

Aristotle, Logic, and QUARC

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Abstract. The goal of this paper is to present a new reconstruction of Aristotle’s assertoric logic as he develops it in *Prior Analytics*, A1-7. This reconstruction will be much closer to Aristotle’s original text than other such reconstructions brought forward up to now. To accomplish this, we will not use classical logic, but a novel system developed by Ben-Yami (2004, 2014) called ‘QUARC’. This system is apt for a more adequate reconstruction since it does not need first-order variables (x , y , ...) on which the usual quantifiers act – a feature also not to be found in Aristotle. Further, in the classical reconstructions, there is also the need for binary connectives (\wedge , \rightarrow) that don’t have a counterpart in Aristotle. QUARC, again, does not need them either to represent the Aristotelian sentence types. However, the full QUARC is also not called for so that I develop a subsystem thereof (‘QUARC_{AR}’) which closely resembles Aristotle’s way of developing his logic. I show that we can prove all of Aristotle’s claims within this systems and, lastly, how it relates to classical logic.

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1 Introduction

The logic of Aristotle has been the topic of many intense studies. In particular, Łukasiewicz (1957) has developed an axiomatic system that is supposed to be a proper representation of Aristotle's logic (as has, following him, Patzig 1968). This approach has been widely criticized and been replaced by the interpretation that Aristotle's logic constitutes a natural deduction system; the main proponents of this interpretation are Corcoran (1972, 1974a, 1974b) and Smiley (1973) who give their own reconstructions.

However, the proposed reconstructions invoke an underlying singular logic, viz., classical logic; for example, Corcoran (1972) has the following semantic principle: 'Let x , y and z be different members of C ' (p. 697). But given Aristotle's original text, such a principle seems not to be a close fit. In particular – and this is the topic of this paper – we can give a much more fitting reconstruction of the text. In doing so, we do not need to enforce the whole modern formalism and way of thinking about it. If we enforce the modern picture on an ancient text, we are prone to obscure the original thoughts and ideas.

In what follows, I will develop a logical system that is more appropriate with respect to the original text. For example, the underlying logic will be a plural one. To this end, I will use the logic QUARC that has been developed by Ben-Yami (2004, 2014). However, just using the system as it is introduced in *Ben-Yami 2014* would suffer from the same problems as other modern reconstructions. Thus, I will develop a new system (called QUARC_{AR}) that is a plural logic and more faithful to the Aristotelian text.

To be faithful to the text, it is helpful to know the text. I will first (Section 2) succinctly present Aristotle's assertoric logic as it is developed in the first few chapters of his *Prior Analytics*. Then (Section 3), I will first briefly motivate and introduce QUARC (Section 3.1), and go on to develop QUARC_{AR} (Section 3.2). In doing so, I will first define the underlying language, develop a semantics, and, lastly, a calculus. The calculus is able to prove all the moods of the three figures – and this in a manner that is close to Aristotle's own proofs. To make this obvious, I include his proofs before giving the formal ones. Having proven the Aristotelian syllogisms (in Section 3.3), I go on to establish soundness and point towards completeness of QUARC_{AR} (Section 3.4). In the last Section 3.5, I extend QUARC_{AR} to include complex terms and relate it to classical logic.

The upshot of this will be that we can translate Aristotle's own proofs word by word and end up with a formal proof within QUARC_{AR} and, similarly, the proposed semantics does not contain more than needed. This, then, has also the potential to shed some light on the usual debates (such as the one mentioned above), even though it is not within the scope of this paper to properly enter into them.

2 Aristotle's Assertoric Logic in the *Prior Analytics*

For our purposes, it is sufficient to focus on chapters A1-7 of the *Prior Analytics*. Aristotle develops his so-called assertoric logic in these chapters. At the heart of his

system lie his three figures which are supplemented by conversion rules. After introducing the three figures and the valid moods of them (the so-called syllogisms),¹ Aristotle goes on to establish some results. For example, he proves that the syllogisms of the first figure are able to prove all syllogisms of the other two figures. The goal of this section is to familiarize ourselves with what and exactly how Aristotle developed his logical system. Major interpretatory questions are not of concern here since they will not have any impact on the topic of this paper which is to give a more fitting formalization of the Aristotelian logic; the framework developed in Section 3.2 is flexible enough to be adapted to different interpretations of the text (as will be exemplified by the question of whether or not all terms are supposed to be non-empty).

The structure of the first chapters of the *Prior Analytics* is as follows. Aristotle starts in chapter 1 by delineating the topic of the writing and by introducing some terminology. In particular, he specifies the structure of the sentences by stipulating² that a sentence ‘affirms or denies something of something, and this is either universal or particular or indeterminate’³ (APr. A1, 24a16f.)⁴. In developing the logic, the ‘indeterminate’ option will not play any further role; thus, I’ll not mention it in the following.⁵ The constituents of a sentence are so-called ‘terms’ (A1, 24b16)⁶ and a copula (A1, 24b17f.)⁷. Every sentence has two terms and a copula and says of the terms in question that the one belongs/does not belong to the other term. This gives us the following sentence forms (where I’ll use ‘*A*’, ‘*B*’, etc. as terms):

- a:** *A* belongs to all *B* (universal-affirmative, AaB)
- i:** *A* belongs to some *B* (particular-affirmative, AiB)
- e:** *A* belongs to no *B* (universal-denying, AeB)
- o:** *A* does not belong to some *B* (particular-denying, AoB).⁸

Note that the notation is ‘reversed’ (as Crivelli 2012, p. 115, calls it); the subject of the sentence is ‘*B*’ and the predicate ‘*A*’. Thus, in a modern formalization, a

¹Cf. also *Read ms*, Section 1.

²This might not be a stipulation at all, but it does not matter in the following. Since Aristotle only considers such sentences, we can interpret it as being stipulated.

³λόγος καταφατικός ἢ ἀποφατικός τινος κατὰ τινος: οὗτος δὲ ἢ καθόλου ἢ ἐν μέρει ἢ ἀδιόριστος.

The Greek text is taken from *Aristotelis 1964*.

⁴All translations of Book A of the *Prior Analytics* are Striker’s as printed in *Aristotle 2009*. In the following, I will suppress the ‘APr.’.

⁵Cf. *Aristotle 2009*, p. 77, and *Crivelli 2012*, p. 115. See, however, A7, 29a27ff.: ‘It is also clear that an indeterminate premiss put in the place of a positive particular premiss will produce the same syllogism in all the figures [δῆλον δὲ καὶ ὅτι τὸ ἀδιόριστον ἀντὶ τοῦ κατηγορικοῦ τοῦ ἐν μέρει ρυθέμενον τὸν αὐτὸν ποιήσει συλλογισμόν ἐν ἅπασιν τοῖς σχήμασιν].’

⁶ὄρον.

⁷προστιθεμένου [ἢ διαφρουμένου] του εἶναι ἢ μὴ εἶναι.

⁸As pointed out by Wedin (1990, pp. 134f., 141), Aristotle’s canonical way of referring to an **o**-statement differs in his writings; in his *De Interpretatione* 17b16–25, he introduces it as explicit negation of the corresponding **a**-statement, whereas in his *Prior Analytics* he lists the following three (instead of two) as particular statements: ‘belonging to some, or not to some, or not to all [τὸ τινὲ ἢ μὴ τινὲ ἢ μὴ παντὶ ὑπάρχειν]’ (A1, 24a19) and then uses it like in ‘*A* does not belong to some of the *B*s [τὸ *A* τινὲ τῶν *B* μὴ ὑπάρχει]’ (A2, 24b22f.). Since the topic of this paper is Aristotle’s *Prior Analytics*, I will stick to the latter.

sentence of the form ' AaB ' is translated as ' $\forall x(B(x) \rightarrow A(x))$ '. Further, since the terms are denoting pluralities, Aristotle's logic can be classified as *plural* (cf., e.g., *Oliver & Smiley 2013*, pp. 6f.).

Aristotle moves on to tell us what a 'syllogism' is:

A syllogism is an argument in which, certain things being posited, something other than what was laid down results by necessity because these things are so. By 'because these things are so' I mean that it results through these, and by 'resulting through these' I mean that no term is required from outside for the necessity to come about.⁹

Even though Aristotle defines syllogism in a way that excludes trivial ones (such as $\frac{AqB}{AqB}$, $q \in \{a, i, e, o\}$), we will relax this to include such implications in our consequence relation (see Definition 8). A possible reason for Aristotle's exclusion of trivial consequences is that his main interest is 'demonstration and demonstrative science'¹⁰ (A1, 24a10f.). For this purpose, he starts with the more general syllogistics which includes demonstration. The difference between a syllogism and a demonstration is roughly the difference between valid and sound arguments (see A1, 24a28-24b1¹¹ and A4, 25b27-31¹²).

So far we have been given some definitions of what is involved in a syllogism. But Aristotle also provides semantic notions to explain why they are valid: something being/not being in something else as in a whole (A1, 24a13f.¹³) and something being predicated of all or of none (A1, 24a14f.¹⁴). The former notion is explained via the latter: let A and B be two terms. Then A is in B as in a whole if, and only if, (iff.) B is predicated of all A (A1, 24b26ff.¹⁵; note, again, the reversed notation); and B is not predicated of all of A iff. 'nothing can be found of the subject of which the other will not be said'¹⁶ (A1, 24b28ff.).

⁹συλλογισμὸς δὲ ἐστὶ λόγος ἐν ᾧ τεθέντων τινῶν ἕτερόν τι τῶν κειμένων ἐξ ἀνάγκης συμβαίνει τῶ ταῦτα εἶναι. λέγω δὲ τῶ ταῦτα εἶναι τὸ διὰ ταῦτα συμβαίνειν, τὸ δὲ διὰ ταῦτα συμβαίνειν τὸ μηδενὸς ἔξωθεν ὄρου προσδεῖν πρὸς τὸ γενέσθαι τὸ ἀναγκαῖον. (A1, 24b18-23)

¹⁰περὶ ἀπόδειξιν καὶ ἐπιστήμης ἀποδεικτικῆς.

¹¹Hence a syllogistic premiss in general will be an affirmation or denial of something about something in the way mentioned, and it will be demonstrative if it is true and accepted on the basis of the initial assumptions [ὥστε ἔσται συλλογιστικὴ μὲν πρότασις ἀπλῶς κατάφασις ἢ ἀπόφασις τινος κατὰ τινος τὸν ἐρημένον τρόπον, ἀποδεικτικὴ δέ, ἐὰν ἀληθὴς ἦ καὶ διὰ τῶν ἐξ ἀρχῆς ὑποθέσεων εἰλημμένη].⁷

¹²We will have to discuss demonstration later. Syllogism must be discussed before demonstration because syllogism is more universal than demonstration, for a demonstration is indeed a kind of syllogism, but not every syllogism is a demonstration [ὑστερον δὲ λεκτέον περὶ ἀποδείξεως. πρότερον δὲ περὶ συλλογισμοῦ λεκτέον ἢ περὶ ἀποδείξεως διὰ τὸ καθόλου μᾶλλον εἶναι τὸν συλλογισμὸν: ἢ μὲν γὰρ ἀπόδειξις συλλογισμὸς τις, ὁ συλλογισμὸς δὲ οὐ πᾶς ἀπόδειξις].⁷

¹³[W]hat it is for this to be or not to be in that as in a whole [τί τὸ ἐν ὅλῳ εἶναι ἢ μὴ εἶναι τόδε τῶδε].⁷

¹⁴'to be predicated of all' or 'of none' [τὸ κατὰ παντὸς ἢ μηδενὸς κατηγορεῖσθαι].⁷

¹⁵For one thing to be in another as in a whole is the same as for the other to be predicated of all of the first [τὸ δὲ ἐν ὅλῳ εἶναι ἕτερον ἐτέρῳ καὶ τὸ κατὰ παντὸς κατηγορεῖσθαι θατέρου θάτερον ταυτὸν ἐστίν].⁷

¹⁶λέγομεν δὲ τὸ κατὰ παντὸς κατηγορεῖσθαι ὅταν μηδὲν ἢ λαβεῖν [τοῦ ὑποκειμένου] καθ' οὗ θάτερον οὐ λεχθήσεται: καὶ τὸ κατὰ μηδενὸς ὡσαύτως.

This means that we can use a simple (set-)inclusion to interpret these notions: A is in B as in a whole iff. BaA [$\forall x(A(x) \rightarrow B(x))$] iff. $A \subseteq B$.¹⁷ For example, every human being is an animal. Thus, human being is in animal as in a whole, and animal is predicated of every human being.

We can similarly understand ‘being predicated of none’, ‘being predicated of some’, and ‘not being predicated of some’, viz., A is predicated of no B (AeB) iff. $B \cap A = \emptyset$, A is predicated of some B (AiB) iff. $B \cap A \neq \emptyset$, and A is not predicated of some B (AoB) iff. $B \not\subseteq A$. This makes it also clear that a sentence of the form AaB has as its *contradictory* a sentence of the form AoB and vice versa, and similarly for the remaining two. Two sentences are contradictories iff. exactly one of them is true (see also Lemma 9 below). The correspondence in modern notation is as follows:

$$\begin{array}{ll} \mathbf{a}: AaB \hat{=} \forall x(B(x) \rightarrow A(x)); & \mathbf{i}: AiB \hat{=} \exists x(B(x) \wedge A(x)); \\ \mathbf{e}: AeB \hat{=} \neg \exists x(B(x) \wedge A(x)); & \mathbf{o}: AoB \hat{=} \neg \forall x(B(x) \rightarrow A(x)). \end{array}$$

Since AoB is the negation of AaB , only one of them can be and one of them has to be true; they are, indeed, contradictories.

Note that I chose the above translations of the sentence-types \mathbf{e} and \mathbf{o} to make conspicuous that they are negations of \mathbf{i} and \mathbf{a} , respectively.¹⁸ Later on, we will use logically equivalent formulations for the four sentence types. Since an \mathbf{e} sentence is ‘universal-denying’, we’ll translate a sentence of the form ‘ AeB ’ as ‘ $\forall x(B(x) \rightarrow \neg A(x))$ ’; and since an \mathbf{o} sentence is ‘particular-denying’, we’ll translate a sentence of the form ‘ AoB ’ as ‘ $\exists x(B(x) \wedge \neg A(x))$ ’.

This brings us to the so-called *conversion rules*. In chapter 2, Aristotle explains and proves three such (where ‘ \rightsquigarrow ’ reads ‘converts to’):

$$(\mathbf{a}\text{-}\mathbf{i}\text{-conv}) AaB \rightsquigarrow BiA$$

$$(\mathbf{i}\text{-}\mathbf{i}\text{-conv}) AiB \rightsquigarrow BiA$$

$$(\mathbf{e}\text{-}\mathbf{e}\text{-conv}) AeB \rightsquigarrow BeA.$$

The conversion rules guarantee the truth of the converted sentence given the truth of the sentences that is converted. The latter two rules are also valid in classical

¹⁷Cf. also Striker’s commentary in *Aristotle 2009*, pp. 83f. The set-inclusion semantics is the orthodox one, but there have been proposed several others; for one alternative, see *Malink 2013*, pp. 63ff.; for others, see *Andrade & Becerra 2008*. Further, see *Andrade-Lotero & Dutilh Novaes 2012* for a discussion of what significance the availability of different semantics has (thanks to an anonymous referee for this reference). Andrade-Lotero & Dutilh Novaes argue that ‘there is as of yet no uncontroversial candidate for the semantic side of a technical analysis of the notion of syllogistic validity, precisely because there are no clear guidelines or criteria of what it means for a semantics to be adequate’ (p. 416); the underlying assumption of this paper is that one such criterion is the *minimal fit* to the original text: even though there are many semantics available, the ‘correct’ one should not rely on techniques that are not to be found in Aristotle (a criterion not discussed in *Andrade-Lotero & Dutilh Novaes 2012*; it is also not a purely semantical criterion). This, of course, does not necessarily lead to a unique best fit, but rules out some of the semantics considered in *Andrade-Lotero & Dutilh Novaes 2012*.

¹⁸This does not mean that we can reduce the sentence types to one another – unless we have a negation in our language.

logic and straightforward given the modern notation: (**i-i-conv**) corresponds to the commutativity of ' \wedge ' and so does (**e-e-conv**). If we substitute the logical equivalent ' $\forall x(B(x) \rightarrow \neg A(x))$ ' for ' $\neg \exists x(B(x) \wedge A(x))$ ', (**e-e-conv**) can also be taken as corresponding to the contraposition of ' \rightarrow ' (assuming double-negation) (but see the end of Section 3.1).

The first conversion rule, (**a-i-conv**), on the other hand, does not hold in a classical setting. The reason is that sentences of the form ' $\forall x\varphi(x)$ ' are also true if there are no objects that satisfy ' $\varphi(x)$ '. In particular, every sentence of the form ' $\forall x\varphi(x)$ ' is true in a model with empty domain. The converted sentence, however, is an existential claim: BiA corresponds to $\exists x(A(x) \wedge B(x))$. Thus, in a model whose domain contains nothing that satisfies ' $B(x)$ ', ' AaB ' is true but ' BiA ' is false.

One way of guaranteeing the validity of (**a-i-conv**) is to ensure that every term/predicate has instances. In such models (**a-i-conv**) is true (and we effectively rule out empty models). Set theoretically speaking, this means that sets of the form ' $\{x|A(x)\}$ ' are non-empty if A is (a simple) term/predicate; this is also one of the assumptions of the QUARC, viz., 'Instantiation' (*Ben-Yami 2014*, p. 130).

Since employing a non-emptiness requirement seems to be the orthodox way of modelling Aristotle's syllogisms,¹⁹ I will likewise stick to this assumption; this will assure easy comparability with the other reconstructions. Nevertheless, I will indicate how to change some clauses to get to a representation that allows for empty terms – a view that finds more and more advocates²⁰ – and comment on which parts of the given reconstructions do not hold anymore (see the footnotes at the relevant places).

Aristotle also proves the conversion rules.²¹ He does not explicitly introduce any of his proof-methods, but only discusses them later on. Aristotle makes use of three methods to show that a conclusion follows from its premises:

- (i) direct proof, (ii) indirect proof, and (iii) proof by 'ecthesis'.

The first two are familiar to a modern reader, so let me just explain the third one. Striker (in *Aristotle 2009*) explains ecthesis as follows:²² 'If A belongs to some B , there is a C such that both B and A belong to C ; and if A does not belong to some B , there is a C such that B belongs to C , but A does not belong to C ' (p. 69). This means that given a sentence of the form ' AiB ', we can choose an appropriate

¹⁹See *Corcoran 1972*, p. 696, *Martin 1997*, p. 6 (note, too, the new footnote added to the reprint cited here), *Smiley 1973*, p. 144, and *Smith 2018*, §5.2.

²⁰See, for example, *Malink 2013*, pp. 81f., and *Wedin 1978, 1990*.

²¹Whether or not he actually proves all of them without circularity does not matter for us. It seems that Aristotle proves first (**e-e-conv**) and uses either something like (**i-i-conv**) (which is not supported by the text and leads to a circularity) or, as Striker argues (in *Aristotle 2009*, pp. 86ff.), *ecthesis*. However, the latter interpretation has its own problems; Striker argues that the term Aristotle uses in the ecthesis is an individual term since otherwise we'd just run into a different circle than with the (**i-i-conv**) option (pp. 87f.). But I don't find it plausible at all that Aristotle would use something like existential specialization and not include it in his logic or even just presuppose it. For a different approach, see *Malink 2013*, pp. 39f. See also the discussion below Theorem 12 below.

²²Cf. also *Malink 2013*, pp. 86–101. I endorse most of what he says there, but, since I reject his semantic reconstructions (they employ, for example, existential instantiation), I don't follow him with respect to his claim that Aristotle is not committed to the **o**-ecthesis as presented here, but to a weaker one.

C such that ' AaC ' and ' BaC ' are true, too; and given a sentence of the form ' AoB ', there is a C such that ' AeC ' and ' BaC ' are true. Thus, an ecthesis provides further premises to use in a proof (but see also the discussion after Theorem 12).

That ecthesis is valid is easily seen. We already noted the correspondence between a sentence of the form ' AiB ' and a set-theoretical statement of the form ' $B \cap A \neq \emptyset$ '. But the latter tells us explicitly that the intersection of B and A is non-empty so that there obviously is a non-empty C such that $C \subseteq B$ and $C \subseteq A$, viz., let $C = B \cap A$ (i.e., $\forall x(C(x) : \leftrightarrow B(x) \wedge A(x))$).

Similarly in the case of ' AoB '. The corresponding set-statement is ' $B \cap A \neq B$ '. Thus, we can find a non-empty C such that $C \subseteq B$ and $C \cap A = \emptyset$, viz., let $C = B \setminus A$ (i.e., $\forall x(C(x) : \leftrightarrow B(x) \wedge \neg A(x))$).

This brings us to the heart of the Aristotelian logic: the syllogisms. Aristotle introduces 3 different figures of syllogisms in chapters 4-6.²³ His goal is to give a complete list of all the valid moods of these three figures (i.e., a complete list of syllogisms). The first figure has the following scheme

$$\text{(First Figure Scheme)} \quad \frac{ApB \quad BqC}{ArC}$$

where $p, q, r \in \{a, i, e, o\}$. In all the figures there are two premises and one conclusion. Since we already ruled out trivial conclusions, certain inferences are not considered to be valid moods of a figure. Overall, Aristotle determines four valid moods (cf. *Crivelli 2012*, p. 128):

$$\begin{array}{cccc} \frac{\text{Barbara}}{Aa\bar{B} \quad \bar{B}aC}{AaC} & \frac{\text{Celarent}}{Ae\bar{B} \quad BaC}{AeC} & \frac{\text{Darii}}{Aa\bar{B} \quad BiC}{AiC} & \frac{\text{Ferio}}{Ae\bar{B} \quad BiC}{AoC} \end{array}$$

The names 'Barbara', 'Celarent', 'Darii', and 'Ferio' have been introduced by the medieval logicians as mnemonic devices and derive from the four kinds of sentence (' a ', ' i ', ' e ', and ' o '; cf. *Malink 2011b*, p. 345). This means that every Barbara syllogism has two universal affirmative premises and a universal affirmative conclusion.

To justify the **Barbara** syllogism, Aristotle argues that it is necessary because of 'what we mean by 'of all''²⁴ (A4, 25b39f.). As all of the first figure syllogisms, **Barbara** is a *perfect*²⁵ syllogism; a perfect syllogism is one that is in no need for further justification or, as Crivelli (2012) puts it, 'evidently valid' (p. 129). Since there are so far no rules for arguments with two premises, Aristotle cannot prove the moods of the first figure, but since they are all declared as perfect, there seems no need to justify them.

²³On the discussion how Aristotle arrives at three (and not four) figures, see *Crivelli 2012*, pp. 125f.

²⁴πῶς τὸ κατὰ παντὸς λέγομεν.

²⁵In chapter 1 of the *Prior Analytics*, Aristotle also introduced the term 'perfect': 'Now I call a syllogism perfect if it requires nothing beyond the things posited for the necessity to be evident; I call a syllogism imperfect if it requires one or more things that are indeed necessary because of the terms laid down, but that have not been taken among the premisses. [τέλειον μὲν οὖν καλῶ συλλογισμὸν τὸν μηδενὸς ἄλλου προσδεόμενον παρὰ τὰ εἰλημμένα πρὸς τὸ φανῆναι τὸ ἀναγκαῖον, ἀτελεῖ δὲ τὸν προσδεόμενον ἢ ἑνὸς ἢ πλείονων, ἃ ἔστι μὲν ἀναγκαῖα διὰ τῶν ὑποκειμένων ὄρων, οὐ μὴν εἴληπται διὰ προτάσεων]' (24b23-26). However, since this does not play any further role, I excluded it from the presentation.

The scheme and the valid moods of the second figure are as follows:

$$\text{(Second Figure Scheme)} \quad \frac{MpN \quad MqX}{NrX}$$

$$\frac{\text{Cesare}}{\frac{MeN \quad MaX}{NeX}}$$

$$\frac{\text{Camestres}}{\frac{MaN \quad MeX}{NeX}}$$

$$\frac{\text{Festino}}{\frac{MeN \quad MiX}{NoX}}$$

$$\frac{\text{Baroco}}{\frac{MaN \quad MoX}{NoX}}$$

Lastly, there is the third figure. Its scheme and valid moods are as follows:

$$\text{(Third Figure Scheme)} \quad \frac{PpS \quad RqS}{PrR}$$

$$\frac{\text{Darapti}}{\frac{PaS \quad RaS}{PiR}}$$

$$\frac{\text{Felapton}}{\frac{PeS \quad RaS}{PoR}}$$

$$\frac{\text{Disamis}}{\frac{PiS \quad RaS}{PiR}}$$

$$\frac{\text{Datisi}}{\frac{PaS \quad RiS}{PiR}}$$

$$\frac{\text{Bocardo}}{\frac{PoS \quad RaS}{PoR}}$$

$$\frac{\text{Ferison}}{\frac{PeS \quad RiS}{PoR}}$$

Before moving on to the formal development of the Aristotelian logic, let me point out that Aristotle does use different letters for the different figures. Thus, if he gives an argument and chooses ‘ P ’, ‘ S ’, and ‘ R ’, we know that he indicates a third figure syllogism. I will stick to this convention in what follows:

Convention 1: In syllogisms,

- A , B , and C indicate the first figure;
- M , N , and X indicate the second figure;
- P , S , and R indicate the third figure.

Further, I will use ‘(a-i-conv)’ (and the like) to denote the informal Aristotelian conversion rule and ‘**Barbara**’ (and the like) for the informal mood. On the other hand, ‘(a-i-conv)’ and ‘(Barbara)’ will be used to denote the corresponding formal (semantic or syntactic) rules of QUARC_{AR}. \dashv

Furthermore, consider, for example, the moods **Darii** and **Datisi**. Both have the same pattern of sentence-types: ‘ a ’ and ‘ i ’ as premises, and ‘ i ’ as conclusion. This means that it is not enough to just know the name of the syllogism; one also has to know to which figure the syllogism belongs.

3 QUARC and QUARC_{AR}

The goal will now be to show that (a subsystem of) QUARC (as developed in *Ben-Yami 2004, 2014, Lanzet & Ben-Yami 2004, Raab 2016*) is capable of capturing the assertoric part of the Aristotelian logic as sketched in Section 2. The relevant reason to use QUARC is that it allows us to reconstruct the Aristotelian logic without the imposition of the classical apparatus. In particular, we do not need variables (‘ x ’, ‘ y ’, ...) and quantifiers binding those. We also do not need binary connectives (‘ \wedge ’, ‘ \rightarrow ’) and their interplay with the quantifiers to capture the Aristotelian sentence

types. All this can be accomplished in a simple way that brings us closer to Aristotle's own system.

In the following, I will first give a brief motivation for QUARC and point to the differences between QUARC and classical logic (Section 3.1). A familiarity with classical logic is assumed throughout what follows. Since the topic is the Aristotelian logic, I will only introduce as much of QUARC as needed to present a subsystem thereof which will be called 'QUARC_{AR}' (Section 3.2). It will be shown to be sufficient for the assertoric syllogistic (Section 3.3) and to be sound and complete (Section 3.4). Lastly, I liberalize the underlying language by allowing *complex terms* and show which part of classical logic this corresponds to (Section 3.5).

3.1 Brief Motivation for QUARC

In a series of publications (*2004*, *2009a*, *2009b*, *2012*, *2014*, *Lanzet & Ben-Yami 2004*), Ben-Yami has developed a logic which he claims is 'closer in some respects to Aristotle's logic than it is to Frege's calculus' (*2014*, p. 120). One of the main differences to the classical Fregean approach is the occurrence of quantified terms in argument position which also motivates the system's name 'QUARC', viz. QUantified ARGument Calculus.²⁶ In particular, the classical system has quantifiers which act on singular variables (' $\forall x$ ', ' $\exists x$ '), and these variables can occur in argument position (' $P(x)$ ', ' $Q(x)$ ', ' $\varphi(x)$ ') and are bound by the quantifiers if the formula is closed. This combines with the connectives such as ' \rightarrow ' and ' \neg ' to give us expressions such as ' $\forall x(P(x) \rightarrow \neg Q(x))$ '. The last expression could, e.g., be the formalization of a sentence like 'all philosophers are not stupid'.

However, in the informal version it rather seems that 'all philosophers' is the subject of the sentence whereas the classically formalized expression obscures this. The classical approach seems insufficient if we want to keep the natural language structure of the sentence. This is why Ben-Yami (*2014*) introduces a different system of logic in which 'quantifiers combine with one-place predicates' (p. 120) such as ' $\forall M$ '; he formalizes the above sentence as ' $(\forall P)\neg Q$ ' and reads it as 'all P are not Q '. This mirrors exactly the natural language structure and grammar.

Ben-Yami (*2014*) introduces, then, the logic of QUARC (pp. 122ff.). The main differences to the classical approach are the aforementioned quantified expressions, the introduction of anaphora, the introduction of permutation to mirror active/passive constructions, and a new ' \neg ' rule to pass from expressions such as ' $\neg(x_1, x_2, \dots, x_n)P$ ' (sentential negation) to ' $(x_1, x_2, \dots, x_n)\neg P$ ' (predicate negation) and vice versa. Note that we now write ' $(x)P$ ' instead of ' $P(x)$ ' to, again, mirror the natural language syntax of which most mention the subject before the predicate (cf. *Ben-Yami 2014*, p. 122). Notably, Ben-Yami (*2014*) argues that quantification presupposes instances of the quantified phrase (see also *Ben-Yami 2004*, pp. 60ff.). To formally guarantee this, he introduces a rule called 'Instantiation' (*2014*, p. 130).

For additional differences to classical logic that are not of interest here, see *Ben-Yami 2014* and, for a more detailed discussion, *Ben-Yami 2004*. Further, QUARC has been proof-theoretically investigated (see *Gratzl & Pavlovic ms*) and also been developed with a three-valued semantics to drop 'Instantiation' (see *Lanzet 2017*).

²⁶Even though the name seems to imply a calculus, I will use 'QUARC' and 'QUARC_{AR}' as names for the logics and not just the calculi.

Generalized Quantifiers & QUARC Let me briefly mention the exchange between Ben-Yami (2009a, 2012) and Westerståhl (2012). The debate is concerned with a comparison of QUARC with the theory of generalized quantifiers. We do not have to go into any detail here, but in his paper, Westerståhl introduces his understanding of determiners as binary quantifiers and compares it to Ben-Yami’s QUARC which he refers to as ‘an alternative account’ (p. 109). The upshot is that Westerståhl (2012) shows ‘how we can go back and forth between’ (p. 114) the two accounts. Thus, using his tools, we can relate QUARC_{AR} to the theory of generalized quantifiers, too, but, for reasons of space, will not do so (but see, e.g., Westerståhl 1989). Let me just mention that we can understand (e-e-conv) as determining what is known as a quantifier being *symmetric* (see, e.g., Westerståhl 2015, p. 18): if we understand the sentence-type e as involving a quantifier, then (e-e-conv) means that we can interchange the arguments of that quantifier and preserve the truth-value. In our representation below, this can also easily be seen. Thus, we don’t have to interpret (e-e-conv) as corresponding to the commutativity of ‘ \wedge ’ or the contraposition of ‘ \rightarrow ’, but as being about a quantifier; see also Westerståhl 1989, pp. 583ff., and 2016, §2; in this sense, we can see QUARC as a nice bridge between classical logic and generalized quantifier theory.

However, just using the theory of generalized quantifiers is not apt for our purposes. For, even though we can move back-and-forth between it and QUARC (as we can between classical logic and QUARC, see Raab 2016), using the theory of generalized quantifiers forces a different conception on us. For example, it works with open formulas which can be closed by quantifiers; as such, there is no negation that operates on predicates, but only on open formulas. Regarding the truth conditions, this does not make a difference (see also Section 3.5 where this is spelled out), but it does in our way of thinking about the matter. In this respect, this paper has similar concerns as, for example, Moss 2015 (cf. his Figure 18.1 on p. 567).

3.2 QUARC_{AR}

Since the QUARC involves much more than is needed to get Aristotle’s syllogistic, we will develop a less rich system called ‘QUARC_{AR}’. The language of QUARC_{AR} only contains the four sentence types and does not have a device for sentence-negation. To still be in a position to prove what we want to prove, we have to introduce different *reductio* rules for the sentence-types to implicitly codify the assumed relations of contradictories.

In a more formalized fashion, the sentences of the Aristotelian logic are of the form ‘ AqB ’ where ‘ A ’ and ‘ B ’ are terms and ‘ q ’ is either ‘ a ’, ‘ i ’, ‘ e ’, or ‘ o ’. In particular, ‘ AaB ’ reads ‘All B are A ’, ‘ AiB ’ reads ‘Some B is A ’, ‘ AeB ’ reads ‘No B is A ’, and ‘ AoB ’ reads ‘Some B is not A ’. This can be captured as ‘ $(\forall B)A$ ’, ‘ $(\exists B)A$ ’, ‘ $(\forall B)\neg A$ ’, and ‘ $(\exists B)\neg A$ ’, respectively.²⁷ As ‘ $(\exists B)\neg A$ ’ suggests, we need the ‘negative predication’. This, then, is enough to capture the four sentence types and shows that the language of QUARC_{AR} is much simpler than the languages of

²⁷Note that we could drop the parentheses and just write, for example, ‘ $\forall AB$ ’ instead of ‘ $(\forall A)B$ ’, but I keep them to indicate that the argument position is written in front of the predicate symbol since the QUARC way of thinking about these formal expressions is unfamiliar to most readers. In particular, it makes conspicuous negative predication: ‘ $(\forall A)\neg B$ ’.

classical logic and, especially, of the full QUARC. As such, we can use QUARC as natural bridge between the Aristotelian term-logic and classical logic without invoking too many concepts foreign to Aristotle. It is in this sense that QUARC is the more appropriate tool to reconstruct Aristotle’s logic compared to, for example, the theory of generalized quantifiers.

In the following, I define the language of QUARC_{AR}, give it a semantics, and, lastly, a calculus. In the next Section 3.3, I show that we can derive the Aristotelian syllogisms within QUARC_{AR}, and indicate in Section 3.4 how to prove its soundness and completeness. The last Section 3.5 extends QUARC_{AR} by allowing complex terms and shows which part of classical logic this extended version corresponds to.

The Language Let us start with the definition of the language of QUARC_{AR} and give the usual definitions for terms, formulas, and sentences. Note that we do not have to include the definition of terms, but I included it (i) to have a resemblance to the way in which classical logic proceeds and (ii) to show that we really have a term-logic, i.e., there is no distinction between arguments and predicates which means that we have plural reference.

DEFINITION 2 (\mathcal{L}_{AR})

The *language* of QUARC_{AR} (\mathcal{L}_{AR}) consists of the following:

- the logical symbols ‘ \neg ’, ‘ \forall ’, and ‘ \exists ’,
- the auxiliary symbols ‘(’ and ‘)’,
- and a set $Pred_{\mathcal{L}_{AR}}$ of unary predicate-symbols. ⊢

Note that the auxiliary symbols are not really necessary; I included them to maintain the readability and to keep the resemblance to QUARC (cf. footnote 27).

DEFINITION 3 (\mathcal{L}_{AR} -Terms)

The set of \mathcal{L}_{AR} -terms ($Term_{\mathcal{L}_{AR}}$) is defined to be $Pred_{\mathcal{L}_{AR}}$. ⊢

As indicated, every predicate is a term. The language does not contain any first-order variables or individual constants (which are the terms of classical logic), but consists solely of the logical symbols and predicates/terms. This also leads to a simplified definition of formula. Since there are no variables, every formula is closed and, thus, a sentence.

Note, at this point, that we only specified in Definition 2 that ‘ \neg ’ is a logical symbol, but not which role it plays. In the classical setting, it would be an operation on sentences; here it will be an operation on terms. If we extended the set of QUARC_{AR}-formulas, the first step would be to introduce also a sentence negation.

DEFINITION 4 (\mathcal{L}_{AR} -Formula)

The set of \mathcal{L}_{AR} -formulas/sentences is defined as follows:

- (F) If A and B are \mathcal{L}_{AR} -terms, then $\ulcorner (\forall A)B \urcorner$, $\ulcorner (\exists A)B \urcorner$, $\ulcorner (\forall A)\neg B \urcorner$,
and $\ulcorner (\exists A)\neg B \urcorner$ are \mathcal{L}_{AR} -formulas.

Let $Form_{\mathcal{L}_{AR}}$ be the set of \mathcal{L}_{AR} -formulas. ⊢

Note that we explicitly allow for formulas of the form ‘ $(qA)A$ ’ ($q \in \{\forall, \exists\}$); there is some textual evidence that Aristotle himself did not dismiss such as formulas (see APr. B15, 63b40–64a4²⁸).

Before developing the semantics, let me briefly comment on the syntax, how to extend it, and what changes in the proof theory such extensions lead to.

Since there is no recursion-clause in the definition of \mathcal{L}_{AR} -formula, all formulas can be taken to be atomic. We have several ways to introduce more complex formulas. For example, as already indicated, we can include a recursion clause that allows for sentence negation. With such an extension, we could introduce typical rules governing it (including the typical negation clause in the semantics). In particular, there could be a *uniform* reduction rule such as

$$\frac{\begin{array}{c} [\psi]^u \\ \vdots \\ \varphi \quad \neg\varphi \\ \hline \neg\psi \end{array}}{\text{(RAA, u)}}$$

instead of the four (types of) reductio rules introduced below (see Definition 19). Further, we would also need a rule for double negation. With these rules alone, i.e., without interaction of negation and the quantifiers, the equivalence classes of formulas are the same as the ones (F) gives rise to (identifying formulas of the form ‘ $\neg((qA)B)$ ’ with ‘ $(q'A)\neg B$ ’, $q \in \{\forall, \exists\}$, $q' \in \{\forall, \exists\} \setminus q$).

The next step would be to allow – again by recursion – for arbitrarily many negation symbols in front of a term to produce formulas such as ‘ $(\forall A)\neg\dots\neg B$ ’. We can introduce rules governing the interaction of quantifiers and negation such as

$$(\neg\forall \rightarrow \exists\neg) \quad \frac{\neg * ((\forall A) *' B)}{*((\exists A)\neg *' B)} \qquad (\forall\neg \rightarrow \neg\exists) \quad \frac{*((\forall A)\neg *' B)}{\neg * ((\exists A) *' B)}$$

where ‘*’ and ‘*’ are (possibly empty) strings of negation symbols, and similarly rules for ‘ \exists ’. These, together with a rule for double negation, allow us then to derive, for example, a formula of the form ‘ $(\forall A)B$ ’ from one of the form ‘ $(\forall A)\neg\neg B$ ’. The resulting equivalence classes of formulas coincide then again with the above ones.

Note that we do, indeed, need both ‘ \forall ’ and ‘ \exists ’ as logical symbols and cannot define the one in terms of the other. Only in case of the second extension mentioned above this is, indeed, possible. The problem is that we would need to define, e.g., $(\forall A)B : \leftrightarrow \neg((\exists A)\neg B)$, but we have no way of moving the negation symbols. Thus, $(\forall A)\neg B$ would not be well-formed since it would be $\neg((\exists A)\neg\neg B)$ which is not a well-formed formula.

All this shows that the language of QUARC_{AR} is very simple; but it leads to rather different problems concerning the relationship of the different formulas. In

²⁸In the middle figure a deduction can be made both of opposites and of contraries. Let A stand for good, let B and C stand for science. If then one assumes that every science is good, and no science is good, A belongs to every B and to no C, so that B belongs to no C; no science, then, is a science [Ἐν δὲ τῷ μέσῳ σχήματι καὶ ἐκ τῶν ἐκ τῶν ἀντικειμένων καὶ ἐκ τῶν ἐναντίων ἐνδέχεται γίνεσθαι συλλογισμόν. ἔστω γὰρ ἀγαθὸν μὲ ἐφ’ οὗ A, ἐπιστήμη δὲ ἐφ’ οὗ B καὶ Γ. εἰ δὴ πᾶσαν ἐπιστήμην σπουδαίαν ἔλαβε καὶ μηδεμίαν, τὸ A τῷ B παντὶ ὑπάρχει καὶ τῷ Γ οὐδενί, ὥστε τὸ B τῷ Γ οὐδενί: οὐδεμία ἄρα ἐπιστήμη ἐπιστήμη ἐστίν.]. The translation is A. J. Jenkinson’s as printed in *Barnes (1995: Vol. 1)*.

particular, Aristotle just states that a sentence of the form ‘ $(\forall A)B$ ’ has as its contradictory the sentence of the form ‘ $(\exists A)\neg B$ ’. In our set-up, we have to encode this in some of the derivation rules as well as enforce it in the semantics. Let’s turn to the latter first.

The Semantics Since the \mathcal{L}_{AR} -formulas arising from Definition 2 are so simple, the interpretation can be simple, too. In particular, QUARC_{AR} does not include individual-terms so that we only need to interpret terms/ predicates. This will be accomplished via sets. This leaves us with the question of how the quantifier and the negation behave. Both will be combinations of intersection and (in-)equality. But first, we need an underlying system of sets that is able to interpret the terms. In particular, we need to ensure that no interpretation of a predicate is empty to ensure that an analogue of (a-i-conv) follows. This move is similar to the one made in QUARC (see Raab 2016) and follows the orthodox way of reconstructing Aristotle’s logic.

Note, in what follows, the footnotes regarding the changes to be made when we drop the non-emptiness requirement (2) of Definition 5.

DEFINITION 5 (\mathcal{L}_{AR} -Structures)

Let \mathcal{L}_{AR} be a language. An \mathcal{L}_{AR} -structure is a tuple $\mathfrak{A} = (D, (A^{\mathfrak{A}})_{A \in \text{Pred}_{\mathcal{L}_{AR}}})$ with the following properties:

- (1) D is a set (the *universe*);²⁹
- (2) if $A \in \text{Pred}_{\mathcal{L}_{AR}}$, then $A^{\mathfrak{A}}$ is a non-empty unary relation (i.e., predicate) on D , viz. $\emptyset \neq A^{\mathfrak{A}} \subseteq D$.³⁰ ⊣

The definition is in principle standard. However, we do not need to enforce the domain to be non-empty, because this follows from clause (2) (except in the rather uninteresting special case that $\text{Pred}_{\mathcal{L}_{AR}} = \emptyset$, i.e., a language without any predicates and, thus, without formulas). This clause also ensures that there are no empty predicates so that every one has instances – whatever and how many they may be. This means, in particular, that it follows from the following Definition that for every A to be B or not to be B , there must be As – see also Lemma 9. With this at

²⁹As Ben-Yami (2004, pp. 59ff., 2012, pp. 49ff.) argues, there is no need for a domain. Lanzet (2017) similarly develops QUARC without one. However, I am not convinced that (i) there is no domain nonetheless and (ii) that one needs no domain (cf. also Westerstähl 2012). Regarding (i), Lanzet uses interpretations that map predicates to extensions; but without a domain, where exactly is the extension? where does the interpretation map to? And, regarding (ii), even if this approach is successful in getting rid of a domain, it seems that the meaning of a predicate becomes context-sensitive (cf. Ben-Yami 2012, p. 50) because the predicates get assigned different extensions. Even though two co-extensional predicates do not have to have the same meaning, they surely don’t have the same meaning if they aren’t. This is why I chose to include a domain. Note also Definition 29 where we need it to assign the proper extensions to complex terms.

³⁰If we drop the non-emptiness requirement, we have to replace (2) with:

(2*) if $A \in \text{Pred}_{\mathcal{L}_{AR}}$, then $A^{\mathfrak{A}} \subseteq D$.

hand, we can easily define satisfaction as follows. I give positive (‘+’) and negative (‘−’) clauses for the four sentence-types, i.e., there are eight clauses. We can cut them in half as noted below (see Lemma 7). However, I chose to formulate all eight clauses to make obvious that it are such with which we work. Aristotle has four sentence types which leads to the positive clauses; the negative ones formalize the assumed relations of contradictory sentence types. Since Aristotle does not reduce the number of sentences (which would need a device for sentence-negation), neither do I.

DEFINITION 6 (*Satisfaction* \models)

Let the *satisfaction-relation* $\mathfrak{A} \models \varphi$ for \mathcal{L}_{AR} -formulas φ and \mathcal{L}_{AR} -structures \mathfrak{A} be defined as follows: Let $A, B \in Pred_{\mathcal{L}_{AR}}$, then

(**a**₊) If φ equals $(\forall A)B$, then $\mathfrak{A} \models \varphi \Leftrightarrow A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = A^{\mathfrak{A}}$.³¹

(**a**_−) If φ equals $(\forall A)B$, then $\mathfrak{A} \not\models \varphi \Leftrightarrow \mathfrak{A} \models (\exists A)\neg B$.

(**i**₊) If φ equals $(\exists A)B$, then $\mathfrak{A} \models \varphi \Leftrightarrow A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$.

(**i**_−) If φ equals $(\exists A)B$, then $\mathfrak{A} \not\models \varphi \Leftrightarrow \mathfrak{A} \models (\forall A)\neg B$.

(**e**₊) If φ equals $(\forall A)\neg B$, then $\mathfrak{A} \models \varphi \Leftrightarrow A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = \emptyset$.

(**e**_−) If φ equals $(\forall A)\neg B$, then $\mathfrak{A} \not\models \varphi \Leftrightarrow \mathfrak{A} \models (\exists A)B$.

(**o**₊) If φ equals $(\exists A)\neg B$, then $\mathfrak{A} \models \varphi \Leftrightarrow A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$.³²

(**o**_−) If φ equals $(\exists A)\neg B$, then $\mathfrak{A} \not\models \varphi \Leftrightarrow \mathfrak{A} \models (\forall A)B$.

⊢

What is interesting about the above clauses is that, even though sentences of type **i** are supposed to be particular-affirmative, the interpretation makes use of a negation; similarly, even though **e** is universal-denying, there is no negation involved. However, if we translate the set-statements into first-order logic, we obtain the right correspondence again: **i** is of the form ‘ $\exists x(A(x) \wedge B(x))$ ’ which is affirmative, and **e** is of the form ‘ $\forall x(A(x) \rightarrow \neg B(x))$ ’ which is negative.

Every sentence type has two clauses – a positive (‘+’) and a negative (‘−’) one. This is to ensure that an **a**-formula has an **o**-formula as its contradictory, and vice

³¹If we assume (2*) rather than (2), this condition becomes instead:

(**a**₊^{*}) If φ equals $(\forall A)B$, then $\mathfrak{A} \models \varphi \Leftrightarrow A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = A^{\mathfrak{A}}$ and $A^{\mathfrak{A}} \neq \emptyset$.

³²Again, using (2*) instead of (2), we get:

(**o**₊^{*}) If φ equals $(\exists A)\neg B$, then $\mathfrak{A} \models \varphi \Leftrightarrow A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$ or $A^{\mathfrak{A}} = \emptyset$.

versa; similarly for **i**- and **e**-formulas. However, we can reduce the number of basic clauses to four; for example, it suffices to have (\mathbf{a}_+) , (\mathbf{a}_-) , (\mathbf{i}_+) , and (\mathbf{i}_-) since the other ones follow from them (see also the following Lemma 7). The reason for this is that we have some kind of double-negation in the meta-language. Whereas it does not follow from $\mathfrak{A} \not\models \varphi$ that $\mathfrak{A} \models \neg\varphi$ since the latter does not even involve a well-formed formula, given the negative clauses, it follows from $\mathfrak{A} \not\models \varphi$ that \mathfrak{A} satisfies the contradictory of φ . This means also that we really need 4 rules to (i) govern the set relations for two sentence types and (ii) ensure which formulas are contradictories; but it does, for example, not suffice to have (\mathbf{a}_+) , (\mathbf{a}_-) , (\mathbf{o}_+) , and (\mathbf{o}_-) since nothing would tell us how **i**- and **e**-formulas behave; also, we can derive (\mathbf{o}_+) and (\mathbf{o}_-) from (\mathbf{a}_+) and (\mathbf{a}_-) .

LEMMA 7 (Reduction of Semantic Clauses)

It suffices to have only 4 of the clauses for satisfaction.³³ ⊢

Proof. I'll only demonstrate three representative cases. Let \mathfrak{A} be an \mathcal{L}_{AR} -structure.

- Suppose that we have (\mathbf{a}_+) and (\mathbf{o}_+) and want to prove (\mathbf{o}_-) .
 - “ \Rightarrow ”: Let $\mathfrak{A} \not\models (\exists A)\neg B$. Then, by (\mathbf{o}_+) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = A^{\mathfrak{A}}$, i.e., $\mathfrak{A} \models (\forall A)B$.
 - “ \Leftarrow ”: Let $\mathfrak{A} \models (\forall A)B$. Then, by (\mathbf{a}_+) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = A^{\mathfrak{A}}$. Suppose that $\mathfrak{A} \models (\exists A)\neg B$. Then, by (\mathbf{o}_+) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$, a contradiction. Therefore, $\mathfrak{A} \not\models (\exists A)\neg B$.
- Suppose we have the clauses for **i** and want to derive (\mathbf{e}_+) :
 - “ \Rightarrow ”: Let $\mathfrak{A} \models (\forall A)\neg B$. Then, by (\mathbf{i}_-) , $\mathfrak{A} \not\models (\exists A)B$. Therefore, by (\mathbf{i}_+) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = \emptyset$.
 - “ \Leftarrow ”: Let $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = \emptyset$. Suppose that $\mathfrak{A} \models (\exists A)B$. Then, by (\mathbf{i}_+) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$, a contradiction. Therefore, $\mathfrak{A} \not\models (\exists A)B$. Thus, by (\mathbf{i}_-) , $\mathfrak{A} \models (\forall A)\neg B$.
- Suppose we have the (\mathbf{a}_+) and (\mathbf{a}_-) and want to derive (\mathbf{o}_+) :³⁴
 - “ \Rightarrow ”: Let $\mathfrak{A} \models (\exists A)\neg B$. By (\mathbf{a}_-) , $\mathfrak{A} \not\models (\forall A)B$, i.e., by (\mathbf{a}_+) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$.
 - “ \Leftarrow ”: Let $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$. Thus, by (\mathbf{a}_+) , $\mathfrak{A} \not\models (\forall A)B$, so, by (\mathbf{a}_-) , $\mathfrak{A} \models (\exists A)\neg B$.

□

Note again that, even though it is implicit, double-negation does not explicitly occur. In the mentioned extension where we allow for negated formulas such as ‘ $\neg((\forall A)B)$ ’, we need the typical negation-clause: $\mathfrak{A} \not\models \varphi$ iff. $\mathfrak{A} \models \neg\varphi$. However, this only further reduces the number of clauses if we ensure the interaction of negation

³³Similarly, this holds if we adopt (\mathbf{a}_+^*) and (\mathbf{o}_+^*) instead of (\mathbf{a}_+) and (\mathbf{o}_+) .

³⁴We can also derive (\mathbf{o}_+^*) from (\mathbf{a}_+^*) and (\mathbf{a}_-) :

“ \Rightarrow ”: Let $\mathfrak{A} \models (\exists A)\neg B$. Then, by (\mathbf{a}_-) , $\mathfrak{A} \not\models (\forall A)B$, so, by (\mathbf{a}_+^*) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$ or $A^{\mathfrak{A}} = \emptyset$.

“ \Leftarrow ”: Let $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$ or $A^{\mathfrak{A}} = \emptyset$. Then, by (\mathbf{a}_+^*) , $\mathfrak{A} \not\models (\forall A)B$, so, by (\mathbf{a}_-) , $\mathfrak{A} \models (\exists A)\neg B$.

and quantifier. As pointed out above, this also means that we need to allow for more negation symbols between the predicates as in ‘ $(\forall A)\neg\neg B$ ’.³⁵

We can also see properly now why Definition 5 (2) does the work it is supposed to do. If there were empty predicates, conditions (\mathbf{a}_+) and (\mathbf{e}_+) would be satisfied if $A = \emptyset$ no matter what B is (cf. also *Malink 2013*, pp. 40f.). These cases are effectively ruled out. Further, since it is a logical truth that $(\exists A)A$ (Theorem 10) and this, by (\mathbf{i}_+) , means that $\emptyset \neq A^{\mathfrak{A}} \cap A^{\mathfrak{A}} = A^{\mathfrak{A}} \subseteq D$, the universe has to be non-empty as expected.³⁶

Note, too, that we did not need to talk about the instances of predicates to interpret the quantifiers; rather, it is interpreted via ‘how much’ the respective predicates have ‘in common’. The predicates A and B can have in common either (i) nothing (\mathbf{e}_+) (exclusion), (ii) something (\mathbf{i}_+) (overlap), (iii) all from the perspective of A (\mathbf{a}_+) (inclusion)³⁷, or (iv) not all from the perspective of A (\mathbf{o}_+) (non-inclusion)³⁸ (cf. *Smiley 1973*, p. 143).

This brings us to the definitions of logical consequence, logically valid and satisfiability which are all standard.

DEFINITION 8

Let $T \subseteq \text{Form}_{\mathcal{L}_{AR}}$.

- (1) φ is a *logical consequence* of T ($T \models \varphi$) $:\Leftrightarrow$ for all \mathcal{L}_{AR} -structures \mathfrak{A} , if $\mathfrak{A} \models \psi$ for all $\psi \in T$, then $\mathfrak{A} \models \varphi$.

If $T = \{\varphi_1, \dots, \varphi_n\}$, we write ‘ $\varphi_1, \dots, \varphi_n \models \varphi$ ’ instead of ‘ $\{\varphi_1, \dots, \varphi_n\} \models \varphi$ ’.

- (2) φ is *logically valid* $:\Leftrightarrow \emptyset \models \varphi$. We will also write this as ‘ $\models \varphi$ ’.

- (3) T is *satisfiable* $:\Leftrightarrow$ there is an \mathcal{L}_{AR} -structure \mathfrak{A} such that $\mathfrak{A} \models \varphi$ for all $\varphi \in T$.

⊥

As an immediate consequence from Definition 5 of \mathcal{L}_{AR} -structure, we get the so-called *square of opposition* (see *Parsons 2017*, §1, and *Peters & Westerståhl 2006*, ch. 1.1.1).

LEMMA 9 (*Square of Opposition*)

The *Square of Opposition* holds:³⁹

Contradictories: $\{(\forall A)B, (\exists A)\neg B\}$ and $\{(\forall A)\neg B, (\exists A)B\}$ are not satisfiable, but any model satisfies one member of the respective sets.

Contraries: $(\forall A)B$ and $(\forall A)\neg B$ are *contraries*, i.e., they cannot both be true (viz., $\{(\forall A)B, (\forall A)\neg B\}$ is not satisfiable) but can both be false.

³⁵This is also true if we drop the non-emptiness requirement, as long as we interpret the negations as predicate-negations and not as term-building operations; see also Lemma 30 in Section 3.5.

³⁶Dropping the non-emptiness requirement, this does not longer hold; see the footnote to Theorem 10.

³⁷In the case of (\mathbf{a}_+^*) , inclusion if A is non-empty.

³⁸In the case of (\mathbf{o}_+^*) , non-inclusion if A is non-empty.

³⁹All of this still holds if we drop the non-emptiness requirement.

Subcontraries: $(\exists A)B$ and $(\exists A)\neg B$ are *subcontraries*, i.e., they cannot both be false, but can both be true.

Subalternation: $(\exists A)B$ is the *subaltern* of $(\forall A)B$ and $(\exists A)\neg B$ of $(\forall A)\neg B$, i.e., if the latter are true, so are the former, and if the former are false, so are the latter:

$$(\mathbf{a} \models \mathbf{i}) \quad (\forall A)B \models (\exists A)B \qquad (\mathbf{e} \models \mathbf{o}) \quad (\forall A)\neg B \models (\exists A)\neg B.$$

⊣

Proof. Let \mathfrak{A} be an \mathcal{L}_{AR} -structure.

Contradictories: • If $\mathfrak{A} \models (\forall A)B$, then, by (\mathbf{o}_-) , $\mathfrak{A} \not\models (\exists A)\neg B$. If $\mathfrak{A} \models (\exists A)\neg B$, then, by (\mathbf{a}_-) , $\mathfrak{A} \not\models (\forall A)B$.

On the other hand, if $\mathfrak{A} \not\models (\forall A)B$, then, by (\mathbf{a}_-) , $\mathfrak{A} \models (\exists A)\neg B$, and, if $\mathfrak{A} \not\models (\exists A)\neg B$, then, by (\mathbf{o}_-) , $\mathfrak{A} \models (\forall A)B$.

• If $\mathfrak{A} \models (\exists A)B$, then, by (\mathbf{e}_-) , $\mathfrak{A} \not\models (\forall A)\neg B$. If $\mathfrak{A} \models (\forall A)\neg B$, then, by (\mathbf{i}_-) , $\mathfrak{A} \not\models (\exists A)B$.

On the other hand, if $\mathfrak{A} \not\models (\exists A)B$, then, by (\mathbf{i}_-) , $\mathfrak{A} \models (\forall A)\neg B$, and, if $\mathfrak{A} \not\models (\forall A)\neg B$, then, by (\mathbf{e}_-) , $\mathfrak{A} \models (\exists A)B$.

Contraries: • Let $\mathfrak{A} \models (\forall A)B$. Then, by (\mathbf{a}_+) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = A^{\mathfrak{A}}$. But, by Definition 5 (2), $A^{\mathfrak{A}} \neq \emptyset$. Therefore, $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$, so, by (\mathbf{i}_+) , $\mathfrak{A} \models (\exists A)B$. Thus, by (\mathbf{e}_-) , $\mathfrak{A} \not\models (\forall A)\neg B$.⁴⁰

On the other hand, let $\mathfrak{A} \models (\forall A)\neg B$. Then, by (\mathbf{e}_+) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = \emptyset$. But, by Definition 5 (2), $A^{\mathfrak{A}} \neq \emptyset$. Thus, $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$. Therefore, by (\mathbf{o}_+) , $\mathfrak{A} \models (\exists A)\neg B$, so, by (\mathbf{a}_-) , $\mathfrak{A} \not\models (\forall A)B$.⁴¹

• To show that they can be both false together, we construct an appropriate model \mathfrak{A} such that $\mathfrak{A} \not\models (\forall A)B$ and $\mathfrak{A} \not\models (\forall A)\neg B$. If there is such a model, then, by (\mathbf{a}_-) and (\mathbf{e}_-) , $\mathfrak{A} \models (\exists A)\neg B$ and $\mathfrak{A} \models (\exists A)B$, i.e., by (\mathbf{o}_+) and (\mathbf{i}_+) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$ and $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$. Choosing, for example, $A^{\mathfrak{A}} = \{x|x \text{ is white}\}$ and $B^{\mathfrak{A}} = \{x|x \text{ is a human being}\}$, we can see that this is indeed satisfiable.

Subcontraries: • If $\mathfrak{A} \not\models (\exists A)B$, then, by (\mathbf{i}_-) , $\mathfrak{A} \models (\forall A)\neg B$. From $(\mathbf{e} \models \mathbf{o})$ (see below) it follows that $\mathfrak{A} \models (\exists A)\neg B$.

On the other hand, if $\mathfrak{A} \not\models (\exists A)\neg B$, then, by (\mathbf{o}_-) , $\mathfrak{A} \models (\forall A)B$. From $(\mathbf{a} \models \mathbf{i})$ (see below) it follows that $\mathfrak{A} \models (\exists A)B$.

• To see that both can be true, just take the model from above.

⁴⁰If we drop the non-emptiness requirement, we have to change the proof as follows: Let $\mathfrak{A} \models (\forall A)B$. Then, by (\mathbf{a}_+^*) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = A^{\mathfrak{A}}$ and $A^{\mathfrak{A}} \neq \emptyset$. Thus, $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$. So, by (\mathbf{i}_+) , $\mathfrak{A} \models (\exists A)B$. Thus, by (\mathbf{e}_-) , $\mathfrak{A} \not\models (\forall A)\neg B$.

⁴¹Dropping the non-emptiness requirement, we get instead: Let $\mathfrak{A} \models (\forall A)\neg B$, i.e., $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = \emptyset$. Suppose that $\mathfrak{A} \models (\forall A)B$. Then, by (\mathbf{a}_+^*) , $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = A^{\mathfrak{A}}$ and $A^{\mathfrak{A}} \neq \emptyset$. Thus, $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$, a contradiction. Therefore, $\mathfrak{A} \not\models (\forall A)B$.

Subalternation: (a \models i): Let $\mathfrak{A} \models (\forall A)B$. Since $(\forall A)B$ and $(\forall A)\neg B$ are contraries, it follows that $\mathfrak{A} \not\models (\forall A)\neg B$. Thus, by (e₋), $\mathfrak{A} \models (\exists A)B$.

(e \models o): Let $\mathfrak{A} \models (\forall A)\neg B$. Since $(\forall A)B$ and $(\forall A)\neg B$ are contraries, it follows that $\mathfrak{A} \not\models (\forall A)B$. Thus, by (a₋), $\mathfrak{A} \models (\exists A)\neg B$.

□

With all this, we can show that predicates are non-empty (Inst) and that every A is A (A-A). These are not (explicit) part of Aristotle's logic but follow from his presuppositions (cf., for the latter, *Malink 2013*, p. 63; see also APr. B15, 63b40–64a4⁴²).

THEOREM 10

The following are logically valid.⁴³

$$(\text{Inst}) \models (\exists A)A$$

$$(\text{A-A}) \models (\forall A)A^{44}$$

+

Proof. (Inst) Since $\emptyset \neq A^{\mathfrak{A}} = A^{\mathfrak{A}} \cap A^{\mathfrak{A}}$ for all $A^{\mathfrak{A}}$ and all \mathfrak{A} , $\models (\exists A)A$.

(A-A) Since $A^{\mathfrak{A}} \cap A^{\mathfrak{A}} = A^{\mathfrak{A}}$ for all $A^{\mathfrak{A}}$ and all \mathfrak{A} , $\models (\forall A)A$.

□

Note also the following:

LEMMA 11 (*Non-Emptiness*)

The following holds:

$$((\forall A)B \models (\exists B)B) \quad (\forall A)B \models (\exists B)B.^{45}$$

□

In the following, I will show that the three figures hold after showing the same for conversion and ecthesis. Thereafter, a calculus will be developed and it is shown

⁴²This passage is quoted above in footnote 28.

⁴³This does not hold if terms can be empty: Let \mathfrak{A} be an \mathcal{L}_{AR} -structure such that $A^{\mathfrak{A}} = \emptyset$. Then, (a₊^{*}) is not satisfied, so $\mathfrak{A} \not\models (\forall A)A$. Further, $A^{\mathfrak{A}} \cap A^{\mathfrak{A}} = \emptyset$, so that (i₊) is likewise not satisfied. Indeed, by (o₊^{*}), $\mathfrak{A} \models (\exists A)\neg A$, and, since $\mathfrak{A} \not\models (\exists A)A$, by (i₋), $\mathfrak{A} \models (\forall A)\neg A$. See also *Read 2015*, pp. 541f.

⁴⁴Even though this is not logically valid without the non-emptiness requirement, we can weaken it as follows:

$$(\text{A-A}^*) \quad (\exists A)B \models (\forall A)A.$$

This holds trivially true under (a₊), but also if we use (a₊^{*}). For, let \mathfrak{A} be an \mathcal{L}_{AR} -structure such that $\mathfrak{A} \models (\exists A)B$. Then, by (i₊), $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$. Therefore, $A^{\mathfrak{A}} \cap A^{\mathfrak{A}} = A^{\mathfrak{A}} \neq \emptyset$, so that, by (a₊^{*}), $\mathfrak{A} \models (\forall A)A$.

⁴⁵This is still valid if we drop the non-emptiness requirement: Let \mathfrak{A} be an \mathcal{L}_{AR} -structure such that $\mathfrak{A} \models (\forall A)B$. Then, by (a₊^{*}), $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = A^{\mathfrak{A}} \neq \emptyset$. Suppose that $B^{\mathfrak{A}} = \emptyset$. Then, $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = \emptyset$, contradiction. Therefore, $B^{\mathfrak{A}} \cap B^{\mathfrak{A}} = B^{\mathfrak{A}} \neq \emptyset$, i.e., by (i₊), $\mathfrak{A} \models (\exists B)B$.

how the figures can be derived. Note, right away, that even though I state ecthesis here, they are not properly part of QUARC_{AR}; see the discussion below.

THEOREM 12 (*Ecthesis*)

Ecthesis holds.

(ec(i)) If $\mathfrak{A} \models (\exists A)B$,
then there is a $C^{\mathfrak{A}} \neq \emptyset$ such that $\mathfrak{A} \models (\forall C)B$ and $\mathfrak{A} \models (\forall C)A$.⁴⁶

(ec(o)) If $\mathfrak{A} \models (\exists A)\neg B$,
then there is a $C^{\mathfrak{A}} \neq \emptyset$ s.t. $\mathfrak{A} \models (\forall C)A$ and $\mathfrak{A} \models (\forall C)\neg B$.⁴⁷

(ec) If $\mathfrak{A} \models (\forall C)A$ and $\mathfrak{A} \models (\forall C)B$, then $\mathfrak{A} \not\models (\forall A)\neg B$.⁴⁸

⊣

Ecthesis will not play a major (or any) role in the semantic proofs of the figures. However, Aristotle explicitly uses ecthesis and so we should say something about it. Note that none of the syntactic proofs that follow need ecthesis either; indeed, it has often been observed that adding ecthesis does not lead to any more deductive power (see, e.g., *Smith 1982*, p. 116). We can prove everything without it – but we cannot account for Aristotle’s own proofs without it. Aristotle’s own use of ecthesis is to establish (*e-e-conv*) and to indicate how to prove certain moods without the use of a reductio rule.⁴⁹

Further, ecthesis is easily seen to follow semantically. The reason is simply that if we have two sets A and B , we also have a unique set which is the intersection of the two sets ($A \cap B$) as well as a unique set that is A without B ($A \setminus B = A \cap (\neg B)$ ⁵⁰). However, this does not show that the intersection is also a term/predicate. What ecthesis, therefore, does is to ensure that our underlying language has enough resources to name the intersections and parts of sets, viz., given two sets A and B , we have $A \setminus B$, $A \cap B$, and $B \setminus A$. This means that we only need it to choose appropriate *subsets* of sets, i.e., that we have appropriate *subterms* for given terms; it is a downwards-looking procedure.⁵¹ Strictly speaking, I have said nothing that enforces this, but it is clear how this can be achieved, viz., by extending the underlying

⁴⁶This is clearly still valid if we drop the non-emptiness requirement.

⁴⁷This is not valid without the non-emptiness requirement: Let \mathfrak{A} be an \mathcal{L}_{AR} -structure such that $A^{\mathfrak{A}} = \emptyset$. Then, by (\mathbf{o}_+^*), $\mathfrak{A} \models (\exists A)\neg B$. However, by (\mathbf{a}_+^*), for any term C , $\mathfrak{A} \models (\forall C)A$ iff. $C^{\mathfrak{A}} \cap A^{\mathfrak{A}} = C^{\mathfrak{A}}$ and $C^{\mathfrak{A}} \neq \emptyset$. Thus, $\mathfrak{A} \models (\forall C)A$ only if $C^{\mathfrak{A}} \cap A^{\mathfrak{A}} \neq \emptyset$. But, since $A^{\mathfrak{A}} = \emptyset$, $C^{\mathfrak{A}} \cap A^{\mathfrak{A}} = \emptyset$. Therefore, $\mathfrak{A} \not\models (\forall C)A$.

To reconcile this, we have to change (ec(o)) to:

(ec(o)^{*}) If $\mathfrak{A} \models (\exists A)\neg B$ and $\mathfrak{A} \models (\exists A)A$,
then there is a $C^{\mathfrak{A}}$ s.t. $\mathfrak{A} \models (\forall C)A$ and $\mathfrak{A} \models (\forall C)\neg B$.

Note that we don’t have to specify that $C^{\mathfrak{A}} \neq \emptyset$ since this is required by (\mathbf{a}_+^*); this is also the reason why we need to include ‘ $(\exists A)A$ ’ because $C^{\mathfrak{A}} = A^{\mathfrak{A}} \setminus B^{\mathfrak{A}}$.

⁴⁸This still holds if we drop the non-emptiness requirement: $(\forall C)A$ is only true in \mathfrak{A} if $C^{\mathfrak{A}} \neq \emptyset$.

⁴⁹Cf. Aristotle’s remarks regarding **Bocardo** at A6, 28b20f.

⁵⁰See Section 3.5 for this notation.

⁵¹Nonetheless, we will be more general in Section 3.5.

language to contain the needed subterms. This also means that the only way for ecthesis to fail to be true is there not being a corresponding term in the language. An alternative approach will become clear in Section 3.5 where we extend QUARC_{AR} to QUARC_{AR}^{*} which includes complex predicates. However, for now, they are not proper part of QUARC_{AR} as developed here.

An alternative approach is to understand ecthesis as a language-extending procedure. In this case, we do not assume the language to include the subterms, but to look at an appropriately extended language that does. To make this alternative formally work, we have to define language extensions and structure extensions. This can easily be done in the standard way.⁵²

To get a proof of (e-e-conv) that is closer to Aristotle’s own, I also include a third kind of ecthesis. This rule is just (Darapti) – a mood of the third figure. We can semantically prove all the moods without circularity, but it seems like Aristotle does not do so himself (at least in the syntactical case) – we have to invoke further logical principles that he does not make explicit. For example, Striker argues (in *Aristotle 2009*, pp. 86ff.) that, since it cannot be **Darapti** that is at play because Aristotle mentions a proof by ecthesis of it, it must be ‘existential generalization’ (p. 88) that is used here. Malink (2013), on the other hand, suggests a different semantics, viz., what he calls ‘preorder semantics’ (pp. 73ff.) to argue that, e.g., (**a-i-conv**) naturally follows from it (p. 67). Martin (2004a)⁵³ shows that we can understand ecthesis as ‘a discharge rule that functions in syllogistic semantics in much the way that disjunction-elimination and existential instantiation function in first-order logic’ (p. 19); a possible way to spell this out is again in terms of language- and model-extensions. Without going into further detail, let me just point out that all of these reconstructions suffer from similar problems as all the others not mentioned here: they invoke a full-blown classical apparatus in the background which includes connectives and quantifier specialization/generalization. Let me just mention, without further arguing for it, that I find it implausible that Aristotle invokes principles such as existential specialization without developing a logic that properly includes it (or even just giving us a hint to suggest that he does). For example, look at Malink’s (2013, p. 67) reconstruction/justification of (**a-i-conv**). He uses \wedge -introduction as well as existential generalization. But neither of these are properly part of Aristotle’s logic – and we can do without them.⁵⁴ For this reason, I do not formally include ecthesis into QUARC_{AR}, even though I called it ‘Theorem’ (which it is (as stated) not for the reasons mentioned in the previous paragraph). Note, however, that we can understand ecthesis as semantic justification of the syntactic conversion rules; given this interpretation, there is no need to include ecthesis into the calculus. In what follows, I will show that (i) we can prove the conversion rules without ecthesis and (ii) if we use ecthesis, we get very close to

⁵²See, for example, *Raab 2016*, Sections 3.4 and 4.3.

⁵³Thanks to an anonymous referee for pointing this out.

⁵⁴This observation loses some of its force if we replace (**a**₊) and (**o**₊) by (**a**₊^{*}) and (**o**₊^{*}) since these rely on conjunction and disjunction. Nonetheless, the point still applies for the specialization/generalization cases. Note also that Malink (2013, p. 89), says explicitly after using ‘rules of classical propositional and quantifier logic’ that ‘[he does] not want to suggest that Aristotle had a clear grasp of all these rules’. However, I also do not want to claim that Aristotle did not have any grasp of it; the point is rather that he did not include them in his formal logic as developed in his *Prior Analytics*, A1–7.

Aristotle's proofs. After that, I will drop ecthesis again and also not include them as derivation rules, but I will indicate syntactic proofs using them.

Proof. Let \mathfrak{A} be an \mathcal{L}_{AR} -structure.

(ec(i)) Let $\mathfrak{A} \models (\exists A)B$ hold, i.e., $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$. Let $C^{\mathfrak{A}} := A^{\mathfrak{A}} \cap B^{\mathfrak{A}}$. Then, $C^{\mathfrak{A}} \neq \emptyset$. Further, since $C^{\mathfrak{A}} \cap A^{\mathfrak{A}} = C^{\mathfrak{A}}$, $\mathfrak{A} \models (\forall C)A$ and similarly for $\mathfrak{A} \models (\forall C)B$.

(ec(o)) Let $\mathfrak{A} \models (\exists A)\neg B$. Then, $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}} \neq \emptyset$. Let $C^{\mathfrak{A}} := A^{\mathfrak{A}} \setminus B^{\mathfrak{A}}$. Since $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$, $C^{\mathfrak{A}} \neq \emptyset$. Then, $C^{\mathfrak{A}} \cap A^{\mathfrak{A}} = C^{\mathfrak{A}}$ so that $\mathfrak{A} \models (\forall C)A$. Further, $C^{\mathfrak{A}} \cap B^{\mathfrak{A}} = \emptyset$ so that $\mathfrak{A} \models (\forall C)\neg B$.⁵⁵

(ec) Let $\mathfrak{A} \models (\forall C)A$ and (i) $\mathfrak{A} \models (\forall C)B$. Then, (ii) $C^{\mathfrak{A}} \cap A^{\mathfrak{A}} = C^{\mathfrak{A}}$ so that $C^{\mathfrak{A}} \cap B^{\mathfrak{A}} \stackrel{(i)}{=} C^{\mathfrak{A}} \stackrel{(ii)}{=} C^{\mathfrak{A}} \cap A^{\mathfrak{A}}$. Thus, intersecting both sides with $A^{\mathfrak{A}}$ gives us $C^{\mathfrak{A}} \cap B^{\mathfrak{A}} \cap A^{\mathfrak{A}} = C^{\mathfrak{A}} \cap A^{\mathfrak{A}} \cap A^{\mathfrak{A}} = C^{\mathfrak{A}} \cap A^{\mathfrak{A}} = C^{\mathfrak{A}} \neq \emptyset$. Therefore, $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$. But then, $\mathfrak{A} \models (\exists A)B$. Thus, by (e₋), $\mathfrak{A} \not\models (\forall A)\neg B$.

□

THEOREM 13 (*Conversion*)

Semantic *conversion* holds:

(a-i-conv) $(\forall A)B \models (\exists B)A$;

(i-i-conv) $(\exists A)B \models (\exists B)A$;

(e-e-conv) $(\forall A)\neg B \models (\forall B)\neg A$;

(o-o~~conv~~) $(\exists A)\neg B \not\models (\exists B)\neg A$.

⊢

This is the point where we reach results that Aristotle proves himself, viz., conversion. I will first quote the original text and then give my proof so that it becomes clear how close we are to the text. I will put numbers (in the form ‘[(n)]’) into the quotation to indicate the parallel moves in QUARC_{AR}. This practice of first quoting will be followed especially when proving the figures in a syntactical fashion.

Proof. Let \mathfrak{A} be an \mathcal{L}_{AR} -structure.

(e-e-conv) First, let the premiss AB be a universal privative. Now if A belongs to none of the Bs, then neither will B belong to any of the As. [(1)] For if it does belong to some, for example, [(2)] to C, [(3)] it will not be true that A belongs to none of the Bs, since C is one of the Bs.⁵⁶

Let $\mathfrak{A} \models (\forall B)\neg A$ hold, i.e., $B^{\mathfrak{A}} \cap A^{\mathfrak{A}} = \emptyset$.

(1) For if it does belong to some,	Suppose $\mathfrak{A} \models (\exists A)B$ by (ec(i)), $\mathfrak{A} \models (\forall C)B$ and $\mathfrak{A} \models$ $(\forall C)A$ by (ec), $\mathfrak{A} \not\models (\forall B)\neg A$
(2) for example, to C,	
(3) it will not be true that A belongs to none of the Bs	

⁵⁵The proof stays essentially the same if we adopt (ec(o)^{*}) instead of (ec(o)): Since $\mathfrak{A} \models (\exists A)A$, $A^{\mathfrak{A}} \neq \emptyset$.

⁵⁶Πρῶτον μὲν οὖν ἔστω στερητικὴ καθόλου ἡ Α Β πρότασις. εἰ οὖν μηδενὶ τῶ Β τὸ Α ὑπάρχει, οὐδὲ τῶ Α οὐδενὶ ὑπάρξει τὸ Β: εἰ γάρ τι, οἷον τῶ Γ, οὐ ἀληθὲς ἔσται τὸ μηδενὶ τῶ Β τὸ Α ὑπάρχειν: τὸ γάρ Γ τῶν Β τί ἐστίν. (A2, 25a14-17)

contradicting the assumption. Therefore, $\mathfrak{A} \not\models (\exists A)B$ so that, by (**i**₋), $\mathfrak{A} \models (\forall A)\neg B$.

A more straightforward proof not relying on ecthesis is this:

Let $\mathfrak{A} \models (\forall B)\neg A$ hold, i.e., $B^{\mathfrak{A}} \cap A^{\mathfrak{A}} = \emptyset \iff A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = \emptyset$. This means that $\mathfrak{A} \models (\forall A)\neg B$.

(a-i-conv) And if A belongs to every B, B also belongs to some A, [(1)] for if it belongs to none, [(2)] neither will A belong to any of the Bs. But it was assumed that it belongs to all.⁵⁷

Let $\mathfrak{A} \models (\forall B)A$.

(1)	for if it belongs to none,		Suppose $\mathfrak{A} \models (\forall A)\neg B$
(2)	neither will A belong to any of the Bs		then, by (e-e-conv), $\mathfrak{A} \models (\forall B)\neg A$

from which it follows, by (**i**₋), that $\mathfrak{A} \not\models (\exists B)A$. But, by (**a** \models **i**), it follows from $\mathfrak{A} \models (\forall B)A$ that $\mathfrak{A} \models (\exists B)A$, contradiction. Therefore, $\mathfrak{A} \not\models (\forall A)\neg B$, i.e., by (**e**₋), $\mathfrak{A} \models (\exists A)B$.

Here, too, there is a more straightforward proof:

Let $\mathfrak{A} \models (\forall B)A$ hold, i.e., $B^{\mathfrak{A}} \cap A^{\mathfrak{A}} = B^{\mathfrak{A}}$. But by Definition 5 (2), $B^{\mathfrak{A}} \neq \emptyset$, so that $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$, i.e., $\mathfrak{A} \models (\exists A)B$.⁵⁸

(i-i-conv) Similarly also in the case of the particular premiss; for if A belongs to some of the Bs, it is necessary that B belong to some of the As. For if it belongs to none, neither does A belong to any of the Bs.⁵⁹

Let $\mathfrak{A} \models (\exists B)A$. Suppose that $\mathfrak{A} \models (\forall A)\neg B$. Then, by (e-e-conv), $\mathfrak{A} \models (\forall B)\neg A$. But, by (**i**₋), this implies $\mathfrak{A} \not\models (\exists B)A$, a contradiction. Therefore, $\mathfrak{A} \not\models (\forall A)\neg B$, and, by (**e**₋), $\mathfrak{A} \models (\exists A)B$.

(o-o~~x~~conv) For it is not the case that, if man does not belong to some animal, then animal also does not belong to some man.⁶⁰

Let $B^{\mathfrak{A}} \subsetneq A^{\mathfrak{A}}$. Then, $\mathfrak{A} \models (\exists A)\neg B$ since $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$. But since $B^{\mathfrak{A}} \cap A^{\mathfrak{A}} = B^{\mathfrak{A}}$, $\mathfrak{A} \models (\forall B)A$. Thus, by (**o**₋), $\mathfrak{A} \not\models (\exists B)\neg A$.⁶¹

□

Before moving on to the figures, let me point out that – if we drop the non-emptiness requirement – we can restrict (o-o~~x~~conv) to a valid version. Let \mathfrak{A} be an

⁵⁷εἰ δὲ παντὶ τὸ A τῶ B, καὶ τὸ B τινὶ τῶ A ὑπάρξει: εἰ γὰρ μηδενί, οὐδὲ τὸ A οὐδενὶ τῶ B ὑπάρξει: ἀλλ' ὑπέκειτο παντὶ ὑπάρχειν. (A2, 25a17ff.)

⁵⁸Dropping the non-emptiness assumption, we can invoke (**a**₊^{*}) instead of the definition of \mathcal{L}_{AR} -structure.

⁵⁹ὁμοίως δὲ καὶ εἰ κατὰ μέρος ἐστὶν ἡ πρότασις. εἰ γὰρ τὸ A τινὶ τῶ B, καὶ τὸ B τινὶ τῶ A ἀνάγκη ὑπάρχειν: εἰ γὰρ μηδενί, οὐδὲ τὸ A οὐδενὶ τῶ B. (A2, 25a20ff.)

⁶⁰οὐ γὰρ εἰ ἄνθρωπος μὴ ὑπάρχει τινὶ ζῴῳ, καὶ ζῴων οὐχ ὑπάρχει τινὶ ἀνθρώπῳ. (A2, 25a12f.)

⁶¹If we choose, as Aristotle indeed does, $B^{\mathfrak{A}}$ to be non-empty, this is also a countermodel with the non-emptiness requirement dropped.

\mathcal{L}_{AR} -structure. Then:

(o-o-conv*) If $\mathfrak{A} \not\models (\exists A)A$, then, if $\mathfrak{A} \models (\exists A)\neg B$, then $\mathfrak{A} \models (\exists B)\neg A$.

For, $\mathfrak{A} \not\models (\exists A)A$ iff. $A^{\mathfrak{A}} = \emptyset$. Thus, if $A^{\mathfrak{A}} = \emptyset$, then, by (\mathbf{o}_+^*) , $\mathfrak{A} \models (\exists A)\neg B$. Also, $B^{\mathfrak{A}} \cap A^{\mathfrak{A}} = \emptyset$. But either (i) $B^{\mathfrak{A}} = \emptyset$ or (ii) $B^{\mathfrak{A}} \neq \emptyset$. If (i), applying (\mathbf{o}_+^*) , $\mathfrak{A} \models (\exists B)\neg A$. And, if (ii), then $B^{\mathfrak{A}} \cap A^{\mathfrak{A}} = \emptyset \neq B^{\mathfrak{A}}$. Thus, by (\mathbf{o}_+^*) , $\mathfrak{A} \models (\exists B)\neg A$.

I will not mention the non-emptiness requirement in what follows. The reason is simply that none of the proofs depends on it, i.e., they still hold if we drop it. This is so because we get existential import in any of the moods of the figures, i.e., at least one of the premises is affirmative in every mood.

THEOREM 14 (First Figure)

The moods of the *first figure* hold:

(Barbara) $(\forall B)A, (\forall C)B \models (\forall C)A$; (Celarent) $(\forall B)\neg A, (\forall C)B \models (\forall C)\neg A$;

(Darii) $(\forall B)A, (\exists C)B \models (\exists C)A$; (Ferio) $(\forall B)\neg A, (\exists C)B \models (\exists C)\neg A$.

⊢

Proof. I will only prove (Barbara) and (Darii), the others have similar proofs.

Let \mathfrak{A} be an \mathcal{L}_{AR} -structure.

(Barbara) Let $\mathfrak{A} \models (\forall B)A$ and $\mathfrak{A} \models (\forall C)B$ hold. This means that (i) $B^{\mathfrak{A}} \cap A^{\mathfrak{A}} = B^{\mathfrak{A}}$ and (ii) $C^{\mathfrak{A}} \cap B^{\mathfrak{A}} = C^{\mathfrak{A}}$. Thus, $C^{\mathfrak{A}} \stackrel{(ii)}{=} C^{\mathfrak{A}} \cap B^{\mathfrak{A}} \stackrel{(i)}{=} C^{\mathfrak{A}} \cap B^{\mathfrak{A}} \cap A^{\mathfrak{A}} \stackrel{(ii)}{=} C^{\mathfrak{A}} \cap A^{\mathfrak{A}}$, i.e., $\mathfrak{A} \models (\forall C)A$.

(Darii) Let $\mathfrak{A} \models (\forall B)A$ and $\mathfrak{A} \models (\exists C)B$. Then, (i) $B^{\mathfrak{A}} \cap A^{\mathfrak{A}} = B^{\mathfrak{A}}$ and (ii) $C^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq \emptyset$. Thus, intersecting (i) with $C^{\mathfrak{A}}$ leads to $B^{\mathfrak{A}} \cap A^{\mathfrak{A}} \cap C^{\mathfrak{A}} \stackrel{(i)}{=} B^{\mathfrak{A}} \cap C^{\mathfrak{A}} \stackrel{(ii)}{\neq} \emptyset$. Therefore, $C^{\mathfrak{A}} \cap A^{\mathfrak{A}} \neq \emptyset$, i.e., $\mathfrak{A} \models (\exists C)A$.

□

THEOREM 15 (Second Figure)

The moods of the *second figure* hold:

(Cesare) $(\forall N)\neg M, (\forall X)M \models (\forall X)\neg N$; (Camestres)
 $(\forall N)M, (\forall X)\neg M \models (\forall X)\neg N$;

(Festino) $(\forall N)\neg M, (\exists X)M \models (\exists X)\neg N$; (Baroco) $(\forall N)M, (\exists X)\neg M \models (\exists X)\neg N$.

⊢

Proof. I will only prove (Baroco).

Let \mathfrak{A} be an \mathcal{L}_{AR} -structure.

(Baroco) Let $\mathfrak{A} \models (\forall N)M$ and $\mathfrak{A} \models (\exists X)\neg M$ hold. Suppose, for contradiction, that $\mathfrak{A} \models (\forall X)N$. Using the first premise and this new one in (Barbara), we get $\mathfrak{A} \models (\forall X)M$, which implies, by (\mathbf{o}_-), $\mathfrak{A} \not\models (\exists X)\neg M$, a contradiction to the second premise. Thus, $\mathfrak{A} \not\models (\forall X)N$ so that, by (\mathbf{a}_-), $\mathfrak{A} \models (\exists X)\neg N$.

□

THEOREM 16 (Third Figure)

The moods of the *third figure* hold:

(Darapti) $(\forall S)P, (\forall S)R \models (\exists R)P$; (Felapton) $(\forall S)\neg P, (\forall S)R \models (\exists R)\neg P$;

(Disamis) $(\exists S)P, (\forall S)R \models (\exists R)P$; (Datisi) $(\forall S)P, (\exists S)R \models (\exists R)P$;

(Bocardo) $(\exists S)\neg P, (\forall S)R \models (\exists R)\neg P$; (Ferison) $(\forall S)\neg P, (\exists S)R \models (\exists R)\neg P$.

⊢

Proof. I will only prove (Darapti), (Bocardo), and (Ferison).

Let \mathfrak{A} be an \mathcal{L}_{AR} -structure.

(Darapti) Let $\mathfrak{A} \models (\forall S)P$ and $\mathfrak{A} \models (\forall S)R$ hold. Applying (a-i-conv) to the second premise and using (Darai), we get $\mathfrak{A} \models (\exists R)P$.

(Bocardo) Let $\mathfrak{A} \models (\exists S)\neg P$ and $\mathfrak{A} \models (\forall S)R$ hold. Suppose, for contradiction, that $\mathfrak{A} \not\models (\exists R)\neg P$. Then, by (\mathbf{o}_-), $\mathfrak{A} \models (\forall R)P$. Applying (Barbara) to it and the second premise, we get $\mathfrak{A} \models (\forall S)P$, which implies, by (\mathbf{o}_-), $\mathfrak{A} \not\models (\exists S)\neg P$ contradicting the first premise. Therefore, $\mathfrak{A} \models (\exists R)\neg P$.

(Ferison) Let $\mathfrak{A} \models (\forall S)\neg P$ and $\mathfrak{A} \models (\exists S)R$ hold. Assume, for contradiction, that $\mathfrak{A} \not\models (\exists R)\neg P$. Then, by (\mathbf{o}_-), $\mathfrak{A} \models (\forall R)P$. Using this and the first premise in a (Camestres), we get that $\mathfrak{A} \models (\forall S)\neg R$. Therefore, by (\mathbf{i}_-), $\mathfrak{A} \not\models (\exists S)R$, which contradicts the second premise. Thus, $\mathfrak{A} \models (\exists R)\neg P$.

□

The Calculus Now that we have seen that the figures are valid, I introduce a natural deduction system. Before giving the rules, we need to say what a derivation is and what assumptions are.⁶² After giving the rules, I prove that all conversion rules can be derived before giving the proofs of the figures in the following Section 3.3.

Ben-Yami (2014) also introduces a natural deduction system, but in Lemmon-style. In this system, all QUARC_{AR}-derivation rules are derivable. Note also Gratzl & Pavlovic (*ms*) who develop the proof theory of QUARC within a sequent calculus, and Lanzet (2017) who also uses a sequent calculus, but for a three-valued QUARC.

⁶²I am following Troelstra & Schwichtenberg 2000.

DEFINITION 17 (*Derivation & Assumption*)

A *derivation* Π in a system \mathbf{S} is a labelled tree where the conclusion is the label of the end-node, and each top-node is labelled by an (open or closed) assumption. \dashv

DEFINITION 18 (*Discharged Assumption*)

An application of an inference rule that is *discharging* will close a class of assumptions A . We write $[A]^u$ to indicate that an assumption class A is closed, and the index u corresponds to the rule application which discharged the assumption. For a derivation Π with end-node B and open assumptions A_1, \dots, A_n , we write $\langle A_1, \dots, A_n \vdash B \rangle$, and say that Π is a derivation of B from *premises* A_1, \dots, A_n . \dashv

DEFINITION 19 (*QUARC_{AR} Derivation Rules*)

The *derivation rules* of QUARC_{AR} are the following:⁶³

$$(\forall \rightarrow \exists) \frac{(\forall A)B}{(\exists A)B}; \quad (\text{A-A}) \frac{}{(\forall A)A}; \quad (\text{e-e-conv}) \frac{(\forall A)\neg B}{(\forall B)\neg A};$$

$$(\text{RAA}_{\mathbf{a-o}}, u) \frac{\begin{array}{c} [(\forall A)B]^u \\ \vdots \\ \pm\varphi \quad \mp\varphi \end{array}}{(\exists A)\neg B}; \quad (\text{RAA}_{\mathbf{o-a}}, u) \frac{\begin{array}{c} [(\exists A)\neg B]^u \\ \vdots \\ \pm\varphi \quad \mp\varphi \end{array}}{(\forall A)B};$$

$$(\text{RAA}_{\mathbf{i-e}}, u) \frac{\begin{array}{c} [(\exists A)B]^u \\ \vdots \\ \pm\varphi \quad \mp\varphi \end{array}}{(\forall A)\neg B}; \quad (\text{RAA}_{\mathbf{e-i}}, u) \frac{\begin{array}{c} [(\forall A)\neg B]^u \\ \vdots \\ \pm\varphi \quad \mp\varphi \end{array}}{(\exists A)B};$$

$$(\text{Barbara}) \frac{(\forall B)A \quad (\forall C)B}{(\forall C)A}; \quad (\text{Celarent}) \frac{(\forall B)\neg A \quad (\forall C)B}{(\forall C)\neg A};$$

$$(\text{Darii}) \frac{(\forall B)A \quad (\exists C)B}{(\exists C)A}; \quad (\text{Ferio}) \frac{(\forall B)\neg A \quad (\exists C)B}{(\exists C)\neg A}.$$

\dashv

⁶³If the terms are not assumed to be non-empty, we replace (A-A) by

$$(\text{A-A}^*) \frac{(\exists A)B}{(\forall A)A}.$$

We have to include the rules $(\forall \rightarrow \exists)$ and (A-A) to ensure the completeness of the logic; otherwise they will not play a major role.⁶⁴ Most notably, we need $(\forall \rightarrow \exists)$ to derive (a-i-conv). This is why it suffices to have (e-e-conv) as basic conversion rule. In the presence of (e-e-conv) and the reductio rules (see Theorem 20), we could have equally well chosen (a-i-conv) instead of $(\forall \rightarrow \exists)$ and derive the latter as follows:

$$\frac{\frac{(\forall A)B}{(\exists B)A} \text{ (a-i-conv)}}{(\exists A)B} \text{ (i-i-conv)} .$$

We also have to include one of the conversion rules to prove the other two where the choice was rather arbitrary. However, we need reductio rules to derive (i-i-conv) from (e-e-conv) or vice versa (see below). In particular, we do not include the ecthesis rules. Here, again, we would need three, where one would essentially be (Darapti). In their presence, we could derive all the conversion rules. However, since Aristotle also wants to reduce all the moods to just (Barbara) and (Celarent), and needs to use conversion rules to do so, it would be rather circular to have one of the reduced rules as a basic derivation rule. In particular, the claim that all the moods are reducible to (Barbara) and (Celarent) would then be plainly false.

To ensure the contradictories again, we have to include four (types of) reductio rules. In the reductio rules, we read ‘ $\pm\varphi$ ’ to ‘ $\mp\varphi$ ’ as contradictories, too, so that the rules are stated somewhat circular. We could spell them out with all the possible pairs to get rid of this circularity, but this would multiply the rules without much gain. For example, we can instantiate (RAA_{i-e}, u) as below to derive (e-e-conv) from (i-i-conv), or (RAA_{a-o}, u) as follows:

$$\frac{\begin{array}{c} [(\forall B)A]^u \\ \vdots \\ (\exists A)\neg B \quad (\forall A)B \end{array}}{(\exists B)\neg A} \text{ (RAA}_{a-o}, u) .$$

Similarly, we could arrive at a different pair of contradictories such as

$$\frac{\begin{array}{c} [(\forall B)A]^u \\ \vdots \\ (\forall C)\neg D \quad (\exists C)D \end{array}}{(\exists B)\neg A} \text{ (RAA}_{a-o}, u) .$$

This also means that for every reductio type, there correspond four instances with different contradictories in place of ‘ $\pm\varphi$ ’ and ‘ $\mp\varphi$ ’. Aristotle himself has more than one reductio rule to conclude the contradictory of a sentence.⁶⁵ This is necessary to pass from a given sentence-type to another one. In an extended language which then

⁶⁴The following remarks on the rules and their interderivability mostly still apply if we allow for empty terms, i.e., drop (A-A). Of course, without (A-A), we cannot drive what is called (Inst) below.

⁶⁵See B11, 61a19f., where Aristotle explains to assume ‘the contradictory of the conclusion [ἡ ἀντίφρασις τεθῆ τοῦ συμπεράσματος]’ to derive a contradiction. Cf. also *Crivelli 2012*, p. 133, and *Malink 2013*, pp. 31ff.

gives rise to rules that govern the interaction of negation and quantifier, the rules can be reduced again; but as long as there is no formal information about when two sentences are contradictories, we actually have to include all of them as rules.⁶⁶

Given these reductio rules, we could have chosen (i-i-conv) as basic rule and derived (e-e-conv) as follows:

$$\frac{(\forall A)\neg B \quad \frac{[(\exists B)A]^u}{(\exists A)B} \text{ (i-i-conv)}}{(\forall B)\neg A} \text{ (RAA}_{i-e}, u) .$$

This also shows that, given the other rules, $(\forall \rightarrow \exists)$ and (a-i-conv) as well as (e-e-conv) and (i-i-conv) are interderivable. Further, we can also derive the following (call it (Inst)):

$$\frac{\frac{(\forall A)A}{(\exists A)A} \text{ (A-A)}}{(\exists A)A} \text{ (a-i-conv)} .$$

Since we use $(\forall \rightarrow \exists)$ in the derivation of (a-i-conv), this proof relies on the former rule. Using (Inst) and (Darii), we can see that (Inst) and $(\forall \rightarrow \exists)$ are interderivable (assuming (A-A)):

$$\frac{(\forall A)B \quad \frac{(\exists A)A}{(\exists A)A} \text{ (Inst)}}{(\exists A)B} \text{ (Darii)} .$$

Thus, we could have chosen (Inst) instead of $(\forall \rightarrow \exists)$.⁶⁷

Since we want to derive (Darii) from (Barbara) and (Celarent) plus the conversion rules, there is a possible circularity in using (Inst) as basic rule. However, to derive (Darii), we only need (e-e-conv) and reductio. Thus, the proof of Theorem 23 shows that the reduction is successful. Further, given (Inst), we can also derive (a-i-conv); just apply (i-i-conv) (whose proof only needs (e-e-conv) and a reductio rule, see below), to the previous derivation.

Lastly, the four moods of the first figure are included as derivation rules. Aristotle does not prove them himself, but notes that they are ‘perfect’ and, thus, not in need for any justification. We will later (in Theorem 23) prove that it, indeed, suffices to have (Barbara) and (Celarent).

The moods of the first figure together with the conversion rules are sufficient to derive all the moods of the second and third figures. Theorem 20 shows the derivability of the remaining two conversion rules. Then, it will be shown that the moods of the second and third figures are derivable as well. To see how adequate the calculus is, I will first quote Aristotle’s own proofs before giving the formal counterparts. As can be seen, the formal derivations are just transcriptions of Aristotle’s own words in the formal language.

⁶⁶An anonymous referee has pointed out that Martin (1997, p. 7) introduces just one general reductio rule which shows that the multiplication of such rules as above is unnecessary. However, Martin is able to reduce it because he introduces a ‘syntactic negation’ (p. 5) which is exactly not done here since Aristotle does not work with such a concept and without such, a more general form of the reductio rules is not viable (Smith 2018, §4, explicitly tells us that Aristotle ‘does not view negations as sentential compounds’). Opting for the mentioned extended language, viz., a language including negation, we can formulate a general reductio rule.

⁶⁷Again, if we allow empty terms, this is no longer the case.

THEOREM 20 (Conversion Rules)

The following *conversion rules* are derivable.

$$\text{(a-i-conv)} \frac{(\forall A)B}{(\exists B)A};$$

$$\text{(i-i-conv)} \frac{(\exists A)B}{(\exists B)A}.$$

⊢

Proof. **(i-i-conv)** $(\exists A)B \vdash (\exists B)A$:

$$\frac{(\exists A)B \quad \frac{[(\forall B)\neg A]^u}{(\forall A)\neg B} \text{(e-e-conv)}}{(\exists B)A} \text{(RAA}_{e-i}, u)$$

(a-i-conv) $(\forall A)B \vdash (\exists B)A$:

$$\frac{(\forall A)B}{(\exists A)B} (\forall \rightarrow \exists) \quad \frac{(\exists A)B}{(\exists B)A} \text{(i-i-conv)}$$

□

3.3 Syllogistic in QUARC_{AR}

Having developed the calculus, we can go on to prove the remaining two figures. This will be done in the following Theorems 21 and 22. After that, I formulate and prove a result also mentioned by Aristotle himself, viz., that we can derive the rules (Darii) and (Ferio) from the remaining ones.

THEOREM 21 (Second Figure)

The moods of the *second figure* are derivable.

$$\text{(Cesare)} \quad (\forall N)\neg M, (\forall X)M \vdash (\forall X)\neg N; \quad \text{(Camestres)} \\ (\forall N)M, (\forall X)\neg M \vdash (\forall X)\neg N;$$

$$\text{(Festino)} \quad (\forall N)\neg M, (\exists X)M \vdash (\exists X)\neg N; \quad \text{(Baroco)} \quad (\forall N)M, (\exists X)\neg M \vdash (\exists X)\neg N.$$

⊢

Proof. **(Cesare)** $(\forall N)\neg M, (\forall X)M \vdash (\forall X)\neg N$:

For let M be predicated of no N and of all X. Now since the privative premiss converts, N will belong to no M; but it was assumed that M belongs to all X, so that N will belong to no X – this was proved before.⁶⁸

$$\frac{\frac{(\forall N)\neg M}{(\forall M)\neg N} \text{(e-e-conv)} \quad (\forall X)M}{(\forall X)\neg N} \text{(Celarent)}$$

(Camestres) $(\forall N)M, (\forall X)\neg M \vdash (\forall X)\neg N$:

⁶⁸κατηγορείσθω γὰρ τὸ M τοῦ μὲν N μηδενός, τοῦ δὲ Ξ παντός. ἐπει οὖν ἀντιστρέφει τὸ στερητικόν, οὐδενὶ τῶ M ὑπάρξει τὸ N: τὸ δὲ γε M παντὶ τῶ Ξ ὑπέικειτο: ὥστε τὸ N οὐδενὶ τῶ Ξ: τοῦτο γὰρ δέδεικται πρότερον. (A5, 27a5-9)

Again, if M belongs to all N and to no X, X will belong to no N. For if M belongs to no X, neither does X belong to any M; but it was assumed that M belongs to all N; therefore, X will belong to no N – for the first figure has come about again. And since the privative premiss converts, neither will N belong to any X, so that there will be the same syllogism.⁶⁹

$$\frac{\frac{(\forall N)M \quad \frac{(\forall X)\neg M}{(\forall M)\neg X} \text{ (e-e-conv)}}{(\forall N)\neg X} \text{ (Celarent)}}{(\forall X)\neg N} \text{ (e-e-conv)}$$

Note that Aristotle's proof explicitly uses conversion to arrive at ' $(\forall X)\neg N$ ', but in stating the syllogism, he says 'X will belong to no N [οὐδὲ τὸ Ξ τῶ N οὐδενὶ ὑπάρξει]' (A5, 27a10) instead of the proven 'N will belong to no X'. The former, however, is not an instantiation of (Second Figure Scheme).

(Festino) $(\forall N)\neg M, (\exists X)M \vdash (\exists X)\neg N$:

For if M belongs to no N but to some X, it is necessary for N not to belong to some X. For since the privative premiss converts, N will belong to no M; but it was assumed that M belongs to some X, so that N will not belong to some X. For a syllogism in the first figure comes about.⁷⁰

$$\frac{\frac{(\forall N)\neg M}{(\forall M)\neg N} \text{ (e-e-conv)} \quad (\exists X)M}{(\exists X)\neg N} \text{ (Ferio)}$$

(Baroco) $(\forall N)M, (\exists X)\neg M \vdash (\exists X)\neg N$:

Again, if M belongs to all N but does not belong to some X, it is necessary for N not to belong to some X. For if it belongs to all X and M is predicated of every N, it is necessary for M to belong to every X. But it was assumed that it did not belong to some.⁷¹

$$\frac{\frac{(\forall N)M \quad [(\forall X)N]^u \text{ (Barbara)}}{(\forall X)M} \quad (\exists X)\neg M}{(\exists X)\neg N} \text{ (RAA}_{\mathbf{a-o}}, u)}$$

□

THEOREM 22 (*Third Figure*)

The moods of the *third figure* are derivable.

⁶⁹πάλιν εἰ τὸ M τῶ μὲν N παντὶ τῶ δὲ Ξ μηδενί, οὐδὲ τὸ Ξ τῶ N οὐδενὶ ὑπάρξει (εἰ γὰρ τὸ M οὐδενὶ τῶ Ξ, οὐδὲ τὸ Ξ οὐδενὶ τῶ M: τὸ δὲ γε M παντὶ τῶ N ὑπῆρχεν: τὸ ἄρα Ξ οὐδενὶ τῶ N ὑπάρχει: γεγένηται γὰρ πάλιν τὸ πρῶτον σχῆμα): ἐπεὶ δὲ ἀντιστρέφει τὸ στερητικόν, οὐδὲ τὸ N οὐδενὶ τῶ Ξ ὑπάρξει, ὥστ' ἔσται ὁ αὐτὸς συλλογισμὸς. (A5, 27a9-14)

⁷⁰εἰ γὰρ τὸ M τῶ μὲν N μηδενί τῶ δὲ Ξ τινὶ ὑπάρχει, ἀνάγκη τὸ N τινὶ τῶ Ξ μὴ ὑπάρχειν. ἐπεὶ γὰρ ἀντιστρέφει τὸ στερητικόν, οὐδενὶ τῶ M ὑπάρξει τὸ N: τὸ δὲ γε M ὑπέκειτο τινὶ τῶ Ξ ὑπάρχειν: ὥστε τὸ N τινὶ τῶ Ξ οὐχ ὑπάρξει: γίνεται γὰρ συλλογισμὸς διὰ τοῦ πρώτου σχήματος. (A5, 27a32-36)

⁷¹πάλιν εἰ τῶ μὲν N παντὶ τὸ M, τῶ δὲ Ξ τινὶ μὴ ὑπάρχει, ἀνάγκη τὸ N τινὶ τῶ Ξ μὴ ὑπάρχειν: εἰ γὰρ παντὶ ὑπάρχει, κατηγορεῖται δὲ καὶ τὸ M παντὸς τοῦ N, ἀνάγκη τὸ M παντὶ τῶ Ξ ὑπάρχειν: ὑπέκειτο δὲ τινὶ μὴ ὑπάρχειν. (A5, 27a36-27b1)

- (Darapti) $(\forall S)P, (\forall S)R \vdash (\exists R)P$; (Felapton) $(\forall S)\neg P, (\forall S)R \vdash (\exists R)\neg P$;
 (Disamis) $(\exists S)P, (\forall S)R \vdash (\exists R)P$; (Datisi) $(\forall S)P, (\exists S)R \vdash (\exists R)P$;
 (Bocardo) $(\exists S)\neg P, (\forall S)R \vdash (\exists R)\neg P$; (Ferison) $(\forall S)\neg P, (\exists S)R \vdash (\exists R)\neg P$.

⊥

Proof. **(Darapti)** $(\forall S)P, (\forall S)R \vdash (\exists R)P$:

If they are universal, then, and when both P and R belong to every S, I say that P will belong to some R of necessity. For since the positive premiss converts, S will belong to some R, so that, since P belongs to all S and R to some S, it is necessary for P to belong to some R, for a syllogism in the first figure comes about.⁷²

$$\frac{(\forall S)P \quad \frac{(\forall S)R}{(\exists R)S} \text{ (a-i-conv)}}{(\exists R)P} \text{ (Darii)}$$

The demonstration can also be carried out through the impossible or by setting out [ecthesis]. For if both terms belong to all S, and one chooses one of the Ss, say N, then both P and R will belong to it, so that P will belong to some R.⁷³

$$\frac{\frac{(\forall S)P}{(\exists S)P} (\forall \rightarrow \exists) \quad \frac{\frac{(\forall S)P}{(\exists S)P} (\forall \rightarrow \exists) \quad \frac{(\forall N)S}{(\forall N)S} \text{ (ecthesis)}}{(\forall N)P} \text{ (ecthesis)}}{(\exists R)P} \quad \frac{\frac{(\forall S)R}{(\exists R)S} \text{ (a-i-conv)} \quad \frac{(\forall N)R}{(\exists R)N} \text{ (a-i-conv)}}{(\exists R)N} \text{ (Darii)}}{(\exists R)P} \text{ (Barbara)}$$

Note that this is a working proof, but it seems rather intricate compared to Aristotle's conclusion that 'both P and R will belong to it, so that P will belong to some R'; in particular, the phrase 'both P and R will belong to it' rather suggests the formalization ' $(qN)P$ ' and ' $(qN)R$ ' and not a mixed case as in the above proof. Taking ' q ' to be ' \exists ' clearly does not help here so that we have to read it as ' $(\forall N)P$ ' and ' $(\forall N)R$ '. But to conclude from this that ' $(\exists R)P$ ' is exactly what is supposed to be proved, viz., (Darapti) (cf. the discussion of ecthesis below the formulation of Theorem 12 as well as the one below Definition 19 of the derivation rules; see also *Malink 2013*, pp. 92ff.).

⁷²Καθόλου μὲν οὖν ὄντων, ὅταν καὶ τὸ Π καὶ τὸ Ρ παντὶ τῷ Σ ὑπάρχη, ὅτι τινὶ τῷ Ρ τὸ Π ὑπάρξει ἐξ ἀνάγκης: ἐπεὶ γὰρ ἀντιστρέφει τὸ κατηγορικόν, ὑπάρξει τὸ Σ τινὶ τῷ Ρ, ὥστ' ἐπεὶ τῷ μὲν Σ παντὶ τὸ Π, τῷ δὲ Π τινὶ τὸ Σ, ἀνάγκη τὸ Π τινὶ τῷ Ρ ὑπάρχειν: γίνεται γὰρ συλλογισμὸς διὰ τοῦ πρώτου σχήματος. (A6, 28a17-22)

⁷³ἔστι δὲ καὶ διὰ τοῦ ἀδυνάτου καὶ τῷ ἐκθέσθαι ποιεῖν τὴν ἀπόδειξιν: εἰ γὰρ ἄμφω παντὶ τῷ Σ ὑπάρχει, ἂν ληφθῇ τι τῷ Σ οἷον τὸ Ν, τούτῳ καὶ τὸ Π καὶ τὸ Ρ ὑπάρξει, ὥστε τινὶ τῷ Ρ τὸ Π ὑπάρξει. (A6, 28a22-26)

(Felapton) $(\forall S)\neg P, (\forall S)R \vdash (\exists R)\neg P$:

And if R belongs to all S and P belongs to none, there will be a syllogism to the effect that P will not belong to some R of necessity. For the demonstration can be carried out in the same way by converting the premiss RS.⁷⁴

$$\frac{(\forall S)\neg P \quad \frac{(\forall S)R}{(\exists R)S} \text{ (a-i-conv)}}{(\exists R)\neg P} \text{ (Ferio)}$$

This could also be proved through the impossible, as in the previous cases.⁷⁵

$$\frac{(\forall S)\neg P \quad \frac{[(\forall R)P]^u \quad (\forall S)R}{(\forall S)P} \text{ (Barbara)}}{(\exists R)\neg P} \frac{(\forall S)P}{(\exists S)P} \text{ (}\forall \rightarrow \exists\text{)} \text{ (RAA}_{\mathbf{a-o}}, u\text{)}$$

(Disamis) $(\exists S)P, (\forall S)R \vdash (\exists R)P$:

For if R belongs to all S and P to some, it is necessary for P to belong to some R. For since the affirmative premiss converts, S will belong to some P, so that since R belongs to all S and S to some P, R will belong to some P and hence P will belong to some R.⁷⁶

$$\frac{(\forall S)R \quad \frac{(\exists S)P}{(\exists P)S} \text{ (i-i-conv)}}{(\exists P)R} \text{ (Darii)} \frac{(\exists P)R}{(\exists R)P} \text{ (i-i-conv)}$$

This can also be demonstrated through the impossible and by setting out, as in the previous cases.⁷⁷

$$\frac{\frac{(\exists S)P}{(\forall N)P} \text{ (ecthesis)} \quad \frac{(\forall S)R \quad \frac{(\exists S)P}{(\forall N)S} \text{ (ecthesis)}}{(\forall N)R} \text{ (Barbara)}}{(\exists R)P} \frac{(\forall N)R}{(\exists R)N} \text{ (a-i-conv)} \text{ (Darii)}$$

(Datisi) $(\forall S)P, (\exists S)R \vdash (\exists R)P$:

⁷⁴καὶ ἂν τὸ μὲν P παντὶ τῷ Σ, τὸ δὲ Π μηδενὶ ὑπάρχη, ἔσται συλλογισμὸς ὅτι τὸ Π τινὶ τῷ P οὐχ ὑπάρξει ἐξ ἀνάγκης: ὁ γὰρ αὐτὸς τρόπος τῆς ἀποδείξεως ἀντιστραφείσης τῆς P Σ προτάσεως. (A6, 28a26-29)

⁷⁵δειχθεῖν δ' ἂν καὶ διὰ τοῦ ἀδυνάτου, καθάπερ ἐπὶ τῶν πρότερον. (A6, 28a29f.)

⁷⁶εἰ γὰρ τὸ μὲν P παντὶ τῷ Σ τὸ δὲ Π τινὶ, ἀνάγκη τὸ Π τινὶ τῷ P ὑπάρχειν. ἐπεὶ γὰρ ἀντιστρέφει τὸ καταφατικόν, ὑπάρξει τὸ Σ τινὶ τῷ Π, ὥστ' ἐπει τὸ μὲν P παντὶ τῷ Σ, τὸ δὲ Σ τινὶ τῷ Π, καὶ τὸ P τινὶ τῷ Π ὑπάρξει: ὥστε τὸ Π τινὶ τῷ P. (A6, 28b7-11)

⁷⁷ἔστι δ' ἀποδείξαι καὶ διὰ τοῦ ἀδυνάτου καὶ τῆ ἐκθέσει, καθάπερ ἐπὶ τῶν πρότερον. (A6, 28b14f.)

Again, if R belongs to some S and P to all S it is necessary for P to belong to some R, for the demonstration proceeds in the same way.⁷⁸

$$\frac{(\forall S)P \quad \frac{(\exists S)R}{(\exists R)S} \text{ (i-i-conv)}}{(\exists R)P} \text{ (DarII)}$$

This can also be demonstrated through the impossible and by setting out, as in the previous cases.⁷⁹

$$\frac{(\forall S)P \quad \frac{(\exists S)R}{(\forall N)S} \text{ (ecthesis)}}{(\forall N)P} \text{ (Barbara)} \quad \frac{(\exists S)R}{(\forall N)R} \text{ (ecthesis)}}{(\exists R)N} \text{ (a-i-conv)}}{(\exists R)P} \text{ (DarII)}$$

(Bocardo) $(\exists S)\neg P, (\forall S)R \vdash (\exists R)\neg P$:

For if R belongs to all S but P does not belong to some S, it is necessary for P not to belong to some R. For if it belongs to all R and R to all S, then P will also belong to all S; but it did not belong to all.⁸⁰

$$\frac{(\exists S)\neg P \quad \frac{[(\forall R)P]^u \quad (\forall S)R}{(\forall S)P} \text{ (Barbara)}}{(\exists R)\neg P} \text{ (RAA}_{\mathbf{a-o}}, u)$$

This can also be proved without the reduction to the impossible if one chooses one of the Ss to which P does not belong.⁸¹

$$\frac{\frac{(\exists S)\neg P}{(\forall T)S} \text{ (ecthesis)} \quad (\forall S)R \text{ (Barbara)}}{\frac{(\forall T)R}{(\exists R)T} \text{ (a-i-conv)}} \quad \frac{(\exists S)\neg P}{(\forall T)\neg P} \text{ (ecthesis)}}{(\exists R)\neg P} \text{ (Ferio)}$$

If we drop the non-emptiness requirement, we have to change the proof to account for the fact that $(ec(\mathbf{o}))$ had to be replaced by $(ec(\mathbf{o})^*)$. The required ecthesis looks now as follows:

$$\frac{(\exists S)\neg P \quad (\exists S)S}{(\forall T)S} \text{ (ecthesis)}$$

and similarly for the conclusion $(\forall T)\neg P$. That we can apply this rule to derive (Bocardo) can be seen by the following:

⁷⁸πάλιν εἰ τὸ μὲν P τινὶ τῶ Σ τὸ δὲ Π παντὶ ὑπάρξει, ἀνάγκη τὸ Π τινὶ τῶ P ὑπάρξει: ὁ γὰρ αὐτὸς τρόπος τῆς ἀποδείξεως. (A6, 28b11-14)

⁷⁹ἔστι δ' ἀποδείξειαι καὶ διὰ τοῦ ἀδυνάτου καὶ τῆ ἐκθέσει, καθάπερ ἐπὶ τῶν πρότερον. (A6, 28b14f.)

⁸⁰εἰ γὰρ τὸ P παντὶ τῶ Σ, τὸ δὲ Π τινὶ μὴ ὑπάρξει, ἀνάγκη τὸ Π τινὶ τῶ P μὴ ὑπάρχειν. εἰ γὰρ παντί, καὶ τὸ P παντὶ τῶ Σ, καὶ τὸ Π παντὶ τῶ Σ ὑπάρξει: ἀλλ' οὐχ ὑπῆρχεν. (A6, 28b17-20)

⁸¹δείκνυται δὲ καὶ ἄνευ τῆς ἀπαγωγῆς, ἐὰν ληφθῆ τι τῶν Σ ᾧ τὸ Π μὴ ὑπάρξει. (A6, 28b20f.)

$$\frac{\frac{\frac{(\forall S)R}{(\exists S)R} (\forall \rightarrow \exists)}{\frac{(\forall S)S}{(\exists S)S} (A-A^*)}}{(\exists S)\neg P} (\forall \rightarrow \exists)}{(\forall T)S} (\text{ecthesis})$$

(Ferison) $(\forall S)\neg P, (\exists S)R \vdash (\exists R)\neg P$:

For if P belongs to no S and R belongs to some S, P will not belong to some R, for there will be the first figure again when the premiss RS is converted.⁸²

$$\frac{(\forall S)\neg P \quad \frac{(\exists S)R}{(\exists R)S} (\text{i-i-conv})}{(\exists R)\neg P} (\text{Ferio})$$

□

As Aristotle notes himself, we can reduce the moods of the first figure to just **Barbara** and **Celarent** (A7, 29b1f.)⁸³. Since we have the appropriate reductio rules, QUARC_{AR} is also capable of establishing this result.

THEOREM 23 (Reduction of the Basic Rules)

(Barbara) and (Celarent) suffice as additional derivation rules. \dashv

Proof. Here, we can improve upon Aristotle's own reduction: 'Now those in the first figure – the particular ones – [...] [can also be proved] through the second figure by reduction to the impossible'⁸⁴ (A7, 29b6ff.). We can also reduce them right away to (Barbara) and (Celarent), where, in the case of (Darii), the proof just contains a proof of (Camestres). This is why I only state (Ferio):

(Ferio) $(\forall B)\neg A, (\exists C)B \vdash (\exists C)\neg A$:

$$\frac{(\exists C)B \quad \frac{\frac{(\forall B)\neg A}{(\forall A)\neg B} (\text{e-e-conv}) \quad [(\forall C)A]^u}{(\forall C)\neg B} (\text{Celarent})}{(\exists C)\neg A} (\text{RAA}_{\mathbf{a-o}}, u)$$

Aristotle's proofs are as follows:

(Darii): For example, if A belongs to every B and B to some C, then A belongs to some C. For if it belongs to none, but to every B, B will belong to no C; this we know from the second figure.⁸⁵

⁸²εἰ γὰρ τὸ Π μηδενὶ τῷ Σ, τὸ δὲ Ρ τινὶ ὑπάρξει τῷ Σ, τὸ Π τινὶ τῷ Ρ οὐχ ὑπάρξει: πάλιν γὰρ ἔσται τὸ πρῶτον σχῆμα τῆς Ρ Σ προτάσεως ἀντιστραφείσης. (A6, 28b33ff.)

⁸³But one can also reduce all syllogisms to the universal ones in the first figure [Ἔστι δὲ καὶ ἀναγαγεῖν πάντας τοὺς συλλογισμοὺς εἰς τοὺς ἐν τῷ πρώτῳ σχήματι καθόλου συλλογισμοὺς].

⁸⁴οἱ δ' ἐν τῷ πρώτῳ, οἱ κατὰ μέρος, ἐπιτελοῦνται μὲν καὶ δι' αὐτῶν, ἔστι δὲ καὶ διὰ τοῦ δευτέρου σχήματος δεικνύναι εἰς ἀδύνατον ἀπάγοντας.

⁸⁵οἷον εἰ τὸ Α παντὶ τῷ Β, τὸ δὲ Β τινὶ τῷ Γ, ὅτι τὸ Α τινὶ τῷ Γ: εἰ γὰρ μηδενί, τῷ δὲ Β παντί, οὐδενὶ τῷ Γ τὸ Β ὑπάρξει: τοῦτο γὰρ ἴσμεν διὰ τοῦ δευτέρου σχήματος. (A7, 29b8-11)

$$\frac{\frac{(\forall B)A \quad [(\forall C)\neg A]^u}{(\forall C)\neg B} \text{ (Camestres)} \quad (\exists C)B}{(\exists C)A} \text{ (RAA}_{\mathbf{e-i}}, u)$$

(Ferio): The demonstration will be similar in the case of the privative syllogism. For if A belongs to no B and B to some C, then A will not belong to some C. For if it belongs to every C but to no B, then B will belong to no C – this was the middle figure.⁸⁶

$$\frac{\frac{(\forall B)\neg A \quad [(\forall C)A]^u}{(\forall C)\neg B} \text{ (Cesare)} \quad (\exists C)B}{(\exists C)\neg A} \text{ (RAA}_{\mathbf{a-o}}, u)$$

□

3.4 Soundness & Completeness of QUARC_{AR}

As is to be expected, QUARC_{AR} is sound and complete; given the simplicity of the language \mathcal{L}_{AR} and its semantics, the proof is rather straightforward.

THEOREM 24 (Soundness and Completeness)

QUARC_{AR} is sound and complete: $\Phi \vdash \varphi$ iff. $\Phi \models \varphi$. ⊣

Proof. Soundness: The proof is an induction on the length of the derivation. The length of a derivation is the length of its longest branch (or any fitting definition will do).

Base Case: The Base Case is a derivation of length 1. The only rule that lead to such derivation is (A-A). Theorem 10 shows its soundness.

Induction Cases: Theorems 13 and 14 show the soundness of the (e-e-conv) and the moods of the first figure. This leaves us with $(\forall \rightarrow \exists)$ and the reductio rules (RAA_{a-o}, u), (RAA_{i-e}, u), (RAA_{e-i}, u) and (RAA_{o-a}, u). I will only prove the soundness of the first of these reductio rules; the others are proven similarly. The soundness of $(\forall \rightarrow \exists)$ follows from Lemma 9.

(RAA_{a-o}, u) Recall the rule:

$$\frac{\begin{array}{c} [(\forall A)B]^u \\ \vdots \\ \pm\varphi \quad \mp\varphi \end{array}}{(\exists A)\neg B}$$

Suppose we have a proof of $(\exists A)\neg B$ from premises $\pm\varphi$ and $(\forall A)B$. Suppose that $\mathfrak{A} \models \pm\varphi$ and $\mathfrak{A} \models (\forall A)B$. Then, by Induction Hypothesis (IH), $\mathfrak{A} \models \mp\varphi$. Since ‘ $\pm\varphi$ ’ and ‘ $\mp\varphi$ ’ are contradictories, they are either of the form ‘ $(\forall C)D$ ’-‘ $(\exists C)\neg D$ ’, or ‘ $(\exists C)D$ ’-‘ $(\forall C)\neg D$ ’. Thus, we have four cases.

⁸⁶ὁμοίως δὲ καὶ ἐπὶ τοῦ στερητικοῦ ἔσται ἡ ἀπόδειξις. εἰ γὰρ τὸ A μηδενὶ τῶ B, τὸ δὲ B τινὶ τῶ Γ ὑπάρξει, τὸ A τινὶ τῶ Γ οὐχ ὑπάρξει: εἰ γὰρ παντὶ, τῶ δὲ B μηδενὶ ὑπάρξει, οὐδενὶ τῶ Γ τὸ B ὑπάρξει: τοῦτο δ’ ἦν τὸ μέσον σχῆμα. (A7, 29b11-15)

- (1) Suppose that $\pm\varphi$ is $(\forall C)D$. Then, by (\mathbf{a}_-) , it follows from $\mathfrak{A} \models (\exists C)\neg D$ that $\mathfrak{A} \not\models (\forall C)D$, a contradiction.
- (2) Suppose that $\pm\varphi$ is $(\exists C)\neg D$. Then, by (\mathbf{o}_-) , it follows from $\mathfrak{A} \models (\forall C)D$ that $\mathfrak{A} \not\models (\exists C)\neg D$, a contradiction.
- (3) Suppose that $\pm\varphi$ is $(\forall C)\neg D$. Then, by (\mathbf{e}_-) , it follows from $\mathfrak{A} \models (\exists C)D$ that $\mathfrak{A} \not\models (\forall C)\neg D$, a contradiction.
- (4) Suppose that $\pm\varphi$ is $(\exists C)D$. Then, by (\mathbf{i}_-) , it follows from $\mathfrak{A} \models (\forall C)\neg D$ that $\mathfrak{A} \not\models (\exists C)D$, a contradiction.

Therefore, $\mathfrak{A} \not\models (\forall A)B$, so, by (\mathbf{a}_-) , $\mathfrak{A} \models (\exists A)\neg B$.

Note that, if incorporated, ecthesis is sound, too, because of $(\text{ec}(\mathbf{i}))$ and $(\text{ec}(\mathbf{o}))$ (and $(\text{ec}(\mathbf{o})^*)$, if the non-emptiness requirement is dropped).

Completeness: I will only indicate how to prove this. Most of the work is already done. We only have the following four formula types:

- (a) $(\forall A)B$ (i) $(\exists A)B$ (e) $(\forall A)\neg B$ (o) $(\exists A)\neg B$.

In the cases in which ‘ B ’ just is ‘ A ’, Theorem 10 shows the first two to be logically true, and we can show the other two to be logically false. These are also the only logical truths. Both are provable due to $(A-A)$ and $(\forall \rightarrow \exists)$.

If we have one premise, the conversion rules apply; by (e-e-conv) and Theorem 20, they are provable.

In the two premise case, the figures apply. Having (Barbara) and (Celarent) plus the other rules, we can derive all the figures (Theorems 21 and 22).

If there are more than two premises, we just iterate the above cases.

In a more formal manner, we can use Martin’s (1997) completeness proof here, too. Martin has two ‘notions of sentence’ (p. 5) where one includes formulas of the form ‘ $\mathbf{A}xx$ ’ which correspond to our ‘ $(\forall A)A$ ’. Further, he interprets terms ‘over non-empty sets’ as did we. In particular then, his semantics just is ours. The completeness of QUARC_{AR} follows then from the intertranslatability of it with Martin’s system. The translation is the obvious one where we identify Martin’s ‘ $\neg\mathbf{A}xy$ ’ with his ‘ $\mathbf{O}xy$ ’ (and similarly for the other cases). □

3.5 QUARC_{AR} and Classical Logic

To get more expressive power into QUARC_{AR} without tinkering with its presupposition that all sentences are of the form ‘ a ’, ‘ i ’, ‘ e ’, and ‘ o ’, we can be more liberal with what counts as a term. In particular, we can introduce rules to form complex terms out of simpler ones. In the following, it will be shown how this works and how much more expressive power we gain by doing so. An immediate consequence is also which part of classical logic QUARC_{AR} actually captures.

First, we have to adjust our underlying language.

DEFINITION 25 (\mathcal{L}_{AR}^*)

The *language* of QUARC_{AR}^{*} (\mathcal{L}_{AR}^*) consists of the following:

- the logical symbols ‘ \neg ’, ‘ \wedge ’, ‘ \vee ’, ‘ \rightarrow ’, ‘ \leftrightarrow ’, ‘ \forall ’, and ‘ \exists ’,
- the auxiliary symbols ‘(’ and ‘)’,
- and a set $Pred_{\mathcal{L}_{AR}^*}$ of unary predicate-symbols. ⊢

We can define the *simple* terms as before.

DEFINITION 26 (*Simple \mathcal{L}_{AR}^* -Terms*)

The set of *simple* \mathcal{L}_{AR}^* -terms is defined to be $Pred_{\mathcal{L}_{AR}^*}$. ⊢

These are the terms that we ensure have a non-empty extension. However, we extend the set of terms now as follows.

DEFINITION 27 (\mathcal{L}_{AR}^* -Terms)

The set of \mathcal{L}_{AR}^* -terms is recursively defined as follows:

- (T1) Every simple term is a term.
- (T2) If A is a term, then so is $(\neg A)$.
- (T3) If A and B are terms, then so are $(A \circ B)$ ($\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$).

Let $Term_{\mathcal{L}_{AR}^*}$ be the set of all \mathcal{L}_{AR}^* -terms. ⊢

The definition of \mathcal{L}_{AR}^* -formula is the same as before, but ranges now over a larger class of terms since $Term_{\mathcal{L}_{AR}} \subsetneq Term_{\mathcal{L}_{AR}^*}$.

DEFINITION 28 (\mathcal{L}_{AR}^* -Formulas/Sentences)

The set $Form_{\mathcal{L}_{AR}^*}$ of \mathcal{L}_{AR}^* -formulas/sentences is defined as follows:

- (F*) If A and B are \mathcal{L}_{AR}^* -terms, then $\ulcorner (\forall A)B \urcorner, \ulcorner (\exists A)B \urcorner, \ulcorner (\forall A)\neg B \urcorner$, and $\ulcorner (\exists A)\neg B \urcorner$ are \mathcal{L}_{AR}^* -formulas. ⊢

Compared to \mathcal{L}_{AR} -formulas, we allow now for formulas of the form ‘ $(qA)\neg\neg C$ ’ (well, strictly speaking, ‘ $(qA)\neg(\neg C)$ ’) since given that the term ‘ B ’ just happens to be ‘ $\neg C$ ’, there is nothing to rule this out as formula. However, strictly speaking, there are two different negations involved. The one is a term-building operation (the ‘ \neg ’ in ‘ $\neg C$ ’), the other not (the first occurrence of ‘ \neg ’ in ‘ $(qA)\neg\neg C$ ’). But since the derivation rules only apply to the latter (and only one such is allowed in a given formula), there arises no problem out of this ambiguity (see Lemma 30). Note, however, that to gain real expressive power, we need these two different negations (taking into account the sentence negation, we actually arrive at three). Intuitively speaking, an \mathcal{L}_{AR}^* -formula of the form ‘ $(\forall A)B$ ’ will correspond to a classical formula

of the form ‘ $\forall x(A(x) \rightarrow B(x))$ ’. But if the ‘ $B(x)$ ’ contains itself negations, e.g., if ‘ $B(x)$ ’ is of the form ‘ $\neg(C(x) \wedge \neg D(x))$ ’, we would have no \mathcal{L}_{AR} -term corresponding to it – but we do have an \mathcal{L}_{AR}^* -term.

Note, too, that even though bracketing does not make a difference in the case of non-empty terms (as witnessed by Lemma 30 below), it does make a difference if we drop this requirement. For, for example, ‘ $(\forall A)\neg B$ ’ and ‘ $(\forall A)(\neg B)$ ’ do come apart then; the former is an **e**-sentence, the latter an **a** one, and, according to (\mathbf{a}_+^*) , the latter is true only if ‘ A ’ is non-empty, whereas this is not the case for the former according to (\mathbf{e}_+) . The latter formula does imply the former, but not vice versa (see Lemma 30 and the footnotes there).

Since here we do not want to make any derivations about the terms themselves (such as $\frac{(\forall(A \vee B))C}{(\forall A)C}$ or $\frac{(\exists A)C}{(\exists(A \vee B))C}$), we do not need any extra rules for them.⁸⁷ However, we have to adjust the semantics to ensure that the complex terms get the correct extensions. To do so, we have to add new clauses:

DEFINITION 29 (*\mathcal{L}_{AR}^* -Structures*)

Let \mathcal{L}_{AR}^* be a language. An \mathcal{L}_{AR}^* -structure is a tuple $\mathfrak{A} = (D, (A^{\mathfrak{A}})_{A \in \text{Term}_{\mathcal{L}_{AR}^*}})$ with the following properties:

- (1) D is a set (the *universe*);
- (2) if $A \in \text{Term}_{\mathcal{L}_{AR}^*}$ is simple, then $A^{\mathfrak{A}}$ is a non-empty unary relation (i.e., predicate) on D , viz. $\emptyset \neq A^{\mathfrak{A}} \subseteq D$;⁸⁸
- (3) if $A \in \text{Term}_{\mathcal{L}_{AR}^*}$ is a complex term of the form ‘ $(\neg B)$ ’ for some $B \in \text{Term}_{\mathcal{L}_{AR}^*}$, then $A^{\mathfrak{A}} = D \setminus B^{\mathfrak{A}}$;
- (4) if $A \in \text{Term}_{\mathcal{L}_{AR}^*}$ is a complex term of the form ‘ $(B \wedge C)$ ’ (‘ $(B \vee C)$ ’) for some $B, C \in \text{Term}_{\mathcal{L}_{AR}^*}$, then $A^{\mathfrak{A}} = B^{\mathfrak{A}} \cap C^{\mathfrak{A}}$ ($A^{\mathfrak{A}} = B^{\mathfrak{A}} \cup C^{\mathfrak{A}}$). ⊣

The clauses involving ‘ \rightarrow ’ and ‘ \leftrightarrow ’ can be defined in terms of the given ones; it would have been sufficient to just give the clause for either ‘ \wedge ’ or ‘ \vee ’, too. With these additional terms and the way to interpret them, we can see that ecthesis properly follows now. In this sense, they are rather structural in nature since they concern the existence and the behaviour of sub-terms of given terms.

That the two different negations are no cause of vicious ambiguity is shown in the following Lemma 30.

LEMMA 30 (*Negation Agreement*)

Let $A, B, C \in \text{Term}_{\mathcal{L}_{AR}^*}$. Let C be of the form ‘ $\neg B$ ’. Then:

$$\mathfrak{A} \models (qA)\neg B \text{ iff. } \mathfrak{A} \models (qA)C. \quad \text{89} \quad \text{⊣}$$

⁸⁷This also means that we do not obtain a completeness result for QUARC_{AR}^{*} as it stands.

⁸⁸Dropping the non-emptiness requirement, we get instead:

(2*) if $A \in \text{Term}_{\mathcal{L}_{AR}^*}$ is simple, then $A^{\mathfrak{A}} \subseteq D$.

⁸⁹Dropping the non-emptiness requirement, we only get $(\forall A)C \models (\forall A)\neg B$ and $(\exists A)C \models (\exists A)\neg B$.

Proof. Let \mathfrak{A} be an \mathcal{L}_{AR}^* -structure. Note that, by Definition 29, $C^{\mathfrak{A}} = (\neg B)^{\mathfrak{A}}$. I will only prove the left-to-right directions for ‘ q ’ being ‘ \forall ’ and for ‘ q ’ being ‘ \exists ’ (since they don’t hold if we opt for (2*) instead of (2) in Definition 29). The other cases are proved analogously.

“ $\stackrel{\forall}{\Rightarrow}$ ”: Let $\mathfrak{A} \models (\forall A)\neg B$. Then, by (\mathbf{e}_+), (i) $\emptyset = A^{\mathfrak{A}} \cap B^{\mathfrak{A}}$. Further, $A^{\mathfrak{A}} = A^{\mathfrak{A}} \cap D \stackrel{D=B^{\mathfrak{A}} \cup (\neg B)^{\mathfrak{A}}}{=} A^{\mathfrak{A}} \cap (B^{\mathfrak{A}} \cup (\neg B)^{\mathfrak{A}}) = (A^{\mathfrak{A}} \cap B^{\mathfrak{A}}) \cup (A^{\mathfrak{A}} \cap (\neg B)^{\mathfrak{A}}) \stackrel{(i)}{=} A^{\mathfrak{A}} \cap (\neg B)^{\mathfrak{A}} \stackrel{C^{\mathfrak{A}} = (\neg B)^{\mathfrak{A}}}{=} A^{\mathfrak{A}} \cap C^{\mathfrak{A}}$. Therefore, by (\mathbf{a}_+), $\mathfrak{A} \models (\forall A)C$.⁹⁰

“ $\stackrel{\exists}{\Rightarrow}$ ”: Let $\mathfrak{A} \models (\exists A)\neg B$. Then, by (\mathbf{o}_+^*), $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \neq A^{\mathfrak{A}}$. Intersecting this with $C^{\mathfrak{A}}$, gets us: $A^{\mathfrak{A}} \cap C^{\mathfrak{A}} \neq A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \cap C^{\mathfrak{A}} \stackrel{C^{\mathfrak{A}} = (\neg B)^{\mathfrak{A}}}{=} A^{\mathfrak{A}} \cap B^{\mathfrak{A}} \cap (\neg B)^{\mathfrak{A}} \stackrel{B^{\mathfrak{A}} \cap (\neg B)^{\mathfrak{A}} = \emptyset}{=} A^{\mathfrak{A}} \cap \emptyset = \emptyset$. Therefore, by (\mathbf{i}_+), $\mathfrak{A} \models (\exists A)C$.⁹¹

□

In similar fashion to Definition 27, we can introduce predicate formation rules to the QUARC.⁹² For the proof theory to keep up, we have to introduce additional derivation rules ($\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$):

$$(C) \frac{(\cdot)P \circ (\cdot)Q}{(\cdot)(P \circ Q)}$$

where the double-line means that both directions (top to bottom and bottom to top) are covered. Introducing these formation and derivation rules for complex unary predicates leads to the intertranslatability of QUARC and classical logic + X , where the X is a set of axioms ensuring that every (simple) unary predicate has instances.⁹³ I will call the latter ‘ FOL_{\exists} ’.

QUARC with these additional predicates and rules that govern them is able to prove everything QUARC_{AR}^* can prove. What interests us now is the question how much of classical logic QUARC_{AR}^* captures.}}

To this end, recall the four sentence-types and their corresponding formalization:

$$\begin{array}{ll} \mathbf{a}: (\forall A)B \hat{=} \forall x(A(x) \rightarrow B(x)); & \mathbf{i}: (\exists A)B \hat{=} \exists x(A(x) \wedge B(x)); \\ \mathbf{e}: (\forall A)\neg B \hat{=} \forall x(A(x) \rightarrow \neg B(x)); & \mathbf{o}: (\exists A)\neg B \hat{=} \exists x(A(x) \wedge \neg B(x)). \end{array}$$

In the classical counterpart, however, formulas can have complex sub-formulas in place of ‘ $A(x)$ ’ and ‘ $B(x)$ ’. Introducing the complex terms to QUARC_{AR} leads to the validation of a part of them, viz., those formulas that are *Aristotelian*:}

⁹⁰Using (\mathbf{a}_+^*) and (\mathbf{o}_+^*) instead of (\mathbf{a}_+) and (\mathbf{o}_+), we can easily construct a countermodel: Let $A^{\mathfrak{A}} = \emptyset$. Then, $A^{\mathfrak{A}} \cap B^{\mathfrak{A}} = \emptyset$, i.e., by (\mathbf{e}_+), $\mathfrak{A} \models (\forall A)\neg B$. However, since $A^{\mathfrak{A}} = \emptyset$, by (\mathbf{o}_+^*), $\mathfrak{A} \models (\exists A)C$, so, by Lemma 9, $\mathfrak{A} \not\models (\forall A)C$.

⁹¹Using (\mathbf{o}_+^*) instead of (\mathbf{o}_+), we can construct a countermodel as before: Let $A^{\mathfrak{A}} = \emptyset$. Then, by (\mathbf{o}_+^*), $\mathfrak{A} \models (\exists A)\neg B$. But $A^{\mathfrak{A}} \cap C^{\mathfrak{A}} = \emptyset$ so that, by (\mathbf{e}_+), $\mathfrak{A} \models (\forall A)\neg C$ and, thus, by (\mathbf{i}_-), $\mathfrak{A} \not\models (\exists A)C$.

⁹²Note that none of the following works if we adopt (2*) instead of (2) in Definition 29. However, we could change QUARC in similar fashion and translate it to classical logic by adding the requirement of non-emptiness in the case of \mathbf{a} -statements, and translating \mathbf{o} -statements to the negation of the corresponding \mathbf{a} -statements.

⁹³See Raab 2016 and, for a similar result using different resources, Lanzet & Ben-Yami 2004.

DEFINITION 31 (Aristotelian Formulas)

A classical formula φ is called *Aristotelian*, if it satisfies the following:

- (1) φ does not contain n -ary predicates for every $n \geq 2$;
- (2) φ is of the form
 - (i) $\forall x\psi$ and ψ has as its main connective ‘ \rightarrow ’, or
 - (ii) $\exists x\theta$ and θ has as its main connective ‘ \wedge ’,

where ψ and θ are quantifier-free formulas whose subformulas all have ‘ x ’ as their only free variable and consist of unary predicates and connectives.

A set of classical formulas Φ is called *Aristotelian* iff. every $\varphi \in \Phi$ is Aristotelian. \dashv

Note that the definition is stated in a redundant fashion; I wanted to include the redundancy to explicitly specify that no proper relations are involved here. In particular, only relatively simple predicates such as ‘ $(A(x) \wedge B(x)) \vee (\neg A(x) \wedge C(x)) \vee (D(x) \rightarrow A(x))$ ’ are subformulas of Aristotelian formulas. With this, we can state and prove which part of classical logic QUARC_{AR}^{*} captures.

THEOREM 32

Let Φ be Aristotelian. Let $\Psi \subseteq \Phi$ and $\varphi \in \Phi$. Then:

$\Psi \models_{FOL_{\exists}} \varphi$ iff. $\Psi \models_{QUARC_{AR}^*} \varphi$. \dashv

We can prove this in different ways (where it is, of course, understood that ‘ $\Psi \models_{QUARC_{AR}^*} \varphi$ ’ concerns the obvious translation of the corresponding formulas). One way is to note that (full) QUARC and FOL_∃ are intertranslatable⁹⁴ so that it is enough to show the equivalence for QUARC and QUARC_{AR}^{*}. But given that the set of Aristotelian formulas reduces QUARC to the formulas of the right form, the result follows where the translation from the classical formula to QUARC looks as follows:

- (1) $\forall x(\psi(x) \rightarrow \theta(x))$ is translated to $(\forall\psi)\theta$.^{95,96}
- (2) $\exists x(\psi(x) \wedge \theta(x))$ is translated to $(\exists\psi)\theta$.

After the translation, ‘ ψ ’ and ‘ θ ’ are unary predicates which can be straightforwardly translated to \mathcal{L}_{AR}^* -terms.

On the other hand, we can reverse the translation to get FOL_∃-formulas from the corresponding QUARC-formulas in the same way.

All this also shows that QUARC_{AR} validates those Aristotelian formulas that are simple in the sense that in ‘ ψ ’ mentioned in Definition 31 the antecedent and consequent are (simple) unary predicates and similarly the conjuncts of the ‘ θ ’.

⁹⁴See Raab 2016. For a translation using a three-valued QUARC, see Lanzet 2017.

⁹⁵Strictly speaking, the general translation is ‘ $(\forall T_{\alpha})T \rightarrow \tau(\psi(x) \rightarrow \theta(x))[\alpha/x]$ ’ where $\tau : FOL_{\exists} \rightarrow QUARC$ is the translation-mapping and ‘ $[\alpha/x]$ ’ means that we replace all the occurrences of the variable ‘ x ’ by the anaphora ‘ α ’ (as soon as the recursive translation bottoms out). However, it can be shown that this is logically equivalent to the given translation.

⁹⁶Again, dropping the non-emptiness requirement for QUARC_{AR}, we have to translate ‘ $\forall x(\psi(x) \rightarrow \theta(x)) \wedge \exists x(\psi(x))$ ’ to ‘ $(\forall\psi)\theta$ ’.

However, to further increase the expressive power of QUARC_{AR} , we have to introduce more complex sentences. But these do not fall into either of the four sentence-types. For example, a sentence of the form $(\forall A)B \wedge (\exists C)\neg D$ has as constituents an **a** and an **o** sentence, but is itself neither **a**, **i**, **e**, or **o**. This means that such sentences are properly out of the scope of Aristotelian syllogistics.

4 Conclusion

In this paper, we have developed a logic (QUARC_{AR}) that is a faithful reconstruction of Aristotle's assertoric logic as presented in his *Prior Analytics*, chapters 1-7. The first step was to introduce the underlying language \mathcal{L}_{AR} that solely consists of the four Aristotelian sentence types. The formal representation thereof has been accomplished by adapting a part of the logic QUARC that allows, as the name suggests, quantified phrases in the argument position of predicates. In this way, the only logical symbols needed to indicate the corresponding sentence type are \forall , \exists , and \neg . In particular, there is no need for variables (x , y) and connectives (\wedge , \rightarrow) as in a representation based on classical logic.

As a consequence, the semantics of QUARC_{AR} does not have to 'go through' the instances of predicates to interpret the sentence types. All that is needed is the relationship of sets, viz., inclusion, overlap, exclusion, and non-inclusion. To further guarantee that the Aristotelian peculiarities such as (**a-i-conv**) follow, a simple adaptation in the definition of a structure was called for and sufficient, viz., that the predicates are assigned non-empty sets.

With this at hand, it is easy to prove valid all the moods that Aristotle singles out as valid. It has not been shown, but is easy enough, to give counterexamples to the moods of the figures that are not valid.

After developing the semantics, a calculus has been developed which is sound and complete with respect to the semantics. Within this calculus, it is possible to prove, once again, all the valid moods of the figures. In particular, it suffices to take Aristotle's own words to produce those proofs. Nothing of the sort of \wedge -introduction/elimination etc. is needed; we can simply follow Aristotle word by word.

There are, on the other hand, certain details of Aristotle's proofs or justifications of the principles he later uses. For example, one of Aristotle's proof techniques is proof by ecthesis. This has not been incorporated in QUARC_{AR} , but can easily be done (at least in $\text{QUARC}_{\text{AR}}^*$). One has to wonder why Aristotle thought it necessary to have this technique if it is not needed, as the proofs of Section 3 show. In particular, he only mentions at points that one can also prove certain things by ecthesis, but does not explicitly do so himself. The only essential use of it is (arguably) to justify certain other principles such as (**e-e-conv**).

One also has to note that this paper does not begin to capture all of Aristotle's logic. In particular, Aristotle's modal logic has not even been touched. It might be interesting to see whether an extension of QUARC_{AR} to a modal logic would be adequate for Aristotle's modal syllogistic.

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References

- Andrade, Edgar José, Becerra, Edward Samuel (2008). Establishing Connections between Aristotle's Natural Deduction and First-Order Logic. *History and Philosophy of Logic* 29(4), 309–325.
- Andrade-Lotero, Edgar, Dutilh Novaes, Catarina (2012). Validity, the Squeezing Argument and Alternative Semantic Systems: the Case of Aristotelian Syllogistic. *Journal of Philosophical Logic* 41(2), 387–418.
- Aristotelis (1964). *Analytica Priora et Posteriora*. Recensvit brevique adnotatione critica instrvxit W. D. Ross, Praefatione et appendice avxit L. Minio-Paluello. Oxford: Oxford University Pres.
- Aristotle (2009). *Prior Analytics, Book I*. Translated with an introduction and commentary by Gisela Striker. Oxford: Clarendon Press.
- Barnes, Jonathan (ed.) (1995: Vol. 1): *The Complete Works of Aristotle, The revised Oxford translation*. Volume one. Princeton: Princeton University Press.
- Barnes, Jonathan (ed.) (1995: Vol. 2): *The Complete Works of Aristotle, The revised Oxford translation*. Volume two. Princeton: Princeton University Press.

- Ben-Yami, Hanoch (2004). *Logic & Natural Language: On Plural Reference and Its Semantic and Logical Significance*. Aldershot/Burlington: Ashgate Publishing.
- Ben-Yami, Hanoch (2009a). Generalized Quantifiers, and Beyond. *Logique et Analyse* 52(208), 306–326.
- Ben-Yami, Hanoch (2009b). Plural Quantification Logic: A Critical Appraisal. *Review of Symbolic Logic* 2(1), 208–232.
- Ben-Yami, Hanoch (2012). Response to Westerståhl. *Logique et Analyse* 55(217), 47–55.
- Ben-Yami, Hanoch (2014). The Quantified Argument Calculus. *The Review of Symbolic Logic* 7, 120–146.
- Corcoran, John (1972). Completeness of an Ancient Logic. *The Journal of Symbolic Logic* 37, 696–702.
- Corcoran, John (1974a). Aristotle’s Natural Deduction System. In: J. Corcoran ed., *Ancient Logic and Its Modern Interpretations*, Dordrecht: Reidel, pp. 85–131.
- Corcoran, John (1974b). Aristotelian Syllogisms - Valid Arguments or True Universalized Conditionals. *Mind* 83(330), 278–281.
- Crivelli, Paolo (2012). *Aristotle’s Logic*. In: Shields (2012), 113–149.
- Gratzl, Norbert, Pavlović, Edi (ms). Proof-Theoretic Analysis of the Quantified Argument Calculus. Manuscript.
- Lanzet, Ran (2017). A Three-Vaues Quantified Argument Calculus: Domain-Free Model-Theory, Completeness, and Embedding of FOL. *The Review of Symbolic Logic* 10(3), 549–582.
- Lanzet, Ran, Ben-Yami, Hanoch (2004). Logical Inquiries into a New Formal System with Plural Reference. In Hendricks, V. F., e.a. (eds.) (2004). *First-Order Logic Revisited*. Berlin: Logos Verlag, 173–223.
- Łukasiewicz, Jan (1957). *Aristotle’s Syllogistic From the Standpoint of Modern Logic*. Second Edition, Enlarged. Oxford: Clarendon Press.
- Malink, Marko (2011b). *Syllogismos*. In: Rapp & Corcilius (2011), 343–348.
- Malink, Marko (2013). *Aristotle’s Modal Syllogistic*, Cambridge: Harvard University Press.
- Martin, John N. (1997). Aristotle’s Natural Deduction Reconsidered. In: Martin (2004), 1–17. Originally published in *History and Philosophy of Logic* 18(1), 1–15.
- Martin, John N. (2004). *Themes in Neoplatonic and Aristotelian Logic. Order, Negation and Abstraction*. Aldershot: Ashgate.
- Martin, John N. (2004a). Ecthesis and Existence in the Syllogistic. In: Martin (2004), 19–23.

- Moss, Lawrence S. (2015). Natural Logic. In Lappin, S., Fox, C. (eds.) (2015). *The Handbook of Contemporary Semantic Theory*. Second Edition. Oxford: Wiley Blackwell, 561–592.
- Oliver, Alex, Smiley, Timothy (2013). *Plural Logic*. Oxford: Oxford University Press.
- Parsons, Terence (2017). The Traditional Square of Opposition. *he Stanford Encyclopedia of Philosophy* (Summer 2017 Edition), Edward N. Zalta (ed.), URL = <<https://plato.stanford.edu/archives/sum2017/entries/square/>>.
- Patzig, Günther (1968). *Aristotle's Theory of The Syllogism. A Logico-Philological Study of Book A of the Prior Analytics*, Translated from the German by Jonathan Barnes. Dordrecht: Springer-Science+Business Media, B. V.
- Peters, Stanley, Westerståhl, Dag (2006). *Quantifiers in Language and Logic*. Oxford: Oxford University Press.
- Raab, Jonas (2016). *The Relationship of QUARC and Classical Logic*. Master Thesis at the Ludwig-Maximilians-Universität München. Available at URL = <http://www.academia.edu/31166336/The_Relationship_of_QUARC_and_Classical_Logic>.
- Rapp, Christof, Corcilius, Klaus (Hrsg.) (2011). *Aristoteles Handbuch. Leben-Werk-Wirkung*. Stuttgart/Weimar: Verlag J. B. Metzler.
- Read, Stephen (2015). Aristotle and Łukasiewicz on Existential Import. *Journal of the American Philosophical Association* 1(3), 535–544.
- Read, Stephen (ms). Aristotle's Theory of the Assertoric Syllogism. Manuscript (dated June 19, 2017). Available at URL = <<https://philpapers.org/archive/REAATO-4.pdf>>. (Last checked on March 19, 2018).
- Shields, Christopher (ed.) (2012). *The Oxford Handbook of Aristotle*. Oxford: Oxford University Press.
- Smiley, Timothy (1973). What is a Syllogism?. *Journal of Philosophical Logic* 2(1), 136–154.
- Smith, Robin (1982). What is Aristotelian Ecthesis?. *History and Philosophy of Logic* 3(2), 113–127.
- Smith, Robin (2018). Aristotle's Logic. *The Stanford Encyclopedia of Philosophy* (Spring 2018 Edition), Edward N. Zalta (ed.), URL = <<https://plato.stanford.edu/archives/spr2018/entries/aristotle-logic/>>.
- Troelstra, Anne Sjerp, Schwichtenberg, Helmut (2000). *Basic Proof Theory*. Second Edition. Cambridge: Cambridge University Press.
- Wedin, Michael V. (1978). Aristotle on the Existential Import of Singular Sentences. *Phronesis* 23(2), 179–196.
- Wedin, Michael V. (1990). Negation and Quantification in Aristotle. *History and Philosophy of Logic* 11(2), 131–150.

- Westerståhl, Dag (1989). Aristotelian Syllogisms and Generalized Quantifiers. *Studia Logica: An International Journal for Symbolic Logic* 48(4), 577–585.
- Westerståhl, Dag (2012). Explaining Quantifier Restriction: Reply to Ben-Yami. *Logique et Analyse* 55(217), 109–120.
- Westerståhl, Dag (2015). Generalized Quantifiers in Natural Language Semantics. In Lappin, S., Fox, C. (eds.) (2015). *The Handbook of Contemporary Semantic Theory*. Second Edition. Oxford: Wiley Blackwell, 9–39.
- Westerståhl, Dag (2016). Generalized Quantifiers. *The Stanford Encyclopedia of Philosophy* (Winter 2016 Edition), Edward N. Zalta (ed.), URL = <<https://plato.stanford.edu/archives/win2016/entries/generalized-quantifiers/>>.