15 (or 1/3), while connecting the unfavorable/red items would change the actual fraction of bounded blue items from 10/20 (or ½) to 10/15 (or 2/3).

While 1/3 and 2/3 both differ from ½ by the same *absolute* amount (1/6), the ratio among these ratios differs markedly: where 1/3 differs from ½ by a ratio of 1.5, ½ differs from 2/3 by a ratio of just 1.3. Given that the ratio among two ratios predicts their discriminability, the conjecture that arrays are represented in a part-whole format, specifying the proportion of favorable items over the value of the superset, predicts that connections among favorable items will exert a greater effect on the perceived favorability of a ratio. As noted, this is precisely what our results revealed, and accords with prior work showing that when adults, infants and children engage in approximate enumeration tasks they automatically encode the quantity of the superset under varied attentional conditions (Halberda, Sires, & Feigenson, 2006; Poltoratski & Xu, 2013).

By contrast, suppose that the proportion of blue circles were represented like a ratio, with a part-part (or subset:subset) structure that specifies the number of blues to the number of reds. Here, connecting blue items would alter the ratio of bounded blue items to bounded red items from a value of 10:10 (or 1:1) to 5:10 (or 1:2); meanwhile connecting the reds would alter the ratio from 10:10 (or 1:1) to 10:5 (or 2:1). In this case the ratio of ratios would be equally different when favorable and unfavorable items are connected. Since ratio discrimination conforms to Weber's Law, connections among favorable and unfavorable items should thereby affect performance equally. To reiterate, this is not what we found. Such considerations thereby reveal that the asymmetry found in our exploratory analyses could be explained by the hypothesis that task performance is underwritten by proportional representations with a part-whole format (like a fraction) as opposed to representations with a part-part structure (like a ratio).

To investigate this possibility further, we calculated the predicted effect on accuracy for the part:part and part:whole formats based on the ratios of ratios or ratios of fractions for all baseline trials compared to their yoked connected favorable and unfavorable trials for each proposed representational format (part:part and part:whole). As described in detail in the Supplementary Results Section 3, we confirmed that if rational numbers are represented in a part-whole format connecting favorable items should have a stronger effect on the perceived favorability of a ratio in all our trials whereas if the representations had a part-part format the effects of connecting favorable and unfavorable items should be equivalent.

We next constructed a Generalized Linear Mixed Model (GLMM) for the part-part and part-whole representational formats respectively, predicting participants' accuracy on each trial following a binomial distribution. The fixed effects include the true ratio of ratios or ratio of fractions and the predicted effect of connectedness under the part-part or part-whole format, while accounting for the random effect of participants. We quantified the predicted effect of connectedness under the part-part format as the deviation of the ratio of ratios of bounded objects (free floating dots and dumbbells) from the true ratio of ratios (individual dots). This was calculated by taking the ratio between the true ratio of ratios and the ratio of ratio of bounded objects (and always ensuring the first value in the ratio is the ratio of ratio of bounded objects and the second one in the ratio is the true ratio of ratios). Similarly, we quantified the predicted effect of connectedness under the part-whole format as the deviation of the ratio of fractions of bounded objects from the true ratios of fractions, which was calculated by taking the ratio between the true ratio of fractions and the ratio of fractions of bounded objects.

In our combined data set from Experiments 1 and 2, the two GLMMs for each of the representational formats both revealed significant fixed effects of the ratio of ratios or ratio of fractions, again indicating that the accuracy of ratio comparisons is predicted by Weber's Law ("part-part": β_{RR}= 0.81, SE = 0.03, Z = 27.21, p < .0001; "part-whole": β_{RF}= 2.13, SE = 0.08, Z = 27.35, p < .0001). The fixed effect of the connectedness effect was also significant for both GLMMs, supporting findings in Experiments 1 and 2 that ratio comparisons were systematically affected by connectedness ("part-part": β_{DiffRR}= 0.23, SE = 0.01, Z = 18.80, p < .0001; "part-whole": β_{DiffRF}= 0.53, SE = 0.03, Z = 20.05, p < .0001).

To evaluate whether a part:part format or part/whole format better captures participants' trial by trial performance, we used Akaike information criterion (AIC) and the Likelihood Ratio Test (LRT) to compare the two models. Lower AIC values indicate a better-fitting model. The LRT statistic follows a chi-squared distribution, and a significant result suggests that one model provides a significantly better fit than the other. We found a significant difference between the model fits, with the part-whole model providing a significantly better fit to the accuracy data compared to the part-part model (AIC_{PP} = 29179, AIC_{PW} = 29106, χ^2 = 73.27, df = 4, p < .0001).

In sum, we propose that our exploratory analyses provided a window into the underlying format of the representations that drive task performance in ratio-comparison tasks. In particular, we suggest that the part/whole format better captures participants' trial-level performance than a part:part format. This supports the conjecture that the ANS's proportional representations are couched in a part/whole format, specifying the quantity of favorable items over the quantity of favorable *and* unfavorable items in the superset, somewhat akin to a fraction.

5. General Discussion

Across two experiments we examined adult humans' capacity to extract, represent, and compare the ratios/fractions of items in two displays. In line with previous findings, the accuracy of participants' discriminations was predicted by Weber's Law, such that the ratio among the ratios under comparison predicted task performance. Our results have extended this observation by showing that these ratio discriminations are (a) susceptible to a connectedness illusion, with connectedness reducing the perceived quantity of items in relevant subsets/supersets (with this then influencing ratio discriminations accordingly) and that they are (b) not significantly affected by the simultaneous performance of a secondary verbal shadowing task. Subsequent exploratory analyses showed that (c) performance in these tasks could not be explained by simpler heuristic strategies that do not involve any sensitivity to the ratios/fractions each display portrayed, with (d) connections among target (or favorable) items in an array having a greater effect on performance than connections between unfavorable items even at the level of trial-by-trial performance. In turn, we suggested that this final observation suggests that people were employing proportional representations, couched in a part/whole rather than part:part format.

How do our findings bear on the hypothesis that the ANS represents rational numbers (Clarke & Beck, 2021a)? Our main finding was that connectedness systematically impacted task performance. This finding indicates that number is the relevant dimension driving task performance, and suggests that the ratios/rational numbers that are being represented concern ratios/fractions among discrete quantities, in the ways we would expect if they have been extracted by the ANS. Thus, our results are consistent with the hypothesis that task performance emerges from a 2-stage process whereby approximate number representations of the natural number of red and blue items are formed first and rational number representations are subsequently formed and compared on the basis of these.

At least two classes of alternative proposals can be identified. In the first, ratio discriminations are achieved by processes which bypass the ANS, and its representations of natural number, entirely. For instance, whole number representations might be "an emergent property of ratio perception (i.e., when the denominator is 1)" (Hubbard & Matthews, 2021: 35). While our findings do not rule out this possibility, this class of alternative explanations does not offer a principled explanation for the effects of connectedness. For while there may be multiple routes to visual number extraction, current evidence indicates that these are not all susceptible to the connectedness illusion. For instance, Fornaciai and Park (2021) found that representations of number in the early visual cortex are not affected by connectedness in the way that approximate number representations implemented in the intraparietal sulcus are known to be. The conjecture that rational numbers are extracted via ANS representations of natural number thus offers a simple and parsimonious explanation for the effects of connectedness documented in our study. Furthermore, positing a distinct ratio system that bypasses the ANS requires not only that there be a duplication of the functions of the ANS but that this duplicative system also operates over bounded entities. That rational number is extracted based on the ANS is also supported by recent work in machine learning, where deep convolutional neural networks that have been trained to compare numerical quantities in a range of distinct formats seem to base their assessment of rational numbers on "ANS style" units responding to natural number (Chuang et al., 2020).

A second class of alternative hypotheses may accept that the ANS underlies some aspects of performance in our task but posits that the ratio or fraction comparison reflects secondary decision stages outside of the purview of the ANS. We contend that our finding that the completion of a secondary verbal distractor task failed to impact task performance (Experiment 2) implies that these ratio discriminations do not depend on verbal comparisons or explicit counting routines. As noted, this is consistent with the view that rational numbers are extracted by the ANS itself, based on its prior extraction of natural number (Clarke & Beck, 2021a). However, we fully acknowledge that it remains possible that ratios or rational numbers are being extracted by a domain-general *ratio processing stage of processing* (RPS), that compares ratios/fractions among both numerical and non-numerical quantities (Matthews et al., 2015; Clarke & Beck 2021b); at least insofar as the RPS avoids placing a burden on verbal working memory in a manner that renders it akin to the ANS proper (see: Cordes et al., 2001).

Support for the RPS proposal comes from studies demonstrating that children and adults can compare ratios across distinct magnitude types, such as line length and number (Binzak, Matthews, & Hubbard, 2019). For instance, Bonn and Cantlon (2017) found that subjects spontaneously compare ratios across magnitude types when judging sequences as more or less similar. On their account, this indicates that ratios are represented in "a highly general code that could qualify as a candidate for a generalized magnitude representation". Meanwhile, Park, Viegut, and Matthews (2020) found that ratio acuity for stimuli in one format (e.g., line ratios) was a better predictor of ratio acuity in other stimulus formats (e.g., numerical ratios), than performance in a non-ratio task with the same stimulus format (e.g., ratio acuity for line length and ordinal comparison for line length). Such findings have been taken to suggest that there are shared cognitive processes involved in the processing of ratios among line lengths and numbers.

On the other hand, these same researchers also found differences in the processing of discrete and continuous magnitudes. For instance, in both ordinal and ratio comparisons, numerical discriminations were found to be harder than size judgments with both circles and irregular shapes (ibid.). Prior work also demonstrates significant differences in the developmental trajectory of proportional reasoning about continuous and discrete quantities (Boyer, Levine, & Huttenlocher, 2008; Jeong et al., 2007; Singer-Freeman, & Goswami, 2001; Spinillo & Bryant, 1999) and suggests that children employ different strategies when *reasoning* about numerical and non-numerical ratios and fractions (Smith, 1986; Hurst & Cordes, 2018; Jeong et al., 2007). These latter results are, thus, suggestive of some degree of psychological independence in the extraction and representation of ratios/fractions among natural numbers and non-numerical dimensions. Given these mixed results, further work is needed to assess the degree of domain generality exhibited by the mechanisms of rational number extraction implicated in ratio comparison tasks.

One of the more intriguing aspects of the present study was the finding that connectedness among favorable items influenced ratio discriminations to a greater extent than connectedness among non-favorable items. As noted earlier, this pattern of results is expected if task performance is driven by rational number representations with a subset-superset structure that is isomorphic to the format of a proportion or fraction, rather than that of a ratio with a subset-subset structure. Moreover, our model comparison results suggest that a part/whole format (akin to a fraction) better captures participants' trial-by-trial performance than a part:part format (akin to a ratio).

Despite these observations, the suggestion that the ANS might represent rational numbers, with a subset/superset format is somewhat surprising given the difficulties children have learning to manipulate and compare symbolic fractions (Lortie-Forgues, Tian, & Siegler, 2015). Furthermore, it could seem to be in tension with older work indicating that children consistently and erroneously rely on subset-subset representations when engaging in proportional reasoning tasks (Smith, 1986; Singer & Resnick, 1992; Spinillo & Bryant, 1999). For instance, Singer and Resnick (1992) tested 6th, 7th, and 8th graders on "fifteen probability problems which varied information content and quantitative relationships between the quantities expressed" (231) and found that children consistently indicated use of a part-part strategy when required to "think-aloud as they tried to solve each problem" (237).

Why then these discrepancies? For the purposes of answering these questions, it is a limitation of the present study that we tested adults not children. However, findings from Szkudlarek and Brannon (2021) already suggested that our task may allow children to avoid some of the biases they fall prone to in other tasks. While 6-8 year-old children did exhibit a tendency to use the "more good" heuristic when their performance deviated from the correct ratio strategy, they did not exclusively rely on such a strategy as they did in other contexts (Jeong et al., 2007). Moreover, many of the paradigms that revealed a strong "more good" strategy in children employed probabilistic reasoning tasks that relied on verbal strategies. For example, Singer and Resnick (1992) required children to explicitly "think-aloud". Verbal reasoning about probability appears to tap fundamentally different cognitive resources from the approximate visual ratio comparisons used in the present study, given that performance in our task was unaffected by verbal shadowing. Thus, even if the ANS represents rational numbers, allowing for intuitive comparisons of proportionality early in development, children nevertheless face a formidable challenge mapping such primitive and intuitive representations onto the conceptual or linguistic representations of fractions employed in math class (this is akin to the more familiar challenge they face mapping natural number words onto ANS representations – see: Carey & Barner, 2021). The fact that the isomorphism between ANS part-whole representations and symbolic fractions is apparently not serving to scaffold the learning of fractions in formal educational settings may reflect architectural constraints or the encapsulation of the algorithms by which they are computed.

It is also important to note that we are not suggesting that adults can *only* represent proportional information in a part/whole format. Our results may suggest that adults spontaneously employ part/whole structures of this sort in some contexts, but it is likely that they can flexibly switch to alternative format types when task demands render these more efficient. In this way, proportional representations might be akin to visuo-spatial representations, which have been said to employ a Polar coordinate format except when stimulus cues render a Cartesian format more effective (Yousif & Keil 2021).

Furthermore, there may be alternative explanations for our finding that connecting favorable items impacted performance more than connecting unfavorable items. One possibility is that the observed asymmetry found in our exploratory analyses instead results from the effects of attention. Visual attention is known to play an important role in visual feature integration (Treisman & Gelade, 1980; Wolfe, 2012). Thus, it is possible that participants attended more to the favorable items in observed displays, and this alone caused the favorable items to be bound into unified dumbbells more consistently than items in the un-favorable subsets. Indeed, this alternative suggestion is consistent with previous work indicating that the deployment of visual attention can affect the strength of the connectedness illusion (Pomé et al., 2021). It is, however, unclear whether this attention-based account of the asymmetry is consistent with findings from our heuristic analyses. For recall: these analyses found that that ratio comparisons were primarily based on ratio or rational number, rather than the absolute number of favorable items (a *more good* strategy) which is what we might expect if participants were simply attending more to the favorable items. Regardless, future work could seek to further disentangle these competing explanations.

6. Conclusion

We have reported the results of two experiments designed to test predictions of the view that the ANS represents rational number. Using a well-known 'connectedness illusion' (He et al., 2009; Franconerri et al., 2009), we manipulated perceived number in a ratio comparison task independently of confounding variables. Our findings suggest that ratio comparisons rely on representations of the natural number of perceived items, and do not place demands on verbal working memory. In addition, we demonstrated that while simple heuristics can explain the errors that adults make in these tasks, people do not rely on any single heuristic to bypass the task instructions and are sensitive to the proportion of favorable items in each collection. Finally, our exploratory analyses provide suggestive evidence that adults may rely on part/whole format rather than part:part format when representing proportional information to solve these tasks. Overall, our findings are consistent with the hypothesis that the ANS supports the representation of full-blown rational numbers based on a prior stage of natural number extraction (Clarke & Beck, 2021a), though further work is necessary to decisively test this hypothesis.

Supplementary materials

The stimulus set and the supplementary data to this article can be accessed at https://osf.io/6gbcz/?view_only=3fa268d3afdc47438d70d967221970cb

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