THE CATEGORY OF MEREOTOPOLOGY AND ITS ONTOLOGICAL CONSEQUENCES

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ABSTRACT. We introduce the category of mereotopology **Mtop** as an alternative category to that of topology **Top**, stating ontological consequences throughout. We consider entities such as boundaries utilizing Brentano's thesis and holes utilizing homotopy theory. Lastly, we mention further areas of study in this category.

CONTENTS

1. INTRODUCTION

The category of mereotopology, though not nearly as developed as topology, has often been of preference to those interested in formal ontology. Where ontology is traditionally defined as the science which deals with nature and the organization of reality, formal ontology deals with formal structures and relations in reality as they are governed in all material domains. This contrasts material ontologies (such

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as chemistry, biology, medicine, etc.) which study the nature and organization of certain sub-regions of reality [4].

The basis for mereotopology is mereology which is a formal theory of parthood. It was first introduced in Husserl's *Logical Investigations*; previously, it had received attention from those such as Plato, Aristotle, Aquinas, Leibniz, and Kant. Specifically, it has proven helpful for disciplines such as natural-language analysis and artificial intelligence where more of an ontological motivation is desired. With the addition of topology, we derive mereotopology and become able to speak of part-whole relations. We, thus, become able to better understand the *a priori* nature of boundaries and holes, and we try to apply this to the questions philosophers and ontologists have been asking about these entities. Is a boundary an independent being? Do we view holes as immaterial particulars or spatiotemporal particulars? How do we address issues of genus? In this paper, we rigorously develop Brentano's thesis, and we introduce homotopy from algebraic topology to further develop the formal ontology of holes. One advantage of this is that we may bypass adopting a predicate 'H' (where 'H' represents the attribute of having a hole), and we may import knowledge of group theory which is already substantially developed to the category of **Mtop**.

The organization of this paper is as follows: we begin by stating the [**G**(**E**)**M**] axiomatic schema in mereology **Mer** and how it is different from ZFC. From there, we introduce the remainder of [**G**(**E**)**M**]**TC**. We mention the rigorous formalization of Brentano's thesis as an important ontological work done in this category. Then, we define a continuous mereological morphism, and we show that Hausdorff topologies are indeed mereotopologies. Lastly, we approach the ontological problem of holes using homotopy theory and mention limitations and areas of further investigation in mereotopology.

2. Mereology

2.1. [**G**(**E**)**M**].

Definition 2.1. The theory of ground mereology (often shortened to [**M**]) concerns the binary predicate P (called "parthood": $\mathrm{P}xy$ is read as "x is a part of y") with the following three axioms:

(P1) $\mathrm{P}xx$, (reflexivity)
(P2) $\mathrm{P}xy \wedge \mathrm{P}yx \rightarrow x = y$, (antisymmetry)
(P3) $\mathrm{P}xy \wedge \mathrm{P}yz \rightarrow \mathrm{P}xz$. (transitivity)

Remark 2.2. . Note the similarity to set theory. Set theory could be defined as the theory surrounding one binary predicate $\in$, where $x \in X$ is read as "x is an element of X," satisfying the axioms of one's choice, quite frequently ZFC. The major difference with mereology is reflexivity: in set theory, $X \in X$ is quite infrequently true.

Though there are some complaints concerning the axioms of ground mereology, these have been, for the most part, dismissed [3]. For example, a common complaint to (P3) is the notion that "the handle is a part of the door" and "the door is a part of the house," yet "the handle is not a part of the house." What we are dealing with here is not parthood in the primitive sense. Instead, we have imposed another condition on parthood. Specifically, we imposed being a"functional part." A part having "function" is, thus, closer to an open formula ϕ (Here, $\phi(x)$ is any open formula in one variable x, $\phi(x, y)$ is the open formula in two variables x, y, and so on. Generally, we say ϕ to mean that the sentence 'ϕ' is true for at least one object x), and no such claim is made that ϕ is transitive.

Thus, the following may or may not hold:

(M1) $(\mathrm{P}xy \wedge \phi(x,y)) \wedge (\mathrm{P}yz \wedge \phi(y,z)) \to (\mathrm{P}xz \wedge \phi(x,z))$

Now that the parthood predicate 'P' is defined, we may define other mereological relations which will be useful to us in the future as follows:

(M2) $\mathrm{PP}xy := \mathrm{P}xy \wedge \neg\mathrm{P}yx$, ($x$ is a proper part of y)
(M3) $\mathrm{O}xy := \exists z(\mathrm{P}zx \wedge \mathrm{P}zy)$, ($x$ overlaps y)
(M4) $\mathrm{U}xy := \exists z(\mathrm{P}xz \wedge \mathrm{P}yz)$, ($x$ underlaps y)
(M5) $\mathrm{D}xy := \neg\mathrm{O}xy$. ($x$ is discrete from y)

Definition 2.3. The theory of extensional mereology [(**E**)**M**] extends [**M**] with the supplementation axiom.

(P4) $\neg\mathrm{P}xy \to \exists z(\mathrm{P}zx \wedge \neg\mathrm{O}zy)$

This is called "strong supplementation," and it allows us to derive another property "weak supplementation" as follows:

(M6) $\mathrm{PP}xy \to \exists z(\mathrm{PP}zy \wedge \neg\mathrm{O}zx)$.

Analogous to extensionality in set theory, strong supplementation allows us to identify when two objects are equal. Two objects would be *equal* in this strong sense if they (1) have the same parts, and (2) are parts of the same objects. Suppose x and y have the same parts. Now, suppose $\neg\mathrm{P}xz \wedge \mathrm{P}yz$. Then, by (P4), we have $\exists w(\mathrm{P}wx \wedge \neg\mathrm{O}wz)$. Thus, by assumption of x and y having the same parts, we have $\mathrm{P}wy$. Then, by (P3), we have $\mathrm{P}wz$, but this is a contradiction of $\neg\mathrm{O}wz$. Therefore, we cannot have both $\neg\mathrm{P}xz \wedge \mathrm{P}yz$, and thus x and y must be part of the same things. Therefore, (P4) tells us that $x = y$ if and only if x and y have the same parts.

Definition 2.4. The theory of closed (extensional) mereology [**C**(**E**)**M**] extends [(**E**)**M**] with the following axioms:

(P5) $\mathrm{U}xy \to \exists z\forall w(\mathrm{O}wz \leftrightarrow (\mathrm{O}wx \vee \mathrm{O}wy))$,
(P6) $\mathrm{O}xy \to \exists z\forall w(\mathrm{P}wz \leftrightarrow (\mathrm{P}wx \wedge \mathrm{P}wy))$,
(P7) $\exists z((\mathrm{P}zx \wedge \neg\mathrm{O}zy) \to \forall w(\mathrm{P}wz \leftrightarrow (\mathrm{P}wx \wedge \neg\mathrm{O}wy)))$.

These three axioms give us what we call sum, product, and difference in mereology. These objects are analogous to union, intersection, and set difference in set theory. However, note that a sum or product only exists when there is an already existing underlap or overlap respectively. Where 'ι' is a description operator for a given language, we have the following:

(M7) $x + y := \iota z\forall w(\mathrm{O}wz \leftrightarrow (\mathrm{O}wx \vee \mathrm{O}wy))$, (Sum)
(M8) $x \times y := \iota z\forall w(\mathrm{P}wz \leftrightarrow (\mathrm{P}wx \wedge \mathrm{P}wy))$, (Product)
(M9) $x - y := \iota z\forall w\ (\mathrm{P}wz \leftrightarrow (\mathrm{P}wx \wedge \neg\mathrm{O}wy))$ (Difference)

Thus, we can restate (P5), (P6), and (P7) as follows:
(P5') $\mathrm{U}xy \rightarrow \exists z(z = x + y)$
(P6') $\mathrm{O}xy \rightarrow \exists z(z = x \times y)$
(P7') $\exists z(\mathrm{P}zx \wedge \neg \mathrm{O}zy) \rightarrow \exists z(z = x - y)$
If we want to be able to, for example, sum arbitrarily many parts, we extend [**C**(**E**)**M**] to [**G**(**E**)**M**] or general (extensional) mereology.

Definition 2.5. The theory of general (extensional) mereology [**G**(**E**)**M**] extends [(**E**)**M**] with the fusion axiom:
(P8) $\exists x\phi \rightarrow \exists z \forall y(\mathrm{O}yz \leftrightarrow \exists x(\phi \wedge \mathrm{O}yx))$

Thus, we can define sums and products in [**G**(**E**)**M**] as follows:
(M10) $\sigma x\phi := \iota z \forall y(\mathrm{O}yz \leftrightarrow \exists x(\phi \wedge\ \mathrm{O}yx))$,
(M11) $\pi x\phi := \sigma z \forall x(\phi \rightarrow \mathrm{P}zx)$.
From here, we may reformulate (P9) as follows:
(P8') $\exists x\phi \rightarrow \exists z(z = \sigma x\phi)$.
This, then yields,
(M12) $\exists x\phi \wedge \exists y \forall x(\phi \rightarrow \mathrm{P}yx) \rightarrow \exists z(z = \pi x\phi)$.
Thus, we have the following definitional equivalences in [**G**(**E**)**M**]:
(M13) $x + y = \sigma z(\mathrm{P}zx \vee\ \mathrm{P}zy)$,
(M14) $x \times y = \sigma z(\mathrm{P}zx \vee\ \mathrm{P}zy)$,
(M15) $x - y = \sigma z(\mathrm{P}zx \wedge \neg\ \mathrm{O}zy)$,
(M16) $x^C = \sigma z(\neg \mathrm{O}zx)$,
(M17) $U = \sigma z(\mathrm{P}zz)$.
We may use these notions to prove the remainder principle:
(M18) $\mathrm{P}xy \wedge x \neq y \rightarrow \exists z(z = y - x)$.

Often the fusion axiom (which states that for any arbitrary number of parts, there exists a sum) is contested as not representing how we define objects colloquially. It brings into question whether there is a such a thing that consists of just my right foot and my left elbow. These types of questions have led some to restrict summations to those objects which are connected. However, an object such as a bikini consists of two disconnected parts, yet it is treated as an individual in our everyday language. The subtlety of the fusion axiom is that there are always summations of arbitrary objects such as my right foot and my left elbow, but these summations are only named as one if they are useful for us to speak about. For example, it is useful for us to speak about a bikini in terms of one object consisting of two disconnected parts. Thus, these hesitations remain at the level of language and raise no serious ontological concern.

Furthermore, we note that there is a semantic difference between $\sigma x\phi$ and simply the extension of ϕ. That is, the sum of some flowers is a bouquet while the bouquet itself is not a flower.

Lastly, we note here that the fusion axiom is stated as a conditional and that the sum itself is unique (which is obvious from (**E**)). In mereology, empty sums do not exist. That is, if ϕ is not satisfied, then $\sigma x\phi$ is undefined. The is precisely because empty sums are not a part of reality.

This then gives rise to the notion of a universe and a complement as follows:

(M19) $\mathcal{U} := \iota z \forall x(\mathrm{P}xz)$,
(M20) $x^C := \mathcal{U} - x$.

Lastly, we notice that we have not assumed atomicity in this formulation as follows:

(M21) $\forall x \exists y \mathrm{PP}yx$.

2.2. **Differences to ZFC.** In comparison to set theory, mereology operates with the parthood predicate 'P.' whereas set theory operates with the set membership predicate '$\in$'. In set theory, a morphism (called a function) maps an element of the domain to an element of the codomain, retaining the set membership predicate. In short, we have $f : X \to Y$ is a set morphism if for each $x \in X$, we associate $f(x) \in Y$. Note, it is helpful to think of parthood as analogous to $\subseteq$ in set theory. Note there is no analogy to '$\in$,' in mereology.

Since we do not operate with this predicate in mereology [**G**(**E**)**M**], we thus have the following definition for a morphism in this category:

Definition 2.6. A *morphism* in mereology is a map from the domain X to codomain Y such that for every part $\mathrm{P}zX$, we associate a part $\mathrm{P}f(z)Y$, and if $\mathrm{P}zw$ with $\mathrm{P}zX$ and $\mathrm{P}wX$, then $\mathrm{P}f(z)f(w)$.

Definition 2.7. An *isomorphism* in mereology is a morphism $f : X \to Y$ such that there exists a morphism $g : Y \to X$ such that $g \circ f = h : X \to X$ and $f \circ g = k : Y \to Y$ are both respectively identity maps, i.e., such that $h(x) = x$ and $k(y) = y$ for all $\mathrm{P}xX$ and $\mathrm{P}yY$.

Secondly, Russell's paradox which posed problems for early set theory does not occur in mereology. Russell's paradox is as follows:

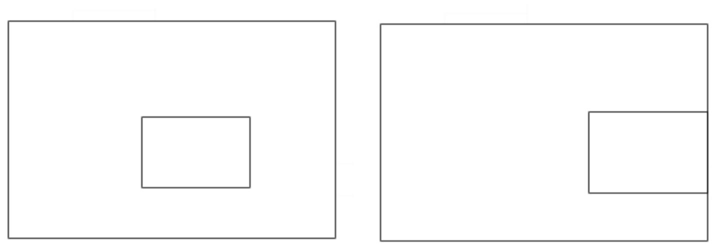
$$\text{Let } R = \{x : x \notin x\}. \text{ Then, } R \in R \leftrightarrow R \notin R.$$

Simply, because we have reflexivity in mereology, we avoid this problem all together.

Lastly, in mereology we have a top but no bottom while in standard set-theory we have the exact opposite.

3. Mereotopology

3.1. [**G**(**E**)**M**]**TC.** Mereology by itself limits us to a theory of parts. For example, mereology does not give us the sufficient language to speak about the distinction between these two parts.



It is clear that though both of these objects are parts of some whole, they are different kinds of parts (specifically, one is a *tangential part* and the other in an *interior part*). Thus, in order to speak more about part-whole relations, we adopt topology. We begin to do so by adding the connection predicate 'C,' understood intuitively as topological connection. We assume that 'C' is reflexive, symmetric, and monotonic with respect to 'P,' giving us ground topology [**T**].

Definition 3.1. The theory of ground topology [**T**] defines the following axioms in relation to the connection predicate 'C.' Adding this theory to [**G**(**E**)**M**] give us [**G**(**E**)**M**]**T**:
(C1) $\mathrm{C}xx$,
(C2) $\mathrm{C}xy \to \mathrm{C}yx$,
(C3) $\mathrm{P}xy \to \forall z(\mathrm{C}zx \to \mathrm{C}zy)$.

From here, we may define other relations as follows:
(MT1) $\mathrm{EC}xy := \mathrm{C}xy \wedge \neg \mathrm{O}xy$, (External Connection)
(MT2) $\mathrm{TP}xy := \mathrm{P}xy \wedge \exists z(\mathrm{EC}zx \wedge \mathrm{EC}zy)$, (Tangential Part)
(MT3) $\mathrm{IP}xy := \mathrm{P}xy \wedge \neg \mathrm{TP}xy$. (Internal Part)
Now, we may define other quasi-topological operators as follows:
(MT4) $\mathrm{i}x := \sigma z\ \mathrm{IP}zx$, (interior)
(MT5) $\mathrm{e}x := \mathrm{i}(x^C)$, (exterior)
(MT6) $\mathrm{c}x := (\mathrm{e}x)^C$, (closure)
(MT7) $\mathrm{b}x := (\mathrm{ix}+\mathrm{ex})^C$. (boundary)
Now, we introduce the language of a self-connected whole as an object which cannot be split into two or more disconnected parts. Notice that this appears similar to the notion of connectedness in topology.
(MT8) $\mathrm{SC}x := \forall y \forall z(x = y + z \to \mathrm{C}yz)$
We can also distinguish between open and closed individuals as follows:
(MT9) $\mathrm{Op}x := x = \mathrm{i}x$ (open),
(MT10) $\mathrm{Cl}x := x = \mathrm{c}x$ (closed).
To receive the full strength of [**G**(**E**)**M**]**TC**, we introduce closure conditions.

Definition 3.2. The theory of [**G**(**E**)**M**]**TC** extends [**G**(**E**)**M**]**T** with closure conditions through the following axioms:
(C4) $\mathrm{Cl}x \wedge \mathrm{Cl}y \to \mathrm{Cl}(x + y)$,
(C5) $\forall x(\phi \to \mathrm{Cl}x) \to (z = \pi x\phi \to \mathrm{Cl}z)$.

Here, we notice that (C5) is given as a conditional because we do not assume a null object.

From here, we may also derive axioms similar to the standard Kuratowski axioms for topological closure. The proofs of these are also identical.
(MT11) $\mathrm{P}x\mathrm{c}x$
(MT12) $\mathrm{c}(\mathrm{c}x) = \mathrm{c}x$
(MT13) $\mathrm{c}(x + y) = \mathrm{c}x + \mathrm{c}y$
(MT14) $\mathrm{P}(\mathrm{i}x)x$
(MT15) $\mathrm{i}(\mathrm{i}x) + \mathrm{i}x$
(MT16) $\mathrm{i}(x \times y) = \mathrm{i}x \times i\mathrm{y}$

(MT17) $\mathrm{b}x = \mathrm{b}(x^C)$
(MT18) $\mathrm{b}(\mathrm{b}x) = \mathrm{b}x$
(MT19) $\mathrm{b}(x \times y) + \mathrm{b}(x + y) = \mathrm{b}x + \mathrm{b}y$

4. Brentano's thesis

4.1. **Boundaries.** One important advancement done in mereotopology is the rigorous formulation of Brentano's thesis which will be explained here. Brentano's Thesis is the ontological notion that a 'boundary' can exist as a matter of necessity only as part of a whole of higher 'dimension' of which it is the boundary. This notion is commonly accepted as the ontological nature of a boundary, and philosophers find benefit in formalizing it through a mereotopological approach rather than a point-set topological one [4]. To quote Smith, "the set-theoretic conception of boundaries [are], effectively, sets of points, each of which can exist though all around it be annihilated" [4]. Whether this motivation is accurate, mereotopology is still the category most ontologists (and those working in AI or cognitive linguistics) work in, and it is beneficial to communicate this notion in this category. Here, we follow Smith in choosing to rigorously formulate a simpler version of Brentano's thesis which does not assume the existence of higher dimensions: every boundary is such that we can find an entity which it bounds, which it is a part of, and which has interior parts. The motivation for such a thing is that we would like to speak of objects such as boundaries in a more ontologically sound manner, formalizing our psychological intuitions of these objects.

In order to do this, we first define crosses 'X.' We define 'Xxy' to be read as 'x' crosses 'y'.

(B1) $\mathrm{X}xy := \neg \mathrm{P}xy \wedge \mathrm{O}xy$

The idea here is that x overlaps with both y and the complement of y. Thus, there exists no entity that crosses itself, and the universe crosses every entity not identical with the universe itself. From here, we define straddles 'ST.'

(B2) $\mathrm{ST}xy := \forall z(\mathrm{IP}xz \rightarrow \mathrm{X}zy)$

Thus, an entity x straddles an entity y whenever every entity of which x is an internal part of crosses y. From here we have the following:

(B3) $\mathrm{ST}xy \rightarrow \neg \mathrm{IP}xy$,

(B4) $\mathrm{P}xy \rightarrow \mathrm{IP}xy \vee \mathrm{ST}xy$.

Note that every part of y is either an internal part of y or straddles y.

Several philosophers have recognized that when we intuitively think about boundaries, there appear to be two different types of boundaries [4]. There are what we will call 'tangents' which include among their parts a 'boundary' of the straddled entity, and there are 'non-tangents' which are not connected and include no such 'boundary'. Then, x does not simply straddle y, but it is a 'boundary' of y. Earlier we defined boundary in (MT17), but here we define the predicates 'B' (where '$\mathrm{B}xy$ is read as 'x is the boundary of y') and tangent 'T' (where $\mathrm{T}xy$ is read as 'x is a tangent of y) as follows:

(B5) $\mathrm{B}xy := \forall z(\mathrm{P}zx \rightarrow \mathrm{ST}zy)$,

(B6) $\mathrm{T}xy := \exists z(\mathrm{P}zy \wedge \mathrm{B}zy)$.

These definitions give rise to the notion that all parts of the boundary of an entity y are not merely straddlers but tangents of y:

(B7) $\mathrm{B}xy \leftrightarrow \forall z(\mathrm{P}zx \to \mathrm{T}zy)$.

Similar to our closure axioms, we have the following properties for boundary:

(B8) $\mathrm{B}xy \wedge \mathrm{B}yz \to \mathrm{B}xz$, (transitivity)

(B9) $\mathrm{P}xy \wedge \mathrm{B}yz \to \mathrm{B}xz$,

(B10) $\mathrm{T}x(y+z) \to \mathrm{T}xy \vee \mathrm{T}xz$. (splitting)

Moreover, we have the following collection principle:

$\forall x(\phi x \to \mathrm{B}xy) \to \sigma x \mathrm{B}(\phi x)y$.

Lastly, we expand on our earlier mention of the predicate 'b' as the predicate 'is a boundary' as follows:

(B11) $\mathrm{b}(x) := \exists y(\mathrm{B}xy)$.

4.2. **Brentanian formulation.** Then, the first Brentanian Thesis is as follows:

$\mathrm{b}(x) \to \exists z \exists t(\mathrm{B}xz \wedge \mathrm{P}xz \wedge \mathrm{IP}tz)$.

However, we want our formulation to capture connectedness such that for connected boundaries, there exist connected wholes of which they bound.

$\mathrm{b}(x)\mathrm{SC}(x) \to \exists z \exists t(\mathrm{P}xz \wedge \mathrm{B}xz \wedge \mathrm{C}(z) \wedge \mathrm{IP}tz)$

Though we have not done it here, we would like to strengthen this formulation by accounting for the intuitive notion that a boundary is somehow associated with a *thing* that is bounds. That is, our formulation of boundary behaves the same way when we consider it around the *thing* it bounds and that *thing*'s complement (since $\mathrm{b}x = \mathrm{b}x^C$). What we would like is to have a formulation which allows for us to capture the difference between the boundary of a stone and the boundary of everything in the universe minus the stone. We also notice that such *things* are not restricted to spacial entities. Indeed, temporal entities such as events, seasons, or entire lives are also *things* of which we may want to define boundary and other mereotopological properties.

5. Topology

Now, we may begin to show that Hausdorff spaces are indeed mereotopological spaces. Previously, it has been shown that $[\mathbf{G}(\mathbf{E})\mathbf{M}]$ is a 'larger' category than **Set** [7] (Note that in order for this to have been done, it must be the case that $P\varnothing A$ is never true, for any A). This means that there exists a "functor" from **Set** to $[\mathbf{G}(\mathbf{E})\mathbf{M}]$ that is injective but not surjective.

Definition 5.1. Let C_1 and C_2 be categories (A category C is defined as that which consists of a class $\mathrm{ob}(C)$ of objects, a class $\hom(C)$ of morphisms between these objects, and for every three objects a, b and c, a binary operation $\hom(a,b) \times \hom(b,c) \to \hom(a,c)$ called a composition of morphisms. Moreover, associativity holds, and an identity exists.) A *functor* F from C_1 and C_2 is then, a mapping which associates to each object X in C_1 an object $F(X)$ in C_2 and associates each morphism $f : X \to Y$ in C_1 a morphism $F(f) : F(X) \to F(Y)$ in C_2 such that $F(\mathrm{id}_X) = \mathrm{id}_{F(x)}$ for every object X in C_1 and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$.

Thus, functors allow us to compare categories. Here, we show that Hausdorff topological space indeed satisfies the necessary axioms of a mereotopological space. To do this, we show the existence of a *functor* from the category of Hausdorff topological spaces to mereotopological spaces by associating each object and morphism that satisfies the axioms of Hausdorff topologies with an object and morphism that satisfies the axioms of mereotopologies respectively. First, however, we begin by introducing the category topology **Top**.

Definition 5.2. A *topology* on a set X is a collection τ of subsets of X, called *open subsets* of X satisfying the following properties:
(T1) $\emptyset$ and X are contained in τ,
(T2) The union of elements of any subcollection of τ is in τ,
(T3) The intersection of elements of any finite subcollection of τ is in τ.

Definition 5.3. A *topological space* (X, τ) is a set X together with a collection of open subsets τ that satisfies the above axioms.

We also consider the definitions of connectedness in topology.

Definition 5.4. A topological space X is said to *connected* if there does not exist a separation of X (where separation of X is a pair of U, V of disjoint non-empty open subsets of X whose union is X).

Definition 5.5. For a *topological space* X, define an equivalence relation $\sim$ on X where $x \sim y$ if there exists a connected subspace of X containing both x and y. These equivalence classes are then called *components* of X.

Next, we prove that this is an equivalence relation on X.

Theorem 5.6. *The relation $\sim$ is an equivalence relation.*

Proof. Reflexivity is obvious since for any point x, there does not exist a separation of x. Thus, $x \sim x$. For symmetry, take two points x and y. If $x \sim y$, then there does not exist a separation of the component they are contained in. Thus, it must be that $y \sim x$. Lastly for transitivity, let A be a connected subspace containing x and y ($x \sim y$). Now, let B be a connected subspace containing y and z ($y \sim z$.) Thus, $A \cup B$ must be connected since they share a common point (To prove that since two sets share a common point that then they must be connected, suppose sets U, V share a common point a. Then $U \cap V \neq \emptyset$. Thus, there does not exist a separation of U, V. Thus, there does not exist a set $W = \{U, V\}$ that is disconnected. Thus, U, V are connected.) Thus, $x \sim z$. □

Definition 5.7. For a topological space X and subspaces $A, B \subseteq X$, we say A *is disconnected from* B if there exist open sets U, V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$. If A and B are not disconnected, then we say A and B are *connected*, and write $tc(A, B)$. As an exception, we say that the empty set is connected to every set.

Quickly, comparing this to our definition of 'C.' Both are reflexive and symmetric, confirming our natural intuitions towards what it means for two objects to be connected. However, monotonicity is slightly different than transitivity.

Remark 5.8. We may 'specify' a mereotopological space by specifying all parts that exist, the P predicate, and the C predicate. Therefore, if the underlying universe is a *set*, this is specified via some subset of the power set, P, and C. If the parts are implicit in the definition of the P predicate, then a mereotopological space may be stated via the universe, $\mathcal{U}$, the parthood predicate, and the connection predicate; therefore, a mereotopological space could be written as a triple $\mathscr{M} = (\mathcal{U}, \mathrm{P}, \mathrm{C})$, where $(\mathcal{U}, P)$ defines a mereological space.

Just as in topology, it is also useful to us to define what it means for a given morphism to be continuous. In topology the definition of a continuous function is as follows:

Definition 5.9. Let X and Y be topological spaces, and let $f : X \to Y$ be a function. If for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X, then we call f a *continuous function*

Definition 5.10. Let X and Y be topological spaces, and let $f : X \to Y$ be a bijection. If f and f^{-1} are both continuous functions, then f is a *homeomorphism*.

In mereotopology, the objects we work with are parts and wholes rather than open sets as in a topology. Thus, it is helpful for us to have a definition of continuity that deals directly with these objects. We make use of the 'C' predicate to define continuity for the category of mereotopology.

Definition 5.11. Let X and Y be mereotopological spaces, and let $f : X \to Y$ be a mereotopological morphism (preserving parthood). Then, for all $\mathrm{P}x_aX$ and $\mathrm{P}x_bX$ where $\mathrm{C}x_ax_b$, we have that $\mathrm{C}f(x_a)f(x_b)$. Then, we call f a *continuous morphism.*

Now, we show that a Hausdorff topological space satisfies the axioms of a mereotopological space.

Theorem 5.12. *If X is a Hausdorff topological space, then $\mathscr{M} = (X, \subseteq, tc)$ is a mereotopological space.*

Proof. First, we show that $(X, \subseteq)$ is a mereological space (A space satisfying the axioms of [M]). Note reflexivity, antisymmetry, and transitivity are trivial.

Next, we show supplementation: suppose $A \not\subseteq B$. Then there exists $z \in A$ such that $z \notin B$. Let $D = \{z\} \subseteq A$. Then, $D \subseteq X$, and $D \cap B = \varnothing$, as desired.

Now, we translate the overlap and underlap predicates. We know that A and B overlap if and only if they *intersect.* Then, we define the sum to be the union. Similarly, A and B always underlap (noting two subsets always both contain the empty set, i.e., we have a *bottom* element). We define the product to be the intersection. The difference, then, is exactly the set difference. Thus, this set exists in **Mtop**, so we have now confirmed (P5), (P6), and (P7).

We now show (P9), recalling that (P8) is an immediate consequence thereof. Formally speaking, (P9) fails to be true in the category of **Set**. However, ZFC has the axiom schema of restricted comprehension: in our current example, we simply restrict the comprehension to X, which is always a set. Thus, we don't run into the problems à la Russell's paradox. Thus, (P9) holds true, and thus $(X, \subseteq)$ is a mereological space.

Second, we check that topologies satisfy the additional axioms of [**G**(**E**)**M**]**TC**.

We show that tc satisfies the axioms of C. Note that if $A \subseteq B$, then every open set containing B contains A, so $A \subseteq U \cap V$. Thus, if $A \neq \varnothing$, we have $A \subseteq B \implies tc(A, B)$. Therefore, $tc(A, A)$. The definition of tc is symmetric. For (C3), suppose $A \subseteq B$. Let D be such that $tc(D, A)$. Let U, V be a pair of open sets such that $D \subseteq U$ and $B \subseteq V$. Then note $A \subseteq V$, so we have $U \cap V \neq \varnothing$ by the assumption that $tc(D, A)$. Therefore, $tc(D, B)$.

Now we need to translate these other predicates into set theoretical formulations. We have $(IPxy \leftrightarrow Pxy \wedge \neg TPxy \leftrightarrow Pxy \wedge \forall z(\neg ECzx \vee \neg ECzy) \leftrightarrow Pxy \wedge \forall z((\neg Czx \vee Ozx) \vee (\neg Czy \vee Ozy).)$ So A is an interior part of B if and only if $A \subseteq B$, and other set D either intersects B, is connected to B, or is connected to A. Being connected to A is harder, so we are left with this.

Thus, suppose $D \cap Y = \varnothing$. Then if $IP(AB)$, $tc(D, A)$ must be false. Thus, there must exist a pair of open sets U, V such that $A \subseteq U$, $D \subseteq V$, and $U \cap V = \varnothing$

Note we assume that $A = iA$ when A is open and $A = cA$ when A is closed. In order to check (C4) and (C5), we must show a set A is open iff $A = iA$ is open and a set A is closed iff $A = cA$ is mereotopologically closed.

Ideally, we would like to prove this for general topologies. However, consider the following space where an open set is not mereotopologically open:

Let X be a space such that there is a point, x, contained in every non-empty closed set. Let A be any open set that is not the entire space. Note the only open set that contains x is the entire space, X. Therefore, every subset is connected to A. Suppose $B \subseteq A$ is an interior part. Then B is non-empty, and every subset is also connected to B. But then $\{x\}$ is connected to B, and $\{x\}$ fails to intersect A. Therefore, B cannot be an interior part. Therefore, A has no interior parts.

The proof for Hausdorff spaces follows since we can separate two points. We would be able to separate $\{x\}$ from points of A and ultimately, avoid the contradiction above.

A similar circumstance occurs when trying to show that a set A is closed iff $A = cA$ is mereotopologically closed. Indeed, the same argument applies, but with complements of open sets instead of open sets.

Now, we check (C4) and (C5). Note that (C4) and (C5) can only be proved when the space is Hausdorff. For two topological spaces A and B, we have the following Kuratowski axiom:

$$cl(A \cup B) = cl(A) \cup cl(B).$$

Since we defined sum to be union, we have (C4).

For (C5), we would like to show that the arbitrary intersection of closed sets is closed. Let X be an open set in the topology τ, and let $Y = (\bigcup_{X \in \tau} X)^C$. Notice that $(\bigcup_{X \in \tau} X)$ is open by the definition of a topology. Thus, Y is closed since it is the complement of something open. Also, by De Morgan's law we have that

$Y = \bigcap_{X \in \tau} X^C$. Since X is an open set, X^C is a closed set, and thus, the intersection of closed sets is a closed set. This translates to mereotopology with set defined as part, union defined as sum, and intersection defined as product. Closed and open sets are then translated to parts equal to their closures and parts equal to their interiors, respectively.

Lastly, we show that if a function is topologically continuous, then it is mereotopogically continuous. Since above we showed that tc in topology is analogous to C in mereotopology, it suffices to prove that the continuous image of a connected space is connected.

We argue by contradiction. Let $f : X \to Y$ be a continuous function, and let X be connected. Assume that $f(X)$ is not connected. Therefore, there exist open sets U, V in Y such that $f(X) \subset U \cup V, (f(X) \cap U) \cap (f(X) \cap V) = \emptyset$, and $f(X) \cap U \neq \emptyset \neq f(X) \cap V$. Now, since f is continuous on X, there are open sets U' and V' in Y such that $X \cap U' = f^{-1}(U)$ and $X \cap V' = f^{-1}(V)$. If $xinX$, then $f(x) \in f(X)$ so that $f(x) \in U$ or $f(x) \in V$. Thus, $x \in U'$ or $x \in V'$, i.e., $X \subset U' \cup V'$. Also, if $x \in V' \cap U' \cap X$, then $f(x) \in U \cap V \in f(X) = \emptyset$. Thus, there is no such possible x. Note also that $U' \cap X = f^{-1}(U) = f^{-1}(U \cap f(X))$. However, since $U \cap f(X) \neq \emptyset$ and f is onto $f(X)$, it must be that $U' \cap X \neq \emptyset$. Similarly, $V' \cap X \neq \emptyset$. Thus, X is disconnected. However, this contradicts our initial assumption. Therefore, it must be that if $f(X)$ is connected.

□

Remark 5.13. I had originally thought that a set was open if and only if it was mereotopologically open for any topological space (X, τ). However, I have not been able to finish the proof of this. So far, I have only been able to prove this to be true assuming Hausdorff which was found necessary for the above proof. Although I suspect a similar result may be true for general spaces, I have yet to determine a modified statement that holds for general spaces. Then a functor may be defined from the category **Top** to **Mtop** that can confirm my suspicion that **Mtop** is a broader category than **Top**.

6. Holes

6.1. **Homotopy.** In order to formalize a conception of circles and holes in **Mtop**, we introduce homotopies from algebraic topology.

Definition 6.1. A *path* is a continuous function $f : I \to A$. A *loop*, then, is a path where $f(0) = f(1)$.

Definition 6.2. The unit interval I is the closed *interval* [0,1].

Definition 6.3. If f and g are continuous maps of the space X into the space Y, we say that f is *homotopic* to g if there is a continuous map $H : X \times I \to Y$ such that

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

for each x. Then, the map H is called a *homotopy* between f and g.

Definition 6.4. Two maps $f, g : X \to Y$ are homotopic if there exists a homotopy between the two maps. We formally denote this by $f \simeq g$. Then, two paths f', g' are path homotopic if $f' \simeq g'$ and $f'(0) = g'(0)$ and $f'(1) = g'(1)$.

Definition 6.5. Two topological spaces X, Y are *homotopy equivalent* if there exist two maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_X$ and $g \circ f \simeq \mathrm{id}_Y$. Then, we write $X \simeq Y$.

Proposition 6.6. *Homotopy is an equivalence relation.*

Proof. Here, we want to show that $\simeq$ is reflexive, symmetric, and transitive.

For reflexivity, we need to show that $f \simeq f$. Here, we use the constant homotopy defined by $h(x,t) = f(x)$ for all t.

For symmetry, we need to show that if $h : f \simeq g$, then $g \simeq f$. Define $H(x,t) = h(x, 1-t)$. This defines a homotopy from g to f.

To prove transitivity, let $f \simeq g$ with $h : X \to Y$ as a homotopy. Now, assume $g \simeq e$ where $h' : g \to e$ is the homotopy between the two. Now, define a function H.

$$H(x,t) = \begin{cases} h(x, 2t) & t \leq \frac{1}{2} \\ h'(x, 2t-1) & t > \frac{1}{2} \end{cases}$$

Since $h(x,1) = g(x) = h'(x,0)$, H is well-defined. Thus, $f \simeq e$.

Thus, $\simeq$ is an equivalence relation. □

Definition 6.7. A space X is *contractible* if it is homotopy equivalent to a single point. That is, $X \simeq \{*\}$.

Now, consider $\mathbb{R}^2 \setminus \{(0,0)\}$. We will prove that this is homotopy equivalent to the circle S^1.

Example 6.8. Let $X = S^1, Y = \mathbb{R}^2 \setminus \{(0,0)\}$, and $i : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ be the standard inclusion.

$$r : \mathbb{R}^2 \setminus \{(0,0)\} \to S^1, x \to \frac{x}{|x|}$$

Then, $r \circ i = \mathrm{id}_{S^1}$. Thus, we take the constant homotopy. Now, $i \circ r : \mathbb{R}^2 \setminus \{(0,0)\}$ is the map f given by $f(x) = \frac{x}{|x|}$.

$$H : (\mathbb{R}^2 \setminus \{(0,0)\}) \times I \to \mathbb{R}^2 \setminus \{(0,0)\}, (x,t) \to \frac{x}{t + (1-t)|x|}$$

Note, this is well defined, as it is impossible for $\frac{x}{t+(1-t)|x|} = 0$. Now, H is a homotopy from $i \circ r$ to $\mathrm{id}_{\mathbb{R}^2 \setminus \{(0,0)\}}$. Thus, i is a homotopy equivalence with homotopy inverse r. Therefore, $S^1 \simeq \mathbb{R}^2 \setminus \{(0,0)\}$.

The equivalence relation $\simeq$ allows us to treat a class of homotopic paths similarly as one equivalence class. As these paths behave similarly, we may uncover universal traits of equivalent paths without identifying those traits in each path individually. Ontologically, this is a very important. What this means for the circle (our primitive object possessing a hole) is that we treat it similarly to a single point removed from

$\mathbb{R}^2$. The use of words such as missing and removed, in this language, suggest that the hole in a circle is an immaterial particular. Specifically, it is an immaterial particular which can be thought of as a point. Thus, this does not exactly formulate the notion that a hole may have parts.

Remark 6.9. In its defense, the word 'hole' is often used in topology as a pictorial learning tool employed to understand the abstract concepts of algebraic topology. In this way, it is not exactly concerned with the ontology of objects such as holes, and it makes no such claim that it is. However, mereotopology possesses more philosophical and ontological motivations, and by slightly altering the crude notion of a hole in topology to this category, there is the opportunity to formally approach this object with more ontological intent.

6.2. **Products in Mereotopology.** Before, we may consider the mereotopological alternative formulation of a hole, we must define the product in mereotopology. We do this through a category theoretic approach with the product being defined.

Definition 6.10. Given two objects X_1 and X_2, we say that (Z, π_1, π_2) is the *product* of X_1 and X_2 if, for all objects Y with maps $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$, there exists a unique map $f : Y \to Z$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f_1}{\swarrow} & \downarrow f & \overset{f_2}{\searrow} & \\ X_1 & \xleftarrow[\pi_1]{} & Z & \xrightarrow[\pi_2]{} & X_2. \end{array}$$

The maps π_1 and π_2 are called the *projection* maps, i.e., π_1 is called projection onto X_1. Frequently, these maps are canonical or in some way assumed, and then, we say that Z is the product of X_1 and X_2, and write $Z = X_1 \times X_2$. Then, the map π_1 is called projection onto the first factor, and map π_2 is called projection onto the second factor. Note that the product is unique.

What we want to show here is that there exists a unique object $X \times Y$ which allows the above diagram to commute and preserves the the below property.

$$(\mathrm{P}zx \wedge \mathrm{P}wy) \to \mathrm{P}(z, w)(x, y) \text{ where } (z, w) = \{z, z, w\}.$$

Though I have not worked out the formal details of existence and uniqueness, there is no obvious contradiction in assuming its existence and uniqueness. From here, we are able to essentially import the definition of homotopy in topology to mereotopology.

6.3. **Homotopy in Mereotopology.** From our earlier discussion of boundaries, what we may want to consider is the notion of the boundary of a hole. It seems intuitive that such a thing exists though we argue for it here. Consider S^1 again, and recall that the boundary of an object is equal to the overlap of the closure of the object and the closure of its complement. Thus, part of the boundary of S^1 is equal to the boundary of the hole. We expect, given that the missing substance of S^1, the hole is a part of the circle's complement. Thus, the hole in S^1 has a boundary that agrees with boundary laws which, from Brentano's thesis, implies the existence of an entity with interior proper parts. Thus, this hole is seen as having proper parts.

An ontological advantage of working in mereotopology comes from not committing ourselves to atomization (despite that we don't assume atomization, something may still be equivalent to an 'atom'). Thus, this hole is not formalized in the language of 'a missing point' in this case. We are able to formalize the intuitive notion that this hole may have parts. To do so, we slightly alter the notion of homotopy for the category of mereotopology (though this is entirely reliant on the notion of a product existing).

First, we must define I in mereotopology, as not all mereotopological objects have points.

Definition 6.11. Define I as a closure of a self connected whole such that if an interior proper part is removed, it becomes disconnected.

Now the same definition for homotopy applies where 0, 1 are analogous to the boundary parts of the mereotopological interval I.

Definition 6.12. If f and g are continuous maps of the space X into the space Y, we say that f is homotopic to g is there is a continuous map $H : X \times I \to Y$ such that

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x)$$

for each x. Then, the map H is called a *homotopy* between f and g.

In this way, we may define a circle and its hole in mereotopology. With homotopy theory, we are able to bypass the formation of a new predicate 'H' (where Hx is read as 'x is holed') which some have chosen to do in mereotopology [9]. Now, we may distinguish the exact property for S^1 to have a hole: the notion that it is homotopy equivalent to a self-connected whole with an interior proper part removed. Indeed, there remains the challenge of formalizing the notion that the hole in S^1 may be a spatiotemporal particular.

7. Conclusion

The main mathematical result contained in this paper was in showing that Hausdorff spaces indeed satisfy the necessary axioms of mereotopological spaces.

From working in an alternate category that is primarily motivated by ontology, we are able to view controversial metaphysical questions such as the nature of boundaries and holes from a slightly different perspective. From here, we can perhaps ponder even broader metaphysical questions. For example, there is the Kantian inclination to believe that all boundaries are *fiat* (those boundaries such as geographic ones between countries whose existence is much more contingent on humans) as opposed to being *bonafide* (existing independent of humans such as bodies of water or mountains) [4]. However, by reasoning formally, some have been able to recognize that most boundaries are a combination of the two.

Concerning our discussion on holes, we were only truly able to speak on the hole in a circle (we did not consider embeddings). However, not all holes are the same.

If extended using notions of the fundamental group which have not been developed here, it seems we will be able to distinguish between T and $T\#T$ (the double torus). This has already been done in topology, but once again, a different mereotopological perspective may be beneficial.

Lastly, it is clear that there remain areas of mereotopological study. Perhaps, we ought to further consider the boundary of other metaphysically contested entities such as shadows or thoughts. We continue to employ mathematics to formalize our intuitions and rid logical contradictions. In this way, mathematics functions as a tool that synthesizes clarity. As Heidegger describes in *The Question Concerning Technology*, the purpose of technology (which includes mathematics) is to be the instrument which reveals truths and knowledge to ourselves [8].

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