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Gödel's Disjunction. The scope and limits of mathematical knowledge.
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Austrian-born Kurt Gödel is widely considered as the greatest logician of the modern times. It is above all his celebrated incompleteness theorems that have deserved for him this fame. These are rigorous mathematical results about the necessary limits of any formalized theory.

There have later been many attempts to draw ambitious philosophical consequences from these results (see Raatikainen 2005; Franzén 2005). The view that the human mind is, in some sense, equivalent to a finite computing machine, is commonly called 'mechanism'. Lucas (1962), for example, has argued that Gödel's results prove for good that human mind can surpass any computing machine, and that mechanism is false. However, his argument has remained controversial; many experts think it is simply flawed.

Gödel himself was quite cautious about drawing strong philosophical consequences from his results. However, he did later suggest a more careful philosophical conclusion with a disjunctive form. There are different formulations, but their common idea is the following:

(GD) Either human mind (even within the realm of pure mathematics) can surpass the power of any finite computing machine, or there are absolutely undecidable mathematical problems.

Gödel characterized this as 'mathematically established fact'. The epithet 'absolutely' here means that 'they would be undecidable, not just within some particular axiomatic system, but by any mathematical proof the human mind can conceive' (Godel 1951, p. 310). Furthermore, Gödel suggested that philosophical implications are, under either alternative, 'very decidedly opposed to materialistic philosophy' (*ibid*.).

The former conclusion is now widely known as 'Gödel's disjunction' (in short: GD), and hence the title of the book at issue. The volume aims to collect together the best up-to-date knowledge related to GD, and illuminate it from various perspectives. The book includes 10 original articles, and a substantial and very helpful introduction by the editors.

Dean seeks to analyze the metaphysical status of algorithms, and in particular the view he calls "algorithmic realism"; this is the view according to which algorithms can be considered as mathematical objects. Though popular, Dean finds this position in the end problematic. The opposite of algorithmic computability is randomness. Moschovakis, in her chapter, provides a survey of the notions of randomness and lawless sequences.

Gödel's second incompleteness theorem concerns, roughly, the unprovability of theory's consistency in the theory itself. It is quite well-known that there are certain technical obstacles for proving a fully general result. The chapter by Visser is a valuable survey of certain more recent research, by e.g. Friedman, Pudlak, Smorynski, and Visser himself, revolving around the second incompleteness theorem and aiming at more general results. This work is not yet widely known, and consequently, this review fills a gap in the literature.

Leach-Krouse compares Gödel's views to those of another pioneer in the field, American logician Emil Post, who achieved independently results similar to Gödel's. These two figures both had interesting philosophical views about mechanism and absolute unprovability, though sometimes pulling to the opposite directions. Leach-Krouse's discussion is not, however, purely historical, but has a systematic aspect too.

The notion of an absolutely undecidable problem, in GD, is grounded on the concept of provability-in-principle, or absolute provability. *Epistemic Arithmetic* is a formal framework, initiated by Shapiro (1985) and Reinhardt (1986), in which a formal arithmetical theory is enriched with a modal operator whose intended interpretation is *absolute provability*. There has been some debate about the usefulness of this approach (see Horsten 1998), but it has proved to be quite a fruitful formal tool for rigorous study of issues such as GD. The chapters by Carlson and by Antonutti and Horsten are contributions to this research program; they examine in particular issues around the so-called Epistemic Church's Thesis (ECT). Carlson goes on to study what he calls 'knowing machines'. Antonutti and Horsten demonstrate an analogue, formulated in terms of ECT, of GD in this context. Also the chapter by Achourioti is related to this general program, and studies an alternative semantic interpretation of the modal operator of absolute provability.

More traditionally, it has been common to take for granted that the notion of absolute provability is sufficiently well-understood. However, Shapiro, Williamson and Koellner all raise, in different but complementing ways, doubts about the very clarity of the concept. Shapiro's contribution is a compact and very clear systematic discussion of GD. He points out that any systematic discussion of mechanism and Gödel's theorems must presuppose a certain amount of idealization. However, Shapiro argues that it is doubtful that there would be any sufficiently sharp and stable notion of idealized human knowability that would support a Gödelian anti-mechanist argument. Williamson argues that we are, with the notion of absolute provability, in a slippery slope: that there is no principled stopping point, and that in closer inspection, any mathematical truth is provable-in-principle.

However, it is Koellner's chapter, above all, which is, at least in reviewer's mind, the true gem of the collection, and really takes the discussion concerning GD to a wholly new level. Gödel himself believed that if we had an adequate paradox-free theory of truth, it would be possible to demonstrate the first anti-mechanist disjunct – he did not consider the existence of absolutely unsolvable problems as plausible. Accordingly, Koellner tries different formal theories of truth: both the Tarskian approach involving a hierarchy of languages, and a hierarchy-free Kripke-Feferman style approach. Building on the earlier work of Reinhardt, and assuming, for the sake of argument, that the notion of absolute provability is well-defined, Koellner shows that GD can be rigorously demonstrated in the setting of Epistemic Arithmetic. However, he also shows that the prospects of demonstrating either disjunct are dim. Koellner also puts forward the possibility that, under some interpretations, both disjuncts of GD may be separately 'absolutely undecidable'.

It is perhaps inevitable that any collection of this kind is at least a bit uneven. All in all, however, this book is a major contribution to this timely topic, and obligatory reading for anyone interested in issues related to GD.

errata

p. 5: ‘Liar sentence which says of itself that it is true’ – should read: ‘…says of itself that it is false’ (or ‘… is not true’).

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