PLENUM THEORY

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1. INTRODUCTION

Plena are large-scale macro-totalities appropriate to the realms of all facts, all truths, and all things. Our attempt here is to take some first technical steps toward an adequate conception of plena.

Why should we care about such objects: what are they good for? The answer lies in the very aim and ambition of abstract thought. Theorizing aspires to universality—to a transcendence of the episodic particularization of this and that, reaching to generalization about totalities of different kinds and—in the end—to the totality of everything-at-large. What plena offer is the prospect of giving substance and structure to this line of thought. Traditional approaches to ontology and universalization exhibit a range of theoretical deficiencies which, it is hoped, this further and decidedly different conception may enable us to overcome.

It seems that there must be a totality of all facts, a totality of all truths, a totality of propositions, of things, and of sets of things. The notion of a set of all sets, however, is well known to lead to contradiction. There can therefore be no such thing. The notion of a totality of truths or of all propositions can be shown to lead to contradiction as well.[1] The notion of a totality of all facts or of all states of affairs will lead to the same result.

Suppose, for example, that there were a set of all facts *F′*. That set will have a power set $\mathsf{P}F'$—the set of all sets of facts. But for each set of facts *F* in $\mathsf{P}F'$, there will be a distinct and specific fact—that *F* contains precisely the facts that it does, for example.

By Cantor's theorem, there must be more elements of the power set of any set than there are elements in the set itself. By the reasoning above, there will then be more facts than in *F′*. But *F′* was designated as the set of *all* facts. Contradiction.[2]

In Cantorean set theory, a set cannot contain its own power set, the latter being of larger cardinality. This disqualifies plenary collectivities from qualifying as Cantorean sets. However, this circumstance affords no sufficient ground for deeming such mega-collectivities impossible, let alone logically inconsistent. In an interesting 1899 letter to Dedekind, Georg Cantor wrote:

> A plurality (*Vielheit*) can be so constituted that the supposition of a "conjoining" ("*Zuzammensein*") of ALL its members leads to contradiction, so that it is impossible to conceive of this plurality as a unity, a compete (*fertig*) object. Such pluralites I term ABSOLUTELY INFINITE or INCONSISTENT…. As is readily shown, the "totality (*Inbegriff*) of everything thinkable" is such an [inconsistent] plurality, and there are further examples as well. On the other hand, when the totality (*Gesammtheit*) of the members of a plurality can be conjoined without contradiction, so that it is possible for them to be taken together as "one single thing" then I call this a CONSISTENT PLURALITY or "set" (*Menge*).[3]

It is—or should be—clear that in equating his "absolutely infinite" with "inconsistent" Cantor was begging a rather large question.

Attempts have been made to deal with intuitive totalities of sets using alternative class theories, but often the result is no more satisfactory: a place is carved for a class of all sets, for example, but with no possibility for a class of all classes. Application of class theories to totalities of truths, propositions, or facts are no more promising.[4]

Following suggestions of George Boolos, further developed in the work of Agustín Rayo and Timothy Williamson, attempts have also been made to carve a place for broad and universal quantification without the broad and universal collections or totalities that would supply a classical model-theoretic domain for that quantification. Our concerns, in contrast, are primarily ontological rather than semantic. We take the totalities themselves to be intuitive. It is these, which the Boolos approach carefully does without, that are precisely the things we want to explore.[5]

In this paper we want to widen consideration of collectivities beyond sets in a way that allows a much more intuitive place for totalities of facts, states of affairs, truths, propositions, and sets. Any attempt of this sort must be non-standard, and ours is non-standard in a number of ways. We offer it as a promising approach not only because it concedes full existence to a range of intuitive metaphysical complexes but also because it opens up a number of unexpected consequences and sets the stage for raising some intriguing questions.

The history of mathematics has witnessed a long series of episodes in which items are finally recognized as appropriate units of concern which previously had either been unacknowledged or dismissed out of hand. Negative integers, zero quantities, imaginary and infinite numbers, and infinite and transfinite sets all exemplify this phenomenon. The present discussion proposes a further enlargement of perspective. Heretofore such supposedly paradoxical items as the "set" of all sets or the "number" of all numbers have been dismissed as incoherent, and self-inconsistent, and absurd. In introducing the idea of plena the present discussion seeks to provide a place in mathematico-logical taxonomy and theory for items of precisely this sort.

2. PLENA

A *collectivity* is a collection of items whose content is specified in terms of conformity to some condition, either extensionally via an inventory of some sort or intensionally through a defining membership feature:

> $[x \mid Kx]$: the collectivity of all items x that meet the condition K.

It must be stressed that K-conformity may be indecisive. In some cases it may neither be determined nor determinate whether or not a given item x does or does not conform to K. The item's conformity is then left problematic with $x \in [x \mid Kx]$ remaining undecided. A collectivity can thus be *amorphous*, with $\in$-bivalence breaking down. It can transpire that neither $x \in C$ nor $x \notin C$ for some item x and for collectivity C.

This distinguishes collectivities at large from those that are specifically sets, a set being a collectivity whose defining condition is sharp-edged. Any item x will either determinately belong to a particular set or not:

> $\{x \mid Kx\}$ = the set of all items x that meet the condition K, where always either $x \in K$ or $x \notin K$.

With sets, unlike collectivities, $\in$-bivalence must hold. A set is such that it is always a matter of fact whether or not a given item belongs.

Collectivities are analogously distinguished from fuzzy sets. For with fuzzy sets, any item *x* will determinately belong or not belong to a specific and precise degree. Such a condition need not hold for collectivities.

Although there will be a collectivity of all sets, it would thus be premature to speak of "the *set* of all sets". There will be a collectivity of all facts, another of all propositions, another of all states of affairs. But it is a further question whether any collectivity will qualify as a set. In each of these cases, we have reason to believe that the answer will be 'no'.

Consider the seemingly innocuous truth that:

(T) Sets require a criterion of membership.

Both intuitively and throughout the model-theoretical tradition, universal claims of the form $(\forall x)(\text{set }(x) \rightarrow \ldots)$ presuppose a realm of discourse and a domain of quantification that encompass all sets. Were this not so, (T) would be incapable of conveying its intended message. But if there is any ontological whole that can serve as a quantificational domain, how are we to characterize it? We cannot say that it is a set—and thereby a set that contains all sets. But what then is it? If we are to have a vocabulary of wholes adequate to serve as quantificational domains, we very much need some functional equivalent of collectivities. Without it, a range of conceptual explorations are blocked at the outset.[6]

There are two kinds of collectivities that will constitute the focus of what follows: indeterminate collectivities and plena.

Indeterminate collectivities are those which violate the set-characterizing condition of membership decisiveness in that it is not the case that every item x will either determinately belong or not. Here it is not merely partial degrees of membership that are at issue, as in the case of fuzzy sets. With an amorphous collectivity *C*, neither $x \in C$ nor $x \notin C$ need apply for some *x*. Further discussion of the limitations thereby imposed on the Law of Excluded Middle is offered below.

A *plenum* is a collectivity that contains distinct entities corresponding to each of its sub-collectivities, where sub-collectivities follow the same pattern as subsets:

$$(\forall s)(s \subset P \leftrightarrow (\forall x)\ (x \in s \rightarrow x \in P)$$

The mark of *plena* is that *every sub-collectivity s of a plenum P is such that there is a unique member of P that represents s*. There will thus be a mapping of all sub-collectivities of a plenum into its members, that is, a mapping *m* such that

$$(\forall s)(s \subset P \rightarrow (\exists x)(x \in P \ \&\ x = m(s)\ \ \&\ \ (\forall s')(s' \neq s \rightarrow m(s') \neq m(s)))$$

Here, as in the indeterminate case, we may note that plena do not answer to the conditions of orthodox set theory by characterizing them as *collectivities* rather than as *sets*. There is no guarantee that smaller collections of their members will qualify as sets, either, and thus we speak of their sub-collectivities rather than their subsets. As we shall soon see, the totality of all objects, or the totality of all ordered sets will constitute a plenum.

There may be various ways in which sub-collectivities of a plenum represent or correspond to a unique member. In the case of a plenum of all truths, for example, truths to the effect that its sub-collectivities have precisely the members they do will be elements of the plenum of truths. Often, however, the relation will be simpler: a plenum may directly contain all of its sub-collectivities as members:

$$\text{(P)} \qquad (\forall x)(x \subset C \rightarrow x \in C)$$

Such a plenum will be characterized specifically as a *membership plenum*, so that *P* is a membership plenum when

$$[x \mid x \subset P] \subset P$$

A number of salient facts regarding plena follow immediately from this characterization.

When viewed from the perspective of anything like sets, plena are *self-amplifying* and indeed explosive in content. Every time one looks to yet another sub-collectivity one automatically finds further members of the plenum itself. Suppose one starts with a set-like estimate of a plenum's membership—an estimate of all the things it contains as members. Because a plenum contains a unique member for each of its sub-collectivities, each collectivity of those things will betoken a further member of the plenum. Plena are so large that we can never bring them into view as a whole. Whatever one's first set-like view, the plenum will contain more—ever adding elements corresponding to all elements of the power set of what is in view. Once one has included *those* in the plenum as well, it is clear that each sub-collectivity of that *larger* compendium will represent a further member of the plenum. The process continues and expands one's view ad infinitum.

It also follows immediately that there can be no merely finite plena. Were we to start by contemplating a finite membership, we would have to recognize every sub-collectivity of that membership as generating a further member, and then every sub-collectivity of that larger group, and so on. As discussed below, the explosiveness of plena puts them not only beyond any finite number but beyond any number at all.

It should be noted that the union of two plena need not itself be a plenum. Although each of the plena combined must contain its own sub-collectivities, there is no guarantee that their union will contain all unions of their sub-collectivities, as would be required for its status as a plenum.

Indeed, adding even a single item to a plenum need not yield another plenum. Think, for example, of conjoining Napoleon's left hand to the totality of all agendas.

The Cartesian product of a plenum with yet another collectivity, however, will always be a plenum.

As collectivities, plena are characterized in terms of their content. The explosive size of plena has been noted. So is what matters with regard to plena their *content* or their *size*? Neither content nor size is alone decisive. For what genuinely distinguishes plena is neither content nor size alone, but structure: the fact that a plenum contains distinct entities corresponding to each of its sub-collectivities.

One indication that it is not merely content that is at issue is the fact that plena can have radically different content. Moreover, we could take a plenum and replace any single item in all its occurrences (including its membership in sub-collectivities) with something else. Consider the collectivity of all sets, for example, in which we replace the set of teaspoons with a single chess piece. If this is done throughout all sets of the collectivity, we will have a collectivity that still contains all its sub-collectivities as members—and is thus a membership plenum—despite the change in content.

What characterizes plena is not merely immense size, on the other hand, because we can add a single item to the elements of a plenum and have a collectivity that is not a plenum as a result. So with plena both numerical size and contentual structure matter. The idea of a heap or pile affords a helpful analogy. One does not have a heap of items unless one has a lot of them. So size matters. But it isn't all. Scatter those items the heap is gone. So structure matters as well.

3. PLENARY TOTALITIES

A *totality* is a collectivity defined by the stipulation that it contains any and all items that belong to a certain specific kind. A totality will clearly be a collectivity of sorts, but it need not necessarily be a set. For the totality at issue need not be sharp edged in the sense specified above as effecting a surgically clean cut between the things that do and those that do not belong.

Georg Cantor argued in 1895 that "the absolutely infinite totality of cardinal numbers" is not a set.[7] For if it were then it would have a cardinality which could not be exceeded by that of any other set. But the set of all of its subsets would indeed have to have a larger cardinality which (ex hypothesi) it cannot. From the angle of set theory totalities can prove to be decidedly problematic. But other angles are available.

A *plenary totality* or *totalistic plenum* is a totality of such a sort that every one of its (distinct) subcollectivities defines yet another (distinct) member of it. With such a plenary totality T we have:

$$(\forall s)(s \subset T \rightarrow (\exists_s x)\ x \in T)$$

Here $\exists_s$ stands for "exists in unique correlation with s." It follows that every plenary totality is a plenum.

Some prime candidates for plenary totalities are those collectivities that answer to the principle that every group of X's (the totality of all X's itself presumably included)[8] itself constitutes a unique X (i.e., one constituted by no different group). This being so, some examples of plena are the totalities of all

- things/entities/individuals/items/objects
- actualities
- ordered sets[9]
- truths
- segmented agendas (for discussion or deliberation)
- structured inventories

All such totalistic collectivities root in the circumstance of what might be called *aggregative closure* in that that every collectivity of items of this kind will itself define—and indeed often *constitute*—a distinct item of that kind. All of these collectivities are accordingly plena. Moreover, all of the preceding are *kind totalities*, in that there is a natural kind K such that the totality in question is:

$$<K> = [x \mid Kx]$$

There are bound to be infinitely many plena. If the totality of all items/objects is a plenum, then so is the totality of all such items except for those of some specific sort (dogs, stars, numbers). If the totality of all truths is a plenum, then so is the totality of all truths that don't mention Y's (dogs, stars, numbers).

It warrants note that kinds are determined intensionally in terms of meaning while totalities are determined extensionally in terms of their membership. Accordingly, we do *not* have it that

$$K \neq K' \rightarrow <K> \neq <K'>$$

As kind specifications, triangles ≠ trilaterals (in a Euclidean context) thanks to the difference in an optimal articulation. But we do indeed have

<Euclidean triangles> = <Euclidean trilaterals>

On the other hand, we cannot have different totalities of the same kind. Thus

$$K = K' \rightarrow <K> = <K'>$$

is perfectly true. If $<K>$ had a member that $<K'>$ lacked, or conversely, then $K = K'$ could not be maintained.

As outlined, a kind totality qualifies as a totalistic plenum if there is a distinct truth for every sub-collectivity of that plenum. Truths therefore form a totalistic plenum simply because there is a distinct truth for each sub-collectivity of truths, including a truth about precisely all the truths. Membership in the totality of truths may remain amorphous in other regards, however. It may remain an open question, for example, whether the collectivity of all truths itself constitutes a truth. Plena are too big for surveyability. That may be one reason why amorphousness, though not definitional of plena, may turn out to be common or even characteristic among plenary totalities.

This kind of amorphousness need not imply plenary totalities being "fuzzy" in the normal sense of the term.[10] To see what is at issue here, note that with regard to a set, collectivity, or property we can have:

- definite ins
- indefinites
- definite outs

Fuzziness characteristically roots in the circumstance that the indefinites can go one way or the other. They are ambivalent: the world hasn't made up its mind about them as between in and out. What is often at issue with plenary totalities, however, is not *ambivalence* but rather *hesitation*. The question is often *not* that of one direction versus another—in or out—but rather whether to stay put as indefinite or to move in. Full exclusion is not an option. Consider for example the problematic totality of items, things, or individuals. One might put this totality itself in the limbo of indefiniteness—as itself only problematically an item, thing, or individual. But it does not appear to be a candidate for definite exclusion. Thus regarded, plenary totalities of this type are not fuzzy but something else again, remotely related. Their characteristic nature renders them not *fuzzy* but *embryonic* in failing to be fully formed but even possessed of growth potential.

Totalities that are amorphous with regard to membership will also be problematic with regard to universal quantification. Specifically, what are we to mean in saying that all members of such a totality have a certain feature when it is not invariably fully and decisively determinate what that membership consists in? The indeterminacy of $x \in P$ will carry that of $(\forall z)(z \in P \rightarrow Fz)$ in its wake. Of course whenever $z \in p$ entails Fz, so that $z \in p \rightarrow Fz$ obtains on logical grounds alone, then $(\forall z)(z \in p \rightarrow Fz)$ must be true. A generalization can be claimed to hold for the entire membership of a totalistic plenum when it holds as a matter of logico-conceptual necessity. In other cases, where $x \in p$ slips into indeterminacy, the status of this generalization can be expected to slip into indeterminacy as well.

4. IS TOTALIZATION LEGITIMATE?

The logical difficulties of a set of all sets are familiar. Logical difficulties for a set of all truths, of all propositions, and of all states of affairs follow a similar pattern. The promise of non-set plena is an alternative approach to such totalities. But there have also been other objections to such totalities that should be noted.

Immanuel Kant deserves note as the philosopher who, in his *Critique of Pure Reason*, first complained about illicit totalization. For he rejected any and all totalities that are not closed as it were, and thereby become impossible to survey *in toto*, since he held that experience is our only pathway to knowledge about existence, and of course we almost never experientially survey certain totalities as such. As Kant saw it, a fundamental fallacy is involved in such totalitarian conceptions:

> The concept of totality is in this case [of the-world-as-a-whole] simply the representation of the completed synthesis of its parts; for we cannot obtain the concept from the apprehension of the whole—that being in this case impossible. . . (CPuR, B456.)

For Kant, such closure-defying, unsurveyable conceptions as that of the world-as-a-whole-one, whose content goes beyond the range of that which could ever be given in experience, is something ill-defined and thereby inappropriate. Only experiential interaction can assure actual existence—description alone can never do the job:

> [It is inappropriate to suppose] an absolute totality of a series that has no beginning or end [such as would be at issue with "the terminus of all successive divisions of a region" or "the initiation of all the causes of an event."] In its empirical meaning, the term "whole" is always only comparative. The absolute whole of quantity (the universe), the whole division [of a line segment], or of [causal] origination or of the condition of existence in general . . . along with all questions as to whether this whole is brought about through finite synthesis or through a synthesis requiring infinite extension . . . [are something altogether inappropriate]. (CPuR, A483-84 = B511-12.)

In particular, Kant held that *experiential* totalization is unachievable because we can never appropriately reify such a totalitarian conception into that of an *object* that has a well-defined identity of its own. Accordingly, such totalizations as "the-physical-world-as-a-whole" represented for Kant an inherently fallacious conception that leads to inconsistency:

> [As a sum-total of existence] the world does not exist in itself, independently of the regressive series of my representations, it exists *in itself* neither as an *infinite* whole nor as a *finite* whole. It exists only in the empirical regress of the series of appearances, and is not to be met with as something in itself. If, then, this series is always conditioned, and therefore can never be given as complete, "the world" is not an unconditioned whole and does not exist as such a whole, either of infinite or of finite magnitude. (CPuR A503-05 = B531-33.)

Kant's objections to totalizations often have a distinctly epistemological flavor: what is absurd often seems to be the notion that there is any possible experience of totalities of this type. In outlining plenary totalities as metaphysical or logical constructs, on the other hand, we feel no such commitment to Kantian restraints.

Kant repeatedly claims, on the other hand, that endorsement of unsurveyable unbounded totalizations will result in logical self-contradiction. It remains open to debate whether his four classic "Antinomies" in the *Critique of Pure Reason* do establish that fact. But logical contradiction will indeed be something we are at pains to avoid.

5. AVERTING RUSSELL'S AND CANTOR'S PARADOXES

Totalities along the contemplated lines may seem to engender Russell's paradox of "the set of all sets that are not members of themselves." But this is not the case, for the distinction between sets and collectivities enters to save the day. One now blocks the problem of "the set of all sets that are not members of themselves" by recognizing that this specification is predicated on a false presupposition, viz. that "the collectivity of all sets that are not members of themselves" qualifies as a set. It just doesn't.

On the other hand, one blocks the problem of "the collectivity of all collectivities that are not members of themselves" by insisting that while such a collectivity indeed exists, it is amorphous in that for *this* collectivity $C^{\sim}$ both $x \in C^{\sim}$ and $x \notin C^{\sim}$ will fail.

For sure, the collectivity of all self-excluding sets is not a set since the supposition of its being so entails a contradiction. But what about "the collectivity of all collectivities that do not include themselves"? This is manageable because collectivity membership is not bivalent with $x \in C^{\sim}$ either definitely true or definitely false. The situation regarding the membership of collectivities can be amorphous. Since collectivities are not sets they can put aside the burdens that sets must bear.

One might think that one could construct a strengthened Russell's paradox here. The collectivities of all collectivities that are not members of themselves can neither be a member of itself or not. But then what of the collectivity C^{s} of all collectivities that either are not members of themselves or are neither members of themselves or not? Won't C^{s} have to be either a member of itself, a non-member, or neither a member or not? And won't any of those three options again give us a contradiction?

The rejection of the Law of Excluded Middle that we envisage here is a strong one. It calls for complete truthvaluelessness rather than simply the prospect of a third expressable value. For indeterminate collectivities, the possibility will remain that neither membership nor non-membership for some items is defined at all. For such collectivities and such items, both $x \in C$ and $x \notin C$ will make no more sense than division by zero or the ratio 3/0. Logical compounds of these will make no more sense than '$10 = 3/0$ or $10 \neq 3/0$'. [11]

With a strong denial of the Law of Excluded Middle, the three options offered as exclusive for the strengthened Russell will be rejected as inappropriate in the same way that the two options offered as exclusive in the case of the collectivity of all non-self-membered sets $C^{\sim}$ are rejected as inappropriate.

Here our model is the propositionalist approach to the Liar. On such an approach, one claims that the Liar sentence expresses no proposition, insisting on the same categorization for strengthened versions such as 'This sentence is either false or expresses no proposition'. One's commitment entails that such a sentence expresses no true proposition, but one resists the move that 'it is therefore true after all' because 'that is what it says.' On the commitment that it expresses no proposition, there is nothing that it says. Here, similarly, we deny that membership and non-membership are defined for all collectivities. Membership and non-membership may also remain undefined for 'strengthened' collectivities specified in terms of non-membership.

Cantor's power set theorem shows that the power set of any set is larger than the set itself, and therefore poses a crucial logical problem for any universal set of all sets. Won't the same problem arise here? It is helpful to encapsulate Cantor's proof:

A set *S'* has greater cardinality than a set *S* if there is a one-to-one mapping of distinct elements of *S* onto distinct elements of *S'* but is no mapping of distinct elements of *S'* onto elements of *S* that does not leave out some element of *S'*. Consider then any set *S* and its power set $\mathcal{P}S$. Distinct elements of *S* can be mapped onto distinct elements of $\mathcal{P}S$ simply by mapping them onto their singleton sets. But consider any mapping *m* proposed from $\mathcal{P}S$ onto *S*. For any such mapping, there will be the diagonal set S^* of precisely those members of *S* which are not elements of that element of $\mathcal{P}S$ mapped onto them by *m*. To what element of *S* can *m* map S^*? For any element $s \in S$ proposed, *s* will have to either be a member of S^* or not. Given the definition of our diagonal set *S**, however, either alternative gives us a contradiction. There can then be no mapping of elements of P*S* into *S*; the power set of any set is larger than the set itself.

Were this to apply to a plenum of all sets, we would have to say that the plenum was larger than itself; that it literally contained more members than it did, or that its members could not be put into one-to-one correspondence with themselves by any relation, presumably including simple identity.

Here again it is the strong rejection of the Law of Excluded Middle that avoids logical contradiction. For a plenum of all sets, there is no guarantee that all its sub-collectivities are themselves sets. If they are not—if, in particular, that which plays the role of *S** is not—then it need not hold for any *x* that $x \in S^*$ or $x \notin S^*$. With the abandonment of sets we lose the presumption of bivalence regarding membership. And without that presumption, we are no longer forced to say that the power set of a plenum is larger than the plenum itself. Indeed it cannot be, since plena are defined so as to contain distinct elements for each element of their power sets.

What this points up is an important relationship between the two non-set categories at issue: between indeterminate collectivities and plena.

6. PLENA AND INDETERMINATE COLLECTIVITIES: DISTINCTIONS AND LINKS

What is the relation between plena and indeterminate collectivities? Must each plenum be an indeterminate collectivity? Will every indeterminate collectivity be a plenum? The answer in each case is negative. Plena and indeterminate collectivities, although related in an important way, are not the same, and neither includes all cases of the other.

Consider the collectivity $C^{\bullet}$ of all self-membered collectivities. Although vast, that collectivity is not a membership plenum. Its sub-collectivities include ø, as does the power collectivity of every collectivity, for the same reasons that every set includes ø as a subset. But ø by definition has no members. It cannot then contain itself as a member, and so cannot be a member of the collectivity $C^{\bullet}$ of all self-membered collectivities. The collectivity $C^{\bullet}$ is not a membership plenum. Whether it is a plenum in any other sense may depend on the other links possible between its members and its sub-collectivities, but at this point there seems to be no reason to think it is a plenum at all.

Consider now the *Russell collectivity*, as invoked above: the collectivity $C^{\sim}$ of all non-self-membered collectivities. This clearly is an indeterminate collectivity. The assumption that it either does or does not contain itself as a member leads to contradiction. For $C^{\sim}$, both $C^{\sim} \in C^{\sim}$ and $C^{\sim} \notin C^{\sim}$ must fail.

Is $C^{\sim}$ a plenum? Interestingly, it neither is nor is not a membership plenum.

If $C^{\sim}$ were a membership plenum, it would contain all of its own sub-collectivities. But since every collectivity is a sub-collectivity of itself, were $C^{\sim}$ to contain all its sub-collectivities it would also contain itself as a member. That, we have seen, it does not.

If $C^{\sim}$ were a membership non-plenum, on the other hand, there would be at least one sub-collectivity that was a non-member of $C^{\sim}$. But consider any such collectivity I. If I is a (true) non-member of the collectivity of all non-self-membered collectivities, it must be self-membered. The collectivity I therefore has itself – I – as a member. But I is a sub-collectivity of $C^{\sim}$. As such, every member of I must be a member of $C^{\sim}$, and must thus be non-self-membered. Here again we have reached a contradiction.

The Russell collectivity of all non-self-membered collectivities is therefore an indeterminate collectivity that is neither a membership plenum nor a membership non-plenum. Here as before, whether it qualifies as a plenum of any other sort depends on links other than membership possible between its elements and its sub-collectivities.

The collectivity $C^{\bullet}$ of all self-membered sets, we have seen, is not a membership plenum. If it is an indeterminate collectivity, it is at least not for the reasons that $C^{\sim}$ is. For $C^{\sim}$, it cannot be maintained either that $C^{\sim} \in C^{\sim}$ or that $C^{\sim} \notin C^{\sim}$. For $C^{\bullet}$, on the other hand, it can be maintained without contradiction either that $C^{\bullet} \in C^{\bullet}$ or that $C^{\bullet} \notin C^{\bullet}$. Self-membership for $C^{\bullet}$, we might say, is underdetermined: either of two options is open, with nothing to direct us to one rather than the other.

What consideration of these two collectivities indicates is that questions of indeterminateness and questions of plenary status are logically independent; answers to one do not necessarily give us an answer to the other. If so, there will be plena that are entirely determinate, as well as plenums that are not. There will be indeterminate collectivities that are plenary, and indeterminate collectivities are not.

There is, however, this important link between the two:

> Every plenum must have an indeterminate collectivity as one of its sub-collectivities.

Consider any collectivity *C* which pairs a unique element with each element of its power collectivity. This can be thought of in terms of a transform Tr: $PC \rightarrow C$ which matches elements of the power collectivity *PC* with unique elements of *C*. For members *p* of *PC*, let

$$\rightarrow\mathrm{Tr}(p)$$

represent the *C*-partner of element *p* of *PC* in such a matching, and let

$$\leftarrow\mathrm{Tr}(c)$$

similarly represent the power-set partner for members *c* of *C*.

Now consider the collectivity *C** of all *c* such that $c \notin \leftarrow\mathrm{Tr}(c)$. *C** is the collectivity of all elements of the original collectivity that are not members of the element of *C*'s power set with which they are partnered.

*C** will be an element of the power collectivity *PC*, since each of its members is a member of *C*. But will $\rightarrow\mathrm{Tr}(C^*) \in C^*$, or not? Will that element of *C* that is the partner of *C** be an element of *C** or not? Both $\rightarrow\mathrm{Tr}(C^*) \in C^*$ and $\rightarrow\mathrm{Tr}(C^*) \notin C^*$ yield a contradiction in precisely the manner of the Russell collectivity.

*C** must accordingly be an indeterminate collectivity. Given the generality of the proof, every plenum *C* must have an indeterminate sub-collectivity of such a form.

Every plenum will thus contain an indeterminate sub-collectivity. Does that entail that plena will themselves necessarily be indeterminate? There does not seem any reason to think so.

Consider again the case of the collectivity $C^{\sim}$ of all non-self-membered collectivities. Such a collectivity can be neither a member nor a non-member of itself. But the grounds of indeterminacy regarding $\tilde{C}$ is its self-membership, not its status as a collectivity. The indeterminacy of $\tilde{C}$ need not, therefore, be contagious upward to a collectivity of all collectivities. There may be other grounds on which we have to conclude that the collectivity of all collectivities is itself indeterminate, but the indeterminacy of $\tilde{C}$ is not one of them. On these grounds, at least, it seems perfectly possible for there to be determinate plena, though all plena must have indeterminate sub-collectivities.

7. NUMBER BEYOND NUMBER

How many *facts* are there? We know, via Georg Cantor's Power Set Theorem, that there will be an uncountably infinite number of subsets of any countably infinite number of items. If each of the subsets of an infinitely complex world's objects gives rise to a fact uniquely characteristic of that particular subset, and if the number of such subsets is more than countably infinite, then too must the number of facts be uncountably infinite.

Other routes to this destination are also available.

The idea of any complete listing of all the facts is manifestly impossible. For consider the following statement. "*The list F of stated facts fails to have this statement on it*." But now suppose this statement to be on the list. Then it clearly does not state a fact, so that the list is after all not a list of the facts (contrary to hypothesis). And so it must be left off the list. But then in consequence that list will not be complete since the statement is true. Facts can never be listed *in toto* because there will always be further facts—facts about the entire list itself—that a supposedly complete list could not manage to register.[12]

We have noted that there can be no merely finite plena. Any plenum must contain all its subsets as elements, which gives us sets of its subsets as further elements, sets of its sets of its subsets, and so on.

Nor can there be a plenum that is merely countably infinite. For suppose any plenum with countably infinite elements. Among its subsets will be all finite and infinite collectivities of those elements. But that gives us a structure reflecting the reals rather than the natural numbers, and thus the plenum must be at least nondenumerably infinite.

The important point is that the argument can be repeated for *any* number proposed for the elements of a plenum, of any order of nondenumerable infinity. The elements of any set of that order will be outnumbered by its subsets, and for each of those subsets there will be a distinct member of the plenum. Plena extend beyond any set-size survey of them. For any notion of number defined in terms of sets, at least, plena are literally beyond number.

Plena are thus *oversize* collectivities in that from the angle of standard set theory they are immeasurably large—so massive as to resist the meaningful assignment of a cardinal number. Plena achieve this condition because, unlike orthodox sets, they are large enough to contain all of their own subsets, something that orthodox sets cannot possibly manage to do in view of the theorem that sets inevitably have more subsets than members.

Can plena be compared in terms of size? Despite the fact that they are beyond set-size number, the answer appears to be 'yes'.

As standard set theory has it, one set is of lesser or equal cardinality than another whenever the membership of the former can be mapped into that of the latter in such a ways that different members are always mapped to different members.

> card (S_1) $\leq$ card (S_2) iff there is a mapping m for S_1 into S_2 such that whenever $x \in S_1$ & $y \in S_2$ & $x \neq y$ then $m(x) \neq m(y)$)

S_1 and S_2 will have the same cardinality whenever both card (S_1) $\leq$ card (S_2) and also conversely card (S_2) $\leq$ card (S_1).

Such a relationship can be implemented by way of logico-conceptual correspondences in terms of plena. Consider, for example, a correspondence between items and sets:

> To any *set* we associate as its paired *item* that set itself, and moreover—
>
> To any *item* we associate as its paired *set* the set containing as its only member that *item* itself.

We could similarly implement a coordination relationship between sets and formulated truths via this correspondence:

> To any formulated *truth* t we associate the *set* $\{t\}$ consisting of that truth alone, and moreover—
>
> To any *set* S we associate the *truth* $S = S$ to the effect that it is self-identical.

With respect to things/items, sets, and formulated truths we obtain the result that these three plena are equinumerous.

Analogously, symbolically formulated *truths* can be coordinated with formulated *possibilities* via the pairing

$$x \approx \Diamond x$$

and formulated *falsehoods* can be coordinated with formulated *truths* via the pairing

$$x \approx \text{not-}x$$

Moreover, for every *truth* (syntactically understood) there is a unique corresponding *item/thing/object*, namely its characterizing formulation. And conversely for every *thing* x there is a unique corresponding *truth*, viz. the thesis $x = x$.

Whether this equinumerosity can be extended across the board for plena—so that one could simply speak of "the cardinality of a plenum"—is an open question. Weak inductive evidence would suggest, however, that an affirmative answer is not implausible.

In addition to numerosity, there is also the matter of structure. Structural order is a key feature of plena. Thus let P_1, P_2, P_3 be sub-collectivities of some plenum P. Then $\{P_1, P_2\}$ correlates with a unique P-member, as does $\{\{P_1, P_2\}, P_3\}$. Moreover $\{P_1, P_3\}$ correlates with a unique P-member, as does $\{P_1, \{P_2, P_3\}\}$. The uniqueness of the correspondence correlation provides the basis for an organizing structure. And the branching at issue here precludes plena from taking on a linear order. The explosive nature of plena blocks the road to the standard Cantorean process for well-ordering sets. As Cantor himself insists in the *Grundlagen*, sets are well-orderable and increasable manifolds, and he insisted that manifolds which fail to conform to these conditions are not sets. But that's exactly what plena do.

Moreover, the Cantorean tradition sees cardinal numbers in terms of sets, a cardinal number being defined by the set of all sets that can be coordinated 1-to-1 with a given set. On this basis plena are non-set collectivities that extend beyond number and there will be no such thing as the number of all numbers. But in this regard the situation will be quite altered when we move from sets to collectivities. With number seen in the light of *collectivities* there is no basis for denying plena access to numerosity.

The inherent structural order of plena means that truth-as-a-whole does not constitute a plenum *with respect to adjunctive compilation*. For it would seem that t_1 & (t_2 & t_3) represents the same truth as (t_1 & t_2) & t_3—and the same with facts. Syntactically construed, however, structurally constituted truth-complexes will indeed constitute a plenum. Thus in this context truths must be understood syntactically in terms of their formulation-structure rather than semantically in terms of their meaning-content. For only then will they constitute a plenum.

8. CONCLUSION

We have to face the facts—and ideally all of them. In doing so, we propose, what we are facing is a plenum of all facts. Propositions, truths, and states of affairs constitute plena as well. The conceptual universe is a plenum.

One salient result of these deliberations is that there is no inherent infeasibility to the idea of totalities of these kinds. One of the authors has argued previously that such totalities prove impossible ‘within any logic we have.’[13] Here we have taken that conclusion as a direct motivation for the exploration of new logical frontiers.

To be sure, plena are extraordinary and anomalous, requiring resort to measures as drastic as restriction in the range of the Law of Excluded Middle.[14] Their accommodation exacts a price in terms of nonstandardness and unorthodoxy. But it is one thing to require extraordinary measures and something decidedly different to be self-contradictory and flat-out impossible. And inherently impossible those grandiose plenary collectivities certainly are not.

Various important questions remain open. Further formal work on the rejection of the Law of Excluded Middle is called for, both here and with regard to other applications.[15] We have noted the inadequacy of set-theoretic model theory when dealing with claims regarding all sets, but full development of a plenary model theory remains. Plena, we have noted, are beyond number in terms of any set-like conception of number. But we have also broached the possibility of realms of plenary scope or magnitude beyond the familiar numerosity of sets.

NOTES

[1] Just as there may be truths that are never known, there may be truths that can be formulated in no human or finite language. It is this sense of "truths," akin to facts and propositions in being something beyond linguistic entities, that we use throughout. The problem of a totality of truths is avoidable if we restrict 'truths' to linguistic entities of some sort (see Christina Schneider, "Totalitäten—ein metaphysisches Problem?," in *Untersuchungen zur Ontologie,* ed. Guido Imaguire and Christina Schneider, Munich: Philosophia Verlag, 2006, pp. 3-22). Essentially the same problem would remain, however, for a totality of facts or states of affairs.

[2] Patrick Grim, "There is no set of all truths." *Analysis* 44 (1984), 206-208.

[3] Georg Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*. Berlin: Springer, p. 443.

[4] For more complete discussion see Grim, *The Incomplete Universe: Totality, Knowledge, and Truth,* (Cambridge, Mass., MIT Press, 1991.

[5] George Boolos, "To Be is to Be a Value of a Variable (or to Be Some Values of Some Variables)," *Journal of Philosophy* 81 (1984), 430-439, and "Nominalistic Platonism," Philosophical Review 94 (1985), 327-344, both reprinted in *Logic, Logic and Logic* (Cambridge, Harvard University Press, 1998). Agustín Rayo and Timothy Williamson, "A Completeness Theorem for Unrestricted First-Order Languages," in JC Beall, ed., *Liars and Heaps* (Oxford: Clarendon Press, 2003), pp. 331-356. See also Vann McGee, "Universal Universal Quantification," pp. 357-364 in the same volume. This paragraph should not be taken, of course, to suggest that semantics and ontology are entirely unrelated.

[6] As noted above, the Boolos approach is precisely an attempt to read quantification without a model-theoretic domain. Our attempt is to follow what we take to be the more intuitive line of recognizing such a domain and asking what it would have to be like. In his introduction to *Liars and Heaps*, with reference to the Boolos tradition, JC Beall notes the open question of whether there are any set-like entities suitable for use as semantic extensions on such an approach. That is another way of the question we are pursuing here.

NOTES

[7] See Georg Cantor "Beitraege zur Begruendung der transfiniten Mengenlehre, I" *Mathematische Annalen*, vol. 46 (1895), pp. 481-512.

[8] The shadow of indecisiveness that prevails here qualifies plenary totalities as *indeterminate* collectivities. However the indeterminateness at issue is not an artifact of "subjective" indecisiveness in our thought but of the indefiniteness at issue in their "objective" constitution.

[9] The function or order is going to be critical here—and also with some of the plena considered below (agenda and inventories, in specific). For joining the pair of sets {a,b},{c,d} and the pair {a,b,c}, {d} will yield one selfsame composite set, {a,b,c,d}. The uniqueness condition for plena is thus validated. However with ordered sets, the situation is resolved, seeing that {a,b}}, {c,d}} and {{a,b,c},{d}} are different.

[10] Nor, as indicated above, does it amount to the specific degree of membership familiar in fuzzy logics.

[11] This strong denial of the law of excluded middle, we think, is the appropriate response to what has been called the "classification problem." If a sentence is neither true or false, what is its classification? If neither p nor ¬p, what is the status of p? Although we will not develop the point here, that question itself assumes a *weak* denial of the law of excluded middle, which does allow a status for those things that fall through the cracks. This is precisely what a strong denial refuses commitment to.

[12] Granted, these deliberations assume a great deal about the nature of facts. But this is not the place for unraveling the relevant complications.

[13] Grim, *The Incomplete Universe*, op. cit..

[14] There are, however, other philosophical forces pushing in the same direction. An adequate treatment of vagueness is one.

[15] The work of Hartry Field is particularly promising in this regard, both for links to problems of vagueness and for an axiomatization that offers a weak and non-truth-functional conditional $P \rightarrow Q$ that flowers into classical material implication with the addition of excluded middle assumptions regarding P and Q. See Hartry Field, "A Revenge-Immune Solution to the Semantic Paradoxes," *Journal of Philosophical Logic* 32 (2003), 139-177, "The Semantic Paradoxes and Paradoxes of Vagueness," in JC Beall, ed., *Liars and Heaps*, pp. 262-311, and Stephen Yablo, "New Grounds for Naïve Truth Theory," pp. 312-330 in the same volume. As both Field and Stephen Yablo emphasize, there may be alternatives that are equally or even more satisfactory. Further formal exploration is called for.