The Concept of Randomness

by

NICHOLAS RESCHER
(Lehigh University Bethlehem, Pennsylvania)

> Though there be no such thing as *chance* in the world, our ignorance of the real causes of any event begets a like species of belief or opinion. HUME

I. *Introduction*

The idea of randomness is an important concept of inductive logic, and has special philosophical interest because of its bearing on the epistemology of prediction and explanation. A selection among alternatives is said to be *strictly random* if a prediction of any one of the possible outcomes is as justified or as well-founded as any other, i.e., if the weight of evidence for the realization of each of the outcomes is equal. Thus the analysis of the concepts of randomness and of random selection is useful, among other things, for the clarification of various epistemological issues, including aspects of inductive logic, of the theory of evidence, and of explanatory reasoning.

If the selection of an alternative is to quality as *strictly random,* the selection situation must be such that the sum-total of the weight of evidence for selecting the chosen alternative as outcome equals the weight of evidence for selection of its competing alternatives. A *strictly* random selection must be such that, on the evidence at hand, the choice actually cannot be justified, i.e. no acçeptable reason can possibly be forthcoming for preferring the predicted outcome to its alternatives. Thus a sequence of 0's and 1's can be called *strictly random* only when we cannot adduce any cogent reason whatever for selecting 0 rather than 1 if we are pressed to guess the digit occupying some (hitherto unexamined) place. These considerations indicate that randomnes obtains, in the first analysis, in the realm of thought: it characterizes states of affairs only obliquely, under the condition that certain items of knowledge regarding them are not to be had.

The relativity of random choice to the information in hand is most graphically illustrated by the case of a person who is placed into the situation of guessing the outcome of the toss of a coin which, wholly unbeknownst to him, actually has two heads. Now *for the subject,* selection of the prediction H or of the prediction T are wholly equivalent; the two choices being wholly indifferent, for him, due to the fact that the available information is entirely symmetric as between the two alternatives. Because of the equivalence of the available information, the selection of H or T is a random matter, despite the non-random character of the actual outcome. Randomness is thus an epistemological concept, not an ontological or a purely "factual" one, in the sense that randomness is relativized to our knowledge and ignorance as to the nature of things, and not to the latter *per se.* (Cf. our motto from Hume.)

The idea of randomness finds applicability primarily in connection with the outcomes of trials of chance events (tosses of a coin, rolls of a die, chance selection of a red-headed man from the population at large, etc.). I propose here to examine this concept of randomness more closely, and to attempt to clarify the conceptual and philosophical issues which are involved.

II. *Random Sequences*[1]

We will here be considering sequences of trial-outcomes of chance events. It will suffice to confine consideration to the case of trials with only two possible outcomes, such as tosses of a coin, or drawing a spade or a non-spade from a deck of cards, because the discussion of this case is readily generalized to cover the rest. Here then we will be considering infinite sequences $a_1, a_2, a_3, \ldots$ of 0's and 1's (say "failure" and "success"), bearing in mind that

[1] The non-mathematical reader is free to omit this section, and proceed immediately to section III.

the discussion based on this case can, *mutatis mutandis,* be generalized without difficulty.

When can we characterize such a sequence $a_1, a_2, a_3, \ldots$ as being random? Clearly only when there is no way of *predicting* the successive elements of the sequences which offers a better prospect of success than we would expect by chance. This notion of a "way of predicting" elements of a sequence is vague and imprecise idea. However, a precise and formal articulation of this idea, endowing it with the logically requisite precision, has been proposed by Alonzo Church.[2] The basis of Church's proposal is, in effect, the idea of what I shall term an *effectively calculable enumerating function.* This is a function ϕ of positive integers fulfilling the following three conditions: (1) for any positive integer i, $\phi(i)$ is defined and is a positive integer, (2) $\phi(i)$ takes on increasing values with i, that is, whenever $n>m$, we have that $\phi(n)>\phi(m)$, and (3) ϕ is an effectively calculable function in that it takes the form of an explicit, effective rule of calculation, i.e. for any integer i we are able to calculate, by means of a finite, complete, step-by-step rules of computation, the corresponding value of ϕ, viz. $\phi(i)$.[3]

On Church's proposal, an infinite sequence of 0's and 1's, $S=a_1, a_2, a_3, \ldots$, is *random* if the following two conditions are satisfied:

(i) The relative frequency of 1's in the sequence approaches a definite limit, say L:

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} a_i = L$$

[2] "On the Concept of a Random Sequence," *Bulletin of the American Mathematical Society,* vol. 46 (1940), pp. 130–135.

[3] Condition 3 could be reformulated as stating that ϕ is a recursive function, if "Church's Thesis", to the effect that every effectively calculable function is general recursive, be accepted. On the theory of recursive functions see S. C. Kleene's *Introduction to Mathematics* (Princeton, 1952).

(ii) Any subsequences of S, say $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ which is defined by means of calculable enumerating function ϕ, that is, which is such that $n_i = \phi(i)$, will have precisely the same limiting frequency of 1's, namely L:

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} a_{n_i} = L.$$

Here the first condition assures that there is a definite probability that a given (uninspected) place in the sequence S will be filled by a 1 (probability L) or by a 0 (probability (1–L)). The second condition rules out the possibility of a better-than-average success rate in effectively enumerable subsequences.

This analysis of randomness by Church thus provides a very neat formal definition for randomness with respect to sequences.

III. *"Randomnly Selected Elements"* and *"Arbitrarily Chosen Individuals"*

The concept of randomness which we have thus far been concerned to examine applies to the manner in which certain entries occur in sequences. It is easily possible also to extend this concept to selections made from a certain group or set of elements or individuals, by considering the outcome of sequences of selections from such a set. The randomness of such a sequence of selections can be considered from the point of view expounded above. This however leads to certain consequences as to the way in which the commonly adopted (and widely misused) terminology "randomnly selected elements" or "arbitrarily chosen individuals" can or cannot, with logical propriety and accuracy, be used. These consequences I propose now to consider.

It is clear that we can never, with logical propriety, speak of a single selection as being made randomnly *per se*. For *randomness* applies in the first instance, only to sequences of selections. And we can speak in a derivative way also of the randomness of a selection *process*, when good reason exists to suppose such a selection process to give rise to random sequence. Now we can also, again at another logical remove, speak of an element of a set as being "randomnly selected" if this element was designated by a random selection process. But it must be stressed that randomness properly speaking characterizes sequences and the selection processes by which sequences can come about; it does not apply to individual selections, and still less to the particular items that are selected. A single example will suffice to provide a graphic illustration of this fact. Consider the set $S = \{1, 2, 7, 8, 13\}$. Is 1 a randomnly selected element of S? Clearly this question is simply not susceptible of a yes or no answer as it stands; we must know the character of the selection process or sequence of selections with which this choice of 1 is associated. Randomness is a property of selection procedures or selection sequences, not of selections.

A conception of randomness which appears on the surface of it to be in conflict with this account of randomness is the use which make of the idea of "arbitrarily" or "randomnly" selected individuals, particularly when they employ this as a device to describe a group or collection of elements in terms of a "randomnly" or "arbitrarily" selected individual member. If this is interpreted in a naive and literal way, to imply that there can be such a thing as an "arbitrarily selected" or a "randomnly chosen" *individual*, the result is nothing but hopeless confusion and contradiction. For consider again the set $S = \{1, 2, 7, 8, 13\}$. What could an "arbitrarily selected" element of S, let us denote it by "s", be like? Is s prime? Is $s > 9$? Is s even? Is s odd? Is $s = 7$? All such questions, it is clear, must be answered negatively, for otherwise s ceases to be "randomnly" or "arbitrarily selected". Thus the idea of "arbitrarily selected" individuals leads to hopeless paradox and confusion.[4]

The simplest, and most cogent and direct line of argument demonstrating the untenability of the conception of "randomnly selected" or "arbitrarily chosen" individuals is the following. Let

[4] This argument against "random" individuals or elements is substantially that presented in the writer's paper "Can There Be Random Individuals?", *Analysis*, vol. 18 (1958), pp. 114–117.

us go directly to the root idea involved in the conception of a "random" or "arbitrary" individual. Assume that we could validly speak of a "randomnly chosen" or "arbitrarily selected" element s of a set S, where s is construed as a "typical" or "representative" element of S, to be so used in our discussion that the only characteristics of S which are to be taken into account are those which it has in common with all other members of the group. We are thus licensed to attribute to s those properties which belong to all members of S. Consequently, two rules govern the attribution of properties to s:

(R1) A property that is attributed to s must characterize all elements of S:

$$\phi s \supset (x)(x \in S \supset \phi x).$$

(R2) A property that characterizes all elements of S can be attributed to s:

$$(x)(x \in S \supset \phi x) \supset \phi s.$$

This seemingly natural conception of a "random" or "arbitrary", and therefore "typical" or "representative" element is readily shown to involve a logical self-contradiction. For let ϕ be $(\lambda x)(x = s)$. Then by (R1) we have,

(1) $$(s = s) \supset (x)(x \in S \supset x = s),$$

and therefore, since the antecedent of (1) is an inevitable tautology, we obtain

(2) $$(x)(x \in S \supset x = s).$$

Now if S has more than one member, then we have,

(3) $$(x)[x \in S \supset (\exists y)(y \in S \;\&\; y \neq x)].$$

Now let ϕ be $(\lambda x)(\exists y)(y \in S \;\&\; y \neq x)$. Then from (3) and (R2) we obtain,

(4) $$(\exists y)(y \in S \;\&\; y \neq s).$$

or equivalently

(5) $$\sim (y)(y \in S \supset y = s).$$

But (5) contradicts (2). Our effort to maintain the concept of a "random" element or individual construed along the lines of the rules (R1) and (R2) thus meets with a decisive logical check.

The cause of the difficulty is readily determined. It is (R1) that must be sacrificed: We cannot expouse the device of so *treating* an individual that "the only characteristics which are taken into account are those which it has in common with all other members of the group." For (R1) is equivalent with,

$$(6) \qquad (\exists x)(x \epsilon S \ \& \ \sim \phi x) \supset \sim \phi s,$$

and this has the consequence that when we find a property ϕ which is not possessed by some S-element, then we must attribute $\sim \phi$ to s, rather than merely not attributing ϕ to s. It is this failure to preserve a distinction between not-attributions and attributions-not which leads to shipwreck for the foregoing construction of the idea of "random" elements or individuals.

Thus, we have shown the concept of a "randomnly selected" or "arbitrarily chosen" individual or element that is to serve in a "typical" or "representative" role (i.e., obeys the principles R1 and R2) leads to outright contradiction. This demonstration decisively re-enforces the previous conclusion that randomness is not a property of individuals but resides in the mode or manner of designation, and not in the object designated.

V. *Philosophical Implications*

I propose now to elaborate some of the consequences of the foregoing argument that randomness is not a property of individuals, but only of sequences and of the processes by which they are generated. The first is the conclusion that randomness is perforce an *intensional* and not an *extensional* concept.

Let us consider the ways in which a random sequence can possibly be presented for our consideration. Infinite sequences can obviously never be presented *explicitly*, that is, they can not be written out *in toto*, with all of their terms overtly exhibited. An infinite sequence can be presented in only two ways: (1) by specifying the rule of calculation by means of which the terms of the sequence can be computed, or (2) by indicating the generating process by means of which as many terms of the sequence as we please can be produced. In virtue of the very nature of randomness, the first of these is excluded for random sequences; for these only the second mode of presentation, viz. non-computational generating processes, afford an adequate vehicle of presentation.

These considerations shift randomness into the family of intensional concepts. A concept is extensional only when it is defined solely and entirely in terms of reference to the items or objects that serve as its constitutent elements. But just this is impossible in the sphere in which randomness has application. For randomness, as we have seen, has reference not to particular objects or elements, but to the generating processes for sequences, or to the manner in which reference to objects is made. Randomness is thus an inevitably intensional concept.

A further epistemological aspect of the concept of randomness is its bearing on the concept of fairness in certain situations of choice in which a preferential selection must be made in the face of equivalent claims. Let us consider this problem of exclusive choice between conflicting, but equally meritorious claims, assuming that we have a case of two conflicting claims of entirely equal strength to an object that cannot possibly be divided between the claimants, but must go to one or another. Precisely because each of the claim is assumed to be equally meritorious, it is essential for fairness to exclude from the mechanism of choice, by deliberate and calculated measure, consideration of any and all "reasons" for preferring any particular claimant. Thus no way remains open but to effect the choice randomnly, for this is the *only* way in which it can be guaranteed that all contesting parties are treated with strict fairness.

Consider, for the sake of an illustration, Bayle's criticism of Spinoza's discussion of the problem of Buridan's Ass.[5] Bayle writes,

> [One] mode of resolution is that of fate or chance. A man is assigned the task of deciding the precedence of two ladies at court. If he finds nothing about them to support a determination, and it is quite neces-

[5] *Ethics II,* final scholion. A detailed study of the logic and of the history of the problem of Buridan's Ass is presented in the writer's paper on "Choice Without Preference", forthcoming in *Kantstudien.*

sary that he must give one precedence over the other, he would not be brought to a stand-still, for he would simply have them draw straws. The same would be done in case a man has engaged himself to play at cards with each of two ladies, and wishes to avoid giving either of them any shade whatever of preference. The short straw would decide with whom he would play first. Thus the equilibrium need not render its victim immobile, as Spinoza would have it. One would be able to find the remedy.[6]

Bayle quite correctly perceives that the problem of choice without preference can take on two forms: (1) selection of one among several (exclusive) alternatives that are essentially identical as regards their desirability-status as objects of possession or realization, i.e. choice without preference among the *objects* involved (the problem of Buridan's Ass), and (2) selection of one among several alternative claimants, whose claims are indivisible and uncompromisable, and whose claims are essentially identical in strength, and must therefore in fairness be treated alike, i.e. choice without preference among the *subjects* involved. Bayle properly recognizes that the device of random selection provides a means of resolution that is entirely appropriate for both cases alike.

Random selection, it is clear, constitutes the sole wholly satisfactory manner of resolving exclusive choice between equivalent claims in a wholly fair and unobjectionable manner. Only random, and thus strictly "unreasoned" choice provides an airtight guarantee that there is no answer forthcoming to the question "why was this alternative, rather than another, selected?". Random choice thus guarantees that the other alternatives *might just as well* (in the strictest of senses) have designated. Where there is no way of *predicting* the outcome in advance no charge of preferential treatment can possibly be substantiated. Thus random choice affords the appropriate avenue of resolution for selection-situations in which considerations of fairness leave no other courses of immediate resolution open as acceptable or as defensible.[7]

[6] *Dictionnaire,* art. "Buridan", my translation.

[7] According to a *New York Times* report (Monday, January 12, 1959, page 6) "chance is the arbiter prescribed by Swedish law for breaking the votes in Parliament." The report states that "a drawing of lots may decide the fate

These considerations serve to indicate yet another philosophical aspect of the concept of randomness. This is its dependence upon the concept of unrealized possibilities. It is clear that a selection among alternatives cannot defensibly be characterized as "random" unless it can justifiably be asserted, after the fact of selection of some one particular alternative outcome, that the remaining alternatives could quite equally as well have been realized. Now it is, of course, true, due to the epistemological status of the randomness concept (as discussed above), that this requirement obtains relatively to the sphere of our knowledge, and amounts to saying that, so far as we are able to determine, i.e. *to the best of our knowledge,* the other alternatives could equally well have been realized. And this, to be sure, is true irrespective of whether in the particular case in question the other alternatives are "in fact" equipossible but unrealized possibilities. But this qualification does not undermine the validity of the point that it is necessary for randomness that the concept of unrealized possibilities be meaningfully applicable. For if it were the case, generally and *a priori* that the concept of realizable-but-unrealized possibilities were never, in any cases whatsoever, correctly applicable, then the concept of randomness would become wholly untenable. Thus randomness requires the concept of unrealized possibilities, in the sense of presupposing the meaningfulness of this concept for its own meaningfulness.

VI. *Conclusion*

In concluding, I wish by way of a summary to survey the principal conclusions which have emerged from the analysis of randomness here presented. It has become clear that randomness is an epistemological concept that takes on meaning relative to knowledge and ignorance: it obtains in the first analysis with respect to the sphere of our knowleadge, and characterizes states of affairs only obliquely, under the (indirect) condition that predictive knowledge about them is not to be had. Furthermore there

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of a controversial pension plan," but goes on to observe that "legislation by lottery has never yet been necessary on any major issue."

are no "randomnly selected" *elements* or "arbitrarily chosen" *individuals*: randomness characterizes only the processes of selection, not its results. There can be such thing as a "random" and thus "representative" or "typical" object. To talk in such terms is to accept what is in fact a highly misleading description of the notational device of ambiguous denotation, which affords a shorthand synopsis for statements about *each and every* particular element or individual.

We have seen that the idea of randomness belongs to the family of intensional concepts. Randomness does not characterize *objects,* but the manner in which reference to objects is made (eg., by the generating processes for sequences). Specifically, the concept of a random sequence is analysable in terms of predictability in such a way as to make possible a formal, mathematically precise concept corresponding to the informal idea of a "random sequence".

But perhaps the main point to have derived from our analysis is the conclusion that randomness presupposes the concept of unrealized possibilities. It requires that even those alternatives which were not obtained can (after the fact) be justly described as *real* possibilities that "could" have obtained, not merely with respect to our knowledge (i.e. "so far as we can tell") but at least in principle with respect to the "facts of the situation". If the idea of realizable-but-unrealized possibilities were never literally applicable, the concept of randomness too would become completely unviable.