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Fragments of quasi-Nelson: residuation

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ABSTRACT

Quasi-Nelson logic (QNL) was recently introduced as a common generalisation of intuitionistic logic and Nelson's constructive logic with strong negation. Viewed as a substructural logic, QNL is the axiomatic extension of the Full Lambek Calculus with Exchange and Weakening by the Nelson axiom, and its algebraic counterpart is a variety of residuated lattices called *quasi-Nelson algebras*. Nelson's logic, in turn, may be obtained as the axiomatic extension of QNL by the double negation (or involutivity) axiom, and intuitionistic logic as the extension of QNL by the contraction axiom. A recent series of papers by the author and collaborators initiated the study of fragments of QNL, which correspond to subreducts of quasi-Nelson algebras. In the present paper we focus on fragments that contain the connectives forming a residuated pair (the monoid conjunction and the so-called *strong Nelson implication*), these being the most interesting ones from a substructural logic perspective. We provide quasi-equational (whenever possible, equational) axiomatisations for the corresponding classes of algebras, obtain twist representations for them, study their congruence properties and take a look at a few notable subvarieties. Our results specialise to the involutive case, yielding characterisations of the corresponding fragments of Nelson's logic and their algebraic counterparts.

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1. Introduction

The introduction of *quasi-Nelson algebras* and their logical counterpart (*quasi-Nelson logic*), recent as it is (Rivieccio and Spinks (2019)), has already resulted in a substantial research output (Liang & Nascimento, 2019; Nascimento & Rivieccio, 2021; Rivieccio, 2020a, 2020b, 2022a; Rivieccio & Jansana, 2021; Rivieccio et al., 2020; Rivieccio & Spinks, 2020). As these papers demonstrate, a particularly fruitful trend has turned out to be the investigation of logics and classes of algebras corresponding to fragments of the quasi-Nelson language. Such a study was pursued in particular in Rivieccio et al. (2020), Rivieccio (2020b), Rivieccio (2020a), Rivieccio and Jansana (2021), Rivieccio (2022a), and Nascimento and Rivieccio (2021). The present paper is also a contribution to this research line.

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As discussed in the papers Rivieccio and Jansana (2021) and Rivieccio (2022a) – to which we also refer the reader for further background and motivation – the propositional language of quasi-Nelson logic/algebras, which coincides with that of Nelson's constructive logic with strong negation (Nelson, 1949), is particularly rich. Indeed, as far as fragments are concerned, the quasi-Nelson setting proved to be even more complex and interesting than that of Nelson logic, because a number of inter-definabilities among connectives are destroyed by the non-involutivity of the negation.

One of the prominent features of Nelson logic and Nelson algebras is that they may be presented in two equivalent propositional/algebraic languages, namely, either (i) the language $\{\wedge, \vee, *, \Rightarrow, 1\}$ of residuated lattices/substructural logics, or (ii) the language $\{\wedge, \vee, \rightarrow\}$ of intuitionistic logic/Heyting algebras enriched with a new (so-called 'strong') involutive negation (here denoted by $\sim$). Accordingly, one may regard Nelson algebras either (i) as a subclass of involutive (commutative, integral, and bounded) residuated lattices, i.e. models of involutive Full Lambek Calculus with Exchange and Weakening (Galatos et al., 2007), or (ii) as a class of Kleene algebras enriched with an intuitionistic-type implication operator. This dual view also applies, *mutatis mutandis*, to quasi-Nelson logic and algebras: but in this case the Nelson negation ($\sim$) is no longer required to be involutive, so for (i) we need to consider non-necessarily involutive residuated lattices (models of Full Lambek Calculus with Exchange and Weakening) and for (ii) we need the class of *quasi-Kleene algebras* (introduced in Rivieccio, 2020b) enriched with an intuitionistic-type implication.

When it comes to fragments of the (quasi-)Nelson language, one is thus able to play with an extended set of basic propositional connectives $\{\wedge, \vee, *, \rightarrow, \Rightarrow, \sim, 0, 1\}$, from which other standard ones may be defined (e.g. the bi-conditionals $\leftrightarrow$ and $\Leftrightarrow$ corresponding to the strong and weak implications $\rightarrow$ and $\Rightarrow$). Within quasi-Nelson logic, all these connectives are related by a number of inter-definabilities, determining a complex landscape of distinct fragments. In the involutive case, the $\{\wedge, \vee, \sim\}$-fragment (corresponding to the variety of *Kleene lattices*) was studied by Monteiro and his school (see e.g. Cignoli, 1986), while the 'two-negation' $\{\wedge, \vee, \sim, \neg\}$-fragment (obtained by adding a second negation given by $\neg x := x \rightarrow 0$, whereas the Nelson negation can be defined by $\sim x := x \Rightarrow 0$) is investigated in Sendlewski (1991) and shown to correspond to the class of *weakly pseudo-complemented Kleene algebras*. These studies were extended to the non-involutive setting of quasi-Nelson algebras in, respectively, Rivieccio (2020a, 2020b); see also Rivieccio et al. (2020).

The more recent papers Rivieccio and Jansana (2021), Rivieccio (2022a), and Nascimento and Rivieccio (2021), generalising earlier work on related structures (Rivieccio, 2014), characterise the $\{\sim, \rightarrow\}$-fragment of quasi-Nelson logic and the corresponding class of algebras. As argued in Rivieccio and Jansana (2021) and Rivieccio (2022a), the interest in this particular fragment is motivated by the observation that the connectives $\sim$ and $\rightarrow$ form a minimal 'algebraizable core' of (quasi-)Nelson logic (see Blok & Pigozzi, 1989); the above-mentioned papers also demonstrate that the corresponding models (*QNI-algebras*) form a well-behaved class of algebras with a rich structure theory.

In the present paper, we shift our attention to fragments of quasi-Nelson logic that contain the two 'substructural' connectives, the *strong* (monoid) *conjunction* ($*$) and the *strong implication* ($\Rightarrow$), which together form a residuated pair on every quasi-Nelson algebra (in this case viewed as a residuated lattice). Such fragments form the core of substructural logics and are therefore of traditional interest within this field (see e.g. Aglianò et al., 2007; Blok & Ferreirim, 1993, 2000; Blok & Raftery, 1997; Esteva et al., 2003); in the case of (quasi-)Nelson logic, they also appear to be key to a deeper understanding of certain peculiar features of the Nelson implication. We note that none of these fragments has been previously considered in the literature on Nelson logic; aside from technical difficulties, this may be explained by the fact that the substructural view on Nelson logic is a relatively recent achievement (Spinks & Veroff, 2008).

In this paper our main focus will be on the $\{*, \Rightarrow, \sim\}$-fragment of quasi-Nelson logic (which coincides with the $\{*, \to, \sim\}$-fragment, as we shall see) and on the $\{\wedge, *, \Rightarrow, \sim\}$-fragment (coinciding with the $\{\wedge, \to, \sim\}$-fragment); however, for technical as well as pedagogical reasons, we shall begin our study by first looking at the $\{*, \sim\}$-fragment. In each case, the principal question we will address is whether the algebraic counterpart of a given fragment of quasi-Nelson logic (i.e. the corresponding class of subreducts of quasi-Nelson algebras) can be axiomatised abstractly by means of identities or quasi-identities. Our main mathematical tool in this investigation will the twist-algebra representation, which will allow us to establish a bridge between the subreducts of quasi-Nelson algebras and more well-known subreducts of Heyting algebras (i.e. algebraic models of fragments of intuitionistic logic); from this connection we shall also derive further information and insight on the classes of algebras of interest.

While the main object of interest in the present paper is not the twist representation itself, it will be apparent as we proceed that it has been neccessary to extend the twist-algebra construction to more general classes of algebras than has been done so far in the literature on (quasi-)Nelson logic. Indeed, if the aim is to obtain a twist-type representation for a given class of algebras (say, within the (quasi-)Nelson family), then the non-involutivity of the negation constitutes a first and major technical difficulty, which has only been overcome thanks to the generalisation of the twist construction introduced in Rivieccio and Spinks (2019). A second difficulty arising in the present context derives from the fact that we will be working with a reduced subset of the algebraic language of (quasi-) Nelson logic. This explains why we will have to deal with somewhat exotic classes of algebras as factors in the twist construction (or will even have to introduce new ones: see e.g. Definitions 3.7, 4.4 and 5.3); these considerations also suggest that the present paper may be regarded as an exploration of the current boundaries of applicability of twist constructions.

Let us stress that, although we shall be dealing almost exclusively with classes of algebras (rather than logical systems), all of them are 'algebras of logic'; indeed, the main motivation for our study derives from non-classical logics, and our algebraic results have a clear logical interpretation. In particular, from our equational presentation of each of the main classes of algebras of interest, one can straightforwardly obtain a Hilbert-style axiomatization for the corresponding logical system by applying the standard methods available for algebraizable logics (Blok & Pigozzi, 1989).

For ease of reference, the classes of subreducts of quasi-Nelson algebras that have been characterised up to now (either in the present paper or elsewhere) are shown in

Table 1. Subreducts of quasi-Nelson algebras characterised so far.

Operations	Subreducts of QNA	Twist factor	Intuitionistic subclass
$\sim, \to$ [$\neg, 0, 1$]	quasi-Nelson implication algebras (Rivieccio, 2022a)	nuclear Hilbert semigroups	bounded Hilbert algebras
$\sim, *$ [$\neg, 0, 1$]	quasi-Nelson monoids	$\rightharpoonup$-semilattices	pseudo-complemented semilattices
$\sim, *, \to$ [$\Rightarrow, \neg, 0, 1$]	quasi-Nelson pocrims	implicative semilattices with $\Box$	bounded implicative semilattices
$\sim, \wedge, \to$ [$*, \Rightarrow, \neg, 0, 1$]	quasi-Nelson semihoops	$\oplus$-implicative semilattices	bounded implicative semilattices
$\sim, \neg, \wedge, \vee$ [$0, 1$]	quasi-Kleene algebras with weak pseudo-complement (Rivieccio, 2020a)	pseudo-complemented lattices with $\Box$	pseudo-complemented lattices
$\sim, \wedge, \vee, 0$ [1]	quasi-Kleene algebras (Rivieccio, 2020b)	distributive lattices with $\Box$	pseudo-complemented lattices

Table 1 (for each fragment, the term definable operations are shown between square brackets).

A systematic classification of fragments of quasi-Nelson (or even Nelson) logic is to this day still lacking, and it is not at all obvious that the techniques employed in the present paper (or in its predecessors) may be successfully adapted to arbitrary fragments; see Section 10 for a discussion of the potential difficulties one is likely to encounter.

Indeed, the very question of 'how many' fragments there are is relative to a chosen set of initial connectives; notice in this respect that the first fragment to be studied (in Sendlewski, 1991) is in the language $\{\wedge, \vee, \sim, \neg\}$ which includes the so-called 'intuitionistic negation' ($\neg$) that is not part of the basic set of connectives in which Nelson logic is usually presented. The same applies to the bi-conditionals $\leftrightarrow$ and $\Leftrightarrow$, which appear nevertheless to be of obvious interest from a logical point of view; indeed, it is easy to see that the fragments corresponding to the languages $\{\leftrightarrow, \sim\}$ and $\{\Leftrightarrow\}$ both form 'algebraizable cores' of quasi-Nelson logic in the sense of Rivieccio (2022a).

The preceding limitations notwithstanding, we believe that the study contained in the present paper constitutes a valuable contribution not only to the theory of (quasi-)Nelson logic/algebras but also, more generally, towards a more satisfactory understanding of the behaviour of connectives/operations (notably $*$ and $\Rightarrow$) that play a central role within substructural logics.

The paper is organised as follows. The next section contains preliminary results on the known classes of algebras we shall be working with, notably those corresponding to fragments of intuitionistic logic (Subsection 2.1) and quasi-Nelson algebras (Subsection 2.3), for which we also recall the fundamental twist representation result (Theorem 2.10). In Subsection 2.2, we introduce and briefly discuss the modal operators known as *nuclei*, which form an essential ingredient in our twist representation of non-involutive algebras.

Section 3 introduces the class of $\{*, \sim\}$-subreducts of quasi-Nelson algebras, which we dub *quasi-Nelson monoids* (Definition 3.2). We define this class through an abstract quasi-equational presentation (Subsection 3.1) and then introduce a corresponding twist construction (Subsection 3.2); the main result of the section is the twist representation (Theorem 3.18), stating that every quasi-Nelson monoid is embeddable into a twist-algebra.

In Section 4, we augment the language of quasi-Nelson monoids with the weak quasi-Nelson implication ($\rightarrow$); in this way the strong implication ($\Rightarrow$) also becomes term definable, so we have a residuated pair $(*, \Rightarrow)$, and the resulting class of algebras is a variety of bounded pocrims (partially ordered commutative residuated integral monoids) which we dub *quasi-Nelson pocrims* (Definition 4.9). In our study of this class of algebras, we rely on previous work (Nascimento & Rivieccio, 2021; Rivieccio, 2022a; Rivieccio & Jansana, 2021) on the class of $\{\rightarrow, \sim\}$-subreducts of quasi-Nelson algebras, known as *quasi-Nelson implication algebras* (Definition 4.1). The main representation result is Theorem 4.16.

Section 5 further expands the language of quasi-Nelson pocrim with the additive conjunction $(\wedge)$, which realises a semilattice operation on the algebras. In this way we obtain a class of bounded *semihoops* (Esteva et al., 2003) which we call *quasi-Nelson semihoops* (Definition 5.1). The corresponding representation result is Theorem 5.9.

In the following Section 6, we consider the question of when the previously defined embeddings may be upgraded to isomorphisms: this is answered positively in the case of quasi-Nelson pocrims (Theorem 6.9) and semihoops (Theorem 6.10), whereas the corresponding problem for quasi-Nelson monoids is open.

Section 7 shows how to embed each algebra in the above-mentioned classes into a quasi-Nelson algebra, thereby justifying the claim of having characterised the corresponding subreducts of quasi-Nelson algebras.

In Section 8, we apply the twist representations to obtain information on congruence-theoretic properties of the classes of algebras under study. The essential result is the existence of an isomorphism between the lattice of congruences of each subreduct of quasi-Nelson algebras and the lattice of congruences of the underlying factor algebra as given by the twist representation. This allows us to establish that quasi-Nelson implication algebras, quasi-Nelson pocrims and quasi-Nelson semihoops are all congruence-distributive varieties (by contrast, we verify that the congruence lattices of quasi-Nelson monoids do not satisfy any non-trivial identity). We further obtain an order-theoretic characterisation of subdirectly irreducible algebras (Corollary 8.10), and establish that all the algebras whose language includes the weak implication possess a (commutative, non-regular) ternary deduction term; hence these varieties also have equationally definable principal congruences and the strong congruence extension property (Proposition 8.14).

In Section 9, we take a look at a few notable subvarieties of the classes of algebras under consideration, focussing in particular on the correspondence between the identities satisfied by a given subvariety and the properties enjoyed by the corresponding factor algebras given by the twist representation.

Section 10 contains a few suggestions for potential directions of future research. To improve readability, the lengthier proofs of a number of results have been grouped together at the end of the paper, in the Appendix.

2. Preliminaries

As mentioned in the introduction, our main tool in the study of algebraic counterparts of fragments of quasi-Nelson logic will be the twist construction. The latter allows one to represent every algebra **A** in a given class K (in our standard example, K is the variety of Nelson algebras: see Definition 2.8) as a subalgebra of a special binary power of (i.e. as a *twist-algebra* over) some algebra **H**, which in the Nelson case is a Heyting algebra (Definition 2.4). The connection between Nelson and Heyting algebras is a fundamental one; discovered already in the 1970s by M.M. Fidel and D. Vakarelov, it would be summarised in the title of a later paper by A. Sendlewski: *Nelson algebras through Heyting ones* (Sendlewski, 1990).

Another paper by the same author (Sendlewski, 1991) showed that, even if **A** is not quite a Nelson algebra but a subreduct thereof (one that lacks, for instance, the implication connective), it may still be possible to represent **A** as a twist-algebra over a subreduct **L** of a Heyting algebra; in the case studied in Sendlewski (1991), the algebra **L** was a pseudo-complemented distributive lattice (i.e. the $\{\wedge, \vee, \neg\}$-subreduct of a Heyting algebra). We shall see that, as one considers weaker and weaker fragments of Nelson logic (corresponding to poorer algebraic languages), establishing a twist representation becomes a harder puzzle, until one reaches a point where the very mechanism of the twist construction seems to break down (regarding this, see the second research direction mentioned in the concluding Section 10).

A difficulty that is somehow orthogonal to the previous one arises if we consider, instead of subreducts, other classes of algebras in the same language as Nelson algebras but being more general in that they may not satisfy certain equational properties: for instance, non-necessarily integral Nelson algebras (i.e. *N4-lattices*) or non-necessarily involutive Nelson algebras (i.e. quasi-Nelson algebras: Definition 2.8). Twist representations covering these two cases have been introduced, respectively, in Odintsov (2004) and Rivieccio and Spinks (2019); both may in fact be seen as special cases of the construction recently proposed in Rivieccio (2022b).

In the present paper we tackle both the above-mentioned difficulties, for we shall be dealing with subreducts of non-necessarily involutive Nelson algebras. We will thus need to consider, as factors in our twist representations, algebraic structures that are related to but, more general, as a rule, than the subreducts of Heyting algebras; following the approach of Rivieccio and Spinks (2019), we will expand these algebras with a modal-like operator which is essentially designed to account for the extra freedom that the non-involutive negation enjoys on quasi-Nelson algebras.

In the next subsections we introduce a few definitions and basic results on the classes of algebras that are most relevant to the present study, beginning with the subreducts of Heyting algebras. We will start from the more general algebras presented in a minimal language, and then gradually add further properties and operations. Regarding the latter, we should already at this point warn the reader that, in order to simplify the notation, we will often overload certain symbols of algebraic operations – e.g. we use $\rightarrow$ to denote both the (Heyting) implication on the factor algebra and the (quasi-Nelson) implication on the twist-algebra, etc. – whenever we believe context will minimise the risk of confusion; in this we are following common practice on twist representations for Nelson algebras and related structures.

We assume familiarity with standard results on universal algebra and (residuated) lattices; for all unexplained terminology, we refer the reader to Burris and Sankappanavar (1981) and Galatos et al. (2007).

2.1. Subreducts of Heyting algebras

The purely implicational subreducts of Heyting algebras are known in the literature as *Hilbert algebras* or *(positive) implication algebras*.

Definition 2.1: A *Hilbert algebra* is an algebra $\langle H; \to, 1\rangle$ of type $\langle 2, 0\rangle$ that satisfies the following (quasi-)identities:

(i) $x \to (y \to x) = 1$.
(ii) $x \to (y \to z) = (x \to y) \to (x \to z)$.
(iii) if $x \to y = y \to x = 1$, then $x = y$.

Every Hilbert algebra has a natural order $\leq$ given, for all $a, b \in H$, by $a \leq b$ iff $a \to b = 1$, having 1 as top element (as a matter of fact, the constant 1 need not be included in the language, for it is term definable by $1 := x \to x$). If the order $\leq$ also has a minimum element (denoted 0), we speak of a *bounded Hilbert algebra*. In such a case we include 0 in the algebraic signature, and one can define a *negation* operation $\neg$ by $\neg x := x \to 0$. Bounded Hilbert algebras correspond to the $\{\to, \neg, 0, 1\}$-subreducts of Heyting algebras. Hilbert algebras that satisfy Peirce's law ($(x \to y) \to x = x$) are known as *Tarski algebras*, and are precisely the subreducts of Boolean algebras. A bounded Tarski algebra is just a Boolean algebra in disguise, for all Boolean operations become term definable.

The subreducts of Heyting algebras obtained by keeping only the infimum and the negation form the class of *pseudo-complemented semilattices* or *p-semilattices* (Frink, 1962; Sankappanavar, 1979).

Definition 2.2: A *pseudo-complemented semilattice* is an algebra $\langle S; \wedge, \neg, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ such that:

(i) $\langle S; \wedge, 0, 1\rangle$ is a bounded semilattice (with order $\leq$).
(ii) $x \leq \neg y$ (i.e. $x \wedge \neg y = x$) if and only if $x \wedge y = 0$.

We shall refer to item (ii) above as to the 'property of the pseudo-complement'. Pseudo-complemented semilattices form a variety (Galatos et al., 2007, p. 26) whose only proper subvariety is the class of Boolean algebras (Sankappanavar, 1979, p. 305); the latter can thus be relatively axiomatized by adding any identity that is not valid on all pseudo-complemented semilattices (for instance the involutive law $\neg\neg x = x$).

If we retain both the meet and the intuitionistic implication, we obtain *implicative semilattices* (also known as *Brouwerian semilattices*).

Definition 2.3: An *implicative semilattice* is an algebra $\langle S; \wedge, \to, 1\rangle$ of type $\langle 2, 2, 0\rangle$ such that:

(i) $\langle S; \wedge, 1\rangle$ is an upper-bounded semilattice (with order $\leq$ and top element 1).
(ii) $x \wedge y \leq z$ if and only if $x \leq y \to z$.

The property in item (ii) is known as *residuation*, and we shall say that $\langle \wedge, \rightarrow \rangle$ form a *residuated pair* (cf. Definition 2.7(iii) below). Implicative meet semilattices are precisely the $\vee$-free subreducts of Heyting algebras; in turn, the $\wedge$-free reduct of every implicative meet semilattice forms a Hilbert algebra. A *bounded implicative semilattice* is one whose semilattice reduct has a least element 0. In such a case, by letting $\neg x := x \rightarrow 0$, one obtains a pseudo-complemented semilattice.

Definition 2.4: A *Heyting algebra* is an algebra $\langle H; \wedge, \vee, \rightarrow, 0, 1 \rangle$ of type $\langle 2, 2, 2, 0, 0 \rangle$ such that:

(i) $\langle H; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice.
(ii) $\langle H; \wedge, \rightarrow, 1 \rangle$ is an implicative semilattice.

The pseudo-complement negation $\neg$ is defined, on every Heyting algebra, by $\neg x := x \rightarrow 0$, as in the case bounded implicative semilattices.

2.2. Nuclei

In this and the following sections we shall consider algebras that result from adding a modal-like operator to the subreducts of Heyting algebras introduced earlier. Such operators are known as *nuclei* (or *modal operators*, or *multiplicative closure operators*), and have been extensively studied in the literature on residuated lattices and Heyting algebras; for our purposes, the results contained in the dissertation by Macnab (1976) will be particularly useful. We shall consider two different but essentially equivalent definitions for a nucleus, which depend on which other operations are available on the algebra.

Definition 2.5: Let **A** be an algebra having a reduct $\langle A; \wedge, 0 \rangle$ that is a (meet-) semilattice with order $\leq$ and minimum 0. We shall say that an operation $\Box \colon A \rightarrow A$ is a *nucleus* on **A** if the following identities are satisfied:

(i) $x \leq \Box x = \Box\Box x$
(ii) $\Box(x \wedge y) = \Box x \wedge \Box y$
(iii) $\Box 0 = 0$.

The third condition is not usually included in the definition of nucleus, and those nuclei that satisfy it are called *dense*. In the present paper, however, we do not need this distinction, for we will only work with dense nuclei. Observe that the above properties entail that, if the order $\leq$ has a maximum element 1, then $\Box 1 = 1$ (so, $\Box$ is indeed a modal-like operator in that it preserves all finite meets).

When the underlying algebra does not have a meet operation, we can define a nucleus as follows.

Definition 2.6 (Rivieccio, 2022a, Def. 4.3): Given an algebra having a bounded Hilbert algebra reduct $\langle H; \rightarrow, 0, 1 \rangle$, we say that an operation $\Box \colon H \rightarrow H$ is a *nucleus* on **H** if:

(i) $x \leq \Box x = \Box\Box x$,
(ii) $\Box(x \to y) \leq \Box x \to \Box y$,
(iii) $\Box 0 = 0$.

Hilbert algebras with nuclei are considered in the recent paper (Celani & Montangie, 2020, Def. 6), but already appeared (under the name of 'positive implication algebras') since at least (Macnab, 1976, Ch. 13). An easy consequence of Macnab (1976, Thm. 13.13) which will be used later on is the following. If a bounded Hilbert algebra **H** satisfies Peirce's law (and is therefore term equivalent to a Boolean algebra), then the only possible nucleus on **H** is the identity map. This is because every nucleus satisfies $x \leq \Box x \leq \neg\neg x$; the same holds for any nucleus (in the sense of Definition 2.5) defined on a Boolean algebra (Macnab, 1976, Thm. 2.2).

Another useful observation is the following (which is essentially Macnab, 1976, Thm. 2.3); but see Rivieccio (2022a, Lemma 4.4) or Celani and Montangie (2020, Thm. 3) for a proof in the setting of Hilbert algebras): the two conditions (i) and (ii) above can be equivalently replaced by the single one:

(iv) $x \to \Box y = \Box x \to \Box y$.

It is easy to verify that both definitions of nucleus introduced above are equivalent on bounded implicative semilattices and Heyting algebras.

2.3. Quasi-Nelson algebras and their twist representation

We now proceed to introduce formally the class of *quasi-Nelson algebras*, which we view as a subvariety of the class of commutative, integral and bounded residuated lattices.

Definition 2.7: A *commutative integral bounded residuated lattice* (CIBRL) is an algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that:

(i) $\langle A; *, 1\rangle$ is commutative monoid, (Mon)
(ii) $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice (with order $\leq$), (Lat)
(iii) $x * y \leq y$ iff $x \leq y \Rightarrow z$. (Res)

On every CIBRL **A**, the presence of the constant 0 allows us to define a *negation* operation ($\sim$) given by $\sim x := x \Rightarrow 0$. A Heyting algebra **H** can be viewed as a CIBRL where the operations $\wedge$ and $*$ coincide (hence, the implication $\Rightarrow$ is the residuum of the meet $\wedge$).

Definition 2.8 (Rivieccio & Spinks, 2019): A *quasi-Nelson algebra* (QN-algebra) is a CIBRL that further satisfies the *Nelson identity*:

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) = x \Rightarrow y. \qquad \text{(Nelson)}$$

A *Nelson algebra* is a quasi-Nelson algebra that satisfies the involutive law $\sim\sim x = x$.

As mentioned earlier, QN-algebras have been introduced only recently, but are the subject of a rapidly growing literature (Liang & Nascimento, 2019; Nascimento & Rivieccio, 2021; Rivieccio, 2020a, 2020b, 2022a; Rivieccio & Jansana, 2021; Rivieccio et al., 2020; Rivieccio & Spinks, 2020). Nelson algebras, on the other hand, have been around for over four decades.[1] Every Heyting algebra satisfies the identity (Nelson), and is therefore an example of a QN-algebra on which the operations $\wedge$ and $*$ coincide (on the other hand, the only Heyting algebras that are also Nelson algebras are the Boolean algebras). As observed earlier, the class of quasi-Nelson algebras can thus be viewed as a common generalisation of Heyting and Nelson algebras.

An alternative language in which (quasi-)Nelson algebras have been traditionally considered is $\{\wedge, \vee, \to, 0, 1\}$, in which the residuated implication $\Rightarrow$ (in this context known as the *strong implication*) is replaced by the *weak implication* $\to$, defining:

$$x \Rightarrow y := (x \to y) \wedge ({\sim} y \to {\sim} x).$$

We shall see yet another and novel way of defining the operation $\Rightarrow$ in Proposition 2.11. On every QN-algebra $\mathbf{A}$, a second negation $\neg$ can be defined by the term $\neg x := x \to 0$, and it is easy to show that $\neg$ only coincides with $\sim$ iff $\mathbf{A}$ is a Heyting algebra. In turn, the weak implication is definable via the strong one by the term $x \to y := x \Rightarrow (x \Rightarrow y)$. Relying on this equivalence, and depending on convenience, we can thus employ either the strong or the weak implication to express the properties of QN-algebras we are interested in.

As discussed earlier, a most fundamental result on quasi-Nelson algebras (and their subreducts, as we shall see) is the twist representation, which we now proceed to introduce.

Definition 2.9: Let $\mathbf{H} = \langle H; \wedge, \vee, \to, \Box, 0, 1\rangle$ be a Heyting algebra with a nucleus. Define the algebra $\mathbf{H}^{\bowtie} = \langle H^{\bowtie}; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ with universe:

$$H^{\bowtie} := \{\langle a_1, a_2\rangle \in H \times H : a_2 = \Box a_2,\ a_1 \wedge a_2 = 0\}$$

and operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in H \times H$, by:

$$\begin{aligned}
1 &:= \langle 1, 0\rangle \\
0 &:= \langle 0, 1\rangle \\
\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &= \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle \\
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle \\
\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle &:= \langle a_1 \vee b_1, a_2 \wedge b_2\rangle \\
\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle &:= \langle (a_1 \to b_1) \wedge (b_2 \to a_2), \Box a_1 \wedge b_2\rangle.
\end{aligned}$$

A *quasi-Nelson twist-algebra over* $\mathbf{H}$ is any subalgebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$ satisfying $\pi_1[A] = H$.

As mentioned earlier, following common practice in the literature on Nelson logics, we overload certain algebraic symbols such as $\wedge$, $\vee$, 0 and 1 (and, later on, $\leq$ and $\to$) to denote both the operations on the Heyting algebra $\mathbf{H}$ and on the twist-algebra $\mathbf{H}^{\bowtie}$.

This greatly improves readability of proofs, and we trust that context will minimise the risk of confusion.

The $\{\wedge, \vee, 0, 1\}$-reduct of every quasi-Nelson twist-algebra is a bounded distributive lattice whose order $\leq$ is given by $\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle$ if and only if ($a_1 \leq b_1$ and $b_2 \leq a_2$). Indeed, every quasi-Nelson twist-algebra is a quasi-Nelson algebra on which the negation is given by $\sim x := x \Rightarrow 0$ and the weak implication by $x \rightarrow y := x \Rightarrow (x \Rightarrow y)$. These definitions give us:

$$\sim\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle \quad \text{and} \quad \langle a_1, a_2\rangle \rightarrow \langle b_1, b_2\rangle = \langle a_1 \rightarrow b_1, \Box a_1 \wedge b_2\rangle.$$

Moreover, every quasi-Nelson algebra is embeddable into a quasi-Nelson twist-algebra, as explained below.

Given a QN-algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ and $a, b \in A$, define:

$$a \equiv b \quad \text{iff} \quad a \rightarrow b = b \rightarrow a = 1.$$

The relation $\equiv$ thus obtained is compatible with the operations $\langle \wedge, \vee, *, \rightarrow\rangle$, though not necessarily with $\Rightarrow$ and $\sim$, giving us a quotient $\langle A/{\equiv}; \wedge, \vee, *, \rightarrow, 0, 1\rangle$. The latter is a Heyting algebra on which the operations $*$ and $\wedge$ coincide. Moreover, since $a \equiv b$ entails $\sim\sim a \equiv \sim\sim b$ for all $a, b \in A$, one can enrich the quotient $\langle A/{\equiv}; \wedge, *, \vee, \rightarrow, 0, 1\rangle$ with a well-defined operation, given by $\Box[a] := [\sim\sim a]$ for each class $[a] \in A/{\equiv}$, which turns out to be a nucleus. Letting $\mathbf{A}_{\bowtie} := \langle A/{\equiv}; \wedge, \vee, \rightarrow, \Box, 0, 1\rangle$, we can construct the twist-algebra $(\mathbf{A}_{\bowtie})^{\bowtie}$ as prescribed by Definition 2.9, obtaining the following result.

Theorem 2.10 (Representation of quasi-Nelson algebras, I): *Every quasi-Nelson algebra* $\mathbf{A}$ *embeds into the quasi-Nelson twist-algebra* $(\mathbf{A}_{\bowtie})^{\bowtie}$ *constructed according to Definition 2.9 through the map* ι *given by* $\iota(a) = \langle [a], [\sim a]\rangle$ *for all* $a \in A$.

Theorem 2.10 specialises to Nelson algebras (i.e. involutive quasi-Nelson algebras), thus allowing us to recover the well-known twist representation due to Fidel and Vakarelov. Indeed, a quasi-Nelson algebra $\mathbf{A}$ is a Nelson algebra if and only if $\mathbf{A}$ can be embedded into a twist-algebra $(\mathbf{A}_{\bowtie})^{\bowtie}$ such that the nucleus on $\mathbf{A}_{\bowtie}$ is the identity map. As in the case of Nelson algebras, Theorem 2.10 also yields a number of interesting consequences, including a characterisation of congruences on each quasi-Nelson algebra in terms of those on the corresponding Heyting algebra factor (see Rivieccio & Jansana, 2021, Sec. 3.2 and Section 8 in the present paper). The twist representation also helps one simplify computations on algebraic terms in the quasi-Nelson language. A first application of this is provided by the following proposition, which has a special significance to the present study.

Proposition 2.11: *Every quasi-Nelson algebra satisfies the following identity:*

$$x \Rightarrow y = (x \rightarrow y) * ((x \rightarrow y) \rightarrow (\sim y \rightarrow \sim x)).$$

Proof: Relying on Theorem 2.10, we work with a quasi-Nelson twist-algebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$. Let $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A$. Recall that

$$\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle = \langle (a_1 \to b_1) \wedge (b_2 \to a_2), \Box a_1 \wedge b_2\rangle.$$

We compute the first component of the expression:

$$(\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle) * ((\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle) \to (\sim\langle b_1, b_2\rangle \to \sim\langle a_1, a_2\rangle)),$$

which is $(a_1 \to b_1) \wedge ((a_1 \to b_1) \to (b_2 \to a_2)) = (a_1 \to b_1) \wedge (b_2 \to a_2)$, as required. The second component is:

$$((a_1 \to b_1) \to (\Box(a_1 \to b_1) \wedge \Box a_1 \wedge b_2)) \\ \wedge (((a_1 \to b_1) \to (b_2 \to a_2)) \to (\Box a_1 \wedge b_2)),$$

which we need to show to be equal to $\Box a_1 \wedge b_2$. Using the nucleus properties (and the assumptions $b_2 = \Box b_2$ and $b_1 \wedge b_2 = 0$), we have $\Box(a_1 \to b_1) \wedge \Box a_1 \wedge b_2 = \Box((a_1 \to b_1) \wedge a_1 \wedge b_2) = \Box(b_1 \wedge a_1 \wedge b_2) = \Box(a_1 \wedge 0) = \Box 0 = 0$. Thus, the above expression reduces to

$$((a_1 \to b_1) \to 0) \wedge (((a_1 \to b_1) \to (b_2 \to a_2)) \to (\Box a_1 \wedge b_2)).$$

Let us verify the inequality:

$$\Box a_1 \wedge b_2 \leq ((a_1 \to b_1) \to 0) \wedge (((a_1 \to b_1) \to (b_2 \to a_2)) \to (\Box a_1 \wedge b_2)).$$

Indeed, we have $\Box a_1 \wedge b_2 \leq ((a_1 \to b_1) \to (b_2 \to a_2)) \to (\Box a_1 \wedge b_2)$ simply by the properties of the Heyting implication. But also $\Box a_1 \wedge b_2 \leq (a_1 \to b_1) \to 0$ because, by residuation, the latter is equivalent to $\Box a_1 \wedge b_2 \wedge (a_1 \to b_1) = 0$. This holds, for we have $\Box a_1 \wedge b_2 \wedge (a_1 \to b_1) \leq \Box a_1 \wedge \Box b_2 \wedge \Box(a_1 \to b_1) = \Box(a_1 \wedge (a_1 \to b_1) \wedge b_2) = \Box(a_1 \wedge b_1 \wedge b_2) = \Box(a_1 \wedge 0) = \Box 0 = 0$. To check the converse inequality, we use residuation. We have:

$$((a_1 \to b_1) \to 0) \wedge (((a_1 \to b_1) \to (b_2 \to a_2)) \to (\Box a_1 \wedge b_2)) \leq \Box a_1 \wedge b_2$$

if and only if

$$((a_1 \to b_1) \to (b_2 \to a_2)) \to (\Box a_1 \wedge b_2) \leq ((a_1 \to b_1) \to 0) \to (\Box a_1 \wedge b_2).$$

The result then follows from this observation: by the properties of the Heyting implication, from $0 \leq b_2 \to a_2$ we have $(a_1 \to b_1) \to 0 \leq (a_1 \to b_1) \to (b_2 \to a_2)$ and, from the latter,

$$((a_1 \to b_1) \to (b_2 \to a_2)) \to (\Box a_1 \wedge b_2) \leq ((a_1 \to b_1) \to 0) \to (\Box a_1 \wedge b_2),$$

as required. ■

Proposition 2.11 is especially significant in the present context because it entails that the $\{*, \to, \sim\}$-fragment of quasi-Nelson logic (which we shall study in Subsection 4.2) is term equivalent to the $\{*, \Rightarrow, \sim\}$-fragment; we note that this observation does not seem to have ever been made before in the literature on Nelson logic.

To conclude the section, we mention a result first established in Rivieccio and Spinks (2020), which is also relevant to the present study. As shown in Rivieccio and Spinks (2020, Sec. 4), Theorem 2.10 can be sharpened so as to establish not just an embedding but a full isomorphism result, in the spirit of (and generalising) Sendlewski's representation of Nelson algebras and of their implication-free subreducts (Sendlewski, 1990, 1991).

Recall that the set $D(\mathbf{H})$ of the *dense elements* of a Heyting algebra $\mathbf{H}$ can be characterised as follows:

$$D(\mathbf{H}) = \{a \in H : \neg a = 0\}$$

where $\neg x := x \to 0$. $D(\mathbf{H})$ is a lattice filter of the lattice reduct of $\mathbf{H}$, and every lattice filter $\nabla \subseteq H$ such that $D(\mathbf{H}) \subseteq \nabla$ is said to be *dense*.

Proposition 2.12 (Rivieccio & Spinks, 2020, Prop. 9): *Let* $\mathbf{H}$ *be a Heyting algebra with a nucleus, and let* $\nabla \subseteq H$ *be a dense filter. Then the set:*

$$Tw(H, \nabla) := \{\langle a_1, a_2 \rangle \in H^{\bowtie} : a_1 \vee a_2 \in \nabla\}$$

is the universe of a twist-algebra over $\mathbf{H}$*, which we denote by* $\mathbf{Tw}(\mathbf{H}, \nabla)$.

An equivalent definition for $Tw(H, \nabla)$ is the following:

$$Tw(H, \nabla) = \{\langle a_1, a_2 \rangle \in H^{\bowtie} : \neg a_1 \to \neg\neg a_2 \in \nabla\}$$

where $\neg x := x \to 0$ (cf. Propositions 6.2 and 6.7).

Given a quasi-Nelson algebra $\mathbf{A}$, we let:

$$A^+ := \{a \in A : {\sim} a \leq a\}.$$

Notice that A^+ can also be characterised as follows:

$$A^+ = \{a \in A : {\sim} a \preceq a\} = \{a \vee {\sim} a : a \in A\}.$$

Proposition 2.13 (Rivieccio & Spinks, 2020, Prop. 10): *Let* $\mathbf{A} \leq \mathbf{H}^{\bowtie}$ *be a quasi-Nelson twist-algebra.*

(i) $A^+ = \{\langle a, 0 \rangle : \langle a, 0 \rangle \in A\}$ *is a lattice filter of* $\mathbf{A}$.
(ii) $\nabla_{\mathbf{A}} := \pi_1[A^+]$ *is a dense filter of* $\mathbf{H}$.
(iii) $A = Tw(H, \nabla)$.

The preceding proposition allows us to state the announced refinement of the representation theorem for quasi-Nelson algebras. Notice that, if $\mathbf{A}$ is any quasi-Nelson algebra (not necessarily identified with a twist-algebra), one can obtain a dense filter $\nabla_{\mathbf{A}}$ by letting $\nabla_{\mathbf{A}} := \{[a] \in A/{\equiv} : [a] = [b] \text{ for some } b \in A^+\}$.

Theorem 2.14 (Representation of quasi-Nelson algebras, II): *Every quasi-Nelson algebra* $\mathbf{A}$ *is isomorphic to the quasi-Nelson twist-algebra* $\mathbf{Tw}(\mathbf{A}_{\bowtie}, \nabla_{\mathbf{A}})$*, constructed according to Proposition 2.12, through the map* ι *given by* $\iota(a) = \langle [a], [{\sim} a] \rangle$ *for all* $a \in A$.

We will see in the next sections that Theorem 2.10 can be extended to all the subreducts of quasi-Nelson algebras under consideration. On the other hand, the proof of Theorem 2.14 relies on the presence of certain algebraic operations in the language (notably the pair $\langle *, \to \rangle$) and it is therefore unclear for the time being whether or how it might be established if these operations are lacking.

3. The $\{*, \sim\}$-fragment: quasi-Nelson monoids

In this section we begin our study by looking at the $\{*, \sim\}$-fragment of the quasi-Nelson language. As mentioned earlier, working initially with such a reduced language appears to be a convenient approach both from a technical and a pedagogical point of view. The idea is to first obtain information and insight on weaker algebras and then proceed by specialisation as we add language, properties and structure.

Observe that the constants 0 and 1 are already term definable in this fragment by $0 := x * \sim x$ and $1 := \sim 0$ (cf. Lemma 3.6). Furthermore, in the involutive case, the strong implication $\Rightarrow$ and the weak $\to$ are also definable by $x \Rightarrow y := \sim(x * \sim y)$ and $x \to y := x \Rightarrow (x \Rightarrow y)$. Thus the $\{*, \sim\}$-fragment of Nelson logic is equivalent to the $\{*, \to, \Rightarrow, \sim\}$-fragment, which we will look at (in full generality) in Subsection 4.2.

In the next subsection we introduce the quasi-variety QNM of *quasi-Nelson monoids* (Definition 3.2). We shall eventually prove that QNM is precisely the class of $\{*, \sim\}$-subreducts of quasi-Nelson algebras (Corollary 7.8); this result shall be established in two steps, the first being a new kind of twist representation whose factors are bounded semilattices having an implication-type operation and a term definable nucleus (Subsection 3.2).

3.1. Quasi-Nelson monoids

Following standard notation on monoids and residuated lattices, given a natural number n, we define the term:

$$x^n := \underbrace{x * \ldots * x}_{n \text{ times}}.$$

We set $x^0 := 1$ and $x^1 := x$.

We say that the operation $*$ is *n-potent* when the identity $x^n = x^{n+1}$ is satisfied.

Definition 3.1: A *3-potent commutative monoid* is an algebra $\mathbf{M} = \langle M; *, 1 \rangle$ of type $\langle 2, 0 \rangle$ such that:

(i) $\langle M; *, 1 \rangle$ is a commutative monoid, (Mon)
(ii) $\mathbf{M} \models x^2 = x^3$. (3-Pot)

In what follows, we shall write $x \preceq y$ as a shorthand for the identity $x^2 = x^2 * y^2$, and $x \equiv y$ as a shorthand for ($x \preceq y$ and $y \preceq x$). We also write $x \leq y$ instead of ($x \preceq y$ and $\sim y \preceq \sim x$), and we use the abbreviation $x \rightharpoonup y := \sim(x * x * \sim y)$.

Definition 3.2: A *quasi-Nelson monoid* (QNM) is an algebra $\mathbf{M} = \langle M; *, \sim, 0, 1 \rangle$ of type $\langle 2, 1, 0, 0 \rangle$ such that:

(i) $\langle M; *, 1\rangle$ is a 3-potent commutative monoid (Definition 3.1), (3-CM)
(ii) the relation induced by $\leq$ is a partial order on A. (PO)
(iii) **M** satisfies the following (quasi-)identities:
  (1) $x * y \leq x$.
  (2) if $x \leq y$, then $x * z \leq y * z$.
  (3) $x \rightharpoonup y \equiv x^2 \rightharpoonup y^2$.
  (4) $(x * y) \rightharpoonup z = x \rightharpoonup (y \rightharpoonup z)$.
  (5) $x \rightharpoonup (y * z) \equiv (x \rightharpoonup y) * (x \rightharpoonup z)$.
  (6) $x \leq \sim y$ if and only if $x * y = 0$.
  (7) $\sim x = \sim\sim\sim x$.
  (8) $\sim 1 = 0$ and $\sim 1 = 0$.
  (9) $\sim\sim(x * y) = \sim\sim x * \sim\sim y$.
  (10) $\sim(x * y) \equiv (x \rightharpoonup \sim y) * (y \rightharpoonup \sim x)$.

The class of all quasi-Nelson monoids will be denoted QNM.

Problem 3.3: It is clear from the definition that QNM is a quasi-variety, but is it in fact a variety?

Using the twist representation, it is straightforward (if tedious) to check that the $\{*, \sim, 0, 1\}$-reduct of every quasi-Nelson algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a quasi-Nelson monoid. Another example of quasi-Nelson monoid is provided by pseudo-complemented semilattices: this is in keeping with the observation that Heyting algebras are special quasi-Nelson algebras, so the same holds for their subreducts.

Proposition 3.4: *Every pseudo-complemented semilattice $\langle S, \wedge, \neg, 0, 1\rangle$ is a quasi-Nelson monoid (where $\wedge = *$ and $\neg = \sim$).*

Notice that Proposition 3.4 may be invoked to show that certain algebraic operations of quasi-Nelson algebras (such as $\vee$ and $\Rightarrow$) are not term definable on quasi-Nelson monoids. For, if they were, then the same terms would define the corresponding operations (join and implication) on Heyting algebras using just the meet and the pseudo-complement, which is well known not to be the case. In other words, as expected, there are quasi-Nelson monoids that are not quasi-Nelson algebras; similar considerations apply to the richer subreducts considered in the next sections. We shall obtain more examples of quasi-Nelson monoids thanks to the twist representation, towards which we now proceed.

Proposition 3.5: *Let* $\mathbf{M} \in$ QNM*, and let $a, b, c, d \in M$ be such that $a \equiv b$ and $c \equiv d$. Then:*

(i) $a * c \equiv b * d$.
(ii) $a \rightharpoonup c \equiv b \rightharpoonup d$.

Proof: (i). Straightforward. Indeed, assuming $a^2 = b^2$ and $c^2 = d^2$, we have $a^2 * c^2 = b^2 * d^2$. By commutativity of $*$, we have $a^2 * c^2 = (a * c)^2$ and likewise $b^2 * d^2 = (b *$

$d)^2$, from which the desired result immediately follows. (ii). From $a^2 = b^2$ and $c^2 = d^2$ we have $a^2 \rightharpoonup c^2 = b^2 \rightharpoonup d^2$. Then the required result follows from Definition 3.2 (iii).3 (and the transitivity of $\equiv$). ■

Proposition 3.5 suggests that we can factor each quasi-Nelson monoid $\mathbf{M} = \langle M; *, \sim, 0, 1\rangle$ by the relation $\equiv$, obtaining a partial quotient algebra $\langle M/{\equiv}; *, 0, 1\rangle$ which is easily verified to be a bounded semilattice. The latter can be further endowed with a binary operation $\rightharpoonup$ and a unary operation $\Box$ given by $\Box x := 1 \rightharpoonup x$, the former acting as an implication and the latter as a nucleus. These properties, which we shall study abstractly in the next subsection, are consequences of the following lemma; for a proof we refer the reader to the Appendix, where we have also collected the lengthier proofs of several subsequent results.

Lemma 3.6: *Every* $\mathbf{M} \in$ QNM *satisfies the following (quasi-)identities:*

- (i) $0 \leq x \leq 1$.
- (ii) $x * {\sim} x = 0$.
- (iii) $x \leq {\sim}{\sim} x$.
- (iv) *If* $x \preceq y$, *then* ${\sim}{\sim} x \preceq {\sim}{\sim} y$ *and* $x \preceq x * y$.
- (v) *If* $x \leq y$, *then* ${\sim} y \leq {\sim} x$.
- (vi) $x \leq y \rightharpoonup x$.
- (vii) $x \preceq {\sim} y$ *if and only if* $x \rightharpoonup {\sim} y = 1$.
- (viii) $x * y \preceq {\sim} z$ *if and only if* $y \preceq x \rightharpoonup {\sim} z$.
- (ix) $x \rightharpoonup y = {\sim}{\sim} x \rightharpoonup {\sim}{\sim} y = x \rightharpoonup {\sim}{\sim} y$.
- (x) ${\sim}(x * y) = {\sim}(x * {\sim}{\sim} y)$.

3.2. Twist-algebras over ⇀-semilattices

We are now going to introduce the class of algebras (⇀-*semilattices*) that shall be needed as factors in the twist representation of quasi-Nelson monoids; these are bounded semilattices having an implication-type operation and a term definable nucleus (Definition 3.7); we then define the twist-algebras arising as binary powers of ⇀-semilattices, which we call *QNM twist-algebras* (Definition 3.14). We will then show that each quasi-Nelson monoid may be identified, up to isomorphism, with a QNM twist-algebra (Theorem 3.18).

Definition 3.7: A ⇀-*semilattice* is an algebra $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ satisfying the following properties (we abbreviate $\Box x := 1 \rightharpoonup x$):

- (i) $\langle S; \wedge, 0, 1\rangle$ is a bounded semilattice (with order $\leq$).
- (ii) $x \rightharpoonup (y \rightharpoonup z) = (x \wedge y) \rightharpoonup z$.
- (iii) $x \rightharpoonup (y \wedge z) = (x \rightharpoonup y) \wedge (x \rightharpoonup z)$.
- (iv) $\Box 0 = 0$.
- (v) $x \leq \Box x$.
- (vi) $x \wedge \Box y = x \wedge (x \rightharpoonup y)$.
- (vii) $x \leq y \rightharpoonup z$ if and only if $x \wedge y \leq \Box z$.

(viii) $x \rightharpoonup y = \Box x \rightharpoonup \Box y$.

Items (ii)–(v) of the preceding Definition entail that, upon defining $\Box x := 1 \rightharpoonup x$, the operation $\Box$ indeed realises a nucleus (in the sense of Definition 2.5) on every $\rightharpoonup$-semilattice $\mathbf{S}$. Therefore, whenever convenient, we shall consider $\rightharpoonup$-semilattices as algebras in the language that includes the nucleus $\Box$ thus defined. The operation $\rightharpoonup$ can be thought of as a generalised (intuitionistic) implication in the following sense. For every algebra having a bounded implicative semilattice reduct $\langle S; \wedge, \to, 0, 1 \rangle$ as per Definition 2.3 and a nucleus $\Box$, we can obtain a $\rightharpoonup$-semilattice by letting $x \rightharpoonup y := x \to \Box y$ (cf. Definition 4.11 and Example 4.12). As a nucleus, we can for example take the double negation (which gives us $x \rightharpoonup y = x \to \neg\neg y$) or the identity function on S (yielding $\rightharpoonup \, = \, \to$). The latter case is clarified in the following proposition.

Proposition 3.8: *Let $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1 \rangle$ be a $\rightharpoonup$-semilattice. The following are equivalent:*

(i) $\mathbf{S} \models \Box x \leq x$.
(ii) $\langle S; \rightharpoonup, 1 \rangle$ *is a Hilbert algebra.*
(iii) $\langle S; \wedge, \rightharpoonup, 1 \rangle$ *is an implicative semilattice.*

Proof: Since every Hilbert algebra satisfies $\Box a = 1 \rightharpoonup a = a$, it suffices to show that (i) entails (iii). Assuming (i), we have that $\Box$ is the identity map. Then Definition 3.7(iii).7 tells us that $\wedge$ and $\rightharpoonup$ form a residuated pair, so $\langle S, \wedge, \rightharpoonup, 1 \rangle$ is an implicative semilattice. ■

As another natural example, consider any pseudo-complemented semilattice $\langle S; \wedge, \neg, 0, 1 \rangle$ as introduced in Definition 2.2. Upon defining $x \rightharpoonup y := \neg(x \wedge \neg y)$, we have that the algebra $\langle S; \wedge, \rightharpoonup, 0, 1 \rangle$ is a $\rightharpoonup$-semilattice. To verify this, we shall use the following (quasi-)identities, which are valid on all pseudo-complemented semilattices (Sankappanavar, 1979, p. 305):

(1) $x \wedge 0 = 0$.
(2) $x \wedge \neg(x \wedge y) = x \wedge \neg y$.
(3) $x \wedge \neg 0 = x$.
(4) $\neg\neg 0 = 0$.
(5) $x \leq \neg\neg x$.
(6) If $x \leq y$, then $\neg y \leq \neg x$.
(7) If $x \leq y$, then $\neg\neg x \leq \neg\neg y$.
(8) $\neg\neg\neg x = \neg x$.
(9) $\neg x \wedge \neg y = \neg\neg(\neg x \wedge \neg y)$.
(10) $\neg(\neg x \wedge \neg y) = \neg(\neg\neg x \wedge \neg\neg y)$.
(11) $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$.

Proposition 3.9: *Given a pseudo-complemented semilattice $\mathbf{P} = \langle P; \wedge, \neg, 0, 1 \rangle$, define $x \rightharpoonup y := \neg(x \wedge \neg y)$. Then $\langle P; \wedge, \rightharpoonup, 0, 1 \rangle$ is a $\rightharpoonup$-semilattice.*

We are going to see that, conversely, on every $\rightharpoonup$-semilattice one can define a pseudo-complement operation (Corollary 3.11). To show this, it is useful to establish a few properties of $\rightharpoonup$-semilattices that will simplify our computations. Given a $\rightharpoonup$-semilattice $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$, we abbreviate $\neg x := x \rightharpoonup 0$.

Proposition 3.10: *Every $\rightharpoonup$-semilattice satisfies the following (quasi-)identities.*

(i) $x \rightharpoonup x = 1$.
(ii) *If* $x \leq y$, *then* $z \rightharpoonup x \leq z \rightharpoonup y$.
(iii) *If* $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$.
(iv) $x \rightharpoonup y = 1$ *if and only if* $x \leq \Box y$.
(v) *If* $x \wedge y \leq z$, *then* $x \leq y \rightharpoonup z$.
(vi) $x \wedge y = 0$ *if and only if* $x \leq \neg y$.
(vii) $x \rightharpoonup \Box y = \Box x \rightharpoonup y = x \rightharpoonup y = \Box(x \rightharpoonup y)$.
(viii) $\neg x = \neg\Box x = \Box\neg x$.
(ix) $\neg(x \wedge y) = x \rightharpoonup \neg y$.
(x) $\neg\neg(x \rightharpoonup y) = x \rightharpoonup \neg\neg y$.
(xi) $x \rightharpoonup \neg y = y \rightharpoonup \neg x$.

The following result is a rephrasing of Proposition 3.10(vi).

Corollary 3.11: *Let* $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ *be a $\rightharpoonup$-semilattice. Then the algebra* $\langle S; \wedge, \neg, 0, 1\rangle$ *is a pseudo-complemented semilattice.*

Given Proposition 3.9 and Corollary 3.11, one might at this point wonder whether $\rightharpoonup$-semilattices as introduced in Definition 3.7 are just pseudo-complemented semilattices under an unusual presentation. This is not the case. Indeed, although we have seen that to every pseudo-complemented semilattice $\langle P; \wedge, \neg, 0, 1\rangle$ one can associate a $\rightharpoonup$-semilattice $\langle P; \wedge, \rightharpoonup, 0, 1\rangle$ by letting $x \rightharpoonup y := \neg(x \wedge \neg y)$, the latter algebra will satisfy $\Box x = 1 \rightharpoonup x = \neg\neg x$. That is, the definition $x \rightharpoonup y := \neg(x \wedge \neg y)$ corresponds to a particular choice (the maximal one in order-theoretic terms) for the nucleus operator on P; all the other possible choices allowed by Definition 3.7 (see e.g. Example 4.12) are not accounted for. This observation is made precise in the following propostion.

Proposition 3.12: *Let* $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ *be a $\rightharpoonup$-semilattice, with the pseudo-comple ment operation $\neg$ given by* $\neg x := x \rightharpoonup 0$. *The following are equivalent:*

(i) $\mathbf{S} \vDash \Box x = \neg\neg x$.
(ii) $\mathbf{S} \vDash x \rightharpoonup y = \neg(x \wedge \neg y)$.

Proof: Let us preliminarily observe that the inequality $a \rightharpoonup b \leq \neg(a \wedge \neg b)$ holds for all $a, b \in S$. Indeed, by the property of the pseudo-complement, the latter is equivalent to $a \wedge \neg b \wedge (a \rightharpoonup b) \leq 0$, which does hold true, for we have:

$$
\begin{aligned}
a \wedge \neg b \wedge (a \rightharpoonup b) &= a \wedge \Box b \wedge \neg b && \text{Definition 3.7(vi)}\\
&\leq a \wedge \Box b \wedge \Box\neg b && x \leq \Box x
\end{aligned}
$$

$$= a \wedge \Box(b \wedge \neg b) \qquad \Box(x \wedge y) = \Box x \wedge \Box y$$
$$= a \wedge \Box 0 \qquad x \wedge \neg x = 0$$
$$= a \wedge 0 \qquad \Box 0 = 0$$
$$= 0.$$

Hence, to see that (i) implies (ii), it suffices to verify the other inequality, $\neg(a \wedge \neg b) \leq a \rightharpoonup b$. By Definition 3.7(vii), the latter is equivalent to $a \wedge \neg(a \wedge \neg b) \leq \Box b$. Thus, assuming (i), we need to show that $a \wedge \neg(a \wedge \neg b) \leq \neg\neg b$. By the property of the pseudo-complement, the latter is equivalent to $\neg b \wedge a \wedge \neg(a \wedge \neg b) \leq 0$. As observed earlier (Sankappanavar, 1979, p. 305), the pseudo-complement also satisfies $\neg b \wedge \neg(a \wedge \neg b) = \neg b \wedge \neg a$, hence we have $\neg b \wedge a \wedge \neg(a \wedge \neg b) = a \wedge \neg b \wedge \neg a = 0 \wedge \neg b = 0$, as required.

Finally, to see that (ii) implies (i), it suffices to instantiate (ii): we have $\Box a = 1 \rightharpoonup a = \neg(1 \wedge \neg a) = \neg\neg a$. ■

Corollary 3.13: *Let* $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ *be a* $\rightharpoonup$*-semilattice. The following are equivalent:*

(i) $\mathbf{S} \models \neg\neg x \leq x$.
(ii) $\langle S; \wedge, \rightharpoonup, 0, 1\rangle$ *is a Boolean algebra.*

Proof: The only non-trivial direction is from (i) to (ii). Observe that $\Box a \leq \neg\neg a$ holds on every $\rightharpoonup$-semilattice $\mathbf{S}$ and for all $a \in S$. Indeed, by Proposition 3.10(vi), $\Box a \leq \neg\neg a$ is equivalent to $\Box a \wedge \neg a = 0$. The latter holds because, using Corollary 3.11, we have $\Box a \wedge \neg a \leq \Box a \wedge \Box\neg a = \Box(a \wedge \neg a) = \Box 0 = 0$. Thus, assuming (i), we have $\Box a \leq \neg\neg a \leq a$ for all $a \in S$. Then we can use Proposition 3.8 to conclude that $\langle S; \wedge, \rightharpoonup, 1\rangle$ is an implicative semilattice. Then, by the assumption that the pseudo-complement $\neg$ is involutive, we have that $\langle S; \wedge, \rightharpoonup, \neg, 0, 1\rangle$ is a Boolean algebra. ■

In the next definition we introduce the twist-algebra construction that will allow us to give a more concrete representation for quasi-Nelson monoids. In this setting, the involutive quasi-Nelson monoids (i.e. $\{*, \sim\}$-subreducts of Nelson algebras) will be obtained by restricting our attention to the twist-algebras whose underlying $\rightharpoonup$-semilattice $\mathbf{S}$ is in fact an implicative semilattice on which the nucleus is the identity map (cf. Proposition 3.8).

Definition 3.14: Let $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ be a $\rightharpoonup$-semilattice. Define the algebra $\mathbf{S}^{\bowtie} = \langle S^{\bowtie}; *, \sim, 0, 1\rangle$ with universe:

$$S^{\bowtie} := \{\langle a_1, a_2\rangle \in S \times S : a_2 = \Box a_2,\ a_1 \wedge a_2 = 0\}$$

and operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in S \times S$, by:

$$1 := \langle 1, 0\rangle$$
$$0 := \langle 0, 1\rangle$$

$$\sim\langle a_1, a_2\rangle := \langle a_2, \Box a_1\rangle$$
$$\langle a_1, a_2\rangle * \langle b_1, b_2\rangle := \langle a_1 \wedge b_1, (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2)\rangle.$$

A *QNM twist-algebra over* **S** is any subalgebra $\mathbf{M} \leq \mathbf{S}^{\bowtie}$ satisfying $\pi_1[M] = S$.

Let us check that the above-defined set $S^{\bowtie}$ is indeed closed under the algebraic operations. The case of the constants is immediate. Assume $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in S^{\bowtie}$. Then $a_2 \wedge \Box a_1 = \Box a_2 \wedge \Box a_1 = \Box(a_2 \wedge a_1) = \Box 0 = 0$ and $\Box\Box a_1 = \Box a_1$. Hence, $S^{\bowtie}$ is closed under the negation. Regarding $*$, using Definition 3.7(vi) and the nucleus properties, we have $a_1 \wedge b_1 \wedge (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2) = a_1 \wedge \Box b_2 \wedge b_1 \wedge \Box a_2 = a_1 \wedge b_2 \wedge b_1 \wedge a_2 = 0 \wedge 0 = 0$. Also, by Proposition 3.10(vii), we have $\Box((a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2)) = \Box(a_1 \rightharpoonup b_2) \wedge \Box(b_1 \rightharpoonup a_2) = (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2)$.

On every QNM twist-algebra $\mathbf{M} \leq \mathbf{S}^{\bowtie}$, we define the following relations:

(i) the pre-order $\preceq$ by $\langle a_1, a_2\rangle \preceq \langle b_1, b_2\rangle$ iff $a_1 = a_1 \wedge b_1$;
(ii) the equivalence relation $\equiv$ by $\langle a_1, a_2\rangle \equiv \langle b_1, b_2\rangle$ iff $(\langle a_1, a_2\rangle \preceq \langle b_1, b_2\rangle$ and $\langle b_1, b_2\rangle \preceq \langle a_1, a_2\rangle)$;
(iii) the partial order $\leq$ by $\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle$ iff $(\langle a_1, a_2\rangle \preceq \langle b_1, b_2\rangle$ and $\sim\langle b_1, b_2\rangle \preceq \sim\langle a_1, a_2\rangle)$.

We overload the symbol $\leq$ to denote both the above-defined partial order and the partial order of the semilattice **S**. Observe that $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$ are (respectively) the least and greatest element of $\leq$. The symbol $\rightharpoonup$ will also be overloaded to define, on each QNM twist-algebra $\mathbf{M} \leq \mathbf{S}^{\bowtie}$, the operation given by the following term:

$$x \rightharpoonup y := \sim(x * x * \sim y).$$

Notice that in the involutive case, i.e. on every (subreduct of a) Nelson algebra, the term $\sim(x * x * \sim y)$ gives us the weak Nelson implication $(\to)$.

For $\mathbf{M} \leq \mathbf{S}^{\bowtie}$ and $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in M$, it is useful to compute the following:

$$\langle a_1, a_2\rangle * \langle a_1, a_2\rangle = \langle a_1, a_1 \rightharpoonup 0\rangle = \langle a_1, \neg a_1\rangle$$
$$\langle a_1, a_2\rangle \rightharpoonup \langle b_1, b_2\rangle = \langle a_1 \rightharpoonup b_1, \Box a_1 \wedge b_2\rangle.$$

The former equality suggests that by letting $\neg x := \sim(x * x)$ we have an alternative way of introducing the 'intuitionistic' negation of quasi-Nelson algebras (defined e.g. in Rivieccio, 2020a by $\neg x := x \to 0$).

Let us justify the preceding equalities. The definition gives us $\langle a_1, a_2\rangle * \langle a_1, a_2\rangle = \langle a_1, a_1 \rightharpoonup a_2\rangle$. Observe that $\neg a_1 = a_1 \rightharpoonup 0 \leq a_1 \rightharpoonup a_2$ holds by Proposition 3.10(ii). On the other hand, by Definition 3.7(vii), we have $a_1 \rightharpoonup a_2 \leq a_1 \rightharpoonup 0$ if and only if $(a_1 \rightharpoonup a_2) \wedge a_1 \leq \Box 0 = 0$. By Definition 3.7(vi) and the requirements $a_1 \wedge a_2 = 0$ and $\Box a_2 = a_2$, we have $(a_1 \rightharpoonup a_2) \wedge a_1 = a_1 \wedge \Box a_2 = a_1 \wedge a_2 = 0$, as desired.

Regarding the second equality, the definition gives us $\langle a_1, a_2\rangle \rightharpoonup \langle b_1, b_2\rangle = \sim(\langle a_1, \neg a_1\rangle * \sim\langle b_1, b_2\rangle) = \langle (a_1 \rightharpoonup \Box b_1) \wedge (b_2 \rightharpoonup \neg a_1), \Box(a_1 \wedge b_2)\rangle$. Using Proposition 3.10(vii), the nucleus properties and the assumption $\Box b_2 = b_2$, we can obtain $\langle (a_1 \rightharpoonup \Box b_1) \wedge (b_2 \rightharpoonup \neg a_1), \Box(a_1 \wedge b_2)\rangle = \langle (a_1 \rightharpoonup b_1) \wedge (b_2 \rightharpoonup \neg a_1), \Box a_1 \wedge b_2\rangle$. It remains to show that $a_1 \rightharpoonup b_1 \leq b_2 \rightharpoonup \neg a_1$. Observe that, by Definition 3.7(ii), we have $b_2 \rightharpoonup \neg a_1 = b_2 \rightharpoonup (a_1 \rightharpoonup 0) = a_1 \rightharpoonup (b_2 \rightharpoonup 0)$. Thus, if we show $b_1 \leq b_2 \rightharpoonup 0$, the result will follow by Proposition 3.10(ii). By Definition 3.7(vii), the inequality $b_1 \leq b_2 \rightharpoonup 0$ is equivalent to $b_1 \wedge b_2 \leq \Box 0 = 0$, which is certainly true.

Proposition 3.15: *Let* $\mathbf{M} \leq \mathbf{S}^{\bowtie}$ *be a QNM twist-algebra. Then:*

(i) $\langle M; *, 1\rangle$ *is a commutative monoid, and* $*$ *is 3-potent.*
(ii) *For all* $a, b, c, d \in M$,
  (1) $a \preceq b$ *iff* $a^2 = a^2 * b^2$.
  (2) $a * b \leq a$.
  (3) *If* $a \leq b$, *then* $a * c \leq b * c$ ($*$ *is compatible with* $\leq$).
  (4) $a \rightharpoonup b \equiv a^2 \rightharpoonup b^2$.
  (5) $(a * b) \rightharpoonup c = a \rightharpoonup (b \rightharpoonup c)$.
  (6) $a \rightharpoonup (b * c) \equiv (a \rightharpoonup b) * (a \rightharpoonup c)$.
  (7) $a \leq {\sim} b$ *iff* $a * b = 0$.
  (8) ${\sim} a = {\sim}{\sim}{\sim} a$.
  (9) ${\sim} 1 = 0$ *and* ${\sim} 1 = 0$.
  (10) ${\sim}{\sim}(a * b) = {\sim}{\sim} a * {\sim}{\sim} b$.
  (11) ${\sim}(a * b) \equiv (a \rightharpoonup {\sim} b) * (b \rightharpoonup {\sim} a)$.

Proposition 3.15 immediately entails the following observation.

Corollary 3.16: *Every QNM twist-algebra (Definition 3.14) is a quasi-Nelson monoid (Definition 3.2).*

Let $\mathbf{M}$ be a quasi-Nelson monoid. By Proposition 3.5, the relation $\equiv$ determines a quotient $A/{\equiv}$ which can be endowed with two binary operations $*$ and $\rightharpoonup$. Let us write $\mathbf{M}_{\bowtie} := \langle M/{\equiv}, *, \rightharpoonup, 0, 1\rangle$. The equivalence class of each $a \in M$ in $\mathbf{M}_{\bowtie}$ will be often denoted by $[a]$.

Proposition 3.17: *For every* $\mathbf{M} = \langle M; *, {\sim}, 0, 1\rangle \in$ QNM, *the algebra* $\mathbf{M}_{\bowtie}$ *is a* $\rightharpoonup$-*semilattice with underlying order* $\leq$. *Moreover,* $[a] \leq [b]$ *iff* $a \preceq b$, *for all* $a, b \in M$.

Theorem 3.18 (Representation of QNM): *Every* $\mathbf{M} \in$ QNM *is embeddable into* $(\mathbf{M}_{\bowtie})^{\bowtie}$ *(constructed according to Definition 3.14) through the map* $\iota \colon M \to M_{\bowtie} \times M_{\bowtie}$ *given by* $\iota(a) := \langle [a], [{\sim} a]\rangle$ *for all* $a \in M$. *In other words, every* $\mathbf{M} \in$ QNM *is isomorphic to a QNM twist-algebra over* $\mathbf{M}_{\bowtie}$.

Proof: First of all, observe that ι is injective. Indeed, given $a, b \in M$ such that $\iota(a) = \iota(b)$, we have $a = b$ by Definition 3.2(ii). Further observe that the direct image $\iota(M)$ satisfies the properties required by Definition 3.14. We obviously have $\pi_1(\iota(M)) = M_{\bowtie}$. Also, for all $\langle [a], [{\sim} a]\rangle \in \iota(M)$, recalling Lemma 3.6(ii), we have $[a] \wedge [{\sim} a] = [a * {\sim} a] = [0]$ and, by Definition 3.2(iii).7, we have $\Box[{\sim} a] = [1] \rightharpoonup [{\sim} a] = [1 \rightharpoonup {\sim} a] = [{\sim}{\sim}{\sim} a] = [{\sim} a]$.

It remains to check that ι respects the operations. The case of the bounds is straightforward. Negation is also easy: one has $\iota(\sim a) = \langle [\sim a], [\sim\sim a] \rangle = \langle [\sim a], [1 \rightharpoonup a] \rangle = \sim\langle [a], [\sim a] \rangle = \sim\iota(a)$. Lastly, regarding the monoid operation, we have:

$$\begin{aligned}
\iota(a * b) &= \langle [a * b], [\sim(a * b)] \rangle \\
&= \langle [a] \wedge [b], [\sim(a * b)] \rangle \\
&= \langle [a] \wedge [b], [(a \rightharpoonup \sim b) * (b \rightharpoonup \sim a)] \rangle && \text{by Definition 3.2(x)} \\
&= \langle [a] \wedge [b], [a \rightharpoonup \sim b] \wedge [b \rightharpoonup \sim a] \rangle \\
&= \langle [a] \wedge [b], ([a] \rightharpoonup [\sim b]) \wedge ([b] \rightharpoonup [\sim a]) \rangle \\
&= \langle [a], [\sim a] \rangle * \langle [b], [\sim b] \rangle \\
&= \iota(a) * \iota(b).
\end{aligned}$$

■

Thanks to Theorem 3.18, we shall from now on assume, whenever convenient, that an arbitrary algebra $\mathbf{M} \in \mathsf{QNM}$ is a subalgebra of a twist-algebra $\mathbf{S}^{\bowtie}$ (we shall briefly write $\mathbf{M} \leq \mathbf{S}^{\bowtie}$), and we can take $\mathbf{S} = \mathbf{M}_{\bowtie}$.

4. Adding implication(s): quasi-Nelson pocrims

Having gained some insight into the behaviour of the quasi-Nelson monoid operation $(*)$, we are now going to look at how it interacts with the quasi-Nelson implications, the strong and the weak one. The latter was extensively studied in the papers (Rivieccio, 2022a; Rivieccio & Jansana, 2021), from which we shall import the relevant results.

4.1. QNI-algebras

The variety of *quasi-Nelson implication algebras* introduced in the above-mentioned papers (Definition 4.1 below) is precisely the class of $\{\to, \sim\}$-subreducts of quasi-Nelson algebras.

Let $\mathbf{A} = \langle A; \to, \sim, 0, 1 \rangle$ be an algebra of type $\langle 2, 1, 0, 0 \rangle$. Following Rivieccio (2022a), we write $x \preceq y$ instead of $x \to y = 1$, and $x \equiv y$ instead of $x \to y = y \to x = 1$. This notation is not really at odds with the one used in the previous sections: see Corollary 4.18. We shall also employ the following abbreviations:

$$x \odot y := \sim(x \to \sim y)$$

$$q(x, y, z) := (x \to y) \to ((y \to x) \to ((\sim x \to \sim y) \to ((\sim y \to \sim x) \to z))).$$

The usefulness of these terms was first demonstrated, in the context of (involutive) twist-algebras, already in Rivieccio (2014).

Definition 4.1 (Rivieccio, 2022a, Def. 3.1, Prop. 3.14): An algebra $\mathbf{A} = \langle A; \to, \sim, 0, 1 \rangle$ of type $\langle 2, 1, 0, 0 \rangle$ is a *quasi-Nelson implication algebra* (QNI-algebra) if the following identities are satisfied:

(i) $1 \to x = x$

(ii) $x \to (y \to x) = x \to x = 0 \to x = 1$
(iii) $x \to (y \to z) = y \to (x \to z) = (x \to y) \to (x \to z)$
(vi) $q(x, y, x) = q(x, y, y)$
(v) $x \odot (y \odot z) \equiv (x \odot y) \odot z$
(vi) $x \odot y \equiv y \odot x$
(vii) $\sim x = \sim\sim\sim x$
(viii) $\sim 1 = 0$ and $\sim 0 = 1$
(ix) $(x \to y) \to (\sim\sim x \to \sim\sim y) = 1$
(x) $x \odot (x \to y) \equiv x \odot y$
(xi) $\sim(x \to y) \equiv \sim\sim x \odot \sim y$.
(xii) $(x \odot y) \to z = \sim\sim x \to (\sim\sim y \to z)$
(xiii) $\sim x \to \sim y \equiv \sim x \to (\sim x \odot \sim y)$
(xiv) $(\sim\sim x \to \sim\sim y) \odot (\sim\sim x \to \sim\sim z) \equiv \sim\sim x \to (y \odot z)$.

The preceding presentation is a slight modification of the one from Rivieccio (2022a, Def. 3.1); we have deleted some redundant conditions and replaced the quasi-equational by equational ones (Rivieccio, 2022a, Prop. 3.14). QNI-algebras form a variety, which we denote by QNI. Every $\mathbf{A} \in$ QNI is partially ordered by the relation $\leq$ defined by $a \leq b$ iff $a \to b = \sim b \to \sim a = 1$ for all $a, b \in A$; the least and greatest element are, respectively, 0 and 1 (Rivieccio & Jansana, 2021, Lemma 22).

In subsequent proofs we shall use the properties of QNI-algebras listed in the following lemma (see Rivieccio, 2022a; Rivieccio & Jansana, 2021 for proofs and further details).

Lemma 4.2: *Every algebra having a QNI-algebra reduct satisfies the following (quasi-)identities:*

(i) $x \to x = 0 \to x = x \to 1 = x \to (\sim x \to y) = 1$.
(ii) $x \to y = x \to (x \to y)$.
(iii) $x \equiv 1$ *if and only if* $x = 1$.

Since Heyting algebras are quasi-Nelson algebras, and the $\{\to, \sim\}$-subreducts of Heyting algebras are bounded Hilbert algebras, it is to be expected that bounded Hilbert algebras constitute examples of QNI-algebras.

Example 4.3 (Rivieccio, 2022a, Prop. 3.11): Let $\langle A; \to, 0, 1\rangle$ be a bounded Hilbert algebra (Definition 2.1), on which the operation $\neg$ is given by $\neg x := x \to 0$. Taking $\sim = \neg$, we have that $\langle A; \to, \sim, 0, 1\rangle$ is a QNI-algebra.

The analogue of Theorem 3.18 for QNI states that every QNI-algebra $\mathbf{A}$ is embeddable into a twist-algebra $\mathbf{S}^{\bowtie}$ where $\mathbf{S}$ is a *bounded nuclear Hilbert semigroup*, a class of algebras which we now proceed to define.

Definition 4.4 (Rivieccio, 2022a, Def. 4.5): A *bounded nuclear Hilbert semigroup* (nH-semigroup for short) is an algebra $\mathbf{S} = \langle S; \odot, \to, 0, 1\rangle$ such that:

(i) $\langle S; \to, 0, 1\rangle$ is a bounded Hilbert algebra.

(ii) $\langle S; \odot \rangle$ is a commutative semigroup.
(iii) The operation $\Box$ given by $\Box x := x \odot x$ is a nucleus on $\langle S; \to, 0, 1 \rangle$ in the sense of Definition 2.6.
(iv) $x \odot y = x \odot (x \to y)$.
(v) $\Box x \to (\Box y \to z) = (x \odot y) \to z$.
(vi) $x \odot 0 = 0$.
(vii) $x \odot 1 = \Box x$.

The following example may provide some first intuition on nH-semigroups.

Example 4.5 (cf. Rivieccio, 2022a, Prop. 4.8): Let $\mathbf{A}$ be any algebra having a bounded Hilbert algebra reduct $\langle A; \to, 0, 1 \rangle$. Define $\neg x := x \to 0$ and $x \odot y := \neg(x \to \neg y)$. Then the algebra $\langle A; \odot, \to 0, 1 \rangle$ is an nH-semigroup where $\Box x = \neg\neg x$.

The twist-algebra construction for QNI-algebras is as follows.

Definition 4.6 (Rivieccio, 2022a, Def. 4.12): Let $\mathbf{S} = \langle S; \odot, \to, 0, 1 \rangle$ be an nH-semigroup. Define the algebra $\mathbf{S}^{\bowtie} = \langle S^{\bowtie}; \to, \sim, 0, 1 \rangle$ with universe:

$$S^{\bowtie} := \{\langle a_1, a_2 \rangle \in S \times S : a_2 = \Box a_2,\ a_1 \odot a_2 = 0\}$$

and operations given, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in S \times S$, by:

$$1 := \langle 1, 0 \rangle,$$
$$0 := \langle 0, 1 \rangle,$$
$$\sim\langle a_1, a_2 \rangle := \langle a_2, \Box a_1 \rangle,$$
$$\langle a_1, a_2 \rangle \to \langle b_1, b_2 \rangle := \langle a_1 \to b_1, a_1 \odot b_2 \rangle.$$

A *QNI twist-algebra over* $\mathbf{S}$ is any subalgebra $\mathbf{A} \leq \mathbf{S}^{\bowtie}$ satisfying $\pi_1[A] = S$.

As before, we write $x \preceq y$ instead of $x \to y = 1$, and $x \equiv y$ instead of $x \to y = y \to x = 1$. We also let $x \odot y := \sim(x \to \sim y)$. We observe that, for $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in S^{\bowtie}$, we have:

$$\langle a_1, a_2 \rangle \odot \langle b_1, b_2 \rangle = \langle a_1 \odot b_1, a_1 \to b_2 \rangle.$$

The following results are Rivieccio (2022a, Prop. 4.15) and Rivieccio (2022a, Thm. 4.16).

Proposition 4.7: *For every* $\mathbf{A} = \langle A; \to, \sim, 0, 1 \rangle \in$ QNI, *the relation* $\equiv$ *is compatible with the operations* $\to$ *and* $\odot$, *and the quotient* $\mathbf{A}_{\bowtie} := \langle A/\equiv; \to, \odot, 0, 1 \rangle$ *is an nH-semigroup.*

Theorem 4.8 (Representation of QNI): *Every algebra* $\mathbf{A} \in$ QNI *is isomorphic to a QNI twist-algebra over the nH-semigroup* $\mathbf{A}_{\bowtie}$ *through the map* $\iota \colon A \to A/\equiv \times A/\equiv$ *given by* $\iota(a) := \langle [a], [\sim a] \rangle$ *for all* $a \in A$.

We are now ready to investigate the interplay between the monoid operation $*$ and the weak implication $\to$ in the framework of (subreducts of) quasi-Nelson algebras; as before, we first introduce the abstract equational definition for the class of algebras of interest and then the twist-algebra construction.

4.2. Quasi-Nelson pocrims

In this subsection we aim at characterising the $\{*, \to, \sim, 0, 1\}$-subreducts of quasi-Nelson algebras; we shall call them *quasi-Nelson pocrims* (QNPs), for indeed, as we are going to see, each QNP is a partially ordered commutative residuated integral monoid. In the light of the previous sections, the class of QNPs will obviously consist of quasi-Nelson monoids structurally enriched with a (weak) implication (Corollary 4.17). By Proposition 2.11 we also know that it will correspond to the class of $\{*, \to, \Rightarrow, \sim, 0, 1\}$-subreducts of QN-algebras, and indeed also to the class of $\{*, \Rightarrow, 0\}$-subreducts (cf. Proposition 4.19 below).

Recall from Blok and Raftery (1997) that a *(commutative integral) pomonoid* is a structure $\langle P; \leq; *, 1\rangle$ such that:

(i) $\langle P; \leq\rangle$ is a partially ordered set having 1 as top element.
(ii) $\langle P; *, 1\rangle$ is a commutative monoid.
(iii) The order $\leq$ is compatible with the monoid operation (i.e. $x \leq z$ and $y \leq w$ entail $x * y \leq z * w$).

Observe that, for every quasi-Nelson monoid $\mathbf{M} = \langle M; *, \sim, 0, 1\rangle$, the structure $\langle M; \leq; *, 1\rangle$ is a commutative integral pomonoid (on which the partial order $\leq$ is given as in item (ii) of Definition 3.2).

A *pocrim* (partially ordered commutative residuated integral monoid) is a structure $\langle P; \leq; *, \Rightarrow, 1\rangle$ such that:

(i) $\langle P; \leq; *, 1\rangle$ is a pomonoid.
(ii) The pair $(*, \Rightarrow)$ is residuated, that is, $x * y \leq z$ if and only if $x \leq y \Rightarrow z$.

A pocrim is said to be *n-potent* when its monoid reduct is.

Definition 4.9: A *quasi-Nelson pocrim* (QNP) is an algebra $\mathbf{A} = \langle A; *, \to, \sim, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that:

(i) $\langle A; *, 1\rangle$ is a 3-potent commutative monoid (Definition 3.1).
(ii) $\langle A; \to, \sim, 0, 1\rangle$ is a QNI-algebra (Definition 4.1).
(iii) The following identities are satisfied:
  (1) $(x * y) \to z = x \to (y \to z)$
  (2) $x \to (y * z) \equiv (x \to y) * (x \to z)$
  (3) $\sim(x \to y) \equiv \sim\sim x * \sim y$
  (4) $\sim(x * y) \equiv (x \to \sim y) * (y \to \sim x)$.

It is convenient to leave for later (Proposition 4.19) a formal proof that quasi-Nelson pocrims are actually pocrims in the sense of Blok and Raftery (1997). By definition, quasi-Nelson pocrims form a variety, henceforth denoted by QNP. Using the twist representation, one can easily check that the $\{*, \to, \sim, 0, 1\}$-reduct of every (quasi-)Nelson algebra is indeed a member of QNP. As in the previous cases, other prominent examples can be found among the intuitionistic algebras. The proof of the following proposition, as well as any subsequent proof which has been omitted in this section, can be found in the Appendix.

Proposition 4.10: *Let* $\mathbf{S} = \langle S; \wedge, \rightarrow, 0, 1\rangle$ *be a bounded implicative semilattice (as per Definition 2.3) where the negation is given by* $\neg x := x \rightarrow 0$. *Taking* $* = \wedge$ *and* $\sim = \neg$, *we have that* $\mathbf{S}$ *is a quasi-Nelson pocrim.*

We now proceed to show that every quasi-Nelson pocrim may be represented as a twist-algebra over an implicative semilattice enriched with a nucleus operator.

Definition 4.11: A *bounded implicative semilattice with a nucleus* is an algebra $\mathbf{S} = \langle S; \wedge, \rightarrow, \Box, 0, 1\rangle$ such that:

(i) $\langle S; \wedge, \rightarrow, 0, 1\rangle$ is a bounded implicative semilattice;
(ii) $\Box$ is a nucleus (Definition 2.6) on the bounded Hilbert algebra reduct $\langle S; \rightarrow, 0, 1\rangle$.

The following example should help clarify the relationship among the new classes of algebras introduced so far.

Example 4.12 (cf. Rivieccio, 2022a, Lemma 4.6): Let $\mathbf{A}$ be any algebra having a reduct $\langle A; \wedge, \rightarrow, \Box, 0, 1\rangle$ that is a bounded implicative semilattice with a nucleus. Define $x \rightharpoonup y := x \rightarrow \Box y$ and $x \odot y := \Box x \wedge \Box y$. Then the algebra $\langle A, \wedge, \rightharpoonup, 0, 1\rangle$ is a $\rightharpoonup$-semilattice (Definition 3.7) and the algebra $\langle A; \odot, \rightarrow 0, 1\rangle$ is an nH-semigroup (Definition 4.4). In particular, by taking $\Box$ to be the identity map, we have that $\langle A, \wedge, \rightarrow, 0, 1\rangle$ is both a $\rightharpoonup$-semilattice and an nH-semigroup.

Proposition 4.7 extends to the class QNP as follows.

Lemma 4.13: *For every* $\mathbf{A} = \langle A; *, \rightarrow, \sim, 0, 1\rangle \in \mathsf{QNP}$, *the relation* $\equiv$ *is compatible with* $*$ *and the quotient* $\mathbf{A}_{\bowtie} := \langle A/\equiv; *, \rightarrow, \Box, 0, 1\rangle$ *is a bounded implicative semilattice with a nucleus given by* $\Box[a] := [\sim\sim a]$ *for all* $a \in A$.

The previous results suggest the following definition for twist-algebras.

Definition 4.14: Let $\mathbf{S} = \langle S; \wedge, \rightarrow, \Box, 0, 1\rangle$ be an implicative semilattice with a nucleus. Define the algebra $\mathbf{S}^{\bowtie} = \langle S^{\bowtie}; *, \rightarrow, \sim, 0, 1\rangle$ with universe:

$$S^{\bowtie} := \{\langle a_1, a_2\rangle \in S \times S : a_2 = \Box a_2,\ a_1 \wedge a_2 = 0\}$$

and operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in S \times S$, by:

$$1 := \langle 1, 0\rangle$$
$$0 := \langle 0, 1\rangle$$
$$\sim\langle a_1, a_2\rangle := \langle a_2, \Box a_1\rangle$$
$$\langle a_1, a_2\rangle * \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, (a_1 \rightarrow b_2) \wedge (b_1 \rightarrow a_2)\rangle$$

$$\langle a_1, a_2 \rangle \to \langle b_1, b_2 \rangle = \langle a_1 \to b_1, \Box a_1 \wedge b_2 \rangle.$$

A *QNP twist-algebra over* **S** is any subalgebra $\mathbf{A} \leq \mathbf{S}^{\bowtie}$ satisfying $\pi_1[A] = S$.

Taking into account the computations performed for the twist-algebras considered earlier, checking that the above-defined $S^{\bowtie}$ is closed under the algebraic operations of the twist-algebra is routine: regarding $\sim$ and $*$, see the proof just after Definition 3.14; regarding $\to$, see the proof following Rivieccio and Jansana (2021, Def. 26). We also omit the proof of the following proposition, which (in the light of Proposition 3.15 and Example 4.12) is a matter of straightforward computation.

Proposition 4.15: *Every QNP twist-algebra is a quasi-Nelson pocrim.*

Theorem 4.16 (Representation of QNP, I): *Every* $\mathbf{A} \in$ QNP *is isomorphic to a QNP twist-algebra over the implicative semilattice with a nucleus* $\mathbf{A}_{\bowtie}$ *(constructed according to Lemma 4.13 and Definition 4.14) through the map* $\iota\colon A \to A_{\bowtie} \times A_{\bowtie}$ *given by* $\iota(a) := \langle [a], [\sim a] \rangle$ *for all* $a \in A$*. In other words, every* $\mathbf{A} \in$ QNP *is isomorphic to a QNP twist-algebra over* $\mathbf{A}_{\bowtie}$*.*

Proof: The injectivity of ι follows from the injectivity of the corresponding map on QNI-algebras (Theorem 4.8). The requirement $\pi_1[\iota[A]] = A_{\bowtie}$ is obviously satisfied. Using the identity $a \to (\sim a \to 0) = (a * \sim a) \to 0$ (item (iii).1 of Definition 4.9), it is easy to check that $[a] \wedge [\sim a] = [a * \sim a] = [0]$, as required by Definition 4.14. It remains to check that ι is a homomorphism. Theorem 4.8 entails that ι preserves the negation. This is anyway straightforward, for we have: $\iota(\sim a) = \langle [\sim a], [\sim\sim a] \rangle = \langle [\sim a], \Box[a] \rangle = \sim\langle [a], [\sim a] \rangle = \sim \iota(a)$. As to the binary operations, we have:

$$\begin{aligned}
\iota(a * b) &= \langle [a * b], [\sim(a * b)] \rangle \\
&= \langle [a * b], [(a \to \sim b) * (b \to \sim a)] \rangle && \text{by Definition 4.9(iii).4} \\
&= \langle [a] \wedge [b], ([a] \to [\sim b]) \wedge ([b] \to [\sim a]) \rangle && \text{by Lemma 4.13} \\
&= \langle [a], [\sim a] \rangle * \langle [b], [\sim b] \rangle \\
&= \iota(a) * \iota(b)
\end{aligned}$$

$$\begin{aligned}
\iota(a \to b) &= \langle [a \to b], [\sim(a \to b)] \rangle \\
&= \langle [a \to b], [\sim\sim a * \sim b] \rangle && \text{by Definition 4.9(iii).3} \\
&= \langle [a] \to [b], [\sim\sim a] \wedge [\sim b] \rangle && \text{by Lemma 4.13} \\
&= \langle [a] \to [b], \Box[a] \wedge [\sim b] \rangle \\
&= \langle [a], [\sim a] \rangle \to \langle [b], [\sim b] \rangle \\
&= \iota(a) \to \iota(b).
\end{aligned}$$

■

With Theorem 4.16 at our disposal, the following statements can be easily verified.

Corollary 4.17: *For every* $\mathbf{A} = \langle A; *, \to, \sim, 0, 1\rangle \in \mathrm{QNP}$*, the reduct* $\langle A; *, \sim, 0, 1\rangle$ *is a quasi-Nelson monoid.*

Proof: Let $\mathbf{A} \in \mathrm{QNP}$, which we view as a QNP twist-algebra $\mathbf{A} \leq \mathbf{S}^{\bowtie}$ with $\mathbf{S} = \langle S; \wedge, \to, \Box, 0, 1\rangle$ an implicative semilattice with a nucleus. As we have seen (Example 4.12), defining $x \rightharpoonup y := x \to \Box y$, we have that $\langle S; \wedge, \rightharpoonup, 0, 1\rangle$ is a $\rightharpoonup$-semilattice (Definition 3.7). In order to be able to view the reduct $\langle A; *, \sim, 0, 1\rangle$ as a QNM twist-algebra (as given in Definition 3.14), it suffices to verify that, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A$, we have $\langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle = \langle a_1 \wedge b_1, (a_1 \to \Box b_2) \wedge (b_1 \to \Box a_2)\rangle$. The latter follows from the requirements $a_2 = \Box a_2$ and $b_2 = \Box b_2$ (in Definition 4.14). ■

The following observation justifies the notation employed so far, in the sense that, on any $\mathbf{A} \in \mathrm{QNP}$, the relation $\preceq$ can be equivalently defined by $\preceq := \{\langle a, b\rangle \in A \times A : a \to b = 1\}$ or by $\preceq := \{\langle a, b\rangle \in A \times A : a^2 * b^2 = a^2\}$.

Corollary 4.18: *Let* $\mathbf{A} = \langle A; *, \to, \sim, 0, 1\rangle \in \mathrm{QNP}$ *and* $a, b \in A$*. The following are equivalent:*

(i) $a \to b = 1$.
(ii) $a^2 = a^2 * b^2$.

Proof: It suffices to verify that, given a QNP twist-algebra $\mathbf{A} \leq \mathbf{S}^{\bowtie}$ and elements $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A$, one has $\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle = \langle 1, 0\rangle$ iff $a_1 \leq b_1$ iff $\langle a_1, a_2\rangle^2 * \langle b_1, b_2\rangle^2 = \langle a_1, a_2\rangle^2$. The latter equivalence has been shown in Proposition 3.15(ii).1, and the former in Rivieccio (2022a). ■

Notice that Corollary 4.18 entails that, on every $\mathbf{A} \in \mathrm{QNP}$, the partial order of the QNI-algebra reduct of $\mathbf{A}$ coincides with the partial order of the quasi-Nelson monoid reduct of $\mathbf{A}$, as was to be expected.

For the purpose of the next proposition, let us abbreviate:

$$x \Rightarrow_* y := (x \to y) * ((x \to y) \to (\sim y \to \sim x)).$$

As the notation suggests, the above-introduced term provides an alternative way to define the strong quasi-Nelson implication (cf. Proposition 2.11). This observation entails, in particular, that the class of $\{*, \to, \sim\}$-subreducts of (quasi-)Nelson algebras is term equivalent to the (in principle more expressive) class of $\{*, \Rightarrow, \sim\}$-subreducts. As far as the author is aware, this was never remarked before in the literature on Nelson logic.

The next proposition extends (Spinks & Veroff, 2008, Thm. 3.7) and (Spinks & Veroff, 2008, Prop. 3.10) to our non-involutive setting.

Proposition 4.19: *For every* $\mathbf{A} = \langle A; *, \to, \sim, 0, 1\rangle \in \mathrm{QNP}$*, we have that the algebra* $\langle A; \leq; *, \Rightarrow_*, 1\rangle$ *is a 3-potent pocrim (hence,* $\langle A; \Rightarrow_*, 1\rangle$ *is a (3-potent) BCK-algebra*[2]*).*

It is easy to check (on a twist-algebra) that the weak implication and the negation can be defined, as in quasi-Nelson algebras, by $x \to y := x \Rightarrow_* (x \Rightarrow_* y)$ and

$\neg x := x \Rightarrow_* 0$. This means that QNP may be viewed (and could be axiomatized) as a subvariety of bounded (3-potent) pocrims; this also allows one to import a number of results from the general theory of pocrims (see e.g. Corollary 8.5 and the subsequent remarks).

Problem 4.20: Consider the class of bounded (3-potent) pocrims satisfying:

$$(x^2 \Rightarrow y) * ((x^2 \Rightarrow y)^2 \Rightarrow ((\sim y)^2 \Rightarrow \sim x)) = x \Rightarrow y.$$

Is this class term equivalent to QNP?

Problem 4.21: Axiomatize the class of BCK-algebras that are the $\{\Rightarrow, 1\}$-subreducts of quasi-Nelson pocrims, and the class of pocrims that are the $\{*, \Rightarrow, 1\}$-subreducts of quasi-Nelson pocrims.

5. Adding meets: quasi-Nelson semihoops

5.1. Quasi-Nelson semihoops and their twist representation

In this section we bring one more connective of (quasi-)Nelson logic into the picture: namely the (additive) conjunction, corresponding to the lattice meet on (quasi-)Nelson algebras. We shall thus be looking at the $\{*, \wedge, \rightarrow, \sim\}$-fragment of quasi-Nelson logic and the corresponding subreducts of quasi-Nelson algebras. Let us begin with two observations:

(1) The $\{*, \wedge, \rightarrow, \sim\}$-fragment quasi-Nelson logic coincides with the $\{\wedge, \rightarrow, \sim\}$-fragment (as well as with the $\{\wedge, \Rightarrow, \sim\}$-fragment), because the monoid operation $*$ can be introduced through the term:

$$x * y := x \wedge y \wedge \sim((x \rightarrow \sim y) \wedge (y \rightarrow \sim x))$$

which can be more concisely rewritten as:

$$x * y := x \wedge y \wedge \sim(x \Rightarrow \sim y)$$

where, as usual,

$$x \Rightarrow y := (x \rightarrow y) \wedge (\sim y \rightarrow \sim x).$$

(2) We further note that the present study is only interesting in a non-involutive setting, because the $\{\wedge, \rightarrow, \sim\}$-fragment of Nelson logic coincides with the full logic. Indeed, the only missing connective (the disjunction $\vee$) can be defined through the De Morgan law by $x \vee y := \sim(\sim x \wedge \sim y)$. Since the disjunction is not definable in the $\{\wedge, \rightarrow, \sim\}$-fragment of intuitionistic logic, the term $\sim(\sim x \wedge \sim y)$ does not define the disjunction (i.e. the lattice-theoretic join) of quasi-Nelson logic either, but rather a 'pseudo-disjunction' which will nevertheless play an important role in the present setting.

In this section, as in the previous ones, we shall proceed from an abstract equational definition of a class of algebras (*quasi-Nelson semihoops*) to the more concrete twist representation. The main result towards which we shall proceed (Proposition 7.7

and Corollary 7.8) states that quasi-Nelson semihoops (Definition 5.1) are precisely the $\{\wedge, \to, \sim\}$-subreducts of quasi-Nelson algebras.

Semihoops were introduced in Esteva et al. (2003, Def. 3.6) as the algebraic counterpart of 0-free fragments of fuzzy logics. A *semihoop* can be defined as an algebra $\langle A; \wedge, *, \Rightarrow, 1\rangle$ of type $\langle 2, 2, 2, 0\rangle$ such that:

(i) $\langle A; \wedge, 1\rangle$ is a semilattice with order $\leq$ and 1 as top element.
(ii) $\langle A; \leq; *, \Rightarrow, 1\rangle$ is a pocrim.

The preceding definition is slightly more informative than the original one, but easily seen to be equivalent. A *hoop* (Esteva et al., 2003, Remark 3.11) may be defined as a semihoop $\langle A; \wedge, *, \Rightarrow, 1\rangle$ that satisfies the *divisibility* identity:

(iii) $x \wedge y = x * (x \Rightarrow y)$.

The meet is thus term definable on hoops, and is often omitted from the signature (for further background on hoops, see Blok & Ferreirim, 1993, 2000).

From now on, we shall abbreviate:

$$x \oplus y := \sim(\sim x \wedge \sim y).$$

As hinted at earlier, the operation $\oplus$ should be viewed as a pseudo-join, interpreting a generalised disjunction.

Definition 5.1: A *quasi-Nelson semihoop* (QNS) is an algebra

$$\mathbf{A} = \langle A; \wedge, *, \to, \sim, 0, 1\rangle$$

of type $\langle 2, 2, 2, 1, 0, 0\rangle$ such that:

(i) $\langle A; *, \to, \sim, 0, 1\rangle$ is a quasi-Nelson pocrim (Definition 4.9).
(ii) $\langle A; \wedge, 0, 1\rangle$ is a bounded semilattice whose partial order coincides with that of the pocrim reduct of $\mathbf{A}$.
(iii) The following identities are satisfied:
  (1) $x \oplus y \equiv x^2 \oplus y^2$
  (2) $\sim\sim(x \wedge y) = \sim\sim x \wedge \sim\sim y$
  (3) $\sim\sim\sim x = \sim x$
  (4) $\sim\sim x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$.

The class of all quasi-Nelson semihoops will be denoted by QNS. It is easy to verify that every member of QNS is, indeed, a semihoop in the terminology of Esteva et al. (2003), though not necessarily a hoop (see Lemma 9.4, Corollary 9.5 and the subsequent observations). As expected, the $\{\vee\}$-free reduct of every quasi-Nelson algebra is an example of a quasi-Nelson semihoop in the above sense. Thus, in particular, the $\vee$-free reduct of every Heyting algebra is a quasi-Nelson semihoop as well (on which $*$ and $\wedge$ coincide).

The 'De Morgan laws' proved in items (ii) and (iii) of the next lemma suggest that the operation $\oplus$ indeed plays the role of a pseudo-join.

Lemma 5.2 (cf. Celani, 2007, Lemma 1.1): *Every* $\mathbf{A} \in \mathsf{QNS}$ *satisfies the following identities:*

(i) $\sim(x \wedge y) = \sim(x \wedge \sim\sim y)$.
(ii) $\sim(x \wedge y) = \sim x \oplus \sim y$.
(iii) $\sim(x \oplus y) = \sim x \wedge \sim y$.
(iv) $\sim\sim x = x \oplus (x \wedge y)$.

Proof: Let all $a, b \in A$.

(i). We have:

$$\begin{aligned} \sim(a \wedge b) &= \sim\sim\sim(a \wedge b) && \text{by Definition 5.1(iii).3} \\ &= \sim(\sim\sim a \wedge \sim\sim b) && \text{by Definition 5.1(iii).2} \\ &= \sim(\sim\sim a \wedge \sim\sim\sim\sim b) && \text{by Definition 5.1(iii).3} \\ &= \sim(a \wedge \sim\sim b). \end{aligned}$$

(ii) Using the commutativity of the meet semilattice operation and the preceding item, we have $\sim(a \wedge b) = \sim(a \wedge \sim\sim b) = \sim(\sim\sim b \wedge a) = \sim(\sim\sim b \wedge \sim\sim a) = \sim(\sim\sim a \wedge \sim\sim b) = \sim a \oplus \sim b$.

(iii). We have:

$$\begin{aligned} \sim(a \oplus b) &= \sim\sim(\sim a \wedge \sim b) \\ &= \sim\sim\sim a \wedge \sim\sim\sim b && \text{by Definition 5.1(iii).2} \\ &= \sim a \wedge \sim b. && \text{by Definition 5.1(iii).3} \end{aligned}$$

(iv). By Lemma 3.6(v), from $a \wedge b \leq a$ we obtain $\sim a \leq \sim(a \wedge b)$. Then $a \oplus (a \wedge b) = \sim(\sim a \wedge \sim(a \wedge b)) = \sim\sim a$, as required. ■

Next we are going to prove that quasi-Nelson semihoops are representable as twist-algebras over the class of implicative semilattice enriched with (a nucleus and) an extra binary operation introduced below. As before, we overload the symbol $\oplus$ in the following definition: notice that this new operation also realises a pseudo-join.

Definition 5.3: A $\oplus$*-implicative semilattice* is an algebra $\mathbf{S} = \langle S; \wedge, \oplus, \to, 0, 1\rangle$ such that:

(i) $\langle S; \wedge, \to, \Box, 0, 1\rangle$ is a bounded implicative semilattice with a nucleus given by $\Box x := x \oplus x$ (Definition 4.11).
(ii) $\langle S; \oplus\rangle$ is a commutative semigroup.

The following identities are satisfied:

(iii) $x \oplus 1 = 1$.
(iv) $\Box x = x \oplus 0 = x \oplus (x \wedge y)$.
(v) $x \leq x \oplus y = \Box x \oplus \Box y$.

(vi) $\Box x \wedge (y \oplus z) = (x \wedge y) \oplus (x \wedge z)$.

Example 5.4: Let $\langle H; \wedge, \vee, \rightarrow, \Box, 0, 1\rangle$ be a Heyting algebra with a nucleus. Then, upon defining $x \oplus y := \Box(x \vee y)$, the algebra $\langle H; \wedge, \oplus, \rightarrow, 0, 1\rangle$ is a $\oplus$-implicative semilattice. Thus, in particular, every Heyting algebra may be viewed as a $\oplus$-implicative semilattice where, taking the nucleus $\Box$ to be the identity map, we have that $\oplus$ coincides with the lattice join, whereas taking $\Box$ to be the double negation map we have $x \oplus y = \neg(\neg x \wedge \neg y)$.

The following lemma will be useful later on.

Lemma 5.5: *Every $\oplus$-implicative semilattice $\mathbf{S} = \langle S; \wedge, \oplus, \rightarrow, 0, 1\rangle$ satisfies the following (quasi-)identities.*

(i) *If $x \leq z$ and $y \leq z$, then $x \oplus y \leq \Box z$.*
(ii) $\Box(x \oplus y) = x \oplus y$.
(iii) $x \rightarrow y \leq (x \oplus z) \rightarrow (y \oplus z)$.
(iv) $\neg(x \oplus y) = \neg x \wedge \neg y$.

Proof: Let $a, b, c, \in S$.

(i). Assuming $a \leq c$ and $b \leq c$, by Definition 5.3(iv) we have $\Box c = c \oplus (c \wedge a) = c \oplus a = c \oplus (c \wedge b) = c \oplus b$. Using the associativity and commutativity of $\oplus$, we then have $\Box c \oplus \Box c = c \oplus a \oplus c \oplus b = c \oplus c \oplus a \oplus b$. But $c \oplus c = \Box c$ and, similarly (using the nucleus properties), $\Box c \oplus \Box c = \Box\Box c = \Box c$. Hence, $\Box c = a \oplus b \oplus \Box c$, which entails $a \oplus b \leq \Box c$ by Definition 5.3(v).

(ii). Using the associativity and commutativity of $\oplus$, we have $\Box(a \oplus b) = a \oplus b \oplus a \oplus b = a \oplus a \oplus b \oplus b = \Box a \oplus \Box b$. Then the required result follows from Definition 5.3(v).

(iii). By Definition 5.3(v), we have $a \wedge (a \rightarrow b) = a \wedge b \leq b \leq b \oplus c$ and $c \wedge (a \rightarrow b) \leq c \leq b \oplus c$. From these two inequalities, using items (i) and (ii) in this lemma, we have $(a \wedge (a \rightarrow b)) \oplus (c \wedge (a \rightarrow b)) \leq \Box(b \oplus c) = b \oplus c$. Using also Definition 5.3(vi), we obtain $(a \wedge (a \rightarrow b)) \oplus (c \wedge (a \rightarrow b)) = \Box(a \rightarrow b) \wedge (a \oplus c) \leq b \oplus c$. From the latter, by residuation, we have $\Box(a \rightarrow b) \leq (a \oplus c) \rightarrow (b \oplus c)$, which immediately entails us the desired result.

(iv). From $a \leq a \oplus b$ (Definition 5.3(v)) and the pseudo-complement properties, we have $\neg(a \oplus b) \leq \neg a$ and, similarly, $\neg(a \oplus b) \leq \neg b$. Hence, by the meet semilattice properties, $\neg(a \oplus b) \leq \neg a \wedge \neg b$. Regarding the other inequality, by the property of the pseudo-complement, we have $\neg a \wedge \neg b \leq \neg(a \oplus b)$ iff $\neg a \wedge \neg b \wedge (a \oplus b) = 0$. By Definition 5.3(vi) and the nucleus properties, we have $\neg a \wedge \neg b \wedge (a \oplus b) \leq \Box(\neg a \wedge \neg b) \wedge (a \oplus b) = (\neg a \wedge \neg b \wedge a) \oplus (\neg a \wedge \neg b \wedge b) = (0 \wedge \neg b) \oplus (\neg a \wedge 0) = 0 \oplus 0 = \Box 0 = 0$, as required. ■

We are now ready to introduce the class of twist-algebras that correspond to quasi-Nelson semihoops.

Definition 5.6: Let $\mathbf{S} = \langle S; \wedge, \oplus, \to, 0, 1\rangle$ be a $\oplus$-implicative semilattice (Definition 5.3). Define the algebra $\mathbf{S}^{\bowtie} = \langle S^{\bowtie}; \wedge, *, \to, \sim, 0, 1\rangle$ with universe:

$$S^{\bowtie} := \{\langle a_1, a_2\rangle \in S \times S : a_2 = \Box a_2,\ a_1 \wedge a_2 = 0\}$$

and operations given, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in S \times S$, by:

$$\begin{aligned}
1 &:= \langle 1, 0\rangle, \\
0 &:= \langle 0, 1\rangle, \\
\sim\langle a_1, a_2\rangle &:= \langle a_2, \Box a_1\rangle, \\
\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, a_2 \oplus b_2\rangle \\
\langle a_1, a_2\rangle * \langle b_1, b_2\rangle &:= \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle \\
\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle &:= \langle a_1 \to b_1, a_1 \wedge b_2\rangle.
\end{aligned}$$

A *QNS twist-algebra over* $\mathbf{S}$ is any subalgebra $\mathbf{A} \le \mathbf{S}^{\bowtie}$ satisfying $\pi_1[A] = S$.

Checking that the set $\mathbf{S}^{\bowtie}$ is closed under the above-operations is straightforward. With regards to the $\wedge$ operation, we need to show that, if $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in \mathbf{S}^{\bowtie}$, then $a_1 \wedge b_1 \wedge (a_2 \oplus b_2) = 0$ and $\Box(a_2 \oplus b_2) = a_2 \oplus b_2$. For the latter equality, see Lemma 5.5(ii). As to the former, using Definition 5.3(vi), we have $a_1 \wedge b_1 \wedge (a_2 \oplus b_2) \le \Box(a_1 \wedge b_1) \wedge (a_2 \oplus b_2) = (a_1 \wedge b_1 \wedge a_2) \oplus (a_1 \wedge b_1 \wedge b_2) = (0 \wedge b_1) \oplus (a_1 \wedge 0) = 0 \oplus 0 = \Box 0 = 0$, as desired.

The proof of the following proposition, as well as any subsequent proof which has been omitted in this section, can be found in the Appendix.

Proposition 5.7: *Every QNS twist-algebra* $\mathbf{A} = \langle A; \wedge, *, \to, \sim, 0, 1\rangle \le \mathbf{S}^{\bowtie}$ *is a quasi-Nelson semihoop (Definition 5.1).*

The twist-algebra construction highlights the formal similarity between the component-wise definition of the pseudo-join operation $\oplus$ on quasi-Nelson semihoops and that of the actual join on quasi-Nelson algebras (Definition 2.9). Indeed, given a QNS twist-algebra $\mathbf{A} \le \mathbf{S}^{\bowtie}$ and $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A$, using Definition 5.3(v) and the usual nucleus properties, we obtain:

$$\langle a_1, a_2\rangle \oplus \langle b_1, b_2\rangle = \langle \Box a_1 \oplus \Box b_1, \Box(a_2 \wedge b_2)\rangle = \langle a_1 \oplus b_1, a_2 \wedge b_2\rangle.$$

As with the previous representations, given $\mathbf{A} \in \mathsf{QNS}$, we consider the quotient $\langle A/{\equiv}; *, \to, \Box, 0, 1\rangle$, which is a bounded implicative semilattice with a nucleus (Lemma 4.13). We enrich this algebra with the operation $\oplus$ given, for all $a, b \in A$, by $[a] \oplus [b] = [a \oplus b] = [\sim(\sim a \wedge \sim b)]$, which is well-defined by Definition 5.1(iii).1. Let $\mathbf{A}_{\bowtie} := \langle A/{\equiv}; *, \oplus, \to, \Box, 0, 1\rangle$.

Proposition 5.8: *For every* $\mathbf{A} = \langle A; \wedge, *, \to, \sim, 0, 1\rangle \in \mathsf{QNS}$*, the algebra*

$$\mathbf{A}_{\bowtie} := \langle A/{\equiv}; *, \oplus, \to, \Box, 0, 1\rangle$$

is a $\oplus$*-implicative semilattice.*

Proof: By Lemma 4.13, it suffices to check that the operation $\oplus$ defined on $A/{\equiv}$ satisfies items (ii)–(vi) of Definition 5.3. Let then $a, b, c \in A$.

To show (ii), observe that the operation $\oplus$ on **A** is obviously commutative, and it is also associative, for we have:

$$\begin{aligned}
(a \oplus b) \oplus c &= \sim(\sim\sim(\sim a \wedge \sim b) \wedge \sim c) \\
&= \sim(\sim c \wedge \sim\sim(\sim a \wedge \sim b)) \\
&= \sim(\sim c \wedge \sim a \wedge \sim b) && \text{by Lemma 5.2(i)} \\
&= \sim(\sim a \wedge \sim b \wedge \sim c) \\
&= \sim(\sim a \wedge \sim\sim(\sim b \wedge \sim c)) && \text{by Lemma 5.2(i)} \\
&= a \oplus (b \oplus c).
\end{aligned}$$

(iii). Recall that the $\{\rightarrow, \sim, 0, 1\}$-reduct of **A** is a QNI-algebra and the $\{*, \sim, 0, 1\}$-reduct of **A** is a quasi-Nelson monoid (Corollary 4.17). Then we have $\sim 1 = 0$, by Definition 4.1(viii). Moreover, by Lemma 3.6(i), we have that 0 is the least element of the reduct $\langle A; \wedge \rangle$. Then $a \oplus 1 = \sim(\sim a \wedge \sim 1) = \sim(\sim a \wedge 0) = \sim 0 = 1$ for all $a \in A$, which immediately entails the desired result.

Regarding (iv), it suffices to observe that the semilattice properties immediately give us $a \oplus 0 = \sim(\sim a \wedge \sim 0) = \sim(\sim a \wedge 1) = \sim\sim a = \sim(\sim a \wedge \sim a) = a \oplus a$, and Lemma 5.2(iv) gives us $\sim\sim a = a \oplus (a \wedge b)$. Hence, in the quotient, we have $\Box[a] = [a \oplus (a \wedge b)]$, as required.

Regarding (v), the equality $a \oplus b = \Box a \oplus \Box b$ follows immediately from Definition 5.1(iii).4. As to the inequality $a \leq a \oplus b$, it is easy to check (e.g. on a QNM twist-algebra) that the negation $\sim$ is order-reversing with respect to $\leq$. Then, from $\sim a \wedge \sim b \leq \sim a$, we obtain $\sim\sim a \leq \sim(\sim a \wedge \sim b) = a \oplus b$. On the quotient, this gives us $\Box[a] \leq [a \oplus b]$. The required result then follows from the observation that $[a] \leq \Box[a]$, which holds true because $\Box$ is a nucleus operator.

Finally, item (vi) is an immediate consequence of Definition 5.1(iii).4. ■

Theorem 5.9 (Representation of QNS, I): *Every* **A** $\in$ QNS *is embeddable into* $(\mathbf{A}_{\bowtie})^{\bowtie}$ *(constructed according to Definition 5.6) through the map* $\iota \colon A \to A_{\bowtie} \times A_{\bowtie}$ *given by* $\iota(a) := \langle [a], [\sim a] \rangle$ *for all* $a \in A$. *In other words, every* **A** $\in$ QNS *is isomorphic to a QNS twist-algebra over* $\mathbf{A}_{\bowtie}$.

Proof: By Theorem 4.16, we only need to verify that the map ι preserves the meet operation of **A**. Let then $a, b \in A$. We have:

$$\begin{aligned}
\iota(a \wedge b) &= \langle [a \wedge b], [\sim(a \wedge b)] \rangle \\
&= \langle [a \wedge b], [\sim(a \wedge \sim\sim b)] \rangle && \text{by Lemma 5.2(i)} \\
&= \langle [a \wedge b], [\sim(\sim\sim b \wedge a)] \rangle \\
&= \langle [a \wedge b], [\sim(\sim\sim b \wedge \sim\sim a)] \rangle && \text{by Lemma 5.2(i)} \\
&= \langle [a \wedge b], [\sim(\sim\sim a \wedge \sim\sim b)] \rangle \\
&= \langle [a \wedge b], [\sim a \oplus \sim b] \rangle
\end{aligned}$$

$$= \langle [a] \wedge [b], [\sim a] \oplus [\sim b] \rangle$$
$$= \langle [a], [\sim a] \rangle \wedge \langle [b], [\sim b] \rangle$$
$$= \iota(a) \wedge \iota(b).$$

■

6. Refining the representations

In this section we introduce more informative twist constructions for quasi-Nelson monoids, pocrims and semihoops, which resemble and generalise the construction for quasi-Nelson algebras considered in Proposition 2.12. In certain cases cases (QNP and QNS) these refined constructions may be used to lift our representation results to equivalences between suitably defined algebraic categories (cf. Section 10); in others (QNM) we do no currently know whether the construction is sufficiently general to obtain a representation (analogous to Theorem 2.14) for all the algebras in a given class (cf. Problem 6.4).

6.1. QNM

Let $\mathbf{M} \in$ QNM be a quasi-Nelson monoid. Define:

$$A^+ := \{a \in A : \sim a \preceq a\} = \{a \in A : (\sim a)^2 = (\sim a)^2 * a^2\}$$

and, for $\mathbf{M} \leq \mathbf{S}^{\bowtie}$, let $\nabla_{\mathbf{A}} := \pi_1[A^+]$.

Keep in mind that (by Corollary 3.11), for every $\rightharpoonup$-semilattice $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1 \rangle$, letting $\neg x := x \rightharpoonup 0$, we obtain a pseudo-complemented semilattice $\langle S; \wedge, \neg, 0, 1 \rangle$ with an operator $\Box$ given by $\Box x := 1 \rightharpoonup x$ which is a nucleus (in the sense of Definition 2.5).

Proposition 6.1 (cf. Proposition 2.13): *Let $\mathbf{M} \leq \mathbf{S}^{\bowtie}$ be a quasi-Nelson monoid. Then:*

(i) $A^+ = \{\langle a, 0 \rangle : \langle a, 0 \rangle \in A\}$.
(ii) *For all $a \in S$, if $\neg a = 0$, then $a \in \nabla_{\mathbf{A}}$ (i.e. $\nabla_{\mathbf{A}}$ is non-empty and contains the set $D(\mathbf{S})$ of the dense elements of $\mathbf{S}$).*
(iii) *For all $a, b \in S$, if $a \in \nabla_{\mathbf{A}}$, then $\neg\neg a \in \nabla_{\mathbf{A}}$ and $b \rightharpoonup a \in \nabla_{\mathbf{A}}$.*
(iv) *For all $a, b \in S$, if $a, b \in \nabla_{\mathbf{A}}$, then $a \wedge \Box b \in \nabla_{\mathbf{A}}$.*

Proof: (i). Let $\langle a_1, a_2 \rangle \in A^+$. Recall that, by Proposition 3.15(ii).1, we have $\sim\langle a_1, a_2 \rangle \preceq \langle a_1, a_2 \rangle$ if and only if $a_2 \leq a_1$. In turn, from the latter we have $0 = a_1 \wedge a_2 = a_2$. Conversely, if $a_2 = 0$, then clearly $a_2 \leq a_1$, so $\langle a_1, a_2 \rangle = \langle a_1, 0 \rangle \in A^+$.

(ii). Let $a \in S$ be such that $\neg a = 0$, Consider an element $b \in H$ such that $\langle a, b \rangle \in A$. Recalling that $a \wedge b = 0$, we have:

$$\begin{aligned} b \rightharpoonup \neg a &= b \rightharpoonup (a \rightharpoonup 0) \\ &= (b \wedge a) \rightharpoonup 0 && \text{Definition 3.7(ii)} \\ &= 0 \rightharpoonup 0 \\ &= 1 && \text{Proposition 3.10(i)} \end{aligned}$$

Hence, using Proposition 3.10(iv) and Definition 3.7(iv), we have $b \leq \Box\neg a = \Box 0 = 0$. We thus have $\langle a, b\rangle = \langle a, 0\rangle \in A^+$, which means that $a \in \nabla_{\mathbf{A}}$, as claimed.

(iii). Assume $a \in \nabla_{\mathbf{A}}$, i.e. $\langle a, 0\rangle \in A^+$. Let us compute $\sim(\langle a, 0\rangle)^2 \rightharpoonup \sim\langle a, 0\rangle = \langle \neg a, \Box a\rangle \rightharpoonup \langle 0, \Box a\rangle = \langle \neg\neg a, \Box\neg a \wedge \Box a\rangle = \langle \neg\neg a, \Box(\neg a \wedge a)\rangle = \langle \neg\neg a, \Box 0\rangle = \langle \neg\neg a, 0\rangle$. Hence, $\neg\neg a \in \nabla_{\mathbf{A}}$, as claimed. Now, let $b \in S$. Then $\langle b, c\rangle \in A$ for some $c \in S$, and we can compute $\langle b, c\rangle \rightharpoonup \langle a, 0\rangle = \langle b \rightharpoonup a, \Box b \wedge 0\rangle = \langle b \rightharpoonup a, 0\rangle$, which gives us $b \rightharpoonup a \in \nabla_{\mathbf{A}}$, as claimed.

(iv). Assume $\langle a, 0\rangle, \langle b, 0\rangle \in A^+$. Let us compute $\langle a, 0\rangle * (\langle a, 0\rangle \rightharpoonup \langle b, 0\rangle) = \langle a, 0\rangle * \langle a \rightharpoonup b, 0\rangle = \langle a \wedge (a \rightharpoonup b), (a \rightharpoonup 0) \wedge ((a \rightharpoonup b) \rightharpoonup 0)\rangle = \langle a \wedge (a \rightharpoonup b), \neg a \wedge \neg(a \rightharpoonup b)\rangle = \langle a \wedge (a \rightharpoonup b), 0\rangle. = \langle a \wedge \Box b, 0\rangle$ With regards to the first component, the last equality holds by Definition 3.7(vi). With regards to the second, by the property of the pseudo-complement, we have $\neg a \wedge \neg(a \rightharpoonup b) = 0$ iff $\neg(a \rightharpoonup b) \leq \neg\neg a$. The latter inequality, in turn, follows from $\neg a = a \rightharpoonup 0 \leq a \rightharpoonup b$ and the fact that the pseudo-complement is order-reversing. ■

If we compare Proposition 6.1 with Proposition 2.13 (see also Proposition 6.6 below), we see that the properties and structure of the set A^+, and consequently those of $\nabla_{\mathbf{A}}$, are determined by the algebraic operations available on **A**. In general, if **A** is not an algebra in the full quasi-Nelson language, we cannot guarantee that $\nabla_{\mathbf{A}}$ is a lattice filter of **S**, or even an increasing set. The properties stated in Proposition 6.1 are nevertheless sufficient for introducing a set ∇ as a parameter in the twist-algebra construction, in the spirit of Proposition 2.12. The proof of the following proposition, as well as any subsequent proof which has been omitted in this section, can be found in the Appendix.

Proposition 6.2: *Let* $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ *be a* $\rightharpoonup$*-semilattice, and let* $\nabla \subseteq S$ *be a non-empty set satisfying items (iii) and (iv) of Proposition 6.1. Then the set:*

$$Tw(S, \nabla) := \{\langle a_1, a_2\rangle \in S^{\bowtie} : \neg a_1 \rightharpoonup \neg\neg a_2 \in \nabla\},$$

with operations given by Definition 3.14, is the universe of a QNM twist-algebra over **S**.

The construction described in the preceding proposition can be used to produce examples of quasi-Nelson monoids which are interesting in the sense that they will not, in general, be reducts of quasi-Nelson algebras (the same will apply to Proposition 6.7, which concerns richer fragments of the quasi-Nelson language).

By comparing Propositions 6.2 and 2.12, one notices that the two definitions of the universe of the twist-algebra differ in that the latter employs the join operation of the underlying Heyting algebra while the former employs the negation and the implication-like operator. This is on the one hand a necessity, for we are looking at factor algebras that need not have a join operation at all; on the other hand, it is not difficult to show that a dense filter ∇ of a Heyting algebra **H** satisfies (for all elements $a, b \in H$) the following property: $\neg a \to \neg\neg b \in \nabla$ if and only if $a \vee b \in \nabla$ (if and only if $\neg a \to b \in \nabla$). Thus Proposition 6.2 (together with its analogue Proposition 6.7, which shall be established later) truly is a generalisation of Proposition 2.12; while the definition employed in the latter is the standard one for Nelson algebras (and

N4-lattices), the one introduced in the former obviously has the advantage of being independent of the existence of the join operation on the factor algebras.

A definition similar to the one appearing in Proposition 6.2 features in the twist representation given in Busaniche and Cignoli (2014) for a class of residuated lattices there dubbed *Kalman lattices*, which are extensively investigated also in the more recent papers (Aglianò & Marcos, 2022a, 2022a; Busaniche et al., 2022). The factor algebras considered in Busaniche and Cignoli (2014); Busaniche et al. (2022) are more general than ours, but the twist construction for (subreducts of) quasi-Nelson algebras cannot be viewed as a special case of any of the above, because the negation is always assumed to be involutive throughout (Aglianò & Marcos, 2022a, 2022a; Busaniche & Cignoli, 2014; Busaniche et al., 2022). Note also that the Kalman construction of Aglianò and Marcos (2022a, 2022a) may be regarded as the paraconsistent companion of the one used in Busaniche et al. (2022) for Nelson algebras, for the former results from dropping the condition corresponding (in e.g. our Definition 2.9) to $a_1 \wedge a_2 = 0$.

Proposition 6.3: *For every quasi-Nelson monoid* $\mathbf{M} \leq \mathbf{S}^{\bowtie}$, $M \subseteq Tw(S, \nabla_{\mathbf{A}})$.

Proof: Given $\langle a_1, a_2 \rangle \in A$, we have:

$$\begin{aligned} \neg(\langle a_1, a_2 \rangle^2) \rightharpoonup {\sim}\langle a_1, a_2 \rangle &= \langle \neg a_1, \Box a_1 \rangle \rightharpoonup \langle a_2, \Box a_1 \rangle \\ &= \langle \neg a_1 \rightharpoonup a_2, \Box \neg a_1 \wedge \Box a_1 \rangle \\ &= \langle \neg a_1 \rightharpoonup a_2, \Box(\neg a_1 \wedge a_1) \rangle \\ &= \langle \neg a_1 \rightharpoonup a_2, \Box 0 \rangle \\ &= \langle \neg a_1 \rightharpoonup a_2, 0 \rangle \in A. \end{aligned}$$

Thus $\neg a_1 \rightharpoonup a_2 \in \nabla_{\mathbf{A}}$. Since $\neg\neg(\neg a_1 \rightharpoonup a_2) = \neg a_1 \rightharpoonup \neg\neg a_2$ (Proposition 3.10.x), we can use Proposition 6.1(iii) to obtain $\neg a_1 \rightharpoonup \neg\neg a_2 \in \nabla_{\mathbf{A}}$. Hence, $\langle a_1, a_2 \rangle \in Tw(S, \nabla)$. ■

Problem 6.4: Can Proposition 6.3 be sharpened so as to establish the equality $M = Tw(S, \nabla_{\mathbf{A}})$? (Cf. Proposition 6.8).

6.2. QNP *and* QNS

As for the algebras in the full quasi-Nelson language (and in contrast to the case of quasi-Nelson monoids discussed in the preceding subsection), also for quasi-Nelson pocrims and semihoops we can indeed refine the result of Theorem 5.9, obtaining an analogue of the representation of quasi-Nelson algebras given in Theorem 2.14. We now proceed towards these results (Theorems 6.9 and 6.10).

The following lemma, which applies to all the intuitionistic factor algebras whose language includes the relevant operations, will be quite helpful in simplifying our computations.

Lemma 6.5 (cf. Rivieccio, 2022a, Lemma 4.7): *Let* $\mathbf{H}$ *be an algebra having a reduct* $\langle H; \rightarrow, \Box, 0, 1 \rangle$ *that is a bounded Hilbert algebra with a nucleus. Letting* $\neg x := x \rightarrow 0$, *the following identities are satisfied:*

(i) $x \to \neg\neg x = 1$.
(ii) $\neg\neg\neg x = \neg x$.
(iii) $x \to \neg y = y \to \neg x$.
(iv) $\neg x = \neg \Box x = \Box \neg x$.
(v) $\neg\neg(x \to y) = x \to \neg\neg y$.

Given an algebra $\mathbf{A}$ having a QNI-algebra reduct, we define: $A^+ := \{a \in A : {\sim} a \preceq a\}$ and, for a twist-algebra $\mathbf{A} \leq \mathbf{S}^{\bowtie}$, we let $\nabla_{\mathbf{A}} := \pi_1[A^+]$.

Proposition 6.6 (cf. Prop. 6.1): *Let $\mathbf{A} \leq \mathbf{S}^{\bowtie}$ be a twist-algebra having a QNI-algebra reduct. Then:*

(i) $A^+ = \{\langle a, 0\rangle : \langle a, 0\rangle \in A\}$.
(ii) *For all $a \in S$, if $\neg a = 0$, then $a \in \nabla_{\mathbf{A}}$ (i.e. $\nabla_{\mathbf{A}}$ contains the set $D(\mathbf{S})$ of the dense elements of $\mathbf{S}$; see Proposition 2.13(ii)).*
(iii) *For all $a, b \in S$, if $a \in \nabla_{\mathbf{A}}$, then $\neg\neg a \in \nabla_{\mathbf{A}}$ and $b \to a \in \nabla_{\mathbf{A}}$.*
(iv) *If $\mathbf{A}$ has a quasi-Nelson pocrim reduct, then $\nabla_{\mathbf{A}}$ is closed under finite meets and satisfies the following condition: for all $a \in S$, if $\neg\neg a \in \nabla_{\mathbf{A}}$, then $a \in \nabla_{\mathbf{A}}$.*
(v) *If $\mathbf{A}$ is a quasi-Nelson algebra, then $\nabla_{\mathbf{A}}$ is a lattice filter.*

Proposition 6.7 (cf. Prop. 6.2): *Let $\mathbf{S} = \langle S; \odot, \to, 0, 1\rangle$ be an nH-semigroup and $\nabla \subseteq S$.*

(i) *If ∇ satisfies item Proposition 6.6(iii), then the set:*

$$Tw(S, \nabla) := \{\langle a_1, a_2\rangle \in S^{\bowtie} : \neg a_1 \to \neg\neg a_2 \in \nabla\}$$

is the universe of a QNI twist-algebra $\langle Tw(S, \nabla); \to, \sim, 0, 1\rangle$ with operations given by Definition 4.6.
(ii) *If $\mathbf{S}$ is a bounded implicative semilattice with a nucleus and ∇ further satisfies Proposition 6.6(iv), then $Tw(S, \nabla)$ is closed under the operation $*$ (given by Definition 4.14), so $\langle Tw(S, \nabla); *, \to, \sim, 0, 1\rangle \in$ QNP.*
(iii) *If $\mathbf{S}$ is a $\oplus$-implicative semilattice and ∇ further satisfies Proposition 6.6(iv), then $Tw(S, \nabla)$ is closed under the operation $\wedge$ (given by Definition 5.6), therefore we have $\langle Tw(S, \nabla); *, \wedge, \to, \sim, 0, 1\rangle \in$ QNS.*
(iv) *If $\mathbf{S}$ is a Heyting algebra with a nucleus and ∇ is a lattice filter that satisfies Proposition 6.6(ii), then $Tw(S, \nabla)$ is closed under the operation $\vee$ (given by Definition 2.9), so $\langle Tw(S, \nabla); *, \wedge, \vee, \Rightarrow, \sim, 0, 1\rangle$ is a quasi-Nelson algebra.*

As noted earlier about quasi-Nelson monoids, the constructions described in the preceding proposition can be employed to produce examples of QNI-algebras and quasi-Nelson semihoops and pocrims which will not, in general, be reducts of quasi-Nelson algebras. Furthermore, we are now ready to prove an analogue of Proposition 2.13 for QNM and QNS.

Proposition 6.8 (cf. Prop. 6.3): *Let $\mathbf{A} \leq \mathbf{S}^{\bowtie}$ be a twist-algebra.*

(i) *If* **A** *is a QNI-algebra, then* $A \subseteq Tw(S, \nabla)$.
(ii) *If* **A** *is a quasi-Nelson pocrim or a quasi-Nelson semihoop, then* $A = Tw(S, \nabla)$.

As hinted at earlier, the proof of preceding proposition entails that, in the case of quasi-Nelson semihoops, the set $Tw(S, \nabla)$ could have been equivalently defined as:

$$Tw(S, \nabla) := \{\langle a_1, a_2\rangle \in S^{\bowtie} : \neg a_1 \rightarrow a_2 \in \nabla\}.$$

Joining Propositions 6.6, 6.7 and 6.8, we obtain the announced refinement of the representation results for quasi-Nelson pocrims and semihoops.

Theorem 6.9 (Representation of QNP, II): *Every quasi-Nelson pocrim* **A** *is isomorphic to a QNP twist-algebra* $\mathbf{Tw}(\mathbf{A}_{\bowtie}, \nabla_{\mathbf{A}})$*, constructed according to Proposition 6.7, through the map* ι *given by* $\iota(a) = \langle [a], [\sim a]\rangle$ *for all* $a \in A$.

Theorem 6.10 (Representation of QNS, II): *Every quasi-Nelson semihoop* **A** *is isomorphic to the QNS twist-algebra* $\mathbf{Tw}(\mathbf{A}_{\bowtie}, \nabla_{\mathbf{A}})$*, constructed according to Proposition 6.7, through the map* ι *given by* $\iota(a) = \langle [a], [\sim a]\rangle$ *for all* $a \in A$.

We close the section by highlighting a few open questions that have emerged so far.

Problem 6.11 (cf. Problem 6.4): Can the representation of Theorem 6.10 be extended to the other classes of algebras considered in the previous sections, namely QNM and QNI?

As observed at the beginning of the section, quasi-Nelson semihoops could be presented in the language $\{\wedge, *, \Rightarrow, 0, 1\}$ of bounded semihoops, or even in the language $\{\wedge, \Rightarrow, 0, 1\}$, which are rich enough to formulate the Nelson identity. This leads to the following conjecture.

Problem 6.12: Does QNS (viewed as a class of algebras in the language of bounded semihoops) coincide with the class of bounded (3-potent) semihoops that satisfy the identity (Nelson)?

A distinct, perhaps harder question is that of characterising the negation-free subreducts of quasi-Nelson semihoops.

Problem 6.13: Axiomatize the class of semihoops that are the $\{\wedge, *, \Rightarrow, 1\}$-subreducts of quasi-Nelson semihoops.

7. Completions and embeddings

In this section we finally verify that the classes of algebras we have been dealing with so far (quasi-Nelson monoids, quasi-Nelson pocrims and quasi-Nelson semihoops) indeed correspond to (respectively) the classes of $\{*, \sim\}$-, $\{*, \rightarrow, \sim\}$- and $\{\wedge, *, \rightarrow, \sim\}$-subreducts of quasi-Nelson algebras. To this end we are going to apply a uniform proof strategy, showing that every quasi-Nelson monoid (pocrim, semihoop) embeds into a quasi-Nelson algebra.

Given a (say) quasi-Nelson monoid $\mathbf{M}$, we will first show that the quotient algebra $\mathbf{M}_{\bowtie}$ embeds into a Heyting algebra with a nucleus $\mathbf{H}$ (say, by a map $e\colon \mathbf{M}_{\bowtie} \to \mathbf{H}$), and then verify that embedding e may be lifted to a map $e^{\bowtie}\colon (\mathbf{M}_{\bowtie})^{\bowtie} \to (\mathbf{H})^{\bowtie}$ between the corresponding twist-algebras which is also an embedding. The same strategy allowed us to establish the corresponding result for QNI algebras in Rivieccio (2022a, Cor. 4.19)

We have seen with Example 4.12 that, given a bounded implicative semilattice with a nucleus $\langle A; \wedge, \to, \Box, 0, 1\rangle$, one can obtain a $\rightharpoonup$-semilattice (Definition 3.7) by letting $x \rightharpoonup y := x \to \Box y$. Thus, one can consider bounded implicative semilattices with nuclei as algebras in this enriched language $\langle A; \wedge, \to, \rightharpoonup, \Box, 0, 1\rangle$, with the operation $\rightharpoonup$ canonically defined as above (in such a case, one can also omit the nucleus from the signature, for it is definable by $\Box x := 1 \rightharpoonup x$). Our next aim is to show that every $\rightharpoonup$-semilattice $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ can be embedded into an algebra $\langle A; \wedge, \to, \rightharpoonup, 0, 1\rangle$ of this type by a map that preserves all the operations of $\mathbf{S}$.

Recall that a semilattice $\langle A; \wedge, 0, 1\rangle$ is *complete* when the meet of every subset $B \subseteq A$ (denoted $\bigwedge B$) exists in A. Given an implicative semilattice $\langle A; \wedge, \to, 0, 1\rangle$ and $b \in A$, we shall consider the operation $\Box_b\colon A \to A$ given by $\Box_b a := (a \to b) \to b$ for every $a \in A$.

Lemma 7.1: *Let $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ be a $\rightharpoonup$-semilattice (with nucleus $\Box$) and let $\mathbf{A} = \langle A; \wedge, \to, 0, 1\rangle$ be a complete implicative semilattice. Suppose $\langle S; \wedge, 0, 1\rangle \leq \langle A; \wedge, 0, 1\rangle$, i.e. suppose that the $\{\wedge, 0, 1\}$-reduct of $\mathbf{S}$ is (isomorphic to) a subalgebra of the corresponding reduct of $\mathbf{A}$. Then:*

(i) $\mathbf{S} \models x \rightharpoonup y = \Box x \to \Box y = x \to \Box y$.
(ii) *Letting $\Box' a := \bigwedge\{\Box_{\Box b} a : b \in S\}$ for all $a \in A$, we have that $\langle A; \wedge, \to, \Box', 0, 1\rangle$ is a bounded implicative semilattice with a nucleus (Definition 4.11) and that $\Box'$ agrees with $\Box$ on S.*

Proof: (i). Let us prove the first equality first. By Definitions 3.7(vi) and 2.5(ii), we have $\Box a \wedge (a \rightharpoonup b) = \Box(a \wedge b) = \Box a \wedge \Box b \leq \Box b$. From $\Box a \wedge (a \rightharpoonup b) \leq \Box b$, by residuation, we obtain $a \rightharpoonup b \leq \Box a \to \Box b$. For the converse inequality, using the properties of the intuitionistic implication, we have $(\Box a \to \Box b) \wedge \Box a = \Box b \wedge \Box a \leq \Box b$. Then, by Definition 3.7(vii), we obtain $\Box a \to \Box b \leq a \rightharpoonup b$.

As to the second equality, recall that the intuitionistic implication is order-reversing in the first argument. Then $a \leq \Box a$ entails $\Box a \to \Box b \leq a \to \Box b$. To show $a \to \Box b \leq \Box a \to \Box b$, observe that, using the properties of the nucleus, we have $\Box a \wedge (a \to \Box b) \leq \Box a \wedge \Box(a \to \Box b) = \Box(a \wedge (a \to \Box b)) = \Box(a \wedge \Box b) = \Box a \wedge \Box\Box b = \Box a \wedge \Box b \leq \Box b$. Thus $\Box a \wedge (a \to \Box b) \leq \Box b$, which gives us $a \to \Box b \leq \Box a \to \Box b$ by residuation.

(ii). Recall from Macnab (1976, Lemma 2.5) that each $\Box_b$ satisfies all the properties postulated in Definition 2.6, except perhaps (i). Moreover, the definition of $\Box' a$ allows us to apply (Macnab, 1976, Thm. 2.8 (ii)) to conclude that $\Box' a$ also satisfies all items in Definition 2.6, except perhaps (i). But the latter is easily seen to be satisfied, for $\Box' 0 \leq \Box_{\Box 0} 0 = \Box_0 0 = 0$. Hence, $\Box'$ is a nucleus in the sense of Definition 2.6. It remains to show that $\Box$ and $\Box'$ agree on S. Let then $a, b \in S$. On the one hand, by item (i) above, we have $a \to \Box b = \Box a \to \Box b$. From the latter, by residuation, we obtain $\Box a \leq (a \to \Box b) \to \Box b$. Thus $\Box a \leq \Box' a$. To show $\Box' a \leq \Box a$, it suffices to observe that $a \leq \Box a$ entails $\Box a = 1 \to \Box a = (a \to \Box a) \to \Box a$. ■

It is well known that every (bounded) meet semilattice $\langle S; \wedge, 0, 1\rangle$ is embeddable into a complete implicative semilattice (for instance, into the lattice of down-sets of $\langle S; \wedge, 0, 1\rangle$, which is a complete Heyting algebra). Thus, given a $\rightharpoonup$-semilattice $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$, we can view the reduct $\langle S; \wedge, 0, 1\rangle$ as a subalgebra of the $\to$-free reduct of some complete implicative semilattice $\langle A; \wedge, \to, 0, 1\rangle$, and apply Lemma 7.1(ii) to extend the operation $\rightharpoonup$ to A. Hence, we reach the following result.

Proposition 7.2: *Every $\rightharpoonup$-semilattice embeds into a complete implicative semilattice with a nucleus.*

The following is an easy consequence of the preceding proposition.

Proposition 7.3: *Every quasi-Nelson monoid (Definition 3.2) embeds into a quasi-Nelson pocrim (Definition 4.9)*

Proof: Let $\mathbf{M} = \langle M; *, \sim_{\mathbf{M}}, 0_{\mathbf{M}}, 1_{\mathbf{M}}\rangle$ be a quasi-Nelson monoid. By the twist representation (Theorem 3.18), we can assume $\mathbf{M} \leq \mathbf{S}^{\bowtie}$ for some $\rightharpoonup$-semilattice $\mathbf{S} = \langle S; \wedge_{\mathbf{S}}, \rightharpoonup_{\mathbf{S}}, 0_{\mathbf{S}}, 1_{\mathbf{S}}\rangle$. By Proposition 7.2, there is an embedding $e\colon S \to A$ of $\mathbf{S}$ into a (complete) implicative semilattice with a nucleus $\mathbf{A} = \langle A; \wedge_{\mathbf{A}}, \to_{\mathbf{A}}, \rightharpoonup_{\mathbf{A}}, \Box_{\mathbf{A}}, 0_{\mathbf{A}}, 1_{\mathbf{A}}\rangle$. It is then easy to verify that the map $e^{\bowtie}\colon S \times S \to A \times A$ given by $e^{\bowtie}\langle a_1, a_2\rangle := \langle e(a_1), e(a_2)\rangle$ for all $a_1, a_2 \in S$ (we write $e^{\bowtie}\langle a_1, a_2\rangle$ instead of $e^{\bowtie}(\langle a_1, a_2\rangle)$ to improve legibility) is an embedding of $\mathbf{M}$ into the quasi-Nelson pocrim $\mathbf{A}^{\bowtie} = \langle A^{\bowtie}; *, \to, \sim, 0, 1\rangle$ constructed according to Definition 4.14. Regarding the monoid operation, we have, in particular (recall that, by Definition 3.14, $\Box_{\mathbf{S}} a_2 = a_2$ and $\Box_{\mathbf{S}} b_2 = b_2$):

$$
\begin{aligned}
& e^{\bowtie}(\langle a_1, a_2\rangle *_{\mathbf{M}} \langle b_1, b_2\rangle) \\
&= e^{\bowtie}(\langle a_1 \wedge_{\mathbf{S}} a_2, (a_1 \rightharpoonup_{\mathbf{S}} b_1) \wedge_{\mathbf{S}} (b_1 \rightharpoonup_{\mathbf{S}} a_2)\rangle) \\
&= \langle e(a_1 \wedge_{\mathbf{S}} a_2), e((a_1 \rightharpoonup_{\mathbf{S}} b_2) \wedge_{\mathbf{S}} (b_1 \rightharpoonup_{\mathbf{S}} a_2))\rangle) \\
&= \langle e(a_1) \wedge_{\mathbf{A}} e(a_2), (e(a_1) \rightharpoonup_{\mathbf{A}} e(b_2)) \wedge_{\mathbf{A}} (e(b_1) \rightharpoonup_{\mathbf{A}} e(a_2))\rangle \\
&= \langle e(a_1) \wedge_{\mathbf{A}} e(a_2), (e(a_1) \to_{\mathbf{A}} \Box_{\mathbf{A}} e(b_2)) \wedge_{\mathbf{A}} (e(b_1) \to_{\mathbf{A}} \Box_{\mathbf{A}} e(a_2))\rangle \\
&= \langle e(a_1) \wedge_{\mathbf{A}} e(a_2), (e(a_1) \to_{\mathbf{A}} e(\Box_{\mathbf{S}} b_2)) \wedge_{\mathbf{A}} (e(b_1) \to_{\mathbf{A}} e(\Box_{\mathbf{S}} a_2))\rangle \\
&= \langle e(a_1) \wedge_{\mathbf{A}} e(a_2), (e(a_1) \to_{\mathbf{A}} e(b_2)) \wedge_{\mathbf{A}} (e(b_1) \to_{\mathbf{A}} e(a_2))\rangle \\
&= \langle e(a_1), e(a_2)\rangle *_{\mathbf{A}^{\bowtie}} \langle e(b_1), e(b_2)\rangle \\
&= e^{\bowtie}\langle a_1, a_2\rangle *_{\mathbf{A}^{\bowtie}} e^{\bowtie}\langle b_1, b_2\rangle.
\end{aligned}
$$
■

We next show that, in turn, every quasi-Nelson pocrim can be embedded into a quasi-Nelson algebra. For this, we only need the following result and the well-known fact that every implicative semilattice embeds into a Heyting algebra (see e.g. Balbes, 1969, Thm. 5.3).

Lemma 7.4 (Macnab, 1976, Thm. 13.15): *Let* $\langle S; \wedge, \to, \Box, 0, 1\rangle$ *a bounded implicative semilattice with a nucleus and let* $\langle A; \wedge, \vee, \to, 0, 1\rangle$ *be a complete Heyting algebra. Suppose* $\langle S; \wedge, \to, 0, 1\rangle \leq \langle A; \wedge, \to, 0, 1\rangle$. *Letting, for all* $a \in A$,

$$\Box' a := \bigwedge\{\Box_{\Box b} a : b \in S\},$$

we have that $\langle A; \wedge, \vee, \to, \Box', 0, 1\rangle$ *is nuclear Heyting algebra and* $\Box'$ *agrees with* $\Box$ *on* S.

Using Lemma 7.4, it is straightforward to mimic the proof of Proposition 7.3; in this case we rely on the observation that every implicative meet semilattice can be embedded into a complete Heyting algebra.

Proposition 7.5: *Every quasi-Nelson pocrim (Definition 4.9) embeds into a quasi-Nelson algebra (Definition 2.8).*

Our next and last goal for this section is to establish an analogue of Proposition 7.5 for quasi-Nelson semihoops. We shall need the following lemma.

Lemma 7.6: *Let* $\mathbf{S} = \langle S; \wedge, \oplus, \to, 0, 1\rangle$ *be a* $\oplus$*-implicative semilattice (Definition 5.3) with nucleus* $\Box$, *and let* $\langle H; \wedge, \vee, \to, 0, 1\rangle$ *be a complete Heyting algebra. Suppose* $\langle S; \wedge, \to, 0, 1\rangle \leq \langle A; \wedge, \to, 0, 1\rangle$. *Then, defining* $\Box'$ *according to Lemma 7.4, we have* $a \oplus b = \Box'(a \vee b)$ *for all* $a, b \in S$, *and* $\Box'$ *agrees with* $\Box$ *on* S.

Proof: Recall that $\langle H; \wedge, \vee, \to, \Box', 0, 1\rangle$ is a nuclear Heyting algebra by Lemma 7.4. It is then easy to check that the operation $\oplus'$ given by $x \oplus' y := \Box'(x \vee y)$ satisfies all the properties required by Definition 5.3, turning $\langle H; \wedge, \oplus', \to, 0, 1\rangle$ into a $\oplus$-implicative semilattice. It remains to prove that $\oplus$ and $\oplus'$ agree on S. To show that

$$a \oplus b \leq \Box'(a \vee b) = \bigwedge\{\Box_{\Box c}(a \vee b) : c \in S\}$$

we proceed as follows. We observe that, for every $c \in S$, we have $a \leq \Box_{\Box c}(a \vee b) = ((a \vee b) \to \Box c) \to \Box c$. Indeed, by the properties of Heyting algebras, we have $((a \vee b) \to \Box c) \to \Box c = ((a \to \Box c) \wedge (b \to \Box c)) \to \Box c$. Thus, by residuation, $a \leq ((a \vee b) \to \Box c) \to \Box c$ is equivalent to $(a \to \Box c) \wedge (b \to \Box c) \wedge a \leq \Box c$, and the latter holds true because $(a \to \Box c) \wedge (b \to \Box c) \wedge a = a \wedge \Box c \wedge (b \to \Box c) = a \wedge \Box c$. Hence, $a \leq ((a \to \Box c) \wedge (b \to \Box c)) \to \Box c$ and, similarly, $b \leq ((a \to \Box c) \wedge (b \to \Box c)) \to \Box c$. Since $((a \to \Box c) \wedge (b \to \Box c)) \to \Box c \in S$, we may apply Lemma 5.5(i) to conclude $a \oplus b \leq \Box(((a \to \Box c) \wedge (b \to \Box c)) \to \Box c) = \Box(((a \vee b) \to \Box c) \to \Box c) = ((a \vee b) \to \Box c) \to \Box c$, where the last equality holds because every nucleus satisfies $\Box(x \to \Box y) = x \to \Box y$ (Macnab, 1976, Lemma 13.8 (ii)). Thus, we have $a \oplus b \leq \Box_{\Box c}(a \vee b)$ for every $c \in S$, which entails $a \oplus b \leq \Box'(a \vee b)$, as required.

For the converse inequality, it suffices to show that $a \oplus b \in \{\Box_{\Box c}(a \vee b) : c \in S\}$, that is, $a \oplus b = ((a \vee b) \to \Box(a \oplus b)) \to \Box(a \oplus b)$. To see this, observe that Definition 5.3(v) and nucleus properties entail $a \leq \Box a \leq \Box(a \oplus b)$ and, similarly, $b \leq \Box(a \oplus b)$. Hence, $(a \vee b) \to \Box(a \oplus b) = (a \to \Box(a \oplus b)) \wedge (b \to \Box(a \oplus b)) = 1$. The latter gives us $((a \vee b) \to \Box(a \oplus b)) \to \Box(a \oplus b) = 1 \to \Box(a \oplus b) = \Box(a \oplus b)$. But $\Box(a \oplus b) = a \oplus b$, by Lemma 5.5(ii). Hence, the required result follows. ■

As before, one can use Lemma 7.6, to mimic the proof of Proposition 7.3, obtaining the following result.

Proposition 7.7: *Every quasi-Nelson semihoop (Definition 5.1) embeds into a quasi-Nelson algebra (Definition 2.8).*

We are finally in a position to join together our main results: the classes of algebras introduced in the previous sections do indeed characterise the corresponding fragments of quasi-Nelson algebras.

Corollary 7.8: *The classes* QNM*,* QNP *and* QNS *are precisely (respectively) the* $\{*, \sim\}$*-, the* $\{*, \to, \sim\}$*- and the* $\{\wedge, *, \to, \sim\}$*-subreducts of quasi-Nelson algebras.*

Let us stress that the preceding result cannot possibly be sharpened by saying that every (say) quasi-Nelson monoid is the *reduct* of a quasi-Nelson algebra. This follows from the results presented in Section 6. Consider, for instance, $\mathbf{S} \in$ QNM and let $\nabla \subseteq S$ be a set that satisfies items (iii) and (iv) of Proposition 6.1 but is *not* a lattice filter. Then, by Proposition 6.2, the set $Tw(S, \nabla)$ is the universe of a quasi-Nelson monoid which (by Proposition 2.13) is not the reduct of any quasi-Nelson algebra. A similar reasoning (using Proposition 6.7) applies to quasi-Nelson QNP and QNS as well.

8. Congruence properties

In this section we try and obtain some information on the congruence lattices of the new classes of algebras we have been dealing with. As we shall see, the twist construction will prove very helpful in this endeavour too, allowing us to establish a link (a lattice isomorphism) between the congruences of a twist-algebra $\mathbf{A}$ and those of the corresponding intuitionistic algebra $\mathbf{A}_{\bowtie}$.

We begin with a lemma that applies to all the factor algebras in our twist representations.

Lemma 8.1: *Let* $\mathbf{H}$ *be an algebra having a Hilbert algebra reduct* $\langle H; \to\rangle$ *with an operator* $\Box$ *satisfying* $a \to b \leq \Box a \to \Box b$ *for all* $a, b \in H$ *(in particular, every nucleus satisfies this). Let* θ *be a congruence of the reduct* $\langle H; \to\rangle$*. Then* θ *is also a congruence of* $\langle H; \to, \Box\rangle$*.*

Proof: We shall use the following observation:

$$\theta = \{\langle a, b\rangle \in H \times H : \langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta\}.$$

As shown in Rivieccio and Jansana (2021, Lemma 36), the property holds for every algebra having a Hilbert algebra reduct and for every congruence θ of $\langle H; \to\rangle$.

Let $\langle a, b\rangle \in \theta$. Then $\langle a \to b, 1\rangle \in \theta$, and also

$$\langle (a \to b) \to (\Box a \to \Box b), 1 \to (\Box a \to \Box b)\rangle = \langle 1, \Box a \to \Box b\rangle \in \theta.$$

In a similar way we obtain $\langle 1, \Box b \to \Box a\rangle \in \theta$. Then, $\langle \Box a, \Box b\rangle \in \theta$, as required. ■

Proposition 8.2 (Rivieccio, 2022a, Prop. 4.14): *For every nH-semigroup* $\mathbf{S} = \langle S; \odot, \to, 0, 1\rangle$*, we have* $\mathrm{Con}(\mathbf{S}) = \mathrm{Con}(\langle S; \to\rangle)$.

As observed in Example 4.12, every bounded implicative semilattice $\langle S; \wedge, \to, 0, 1\rangle$ may be viewed as an nH-semigroup (on which $\odot = \wedge$). Thus, Proposition 8.2 also applies to congruences of bounded implicative semilattices (see e.g. Celani & Jansana, 2012). That is, for every bounded implicative semilattice $\langle S; \wedge, \to, 0, 1\rangle$, we have $\mathrm{Con}(\langle S, \wedge, \to\rangle) = \mathrm{Con}(\langle S, \to\rangle)$.

The proof of the following proposition, as well as any subsequent proof which has been omitted in this section, can be found in the Appendix.

Proposition 8.3: *For every* $\oplus$*-implicative semilattice* $\mathbf{S} = \langle S; \wedge, \oplus, \to, 0, 1\rangle$*, we have* $\mathrm{Con}(\mathbf{S}) = \mathrm{Con}(\langle S; \to\rangle)$.

Proposition 8.4 (Rivieccio & Jansana, 2021, Thm. 6, Prop. 37): *For every* $\mathbf{A} \in \mathrm{QNI}$*, one has* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A}_{\bowtie})$ *via the mutually inverse maps* $(.)^{\bowtie}$ *and* $(.)_{\bowtie}$ *defined, for all* $\theta \in \mathrm{Con}(\mathbf{A})$ *and* $\eta \in \mathrm{Con}(\mathbf{A}_{\bowtie})$*, as follows:*

$$\theta_{\bowtie} := \{\langle [a], [b]\rangle \in A/{\equiv} \times A/{\equiv} : \langle a \to c, b \to c\rangle \in \theta \text{forallc} \in A\}.$$
$$\eta^{\bowtie} := \{\langle a, b\rangle \in A \times A : \langle [a], [b]\rangle, \langle [\sim a], [\sim b]\rangle \in \eta\}.$$

Hence (by Proposition 8.2), we have $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\langle A/{\equiv}; \to, 1\rangle)$*, where* $\langle A/{\equiv}; \to, 1\rangle$ *is the Hilbert algebra reduct of* $\mathbf{A}_{\bowtie}$ *(cf. Proposition 4.7).*

Since the lattice of congruences of every Hilbert algebra is distributive (Celani et al., 2009, p. 477), Proposition 8.4 entails that the variety QNI is congruence-distributive as well. The latter property (being characterisable by a Maltsev term) is preserved by language expansions. Thus, we have the following.

Corollary 8.5 (cf. Rivieccio & Jansana, 2021, Cor. 38): *The varieties* QNI*,* QNP *and* QNS *are congruence-distributive.*

Since QNP is a variety of 3-potent pocrims (Proposition 4.19), the preceding result on QNP could also be derived from Blok and Raftery (1997). Indeed, Blok and Raftery (1997, Ex. IV) further implies that QNP is 3-permutable, and Blok and Raftery (1997, Prop. 3.1 (iv)) that QNP is 1-regular; regarding the congruence extension property, see Proposition 8.14 below. By contrast, the congruence lattices of QNM do not satisfy distributivity, nor indeed any non-trivial lattice identity. This follows from the observation that the same holds for pseudo-complemented semilattices (Sankappanavar (1979, Cor. 4.14)), which may be viewed as a subvariety of QNM (see Proposition 3.4).

Proposition 8.6: *For every* $\mathbf{A} \in \mathrm{QNP}$*, we have* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A}_{\bowtie})$ *via the maps* $(.)^{\bowtie}$ *and* $(.)_{\bowtie}$ *defined in Proposition 8.4.*

Proof: In the light of Proposition 8.4, it suffices to show that $\eta^{\bowtie} \in \mathrm{Con}(\mathbf{A})$ for all $\eta \in \mathrm{Con}(\mathbf{A}_{\bowtie})$ and $\theta_{\bowtie} \in \mathrm{Con}(\mathbf{A}_{\bowtie})$ for all $\theta \in \mathrm{Con}(\mathbf{A})$. Regarding the latter, recall that, from our earlier observations and Lemma 8.1, we have $\mathrm{Con}(\langle \mathbf{A}/{\equiv}; \wedge, \to, \Box\rangle) = \mathrm{Con}(\langle \mathbf{A}/{\equiv}; \to, \Box\rangle) = \mathrm{Con}(\langle \mathbf{A}/{\equiv}; \to\rangle)$. Then $\theta_{\bowtie} \in \mathrm{Con}(\mathbf{A}_{\bowtie})$ simply holds by Proposition 8.4. To show $\eta^{\bowtie} \in \mathrm{Con}(\mathbf{A})$, it suffices to verify that $\eta^{\bowtie}$ is compatible with the monoid operation.

Let then $a, b \in A$ be such that $\langle a, b\rangle \in \eta^{\bowtie}$. We shall write $[a], [b]$ etc. for the elements of $\mathbf{A}_{\bowtie}$ instead of $a/{\equiv}, b/{\equiv}$ etc. By assumption, $\langle [a], [b]\rangle, \langle [\sim a], [\sim b]\rangle \in \eta$. Let $c \in A$. From $\langle [a], [b]\rangle \in \eta$ we have $\langle [a] \wedge [c], [b] \wedge [c]\rangle = \langle [a * c], [b * c]\rangle \in \eta$. Also from $\langle [a], [b]\rangle \in \eta$ we have $\langle [a] \to [\sim c], [b] \to [\sim c]\rangle = \langle [a \to \sim c], [b \to \sim c]\rangle \in \eta$. Likewise, $\langle [\sim a], [\sim b]\rangle \in \eta$ gives us $\langle [c] \to [\sim a], [c] \to [\sim b]\rangle = \langle [c \to \sim a], [c \to \sim b]\rangle \in \eta$. Then $\langle [a \to \sim c] \wedge [c \to \sim a], [b \to \sim c] \wedge [c \to \sim b]\rangle = \langle [(a \to \sim c) * (c \to \sim a)], [(b \to \sim c) * (c \to \sim b)]\rangle \in \eta$. Observe that $\langle [(a \to \sim c) * (c \to \sim a)], [(b \to \sim c) * (c \to \sim b)]\rangle = \langle [\sim(a * c)], [\sim(b * c)]\rangle$, by Definition 4.9(iii).4. We have thus $\langle [a * c], [b * c]\rangle, \langle [\sim(a * c)], [\sim(b * c)]\rangle \in \eta$, i.e. $\langle a * c, b * c\rangle \in \eta^{\bowtie}$. Since the monoid operation is commutative, this entails $\eta^{\bowtie} \in \mathrm{Con}(\mathbf{A})$. ■

Proposition 8.7: *For every* $\mathbf{A} \in \mathsf{QNS}$, *we have* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A}_{\bowtie})$ *via the maps* $(.)^{\bowtie}$ *and* $(.)_{\bowtie}$ *defined in Proposition 8.4.*

The preceding propositions imply, for instance, that every algebra $\mathbf{A} \in \mathsf{QNP} \cup \mathsf{QNS}$ is subdirectly irreducible (resp. simple) if and only if $\mathbf{A}_{\bowtie}$ (viewed as either an implicative semilattice or as a Hilbert algebra) is subdirectly irreducible (resp. simple). The following is also an immediate consequence.

Corollary 8.8: *For every* $\mathbf{A} = \langle A; *, \to, \sim, 0, 1\rangle \in \mathsf{QNP} \cup \mathsf{QNS}$, *we have* $\mathrm{Con}(\mathbf{A}) = \mathrm{Con}(\langle A; \to, \sim\rangle)$.

Given an algebra $\mathbf{A}$ with a partial order $\leq$ and maximum 1, we shall say that an element $c \in A$ is the *penultimate element* of A if $c \neq 1$ and, for all $a \in A$ such that $a < 1$, it holds that $a \leq c$.

Lemma 8.9 (Rivieccio, 2022a, Thm. 4.23): *An algebra* $\mathbf{A} \in \mathsf{QNI}$ *is subdirectly irreducible if and only if the order* $\leq$ *on* $\mathbf{A}$ *has a penultimate element.*

Lemma 8.9 and Corollary 8.8 give us the following.

Corollary 8.10: *An algebra* $\mathbf{A} \in \mathsf{QNP} \cup \mathsf{QNS}$ *is subdirectly irreducible if and only if the order* $\leq$ *on* $\mathbf{A}$ *has a penultimate element.*

Recall from earlier the term:

$$q(x, y, z) := (x \to y) \to ((y \to x) \to ((\sim x \to \sim y) \to ((\sim y \to \sim x) \to z))).$$

Proposition 8.11 (Rivieccio & Jansana, 2021, Cor. 34): *The term* $q(x, y, z)$ *is a (commutative, non-regular) ternary deduction term for QNI-algebras in the sense of Blok and Pigozzi (1994). Hence, the variety* QNI *has equationally definable principal congruences and the strong congruence extension property (Blok & Pigozzi, 1994, Thm. 2.12).*

Proposition 8.6 suggests that $q(x,y,z)$ may be a ternary deduction term for QNP as well. We give a direct proof of this below. To this end, observe that, by Definition 4.9(iii).1, instead of the term $q(x,y,z)$ we can equivalently use the following one, which is easier to handle in computations:

$$r(x,y,z) := ((x \to y) * (y \to x) * ({\sim}x \to {\sim}y) * ({\sim}y \to {\sim}x)) \to z.$$

Lemma 8.12: *Every* **A** $\in$ QNP *satisfies the following identities:*

(i) $r(x,x,y) = y$.
(ii) $r(x,y,x) = r(x,y,y)$.
(iii) $r(x,y,{\sim}z) = r(x,y,{\sim}r(x,y,z))$.
(iv) $r(x,y,z \to w) = r(x,y,r(x,y,z) \to r(x,y,w))$.
(v) $r(x,y,z * w) = r(x,y,r(x,y,z) * r(x,y,w))$.

Lemma 8.13: *Every* **A** $\in$ QNS *satisfies the following identity:*

$$r(x,y,z \wedge w) = r(x,y,r(x,y,z) \wedge r(x,y,w)).$$

Proof: As before, we let $\alpha := (a_1 \to b_1) \wedge (b_1 \to a_1) \wedge (a_2 \to b_2) \wedge (b_2 \to a_2)$. Then

$$r(a,b,c) \wedge r(a,b,d) = \langle (\alpha \to c_1) \wedge (\alpha \to d_1), (\Box\alpha \wedge c_2) \oplus (\Box\alpha \wedge d_2)\rangle,$$

which, using Definition 5.3(vi) as well as Definition 2.5(i) and the properties of the Heyting implication, we can reduce to $\langle \alpha \to (c_1 \wedge d_1), \Box\alpha \wedge (c_2 \oplus d_2)\rangle$.

The first component of $r(a,b,r(a,b,c) \wedge r(a,b,d))$ is thus:

$$\alpha \to (\alpha \to (c_1 \wedge d_1)) = \alpha \to (c_1 \wedge d_1)$$

and the second is $\Box\alpha \wedge \Box\alpha \wedge (c_2 \oplus d_2) = \Box\alpha \wedge (c_2 \oplus d_2)$. Since we have

$$r(a,b,c \wedge d) = \langle \alpha \to (c_1 \wedge d_1), \Box\alpha \wedge (c_2 \oplus d_2)\rangle,$$

the desired result follows. ■

The preceding lemmas immediately give us the following result (cf. Blok & Raftery, 1997, Thm. 4.2).

Proposition 8.14 (cf. Rivieccio & Jansana, 2021, Cor. 34): *The term $r(x,y,z)$, or equivalently $q(x,y,z)$, is a (commutative, non-regular) ternary deduction term for* QNP *as well as for* QNS. *Hence, the varieties* QNP *and* QNS *have equationally definable principal congruences and the strong congruence extension property (Blok & Pigozzi, 1994, Thm. 2.12).*

Problem 8.15: The variety of quasi-Nelson algebras (as a subvariety of residuated lattices) is congruence-permutable, with Maltsev term $((x \Rightarrow y) \Rightarrow z) \wedge ((z \Rightarrow y) \Rightarrow x)$. The same term witnesses the congruence-permutability of the variety QNS and of the variety of implicative semilattices (Blok et al., 1984, p. 367). On the other hand, since

Hilbert algebras are not congruence-permutable (Blok et al., 1984, p. 368), the class QNI cannot be congruence-permutable either (cf. Example 4.3). The same holds for QNM. Indeed, as observed earlier, the congruence lattices of members of QNM do not satisfy any non-trivial lattice identity: in particular, QNM is not congruence-modular, which entails that it cannot be congruence-permutable (Burris & Sankappanavar, 1981, Thm. II.5.10). We have seen that QNP has 3-permutable congruences, but the above considerations leave the following question open: is QNP congruence-permutable?

Problem 8.16: It was shown in Rivieccio and Spinks (2020) that quasi-Nelson algebras can be characterised as the class of $(0, 1)$-*congruence orderable* commutative integral bounded residuated lattices (see Rivieccio & Spinks, 2020 for the relevant definitions). Similar questions may be asked concerning quasi-Nelson pocrims and semihoops, namely: is QNP precisely the class of $(0, 1)$-congruence orderable bounded pocrims? Is QNS precisely the class of $(0, 1)$-congruence orderable bounded semihoops? We notice that the lemmas leading to the characterisation of quasi-Nelson algebras obtained in Rivieccio and Spinks (2020) rely essentially on the presence of certain operations in the language, especially the lattice join, and therefore do not seem to be easily adaptable to the case of pocrims and semihoops.

9. Subvarieties

In the preceding sections, we have often mentioned and used the observation that any pseudo-complemented semilattice may be viewed as a quasi-Nelson monoid (Proposition 3.4); similarly, any bounded Hilbert algebra may be viewed as a QNI-algebra (Example 4.3), every bounded implicative semilattice is an example of a quasi-Nelson pocrim (Proposition 4.10), and every Heyting algebra is a quasi-Nelson semihoop (Example 5.4). It is therefore natural to ask (if and) how the above-mentioned classes of algebras can be obtained as subvarieties of QNM, QNI, QNP and QNS. For QNI, some of these questions have been settled in Rivieccio (2022a), from which we cite a few results below.

Proposition 9.1 (Rivieccio, 2022a, Prop. 4.26): *Let* $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle \leq \mathbf{S}^{\bowtie}$ *be a QNI twist-algebra, with* $\mathbf{S} = \langle S; \odot, \to, 0, 1\rangle$ *an nH-semigroup.*

- (i) $\mathbf{A} \vDash {\sim}{\sim} x \to x = 1$ *iff* $\mathbf{A} \vDash {\sim}{\sim} x = x$ *iff* $\mathbf{A} \vDash x = 1 \odot x$ *(i.e. 1 is the neutral element for* $\odot$ *on the left) iff* $\mathbf{A} \vDash (x \odot x) \to x = 1$ *iff* $\mathbf{S} \vDash \Box x \to x = 1$ *iff* $\mathbf{S} \vDash \Box x = x$ *(i.e.* $\Box$ *is the identity map) iff the natural order of the Hilbert algebra reduct of* $\mathbf{S}$ *forms a bounded meet-semilattice with* $\odot$ *as meet.*
- (ii) $\mathbf{A} \vDash ({\sim}{\sim} x \to {\sim}{\sim} y) \to {\sim}{\sim}(x \to y) = 1$ *iff* $\mathbf{A} \vDash {\sim}{\sim} x \to {\sim}{\sim} y = {\sim}{\sim}(x \to y)$ *iff* $\mathbf{S} \vDash (\Box x \to \Box y) \to \Box(x \to y) = 1$ *iff* $\mathbf{S} \vDash \Box x \to \Box y = \Box(x \to y)$.
- (iii) *The operation* $\odot$ *is commutative iff* $\mathbf{S} \cong \mathbf{A}$ *via the map* $a \mapsto \langle a, a \to 0\rangle$ *for all* $a \in S$*, iff* $\mathbf{A}$ *is a bounded Hilbert algebra.*
- (iv) $\mathbf{A} \vDash x = x \odot x$ *(i.e. the operation* $\odot$ *is idempotent) iff* $\mathbf{A} \vDash x = x \odot 1$ *(i.e. 1 is the neutral element for* $\odot$ *on the right) iff* $\mathbf{A} \vDash ({\sim} x \to {\sim} y) \to (y \to x) = 1$ *iff* $\mathbf{A} \vDash {\sim} x \to {\sim} y = y \to x$ *iff* $\mathbf{A}$ *is a Boolean algebra (on which* $\odot$ *is the meet) iff* $\mathbf{S}$ *is a Boolean algebra and* $\mathbf{S} \cong \mathbf{A}$ *via the map* $a \mapsto \langle a, a \to 0\rangle$ *for all* $a \in S$.

(v) $\mathbf{A} \models ((x \to y) \to x) \to x = 1$ *iff* $\mathbf{S}$ *is a Boolean algebra.*

We write $x \preceq y$ instead of $x^2 = x^2 * y^2$, and we let $\neg x := {\sim}(x * x)$. This notation is justified by the observation that every QN-algebra satisfies $x \to 0 = {\sim}(x * x)$.

The proof of the following proposition, as well as any subsequent proof which has been omitted in this section, can be found in the Appendix.

Proposition 9.2: *Let* $\mathbf{M} = \langle M; *, \sim, 0, 1\rangle \leq \mathbf{S}^{\bowtie}$ *be a QNM twist-algebra, with* $\mathbf{S} = \langle S; \wedge, \rightharpoonup, 0, 1\rangle$ *a* $\rightharpoonup$*-semilattice.*

(i) $\mathbf{M} \models {\sim}{\sim}x = x$ *(i.e.* $\mathbf{M}$ *is the subreduct of a Nelson algebra) iff* $\mathbf{S} \models \Box x = x$ *(i.e.* $\Box$ *is the identity map) iff* $\mathbf{S} = \langle S, \wedge, \rightharpoonup, 0, 1\rangle$ *is a bounded implicative semilattice (with* $\rightharpoonup$ *as implication).*
(ii) $\mathbf{M} \models x = x * x$ *(i.e.* $*$ *is a semilattice operation) iff* $\langle S; \wedge, \neg, 0, 1\rangle \cong \mathbf{M}$ *via the map* $a \mapsto \langle a, \neg a\rangle$ *for all* $a \in S$ *iff* $\mathbf{M}$ *is a pseudo-complemented semilattice.*
(iii) $\mathbf{M} \models {\sim}{\sim}x = x * x$ *iff* $\mathbf{M}$ *is a Boolean algebra (with* $*$ *as meet and* $\sim$ *as Boolean complement) iff* $\mathbf{S}$ *is a Boolean algebra and* $\langle S; \wedge, \neg, 0, 1\rangle \cong \mathbf{M}$ *via the map* $a \mapsto \langle a, \neg a\rangle$ *for all* $a \in S$.
(iv) $\mathbf{M} \models (x \rightharpoonup 0) \rightharpoonup 0 \preceq x$ *iff* $\mathbf{S}$ *is a Boolean algebra (with* $\rightharpoonup$ *as implication).*
(v) $\mathbf{M} \models (\neg{\sim}x)^2 \preceq \neg\neg x$ *iff* $\neg a_1 = \neg\neg a_2$ *for all* $\langle a_1, a_2\rangle \in M$ *(i.e.* $\mathbf{M}$ *is normal in the terminology of Goranko (1985)).*

The correspondences stated in Propositions 9.1 and 9.2 extend to the classes QNP and QNS in the obvious way. Moreover, with a richer language at our disposal, we can formulate more properties and establish further correspondences. Recall from Font et al. (1984) that a *Wajsberg algebra* is an algebra $\langle A; \Rightarrow, \sim, 1\rangle$ that satisfies the following identities:

(w1) $1 \Rightarrow x = x$.
(w2) $(x \Rightarrow y) \Rightarrow ((y \Rightarrow z) \Rightarrow (x \Rightarrow z)) = 1$.
(w3) $(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$.
(w4) $({\sim}x \Rightarrow {\sim}y) \Rightarrow (y \Rightarrow x) = 1$.

Wajsberg algebras (also known as *MV-algebras* when presented in an alternative language) are the algebraic counterpart of infinite-valued Łukasiewicz logic. Interpreting the operations $\Rightarrow, \sim$ and 1 as, respectively, the strong implication, the negation and the top element of the partial order, it can be easily verified that every subreduct of a quasi-Nelson algebra satisfies (w1) and (w2) but not necessarily (w3) or (w4); Nelson algebras further satisfy (w4) but not necessarily (w3). It is well known that the subvariety of Nelson algebras defined by (w3) coincides with the class of three-valued Wajsberg algebras; this is the algebraic counterpart of the observation that the least common extension of of Nelson logic and infinite-valued Łukasiewicz logic is three-valued Łukasiewicz logic (see e.g. Vakarelov, 1977, Thm. 11). The first item of Proposition 9.3 below shows that the result of requiring a quasi-Nelson pocrim to satisfy (w3) yields a similar result, that is, also in the non-involutive setting we obtain three-valued Wajsberg algebras.

The identity considered in the second item of Proposition 9.3 corresponds, in the setting of residuated lattices, to one of the possible formulations of the *prelinearity* property (see Rivieccio & Flaminio, 2022; Rivieccio et al., 2020 for an extensive study of prelinearity in the quasi-Nelson context). When the algebraic language includes the lattice join, prelinearity is usually expressed by the less opaque identity $(x \to y) \vee (y \to x) = 1$. Classes of algebras satisfying prelinearity are of special interest in fuzzy logics, and are particularly easy to work with, because each prelinear algebra is well known to be representable as a subdirect product of linearly ordered algebras. In the quasi-Nelson setting we could alternatively formulate the prelinearity property, for instance, through the identity:

$$(x \Rightarrow y) \Rightarrow z \leq ((y \Rightarrow x) \Rightarrow z) \Rightarrow z$$

which uses the strong rather than the weak quasi-Nelson implication. It turns out, however, that both options are equivalent (see Rivieccio et al., 2020; it is easy to see that the same proof works in the non-involutive setting as well).

Proposition 9.3: *Let* $\mathbf{A} = \langle A; *, \to, \sim, 0, 1\rangle \leq \mathbf{S}^{\bowtie} \in \mathsf{QNP}$, *with* $\mathbf{S} = \langle S; \wedge, \to, \Box, 0, 1\rangle$ *an implicative semilattice with a nucleus.*

(i) $\mathbf{A} \vDash (x \to y) \to y = (y \to x) \to x$ *iff* $\langle S; \wedge, \to, 0, 1\rangle$ *is a Boolean algebra (on which* $\Box$ *is the identity map) iff* $\langle A; \Rightarrow, \sim, 1\rangle$ *is a (three-valued) Wajsberg algebra iff* $\mathbf{A} \vDash (x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$.

(ii) $\mathbf{A} \vDash (x \to y) \to z \preceq ((y \to x) \to z) \to z$ *iff* $\langle S; \wedge, \to, 0, 1\rangle$ *is a Gödel algebra.*[3]

Proof: (i). It is clear that $\mathbf{A} \vDash (x \to y) \to y = (y \to x) \to x$ if and only if $\mathbf{S} \vDash (x \to y) \to y = (y \to x) \to x$. Observe that, if the latter identity holds, then we can instantiate it as $(x \to 0) \to 0 = (0 \to x) \to x$. Since $(0 \to x) \to x = x$ holds on every implicative semilattice (or Hilbert algebra), we conclude that $\mathbf{S}$ satisfies $\neg\neg x = x$. Hence, $\langle S; \wedge, \to, 0, 1\rangle$ is a Boolean algebra on which $x \vee y := \neg(\neg x \wedge \neg y)$ and, as observed earlier, the nucleus $\Box$ is necessarily the identity map.

This shows that the first two statements in item (i) are equivalent. The equivalence between the second statement and the third is well known from the literature on Nelson logic. It is also clear that the third statement implies the fourth.

To conclude the proof, assume $\mathbf{A} \vDash (x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$. Then $\mathbf{A}$ is involutive, for we have $\sim\sim a = (a \Rightarrow 0) \Rightarrow 0 = (0 \Rightarrow a) \Rightarrow a = 1 \Rightarrow a = a$ for all $a \in A$. This means that $\mathbf{A}$ is the subreduct of a Nelson algebra, and $\mathbf{A}$ satisfies all four identities (w1)–(w4) which define Wajsberg algebras. Hence, $\langle A; \Rightarrow, \sim, 1\rangle$ is a (three-valued) Wajsberg algebra, as required.

(ii). It is clear that $\mathbf{A} \vDash (x \to y) \to z \preceq ((y \to x) \to z) \to z$ if and only if $\mathbf{S} \vDash ((x \to y) \to z) \to (((y \to x) \to z) \to z) = 1$. The latter means that $\langle S; \wedge, \to, 0, 1\rangle$ is a bounded Gödel hoop, i.e. a Gödel algebra (where the join is defined by $x \vee y := ((x \to y) \to y) \wedge ((y \to x) \to x)$ Aglianò et al., 2007, Thm. 1.7). ■

In the next lemma we look at conditions for a quasi-Nelson pocrim to have a hoop (hence a meet semilattice) reduct. As the lemma illustrates, the latter amounts to imposing that the nucleus be the maximal one on the underlying implicative semilattice.

Lemma 9.4: *Let* $\mathbf{A} = \langle A; *, \rightarrow, \sim, 0, 1\rangle \leq \mathbf{S}^{\bowtie} \in \mathsf{QNP}$, *with* $\mathbf{S} = \langle S; \wedge, \rightarrow, \Box, 0, 1\rangle$ *an implicative semilattice with a nucleus. The following conditions are equivalent:*

(i) $\mathbf{A} \vDash x * (x \Rightarrow y) = y * (y \Rightarrow x)$.
(ii) $\mathbf{A} \vDash x * (x \Rightarrow {\sim}x) = {\sim}x * ({\sim}x \Rightarrow x)$.
(iii) $\mathbf{A} \vDash \neg\neg x \preceq {\sim}{\sim}x$.
(iv) $\mathbf{S} \vDash \Box x = \neg\neg x$.

Corollary 9.5: *Let* $\mathbf{A} = \langle A; *, \rightarrow, \sim, 0, 1\rangle \leq \mathbf{S}^{\bowtie} \in \mathsf{QNP}$, *with* $\mathbf{S} = \langle S; \wedge, \rightarrow, \Box, 0, 1\rangle$ *an implicative semilattice with a nucleus. Assume* $\mathbf{A}$ *is involutive, i.e.* $\mathbf{A} \vDash {\sim}{\sim}x = x$. *The following conditions are equivalent:*

(i) *Any of the statements in Proposition 9.3(i) holds.*
(ii) $\mathbf{A} \vDash x * (x \Rightarrow y) = y * (y \Rightarrow x)$.

Proof: It is clear that (i) entails $\mathbf{S} \vDash \Box x = \neg\neg x$, which gives us (ii) by Lemma 9.4. For the converse, recall that $\mathbf{A}$ is involutive if and only if $\Box$ is the identity map on $\mathbf{S}$. Then, assuming (ii), we have $\mathbf{S} \vDash x = \neg\neg x$, by Lemma 9.4. We conclude that $\mathbf{S}$ is Boolean, as required. ■

We have observed right after Definition 5.1 that a quasi-Nelson semihoop $\mathbf{A}$ need not be a hoop. Indeed, more precisely, $\mathbf{A}$ is a hoop if and only if (any of) the conditions in Lemma 9.4 are satisfied; and Corollary 9.5 tells us that a quasi-Nelson semihoop that is not a hoop can be obtained by considering the reduct of a Nelson algebra constructed as a twist-algebra over any non-Boolean Heyting algebra. We formalise these as well as a few other interesting observations in the next and last proposition (we omit the proof, which is analogous to the previous ones).

Proposition 9.6: *Let* $\mathbf{A} = \langle A; \wedge, *, \rightarrow, \sim, 0, 1\rangle \leq \mathbf{S}^{\bowtie}$ *be an algebra in* QNS, *with* $\mathbf{S} = \langle S; \wedge, \oplus, \rightarrow, 0, 1\rangle$ *a* $\oplus$*-implicative semilattice.*

(i) $\mathbf{A} \vDash {\sim}{\sim}x = x$ *iff* $\langle A; \wedge, \oplus, \rightarrow, \sim, 0, 1\rangle$ *is a Nelson algebra* ($\oplus$ *being the lattice join) iff the operation* $\oplus$ *is idempotent on* $\mathbf{S}$ *(hence,* $\oplus$ *is a join on* $\mathbf{S}$ *and* $\Box$ *is the identity map) iff* $\langle S; \wedge, \oplus, 0, 1\rangle$ *is a bounded (distributive) lattice.*
(ii) $\mathbf{A} \vDash x \wedge y = x * y$ *iff* $\mathbf{A}$ *is a bounded implicative semilattice iff*

$$\langle S; \wedge, \rightarrow, \neg, 0, 1\rangle \cong \langle A; \wedge, \rightarrow, \sim, 0, 1\rangle$$

via the map $a \mapsto \langle a, \neg a\rangle$ *for all* $a \in S$.
(iii) $\mathbf{A} \vDash x \wedge y = x * (x \Rightarrow y)$ *iff any of the conditions of Lemma 9.4 applies.*

10. Future work

We have presented a first algebraic study of certain fragments of (quasi-)Nelson logic, most of which had never been considered in the literature so far. Several further topics of potential interest remain to be explored. Besides the numbered Problems mentioned in the previous sections, we list a few ideas below.

1. As we have seen with Theorem 2.14, the twist representation of quasi-Nelson algebras associates, to each quasi-Nelson algebra **A**, a pair $\langle \mathbf{H}, \nabla \rangle$ such that **H** is a Heyting algebra with a nucleus and $\nabla \subseteq H$ is a dense lattice filter of **H**. Since each pair $\langle \mathbf{H}, \nabla \rangle$ determines a unique twist-algebra, this establishes a one-to-one correspondence which can be easily formulated as a categorical equivalence between two naturally associated algebra-based categories (for details as well as potential applications of such an equivalence, see e.g. Rivieccio et al., 2020). We have established with Theorems 6.9 and 6.10 that, similarly to the algebras in the full language, every quasi-Nelson pocrim (or quasi-Nelson semihoop) may be represented as a pair $\langle \mathbf{S}, \nabla \rangle$ such that **S** is a bounded ($\oplus$-)implicative semilattice with a nucleus and $\nabla \subseteq S$ is a special subset of **S**. By contrast, the representations obtained so far for quasi-Nelson monoids (Theorem 3.18) and for QNI-algebras (Theorem 4.8) only allow us to associate a family of distinct algebras to a given factor algebra **S**, namely, all the twist-algebras over **S**; such a correspondence can also be formulated categorically, and results in an adjunction rather than an equivalence.[4] The reason for this is that the representation of algebras as pairs (such as $\langle \mathbf{H}, \nabla \rangle$ or $\langle \mathbf{S}, \nabla \rangle$) relies heavily on the presence of certain operations in the algebraic language (the monoid opreation and the implication in the case of quasi-Nelson algebras, pocrims and semihoops, the meet, the join and the two negations in the case of the $\{\wedge, \vee, \sim, \neg\}$-subreducts considered in Rivieccio, 2020a and Sendlewski, 1991. As mentioned in Problems 6.4 and 6.11, it is therefore an open question whether the representations of quasi-Nelson monoids and QNI-algebras can be refined by adding further structure to the twist factors, thus obtaining a one-to-one correspondence similar to the one we have for quasi-Nelson algebras.

2. Can the techniques employed in the present paper be successfully applied to obtain characterisations of other fragments of quasi-Nelson logic/algebras? Previous experience suggests that we may be optimistic with regards to certain fragments, for instance the ones corresponding to the $\{*, \wedge, \sim\}$-, the $\{*, \vee, \sim\}$- and the $\{\vee, \to, \sim\}$-subreducts of quasi-Nelson algebras. Other fragments, unfortunately, seem to lie beyond the scope of our current techniques, either because the language is too weak to express the key properties of quasi-Nelson algebras or because the relevant operations do not behave well with respect to the twist construction. Examples of the former kind are the negation-less fragments (including e.g. the $\{\Rightarrow\}$-fragment, which is of obvious logical significance; see also the Problems 4.21 and 6.13 mentioned earlier); an example of the latter is the $\{\Rightarrow, \sim\}$-fragment.

3. Although in the present paper we have not dealt with logical systems as such, most of our results have a straightforward logical interpretation. The classes QNP and QNS (as well as the subvarieties mentioned in Section 9), in particular, are obviously the algebraic counterparts of algebraizable logical systems. We leave for a future publication the study of these systems, as well as the investigation of their relations with other (fragments of) substructural logics.

4. The recent paper by Busaniche et al. (2022) introduces a very general twist construction based on the notion of *Nelson conucleus*, which allows one to obtain virtually all the known examples in the literature as special cases. The main idea is that the various twist representations may be obtained uniformly by employing a unary function

that realises, on every algebra, a special interior operator (a *conucleus*). In many of the known cases, the conucleus is term definable from the basic algebraic operations (for instance on Nelson algebras it may be given by the map $x \mapsto x * x$), but (Busaniche et al., 2022) consider in general enriched algebras where the conucleus is explicitly added as primitive. This approach is extended to quasi-Nelson and other non-necessarily involutive algebras in Busaniche and Rivieccio (202x) , suggesting that it might be successfully applied also to the setting of subreducts considered in the present paper.

5. As we have seen, some of the twist constructions introduced in the present paper involve, as factors, algebras that have never been previously considered in the literature as such ($\rightharpoonup$-semilattices, nH-semigroups, $\oplus$-implicative semilattices). This may be seen as a downside of our approach, for one of the main advantages of twist representations is that they allow one to work with fairly well-known factor algebras (e.g. Heyting algebras) instead of (e.g.) the more exotic Nelson algebras. On the other hand, the new classes of algebras introduced here turn out to be easy to handle, for they may be viewed as 'term definable subreducts' of Heyting algebras expanded with a modal operator (the latter, known as *nuclear Heyting algebras*, have been extensively investigated in different contexts since at least the dissertation by Macnab (1976)). The present study suggests that these subreducts of nuclear Heyting algebras may also have an independent interest as objects of algebraic investigation; in fact we have already started pursuing this line of research (Celani & Rivieccio, 202x).

Notes

1. In the papers (Rivieccio & Spinks, 2019, 2020), QN-algebras are also called *quasi-Nelson residuated lattices*: the two terms refer to the two presentations (using either the strong or the weak implication as primitive) of the 'same' class of algebras. In the present paper we shall refrain from employing the term 'quasi-Nelson residuated lattices' and the alternative abstract presentation appearing in Rivieccio and Spinks (2019) and Rivieccio and Spinks (2020).
2. BCK-algebras are the algebraic counterpart of Meredith's BCK-logic. In the literature one can find two alternative definitions (one dual to the other) of BCK-algebras; for a comparison with (quasi-)Nelson algebras it is best to work with the definition given e.g. in Spinks and Veroff (2008, pp. 331–332).
3. *Gödel algebras* are precisely the prelinear Heyting algebras, i.e. those satisfying $(x \to y) \vee (y \to x) = 1$.
4. Results of this kind for algebras related to Nelson logic are established e.g. in Busaniche and Cignoli (2014, Sec. 4) and Rivieccio (2010, Ch. 5).

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Appendix. Proofs

Proof of Proposition 3.4: Let us take a look at the requirements postulated by Definition 3.2. Regarding (i), we certainly have that $\langle S, \wedge, 1\rangle$ is a 3-potent monoid. Regarding (ii), observe that $\preceq = \leq$, the latter being the semilattice order on S. Using this observation and the properties of the pseudo-complement, it is easy to verify items (iii).1–(iii).3 and (iii).6–(iii).9. Since $x \rightharpoonup y = \neg(x \wedge \neg y)$, items (iii).4 and (iii).5 have been already verified in the proof of Proposition 3.9. Finally, item (iii).10 translates as $\neg(a \wedge b) = \neg(a \wedge \neg\neg b) \wedge \neg(b \wedge \neg\neg a)$, which is an easy consequence of the properties of pseudo-complemented semilattices listed earlier (in particular, (8) and (11) from Sankappanavar (1979, p. 305)). ■

Proof of Lemma 3.6: Let $a, b, c \in M$.

(i). From Definition 3.2(iii).1 and the commutativity of $*$, we have $\sim a * a = a * \sim a \leq a$. Also, from Definition 3.2(iii).6 and $\sim a \leq \sim a$ we have $\sim a * a = 0$. Thus $0 \leq a$ for all $a \in M$. To show $a \leq 1$, we use Definition 3.2(ii) and check that $a \preceq 1$ and $\sim 1 \preceq \sim a$. As to the former, by definition of $\preccurlyeq$ we have $a \preceq 1$ iff $a^2 * 1^2 = a^2$ which follows easily from the monoid properties. Regarding $\sim 1 \preceq \sim a$, we have $\sim 1 = 0$ by Definition 3.2(iii).8. By definition of $\preccurlyeq$ we have $0 \preceq \sim a$ iff $0^2 = 0^2 * (\sim a)^2$. The desired result then follows from the observation that $0 * b = 0$ for all $b \in M$ (recall that 0 is the least element in the $\leq$ order, and that item (iii).1 entails $0 * b \leq 0$).

(ii). By Definition 3.2(iii).6 we have $\sim a \leq \sim a$ iff $\sim a * a = 0$. Thus by commutativity of $*$ we also have $a * \sim a = 0$.

(iii). By Definition 3.2(iii).6 we have $a \leq \sim\sim a$ iff $a * \sim a = 0$, which we have proven in the preceding item.

(iv). Assume $a \preceq b$, i.e. $a^2 = a^2 * b^2$. Using Definition 3.2(iii).9, we have $(\sim\sim a)^2 = \sim\sim(a^2) = \sim\sim(a^2 * b^2) = \sim\sim(a^2) * \sim\sim(b^2) = (\sim\sim a)^2 * (\sim\sim b)^2$. Hence, $\sim\sim a \preceq \sim\sim b$. Also, by the commutativity of $*$ and 3-potency, we have $a^2 * (a * b)^2 = a^2 * a^2 * b^2 = a^2 * b^2 = a^2$. Hence, $a \preceq a * b$.

(v). Assuming $a \leq b$, we have $a \preceq b$ and $\sim b \preceq \sim a$. By the preceding item, from $a \preceq b$ we obtain $\sim\sim a \preceq \sim\sim b$. Hence, $\sim b \leq \sim a$, as required.

(vi). Recall that $b \rightharpoonup a = \sim(b^2 * \sim a)$. By Definition 3.2(iii).1 (and the commutativity of $*$), we have $b^2 * \sim a \leq \sim a$. Thus, we can apply the preceding item to obtain $\sim\sim a \leq \sim(b^2 * \sim a)$. The result then follows from the inequality $a \leq \sim\sim a$, which we have shown in item (iii) above.

(vii). Assume $a \preceq \sim b$, that is $a^2 * (\sim b)^2 = a^2$. By Definition 3.2(iii).1, we have $a^2 * (\sim b)^2 = (a^2 * \sim b) * \sim b \leq \sim b$. Thus $a^2 \leq \sim b$. By Definition 3.2(iii).7, we have $\sim b = \sim\sim\sim b$ and so $a^2 \leq \sim\sim\sim b$. By Definition 3.2(iii).6, we have $a^2 \leq \sim\sim\sim b$ iff $a^2 * \sim\sim b = 0$. Then, using Definition 3.2(iii).8, we conclude $1 = \sim 0 = \sim(a^2 * \sim\sim b) = a \rightarrow \sim b$, as required.

Conversely, assume $a \rightarrow \sim b = \sim(a^2 * \sim\sim b) = 1$. Notice that $a^2 * (\sim b)^2 \leq a^2$ always holds (Definition 3.2(iii).1) so it suffices to show $a^2 \leq a^2 * (\sim b)^2$. From $\sim(a^2 * \sim\sim b) = 1$ we have $\sim\sim(a^2 * \sim\sim b) = \sim 1 = 0$, the last equality holding by Definition 3.2(iii).8. By item (iii) above we have $a^2 * \sim\sim b \leq \sim\sim(a^2 * \sim\sim b) = 0$ and so $a^2 * \sim\sim b = 0$. Then we can use Definition 3.2(iii).6 to obtain $a^2 \leq \sim\sim\sim b = \sim b$, the last equality holding by Definition 3.2(iii).7. From $a^2 \leq \sim b$, using 3-potency, we have $a^2 * a^2 = a^2 \leq \sim b * \sim b = (\sim b)^2$. Similarly, using Definition 3.2(iii).2, from $a^2 \leq (\sim b)^2$ we obtain $a^2 * a^2 = a^2 \leq a^2 * (\sim b)^2$.

(viii). Assuming $a * b \preceq \sim c$, we can use the preceding item to obtain $(a * b) \rightharpoonup \sim c = 1$. By Definition 3.2(iii).4 (and the commutativity of $*$) we have $1 = (a * b) \rightharpoonup \sim c = (b * a) \rightharpoonup \sim c = b \rightharpoonup (a \rightharpoonup \sim c)$. Recall that $a \rightharpoonup \sim c = \sim(a^2 * \sim\sim c)$. Then, letting $a' = b$ and $b' = (a^2 * \sim\sim c)$, we have $a' \rightharpoonup \sim b' = 1$. Then we can apply the preceding item again to conclude $b = a' \preceq \sim b' = a \rightharpoonup \sim c$, as required.

Conversely, assume $b \preceq a \rightharpoonup \sim c$. Reasoning as before, we can the preceding item to obtain $b \rightharpoonup (a \rightharpoonup \sim c) = 1$. By Definition 3.2(iii).4 (and the commutativity of $*$), we have $1 = b \rightharpoonup (a \rightharpoonup \sim c) = (b * a) \rightharpoonup \sim c = (a * b) \rightharpoonup \sim c$. Then, again the preceding item, we obtain $a * b \preceq \sim c$, as required.

(ix). We have:

$$\begin{aligned}
{\sim}{\sim} a \rightharpoonup {\sim}{\sim} b &= {\sim}({\sim}{\sim} a * {\sim}{\sim} a * {\sim}{\sim}{\sim}{\sim} b) \\
&= {\sim}({\sim}{\sim} a * {\sim}{\sim}(a * {\sim} b)) && \text{by Definition 3.2(iii).9} \\
&= {\sim}{\sim}{\sim}(a * a * {\sim} b) && \text{by Definition 3.2(iii).9} \\
&= {\sim}(a * a * {\sim} b) && \text{by Definition 3.2(iii).7} \\
&= a \rightharpoonup b.
\end{aligned}$$

Since ${\sim} b = {\sim}{\sim}{\sim} b$, the last passage also easily entails $a \rightharpoonup b = a \rightharpoonup {\sim}{\sim} b$.

(x). We have:

$$\begin{aligned}
{\sim}(a * b) &= {\sim}{\sim}{\sim}(a * b) && \text{by Definition 3.2(iii).7} \\
&= {\sim}({\sim}{\sim} a * {\sim}{\sim} b) && \text{by Definition 3.2(iii).9} \\
&= {\sim}({\sim}{\sim} a * {\sim}{\sim}{\sim}{\sim} b) && \text{by Definition 3.2(iii).7} \\
&= {\sim}{\sim}{\sim}(a * {\sim}{\sim} b) && \text{by Definition 3.2(iii).9} \\
&= {\sim}(a * {\sim}{\sim} b). && \text{by Definition 3.2(iii).7}
\end{aligned}$$

■

Proof of Proposition 3.9: We proceed to verify that items (ii)–(viii) of Definition 3.7 are satisfied. Let $a, b, c \in P$.

(ii). Using (11) and (8), we have $a \rightharpoonup (b \rightharpoonup c) = \neg(a \wedge \neg\neg(b \wedge \neg c)) = \neg(a \wedge \neg\neg b \wedge \neg\neg\neg c) = \neg(a \wedge b \wedge \neg c) = (a \wedge b) \rightharpoonup c$.

(iii). Let us compute $a \rightharpoonup (b \wedge c) = \neg(a \wedge \neg(b \wedge c))$ and $(a \rightharpoonup b) \wedge (a \rightharpoonup c) = \neg(a \wedge \neg b) \wedge \neg(a \wedge \neg c)$. Observe that, by (6), from $b \wedge c \leq b$ we have $\neg b \leq \neg(b \wedge c)$, so $a \wedge \neg b \leq a \wedge \neg(b \wedge c)$ by the semilattice properties. Hence, again by (6), we have $\neg(a \wedge \neg(b \wedge c)) \leq \neg(a \wedge \neg b)$. A similar reasoning shows that $\neg(a \wedge \neg(b \wedge c)) \leq \neg(a \wedge \neg c)$. Hence, $\neg(a \wedge \neg(b \wedge c)) \leq \neg(a \wedge \neg b) \wedge \neg(a \wedge \neg c)$. To show the other inequality, we resort to the property of the pseudo-complement. We have $\neg(a \wedge \neg b) \wedge \neg(a \wedge \neg c) \leq \neg(a \wedge \neg(b \wedge c))$ iff $a \wedge \neg(b \wedge c) \wedge \neg(a \wedge \neg b) \wedge \neg(a \wedge \neg c) = 0$. Using (2) and (11), we have $a \wedge \neg(b \wedge c) \wedge \neg(a \wedge \neg b) \wedge \neg(a \wedge \neg c) = a \wedge \neg(b \wedge c) \wedge \neg\neg b \wedge \neg\neg c = a \wedge \neg(b \wedge c) \wedge \neg\neg(b \wedge c)$. The result then follows from the observation that $d \wedge \neg d = 0$ holds on every pseudo-complemented semilattice, for one has $a \wedge \neg(b \wedge c) \wedge \neg\neg(b \wedge c) = a \wedge 0 = 0$.

(iv). Since $\neg\neg 0 = 0$ by (4), it suffices to compute $\Box 0 = \neg(1 \wedge \neg 0) = \neg\neg 0 = 0$.

(v). Observe that $\Box a = 1 \rightharpoonup a = \neg(1 \wedge \neg a) = \neg\neg a$. Then the result follows from the identity (5), i.e. $x \leq \neg\neg x$.

(vi). Using (11) and (2), we have $\Box(a \wedge b) = \neg\neg(a \wedge b) = \neg\neg a \wedge \neg\neg b == \neg\neg a \wedge \neg(a \wedge \neg b) = \Box a \wedge (a \rightharpoonup b)$.

(vii). Using the property of the pseudo-complement and (8), we have $a \leq \neg(b \wedge \neg c) = b \rightharpoonup c$ iff $a \wedge b \wedge \neg c = 0$ iff $a \wedge \neg c \leq \neg b = \neg\neg\neg b$ iff $a \wedge \neg c \wedge \neg\neg b = 0$ iff $a \wedge \neg\neg b = a \wedge \Box b \leq \Box c = \neg\neg c$.

(viii). Let us compute $\Box a \rightharpoonup \Box b = \neg\neg a \rightharpoonup \neg\neg b = \neg(\neg\neg a \wedge \neg\neg\neg b)$. Using (11) and (8), we have $\neg(\neg\neg a \wedge \neg\neg\neg b) = \neg\neg\neg(a \wedge \neg b) = \neg(a \wedge \neg b) = a \rightharpoonup b$, as required. ■

Proof of Proposition 3.10: Let $a, b, c \in S$.

(i). Since $1 \wedge a \leq \Box a$, we may apply Definition 3.7(vii) to obtain $1 \leq a \rightharpoonup a$.

(ii). If $a \leq b$, then using Definition 3.7(iii) we have

$$c \to a = c \to (a \wedge b) = (c \to a) \wedge (c \to b).$$

So, $c \to a \leq c \to b$.

(iii). Assume $a \leq b$, so $a = a \wedge b$. Observe that:

$$(b \to c) \wedge a = (b \to c) \wedge a \wedge b$$

$$= b \wedge \Box c \wedge a \qquad \text{by Definition 3.7(vi)}$$
$$= \Box c \wedge a$$
$$\leq \Box c.$$

Then, by Definition 3.7(vii), we obtain $b \to c \leq a \to c$.

(iv). If $1 \leq a \rightharpoonup b$, then $1 \wedge a = a \leq \Box b$ Definition 3.7(vii). Conversely, if $a \leq \Box b$, then Definition 3.7(vii) gives us $1 \leq a \rightharpoonup \Box b$. But, using Definition 3.7(viii) and the nucleus properties, we have $a \rightharpoonup \Box b = \Box a \rightharpoonup \Box\Box b = \Box a \rightharpoonup \Box b = a \rightharpoonup b$, as required (see also item (vii) below).

(v). Observe that $a \wedge b \leq c$ entails $a \wedge b \leq \Box c$. Then we may apply Definition 3.7(vii) to obtain $a \leq b \rightharpoonup c$.

(vi). The 'only if' part follows from the preceding item. Conversely, assume $a \leq b \rightharpoonup 0$. Then $a \wedge b \leq \Box 0 = 0$ (by Definition 3.7(vii)).

(vii). Using Definition 3.7(viii) and the properties of the nucleus, we have $\Box a \rightharpoonup b = \Box\Box a \rightharpoonup \Box b = \Box a \rightharpoonup \Box b = a \rightharpoonup b$. Similarly, $a \rightharpoonup \Box b = \Box a \rightharpoonup \Box\Box b = \Box a \rightharpoonup \Box b = a \rightharpoonup b$. To conclude the proof, it suffices to show that $a \rightharpoonup b = \Box(a \rightharpoonup b)$. The inequality $a \rightharpoonup b \leq \Box(a \rightharpoonup b)$ holds because $\Box$ is a nucleus. To show that $\Box(a \rightharpoonup b) \leq a \rightharpoonup b$, we begin by noticing that, using Definition 3.7(vi) and the nucleus properties, we have $\Box(a \rightharpoonup b) \wedge a \leq \Box(a \rightharpoonup b) \wedge \Box a = \Box((a \rightharpoonup b) \wedge a) = \Box(a \wedge \Box b) = \Box a \wedge \Box\Box b = \Box a \wedge \Box b \leq \Box b$. From $\Box(a \rightharpoonup b) \wedge a \leq \Box b$, by Definition 3.7(vii), we obtain $\Box(a \rightharpoonup b) \leq a \rightharpoonup b$, as required.

(viii). We have seen with item (vi) that $\neg$ is a pseudo-complement operation (Corollary 3.11), hence we can rely on the general properties of pseudo-complements stated earlier. Keeping this in mind, observe that $a \leq \Box a$ we have $\neg\Box a \leq \neg a$. Also by item (vi), the inequality $\neg a \leq \neg\Box a$ is equivalent to $\Box a \wedge \neg a = 0$, and the latter holds true because, by the nucleus properties, we have $\Box a \wedge \neg a \leq \Box a \wedge \Box\neg a = \Box(a \wedge \neg a) = \Box 0 = 0$. Hence, $\neg a = \neg\Box a$.

Similarly to the preceding case, we have $\neg a \leq \Box\neg a$ simply as a consequence of $x \leq \Box x$. The other inequality, $\Box\neg a \leq \neg a$, is equivalent to $\Box\neg a \wedge a = 0$, which holds true because $\Box\neg a \wedge a \leq \Box\neg a \wedge \Box a = \Box(\neg a \wedge a) = \Box 0 = 0$. Hence, $\Box\neg a = \neg a$, concluding our proof.

(ix). By item (vi) above, we have $a \rightharpoonup \neg b \leq \neg(a \wedge b)$ iff $a \wedge b \wedge (a \rightharpoonup \neg b) = 0$. The latter equality holds because, using Definition 3.7(vi) and the nucleus properties, we have $a \wedge b \wedge (a \rightharpoonup \neg b) = b \wedge a \wedge \Box\neg b \leq \Box b \wedge a \wedge \Box\neg b = a \wedge \Box(b \wedge \neg b) = a \wedge \Box 0 = a \wedge 0 = 0$.

The other inequality, $\neg(a \wedge b) \leq a \rightharpoonup \neg b$, is equivalent (by Definition 3.7(vii)) to $a \wedge \neg(a \wedge b) \leq \Box\neg b$. By item (viii) above, the latter can be rewritten as $a \wedge \neg(a \wedge b) \leq \neg b$, which is equivalent to the (obviously true) equality $b \wedge a \wedge \neg(a \wedge b) = 0$.

(x). By item (ix) above, we have $a \rightharpoonup \neg\neg b = \neg(a \wedge \neg b)$. Thus, it suffices to show that $\neg(a \wedge \neg b) = \neg\neg(a \rightharpoonup b)$.

We first tackle the inequality $\neg(a \wedge \neg b) \leq \neg\neg(a \rightharpoonup b)$. Observe that $\neg a \leq a \rightharpoonup b$. Indeed, by Definition 3.7(vii), the latter inequality is equivalent to $a \wedge \neg a \leq \Box b$, which does hold because $a \wedge \neg a = 0$. From $\neg a \leq a \rightharpoonup b$, by the properties of the pseudo-complement ($\neg$ is order-reversing), we obtain $\neg(a \rightharpoonup b) \leq \neg\neg a$. A similar reasoning allows us to establish that $\neg(a \rightharpoonup b) \leq \neg b$. Indeed, $b \leq a \rightharpoonup b$ holds because it is equivalent (again by Definition 3.7(vii)) to $b \wedge a \leq \Box b$, which is certainly true. From $b \leq a \rightharpoonup b$ we obtain $\neg(a \rightharpoonup b) \leq \neg b$. Then, by the semilattice properties, we have $\neg(a \rightharpoonup b) \leq \neg\neg a \wedge \neg b$. Observe that, by the properties of the pseudo-complement stated earlier, we have $\neg\neg a \wedge \neg b = \neg\neg a \wedge \neg\neg\neg b = \neg\neg(a \wedge \neg b)$. Hence, $\neg(a \rightharpoonup b) \leq \neg\neg(a \wedge \neg b)$, which gives us $\neg\neg\neg(a \wedge \neg b) = \neg(a \wedge \neg b) \leq \neg\neg(a \rightharpoonup b)$, as desired.

The other inequality, $\neg\neg(a \rightharpoonup b) \leq \neg(a \wedge \neg b)$, can be obtained by establishing $a \wedge \neg b \leq \neg(a \rightharpoonup b)$. By item (vi) above, the latter is equivalent to $(a \rightharpoonup b) \wedge a \wedge \neg b = 0$. To see that the latter equality holds, we use Definition 3.7(vi) and the nucleus properties to compute $(a \rightharpoonup b) \wedge a \wedge \neg b = a \wedge \Box b \wedge \neg b \leq a \wedge \Box b \wedge \Box\neg b = a \wedge \Box(b \wedge \neg b) = a \wedge \Box 0 = a \wedge 0 = 0$.

(xi). By symmetry, it suffices to show that $a \rightharpoonup \neg b \leq b \rightharpoonup \neg a$. By Definition 3.7(vii), we have $a \rightharpoonup \neg b \leq b \rightharpoonup \neg a$ iff $b \wedge (a \rightharpoonup \neg b) \leq \Box\neg a = \neg a$ (the latter equality holding by item (viii) above) iff (by the property of the pseudo-complement) $a \wedge b \wedge (a \rightharpoonup \neg b) \leq 0$. The latter inequality holds because, Definition 3.7(vi) and item (viii) above, we have $a \wedge b \wedge (a \rightharpoonup \neg b) = a \wedge \Box\neg b \wedge b = a \wedge \neg b \wedge b = a \wedge 0 = 0$. ■

Proof of Proposition 3.15: Throughout the proof we let $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$ etc.

(i). To check associativity, observe that we only need to worry about the second components of $(\langle a_1, a_2\rangle * \langle b_1, b_2\rangle) * \langle c_1, c_2\rangle$ and $\langle a_1, a_2\rangle * (\langle b_1, b_2\rangle * \langle c_1, c_2\rangle)$. These are, respectively, $((a_1 \wedge b_1) \rightharpoonup c_2) \wedge (c_1 \rightharpoonup ((a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2)))$ and $(a_1 \rightharpoonup ((b_1 \rightharpoonup c_2) \wedge (c_1 \rightharpoonup b_2))) \wedge ((b_1 \wedge c_1) \rightharpoonup a_2)$. We have:

$$
\begin{aligned}
&((a_1 \wedge b_1) \rightharpoonup c_2) \wedge (c_1 \rightharpoonup ((a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2)))\\
&\quad = (a_1 \rightharpoonup (b_1 \rightharpoonup c_2)) \wedge (c_1 \rightharpoonup ((a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2))) && \text{by Definition 3.7(ii)}\\
&\quad = (a_1 \rightharpoonup (b_1 \rightharpoonup c_2)) \wedge (c_1 \rightharpoonup (a_1 \rightharpoonup b_2)) \wedge (c_1 \rightharpoonup (b_1 \rightharpoonup a_2)) && \text{by Definition 3.7(iii)}\\
&\quad = (a_1 \rightharpoonup (b_1 \rightharpoonup c_2)) \wedge (a_1 \rightharpoonup (c_1 \rightharpoonup b_2)) \wedge (c_1 \rightharpoonup (b_1 \rightharpoonup a_2)) && \text{by Definition 3.7(ii)}\\
&\quad = (a_1 \rightharpoonup (b_1 \rightharpoonup c_2)) \wedge (a_1 \rightharpoonup (c_1 \rightharpoonup b_2)) \wedge ((b_1 \wedge c_1) \rightharpoonup a_2) && \text{by Definition 3.7(ii)}\\
&\quad = (a_1 \rightharpoonup ((b_1 \rightharpoonup c_2) \wedge (c_1 \rightharpoonup b_2))) \wedge ((b_1 \wedge c_1) \rightharpoonup a_2). && \text{by Definition 3.7(iii)}
\end{aligned}
$$

The commutativity of $*$ follows directly from the commutativity of $\wedge$. Let us check that $\langle 1, 0\rangle$ is the neutral element. Recall that, for all $\langle a_1, a_2\rangle \in A$, we have $1 \rightharpoonup a_2 = \Box a_2 = a_2$. Moreover, from $\Box a_1 \wedge a_2 = \Box(a_1 \wedge a_2) = \Box 0 = 0$, using items (vii) and (viii) of Definition 3.7, we obtain $a_2 \leq \Box a_1 \rightharpoonup \Box 0 = a_1 \rightharpoonup 0$. Hence, $\langle a_1, a_2\rangle * \langle 1, 0\rangle = \langle a_1 \wedge 1, (a_1 \rightharpoonup 0) \wedge (1 \rightharpoonup a_2)\rangle = \langle a_1, (a_1 \rightharpoonup 0) \wedge a_2\rangle = \langle a_1, a_2\rangle$, as required. To check 3-potency, recall that $\langle a_1, a_2\rangle * \langle a_1, a_2\rangle = \langle a_1, a_1 \rightharpoonup 0\rangle$. Then $\langle a_1, a_2\rangle * \langle a_1, a_2\rangle * \langle a_1, a_2\rangle = \langle a_1, a_2\rangle * \langle a_1, a_1 \rightharpoonup 0\rangle = \langle a_1, a_1 \rightharpoonup (a_1 \rightharpoonup 0)\rangle = \langle a_1, (a_1 \wedge a_1) \rightharpoonup 0\rangle = \langle a_1, a_1 \rightharpoonup 0\rangle$.

(ii).1. Recall that $\langle a_1, a_2\rangle^2 = \langle a_1, a_1 \rightharpoonup 0\rangle$. Using Definition 3.7(ii), we have $\langle a_1, a_2\rangle^2 * \langle b_1, b_2\rangle^2 = \langle a_1 \wedge b_1, (a_1 \rightharpoonup (b_1 \rightharpoonup 0)) \wedge (b_1 \rightharpoonup (a_1 \rightharpoonup 0))\rangle = \langle a_1 \wedge b_1, (a_1 \wedge b_1) \rightharpoonup 0\rangle$, which immediately gives us the 'only if' part. Conversely, assuming $a_1 \wedge b_1 = a_1$, we have $\langle a_1, a_2\rangle^2 * \langle b_1, b_2\rangle^2 \langle a_1 \wedge b_1, (a_1 \wedge b_1) \rightharpoonup 0\rangle = \langle a_1, a_1 \rightharpoonup 0\rangle = \langle a_1, a_2\rangle^2$, as required.

(iii).2. It is clear that $\langle a_1, a_2\rangle * \langle b_1, b_2\rangle \preceq \langle a_1, a_2\rangle$. We also have ${\sim}\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle \preceq \langle (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2), \Box(a_1 \wedge b_1)\rangle = {\sim}(\langle a_1, a_2\rangle * \langle b_1, b_2\rangle)$ because of the following reasoning. On the one hand, $a_2 \leq b_1 \rightharpoonup a_2$ follows easily from Definition 3.7(vii). On the other hand, by Proposition 3.10(vi), from $a_1 \wedge a_2 = 0$ we obtain $a_2 \leq a_1 \rightharpoonup 0 \leq a_1 \rightharpoonup b_2$. Thus, $a_2 \leq (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2)$, as required.

(ii).3. Assume $\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle$, i.e. $a_1 \leq b_1$ and $b_2 \leq a_2$. From the former, we have $a_1 \wedge c_1 \leq b_1 \wedge c_1$. It remains to show that $(b_1 \rightharpoonup c_2) \wedge (c_1 \rightharpoonup b_2) \leq (a_1 \rightharpoonup c_2) \wedge (c_1 \rightharpoonup a_2)$. By Proposition 3.10(i), from $b_2 \leq a_2$ we obtain $c_1 \rightharpoonup b_2 \leq c_1 \rightharpoonup a_2$. By Proposition 3.10(ii), from $a_1 \leq b_1$ we obtain $b_1 \to c_2 \leq a_1 \to c_2$. Hence, $(b_1 \rightharpoonup c_2) \wedge (c_1 \rightharpoonup b_2) \leq (a_1 \rightharpoonup c_2) \wedge (c_1 \rightharpoonup a_2)$, as required.

(ii).4. Follows immediately from the component-wise definition of $\rightharpoonup$.

(ii).5. Let us compute:

$$
\begin{aligned}
(\langle a_1, a_2\rangle * \langle b_1, b_2\rangle) \to \langle c_1, c_2\rangle &= \langle (a_1 \wedge b_1) \rightharpoonup c_1, \Box(a_1 \wedge b_1) \wedge c_2\rangle\\
&= \langle a_1 \rightharpoonup (b_1 \rightharpoonup c_1), \Box a_1 \wedge \Box b_1 \wedge c_2\rangle\\
&= \langle a_1, a_2\rangle \to (\langle b_1, b_2\rangle \to \langle c_1, c_2\rangle)
\end{aligned}
$$

(ii).6. We are only concerned with the first components of $\langle a_1, a_2\rangle \rightharpoonup (\langle b_1, b_2\rangle * \langle c_1, c_2\rangle)$ and $(\langle a_1, a_2\rangle \rightharpoonup \langle b_1, b_2\rangle) * (\langle a_1, a_2\rangle \rightharpoonup \langle c_1, c_2\rangle)$. These are, respectively, $a_1 \rightharpoonup (b_1 \wedge c_1)$ and $(a_1 \rightharpoonup b_1) \wedge (a_1 \to c_1)$. The result then follows from Definition 3.7(iii).

(ii).7. Assume $\langle a_1, a_2\rangle \leq \langle b_2, \Box b_1\rangle = {\sim}\langle b_1, b_2\rangle$. Then $a_1 \leq b_2$ and $\Box b_1 \leq a_2$. From the former we have $a_1 \wedge b_1 \leq b_1 \wedge b_2 = 0$; from the latter, by items (iv) and (vii) of Proposition 3.10, we obtain $\Box b_1 \rightharpoonup a_2 = b_1 \rightharpoonup a_2 = 1$. Hence, $\langle a_1, a_2\rangle * \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2)\rangle = \langle 0, 1\rangle$. Conversely, assume the latter holds. From $a_1 \rightharpoonup b_2 = 1$, by Proposition 3.10(iv), we have $a_1 \leq \Box b_2 = b_2$. By the same token, from $b_1 \rightharpoonup a_2 = 1$ we have $\Box b_1 \leq a_2$, as required.

Items (ii).8 and (ii).9 follow easily from the nucleus properties of $\Box$.

(ii).10. It suffices to verify that the first components of ${\sim}{\sim}(\langle a_1, a_2\rangle * \langle b_1, b_2\rangle)$ and ${\sim}{\sim}\langle a_1, a_2\rangle * {\sim}{\sim}\langle b_1, b_2\rangle$ are equal, and this follows easily from the observation that $\Box$ preserves finite meets.

(ii).11. We are only concerned with the first components of $\sim(\langle a_1, a_2\rangle * \langle b_1, b_2\rangle)$ and $(\langle a_1, a_2\rangle \rightharpoonup \sim\langle b_1, b_2\rangle) * (\langle b_1, b_2\rangle \rightharpoonup \sim\langle a_1, a_2\rangle)$, which are easily seen to be equal. ■

Proof of Proposition 3.17: Proposition 3.5 implies that the quotient $\langle M/{\equiv}, *, \rightharpoonup, 0, 1\rangle$ is indeed well-defined. To show that $*$ is a semilattice operation on $M/{\equiv}$, it suffices to show idempotency, i.e. that $[a] * [a] = [a]$ for all $a \in M$. This easily follows from 3-potency, for one has $[a] * [a] = [a * a] = [a]$ if and only if $(a * a)^2 = a^2$. That $[1]$ is the top element of $M/{\equiv}$ follows from the identity $a * 1 = a$, which holds on every monoid. To show that $[0]$ is the least element (i.e. that $[0] * [a] = [0 * a] = [0]$ for all $a \in M$) we reason as follows. On the one hand we have $0 * a \leq 0$ by Definition 3.2(iii) (i). Thus, in particular, $0 * a \preceq 0$. On the other hand, as shown earlier, 0 is the bottom element of the order $\leq$. This entails $0 \leq a$ and, a fortiori, $0 \preceq a$. Hence, $[0 * a] = [0]$, as required. Thus $\langle M/{\equiv}, *, 0, 1\rangle$ is a bounded semilattice.

Let us now show the last claim. Let $a, b \in M$. By definition of $\preccurlyeq$, we have $a \preceq b$ iff $a^2 * b^2 = a^2$. Since $a^2 * b^2 = (a * b)^2$, this means that $[a * b] = [a] * [b] = [a]$, i.e. $[a] \leq [b]$.

Keeping the above in mind, we proceed to show that the remaining properties of Definition 3.7 are satisfied.

(ii). Follows immediately from Definition 3.2(iii).4.

(iii). Follows immediately from Definition 3.2(iii).5.

(iv). It suffices to show $1 \rightharpoonup 0 \preceq 0$. This is easy, for we have $\sim(1^2 * \sim 0) = \sim\sim 0 = \sim 1 = 0$.

(v). Observe that $1 \rightharpoonup a = \sim(1^2 * \sim a) = \sim\sim a$. Thus, we need to show that $a \preceq \sim\sim a$ and $\sim\sim\sim a \preceq \sim a$. The latter follows from Definition 3.2(iii).7. Regarding the former, recall that $a \leq \sim\sim a$ by Lemma 3.6(iii). Hence, $a \leq \sim\sim a$, as required.

(vi). Let us compute $a * (1 \rightharpoonup b) = a * \sim\sim b$ and $a * (a \rightharpoonup b) = a * \sim(a^2 * \sim b)$. We have seen in the proof of Lemma 3.6(vi) that $\sim\sim b \leq \sim(a^2 * \sim b)$. Then, by Definition 3.2(iii).2, we have $a * \sim\sim b \leq a * \sim(a^2 * \sim b)$, which entails $a * \sim\sim b \preceq a * \sim(a^2 * \sim b)$. To show that $a * \sim(a^2 * \sim b) \preceq a * \sim\sim b$, we begin by observing that $a * \sim(a^2 * \sim b) \preceq \sim\sim b$. Indeed, by Lemma 3.6(ix), we have $\sim(a^2 * \sim b) = a \rightharpoonup b \preceq a \rightharpoonup \sim\sim b$. Then, by Lemma 3.6(viii), from $\sim(a^2 * \sim b) \preceq a \rightharpoonup \sim\sim b$ we obtain $a * \sim(a^2 * \sim b) \preceq \sim\sim b$. Observe that 3-potency entails $x * y \preceq y$, so, in particular, $a * \sim(a^2 * \sim b) \preceq a$. Then, by Lemma 3.6(iv), from $a * \sim(a^2 * \sim b) \preceq \sim\sim b$ we have $a * \sim(a^2 * \sim b) \preceq a * \sim(a^2 * \sim b) * \sim\sim b \preceq a * \sim\sim b$, as required.

(vii). Assume $[a] \leq [b] \rightharpoonup [c]$. Then, $a \preceq b \rightharpoonup c = b \rightharpoonup \sim\sim c$ (by item (ix) of Lemma 3.6). We can then apply Lemma 3.6(viii) to obtain $a * b \sim\sim c = \preceq 1 \rightharpoonup c$. Then $[a] \wedge [b] \leq [1] \rightharpoonup [c]$, as required. Conversely, assuming $[a] \wedge [b] \leq [1] \rightharpoonup [c]$, we have $a * b \preceq \sim\sim c$. Then, by items (viii) and (ix) of Lemma 3.6, we obtain $a \preceq b \rightharpoonup \sim\sim c = b \rightharpoonup c$. Hence $[a] \leq [b] \rightharpoonup [c]$, as required.

(viii). Using Definition 3.2(iii).7 and (iii).9, it suffices to compute $(1 \rightharpoonup a) \rightharpoonup (1 \rightharpoonup b) = \sim\sim a \rightharpoonup \sim\sim b = \sim(\sim\sim a * \sim\sim a * \sim\sim\sim b) = \sim\sim\sim(a * a * \sim b) = \sim(a * a * \sim b) = a \rightharpoonup b$. ■

Proof of Proposition 4.10: Let $\mathbf{S} = \langle S; \wedge, \to, 0, 1\rangle$ be a bounded implicative semilattice. Then $\langle S; \wedge, 1\rangle$ is certainly 3-potent commutative monoid, and we have seen with Example 4.3 that $\langle S; \to, \sim, 0, 1\rangle$ is a QNI-algebra. Thus, items (i) and (ii) of Definition 4.9 are satisfied. The equalities corresponding to items (iii).1 and (iii).2 are well known to hold on every implicative semilattice. To verify the remaining ones, let $a, b \in S$.

(iii).3. To show that $\neg(a \to b) \leq \neg\neg a \wedge \neg b$, we verify that $\neg(a \to b) \leq \neg\neg a$ and $\neg(a \to b) \leq \neg b$. Recall that every bounded implicative semilattice (or indeed every pseudo-complemented semilattice) satisfies $x \wedge \neg x = 0$ and $x \leq \neg\neg x$. By residuation, from $0 = a \wedge \neg a \leq b$ we have $\neg a \leq a \to b \leq \neg\neg(a \to b)$. Also by residuation, from $\neg a \leq \neg\neg(a \to b)$ we have $\neg a \wedge \neg(a \to b) = 0$, and from the latter $\neg(a \to b) \leq \neg\neg a$. By a similar reasoning, from $b \leq a \to b \leq \neg\neg(a \to b)$ we have $b \wedge \neg(a \to b) = 0$ and, from the latter, $\neg(a \to b) \leq \neg b$. It remains to show $\neg\neg a \wedge \neg b \leq \neg(a \to b)$. We shall use the identities $x \wedge (x \to y) = x \wedge y$ and $\neg\neg\neg x = \neg x$, which hold on every bounded implicative semilattice. Using the former, we have $a \wedge (a \to b) \wedge \neg b = a \wedge b \wedge \neg b = a \wedge 0 = 0$. By residuation, from $a \wedge (a \to b) \wedge \neg b = 0$ we obtain $(a \to b) \wedge \neg b \leq \neg a = \neg\neg\neg a$. From the latter, again by residuation, we have $(a \to b) \wedge \neg\neg a \wedge \neg b = 0$, which gives us $\neg\neg a \wedge \neg b \leq \neg(a \to b)$, as required.

(iii).4. To show that $\neg(a \wedge b) \le (a \to \neg b) \wedge (b \to \neg a)$, it suffices, by symmetry, to verify that $\neg(a \wedge b) \le a \to \neg b$. For this, it suffices to observe that, by residuation, we have $\neg(a \wedge b) \le a \to \neg b$ iff $a \wedge \neg(a \wedge b) \le \neg b$ iff $b \wedge a \wedge \neg(a \wedge b) = 0$. To show $(a \to \neg b) \wedge (b \to \neg a) \le \neg(a \wedge b)$, observe that $a \wedge b \wedge (a \to \neg b) \wedge (b \to \neg a) = a \wedge \neg b \wedge b \wedge \neg a = 0$. From $a \wedge b \wedge (a \to \neg b) \wedge (b \to \neg a) = 0$, by residuation, we have $(a \to \neg b) \wedge (b \to \neg a) \le \neg(a \wedge b)$, as required. ∎

Proof of Lemma 4.13: By the above-mentioned results, it suffices to verify that:

(i) $\equiv$ is compatible with the monoid operation,
(ii) the quotient $\langle A/{\equiv}; *, 0, 1\rangle$ is a bounded semilattice,
(iii) the pair $(*, \to)$ is residuated on $A/{\equiv}$,
(iv) the above-defined $\Box$ is a nucleus on the semilattice $\langle A/{\equiv}; *, 0, 1\rangle$.

(i). We shall rely on the properties of QNI-algebras listed in Lemma 4.2. Assume $a \equiv b$ for some $a, b \in A$. Thus, in particular, $a \to b = 1$. We have:

$$
\begin{aligned}
(a * c) \to (b * c) &\equiv ((a * c) \to b) * ((a * c) \to c) && \text{by Definition 4.9(iii).2}\\
&= ((c * a) \to b) * (a \to (c \to c)) && \text{by Definition 4.9(i) and (iii).1}\\
&= (c \to (a \to b)) * (a \to (c \to c)) && \text{by Definition 4.9(iii).1}\\
&= (c \to 1) * (a \to 1)) && x \to x = 1\\
&= 1 * 1 && x \to 1 = 1\\
&= 1 && \text{by Definition 4.9(i).}
\end{aligned}
$$

Thus, $(a * c) \to (b * c) \equiv 1$, which (by item (iii) of Lemma 4.2) entails $(a * c) \to (b * c) = 1$. Hence, $(a * c) \preceq (b * c)$. A similar reasoning shows that $(b * c) \preceq (a * c)$, so $a * c \equiv b * c$. This observation (and the commutativity of $*$) immediately entail that $\equiv$ is compatible with the monoid operation.

(ii). Obviously $\langle A/{\equiv}; *, 1\rangle$ is a monoid. Idempotency follows from the following computations. On the one hand, we have $(a * a) \to a = a \to (a \to a) = a \to 1 = 1$ by Definition 4.9(iii).1. On the other hand, $a \to (a * a) \equiv (a \to a) * (a \to a) = 1 * 1 = 1$. Hence, $a \to (a * a) = 1$, and $a \equiv a * a$. Thus $\langle A/{\equiv}; *\rangle$ is a semilattice with greatest element 1. Further observe that, by Definition 4.9(iii).1, we have $(a * 0) \to 0 = a \to (0 \to 0) = a \to 1 = 1$ and $0 \to (a * 0) \equiv (0 \to a) * (0 \to 0) = 1 * 1 = 1$. Thus $a * 0 \equiv 0$, i.e. 0 is the least element of the semilattice $\langle A/{\equiv}; *\rangle$.

(iii). Observe that, denoting by $\le$ the semilattice order of $\langle A/{\equiv}; *\rangle$, we have $a \preceq b$ iff $a/{\equiv} \le b/{\equiv}$, for all $a, b \in A$. Indeed, $a/{\equiv} \le b/{\equiv}$, by definition, means that $a * b \equiv a$. If the latter holds, then $a \to b = 1 * (a \to b) = (a \to a) * (a \to b) \equiv a \to (a * b) = 1$. Hence, $a \to b = 1$, i.e. $a \preceq b$. Conversely, if the latter holds, we have $a \to (a * b) \equiv (a \to a) * (a \to b) = 1 * 1 = 1$, so $a \preceq a * b$. Since $a * b \preceq a$ always holds (because $(a * b) \to a = b \to (a \to a) = b \to 1 = 1$), we have $a \equiv a * b$, as claimed.

Now, in order to show that $(*, \to)$ form a residuated pair on $A/{\equiv}$, we need to check that $a/{\equiv} * b/{\equiv} \le c/{\equiv}$ iff $a/{\equiv} \le b/{\equiv} \to c/{\equiv}$, for all $a, b, c \in A$. In the light of the preceding considerations, this amounts to showing that $a * b \preceq c$ iff $a \preceq b \to c$, which is an easy consequence of Definition 4.9(iii).1.

(iv). By Rivieccio (2022a, Prop. 4.15), the above-defined operation $\Box$ is a nucleus on the bounded Hilbert algebra $\langle A/{\equiv}; \to, 0, 1\rangle$. By the preceding item, the natural order on $\langle A/{\equiv}; \to, 0, 1\rangle$ coincides with the semilattice order of $\langle A/{\equiv}; *\rangle$. This immediately entails that the $\Box$ satisfies items (i), (iii) and (iv) of Definition 2.5. Regarding item (iv), the inequality $\Box(x \wedge y) \le \Box x \wedge \Box y$ holds because $\Box$ is monotonic. As to the other inequality, using the identities $(x * y) \to z = x \to (y \to z)$ and $x \to \Box y = \Box x \to \Box y$ (see Rivieccio, 2022a, Lemma 4.4), we can obtain $(\Box x * \Box y) \to \Box(x * y) = \Box x \to (\Box y \to \Box(x * y)) = x \to (y \to \Box(x * y)) = (x * y) \to \Box(x * y)$. Then Definition 2.5(iv) entails the required result. ∎

Proof of Proposition 4.19: Corollary 4.17 entails that $\langle A; \leq; *, 1\rangle$ is a commutative integral pomonoid. It remains to check that the pair $(*, \Rightarrow_*)$ is residuated. This follows from the observation that $\Rightarrow$ is given (on twist-algebras) as the strong implication of quasi-Nelson algebras Proposition 2.11.

Indeed, relying on Theorem 4.16, we consider a QNP twist-algebra $\mathbf{A} \leq \mathbf{S}^{\bowtie}$ and elements $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$, $c = \langle c_1, c_2\rangle \in A$. Suppose $\langle a_1, a_2\rangle * \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (b_1 \to a_2)\rangle \leq \langle c_1, c_2\rangle$, i.e. $a_1 \wedge b_1 \leq c_1$ and $c_2 \leq a_1 \to b_2, b_1 \to a_2$. From the former, by residuation (of $\to$ w.r.t $\wedge$), we have $a_1 \leq b_1 \to c_1$. From the latter, also by residuation, we have $a_1 \leq c_2 \to b_2$. Thus $a_1 \leq (b_1 \to c_1) \wedge (c_2 \to b_2)$. Likewise, from $c_2 \leq b_1 \to a_2$ we have $b_1 \wedge c_2 \leq a_2$. Observe that, using the identity $x \to \Box y = \Box x \to \Box y$ (item (iv) after Definition 2.6) and the requirement $\Box a_2 = a_2$, we have $\Box b_1 \to a_2 = \Box b_1 \to \Box a_2 = b_1 \to \Box a_2 = b_1 \to a_2$. Thus, from $c_2 \leq b_1 \to a_2 = \Box b_1 \to a_2$, by residuation we obtain $\Box b_1 \wedge c_2 \leq a_2$. Hence, $\langle a_1, a_2\rangle \leq \langle (b_1 \to c_1) \wedge (c_2 \to b_2), \Box b_1 \wedge c_2\rangle = \langle b_1, b_2\rangle \Rightarrow \langle c_1, c_2\rangle$. The converse implication (from $a \leq b \Rightarrow c$ to $a * b \leq c$) can be easily established through a similar reasoning. ■

Proof of Proposition 5.7: The first item of Definition 5.1 is satisfied by construction. We proceed to check (ii).

Let us first verify that $\langle A, \wedge, 0, 1\rangle$ is a bounded semilattice. The commutativity and associativity of $\wedge$ are consequences of Definition 5.3(ii). Let $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle$, $c = \langle c_1, c_2\rangle \in A$. Idempotency is easy: we have $\langle a_1, a_2\rangle \wedge \langle a_1, a_2\rangle = \langle a_1 \wedge a_1, a_2 \oplus a_2\rangle = \langle a_1, \Box a_2\rangle = \langle a_1, a_2\rangle$. As for the bounds, by Definition 5.3(iii), we have $\langle a_1, a_2\rangle \wedge \langle 0, 1\rangle = \langle a_1 \wedge 0, a_2 \oplus 1\rangle = \langle 0, 1\rangle$. Similarly, using Definition 5.3(iv), we obtain $\langle a_1, a_2\rangle \wedge \langle 1, 0\rangle = \langle a_1 \wedge 1, a_2 \oplus 0\rangle = \langle a_1, \Box a_2\rangle = \langle a_1, a_2\rangle$.

To conclude the proof of (ii), let $\leq$ be the order of the pocrim reduct of $\mathbf{A}$. Let $a = \langle a_1, a_2\rangle$, $b = \langle b_1, b_2\rangle \in A$ be such that $\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle$. Then, $a_1 \leq b_1$ and $b_2 \leq a_2$. The former gives us $a_1 = a_1 \wedge b_1$. Also, by Definition 5.3(iii), we have and $a_2 = \Box a_2 = a_2 \oplus (a_2 \wedge b_2) = a_2 \oplus b_2$. Hence, $a = a \wedge b$. Conversely, assume the latter holds, i.e. $a_1 = a_1 \wedge b_1$ and $a_2 = a_2 \oplus b_2$. The former immediately gives us $a_1 \leq b_1$. From the latter, using Definition 5.3(iv), we obtain $b_2 = \Box b_2 \leq b_2 \oplus a_2 = a_2 \oplus b_2 = a_2$. Hence, $\langle a_1, a_2\rangle \leq \langle b_1, b_2\rangle$.

Let us now look at the conditions in item (iii) of Definition 5.1. The component-wise definition $\langle a_1, a_2\rangle \oplus \langle b_1, b_2\rangle = \langle a_1 \oplus b_1, a_2 \wedge b_2\rangle$ immediately entails that (iii).1 is satisfied. Item (iii).2 follows easily from the observation that the nucleus preserves binary meets. Item (iii).3 easily follows from the nucleus properties (item (i) of Definition 2.5) and the requirement that $\Box a_2 = a_2$ for all $\langle a_1, a_2\rangle \in A$. Finally, Item (iii).4 is easily proved using Definition 5.3(vi). ■

Proof of Proposition 6.2: Observe that $1 \in \nabla$. Indeed, since ∇ is non-empty, let $a \in \nabla$. Then Propositions 6.1(iii) and 3.10(i) give us $a \rightharpoonup a = 1 \in \nabla$. We further claim that $\langle a, \neg a\rangle \in Tw(S, \nabla)$ for all $a \in S$. Indeed, we have $\Box\neg a = \neg a$ (by Proposition 3.10(viii)) and $a \wedge \neg a = 0$ by the property of the pseudo-complement. Furthermore, recalling again Proposition 3.10(i), we have $\neg a \rightharpoonup \neg\neg\neg a = \neg a \rightharpoonup \neg a = 1 \in \nabla$. Hence $\pi_1[Tw(S, \nabla)] = S$, as required by Definition 3.14. For the rest of the proof, assume $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Tw(S, \nabla)$, so $a_1 \wedge a_2 = b_1 \wedge b_2 = 0$ and $\neg a_1 \rightharpoonup \neg\neg a_2, \neg b_1 \rightharpoonup \neg\neg b_2 \in \nabla$. Let us show that $Tw(S, \nabla)$ is closed under the QNM twist-algebra operations. The case of the constants is easy, for we have $\neg 1 \rightharpoonup \neg\neg 0 = 0 \rightharpoonup 0 = 1 = \neg 0 \rightharpoonup \neg\neg 1 = 1 \rightharpoonup 1$.

($\sim$). To show that $\sim\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle \in Tw(S, \nabla)$, recall that $\neg a_1 \rightharpoonup \neg\neg a_2 = \neg a_2 \rightharpoonup \neg\neg a_1$ (Proposition 3.10.xi) and $\neg\Box a_1 = \neg a_1$ (Proposition 3.10 (viii)). Then, $\neg a_2 \rightharpoonup \neg\neg\Box a_1 = \neg a_2 \rightharpoonup \neg\neg a_1 = \neg a_1 \rightharpoonup \neg\neg a_2 \in \nabla$, as required.

($*$). We need to check that $\neg(a_1 \wedge b_1) \rightharpoonup \neg\neg((a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2)) \in \nabla$. Using the properties of the pseudo-complement, we have:

$$
\begin{aligned}
&\neg(a_1 \wedge b_1) \rightharpoonup \neg\neg((a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2)) \\
&\quad = (a_1 \rightharpoonup \neg b_1) \rightharpoonup (\neg\neg(a_1 \rightharpoonup b_2) \wedge \neg\neg(b_1 \rightharpoonup a_2)) \\
&\quad = (a_1 \rightharpoonup \neg b_1) \rightharpoonup ((a_1 \rightharpoonup \neg\neg b_2) \wedge (b_1 \rightharpoonup \neg\neg a_2)) && \text{by Proposition 3.10(x)} \\
&\quad = ((a_1 \rightharpoonup \neg b_1) \rightharpoonup (a_1 \rightharpoonup \neg\neg b_2)) \wedge ((a_1 \rightharpoonup \neg b_1) \rightharpoonup (b_1 \rightharpoonup \neg\neg a_2)) && \text{by Definition 3.7(iii)}
\end{aligned}
$$

$$= (((a_1 \rightharpoonup \neg b_1) \wedge a_1) \to \neg\neg b_2) \wedge (((b_1 \rightharpoonup \neg a_1) \wedge b_1) \rightharpoonup \neg\neg a_2) \quad \text{by Definition 3.7(ii)}$$
$$= ((\Box\neg b_1 \wedge a_1) \to \neg\neg b_2) \wedge ((\Box\neg a_1 \wedge b_1) \rightharpoonup \neg\neg a_2) \quad \text{by Definition 3.7(vi)}$$
$$= ((a_1 \wedge \neg b_1) \rightharpoonup \neg\neg b_2) \wedge ((b_1 \wedge \neg a_1) \rightharpoonup \neg\neg a_2) \quad \text{by Proposition 3.10(viii)}$$
$$= (a_1 \rightharpoonup (\neg b_1 \rightharpoonup \neg\neg b_2)) \wedge (b_1 \rightharpoonup (\neg a_1 \rightharpoonup \neg\neg a_2)) \quad \text{by Definition 3.7(ii).}$$

Now observe that $a_1 \rightharpoonup (\neg b_1 \rightharpoonup \neg\neg b_2) \in \nabla$ and $b_1 \rightharpoonup (\neg a_1 \rightharpoonup \neg\neg a_2) \in \nabla$ follow from the assumptions together with the property stated in Proposition 6.1(iii). Then, using Proposition 6.1(iv), we can obtain $(a_1 \rightharpoonup (\neg b_1 \rightharpoonup \neg\neg b_2)) \wedge \Box(b_1 \rightharpoonup (\neg a_1 \rightharpoonup \neg\neg a_2)) \in \nabla$. To conclude the proof, it suffices to observe that $\Box(b_1 \rightharpoonup (\neg a_1 \rightharpoonup \neg\neg a_2)) = b_1 \rightharpoonup (\neg a_1 \rightharpoonup \neg\neg a_2)$ by Proposition 3.10(vii). ■

Proof of Lemma 6.5: The only items not proved in Rivieccio (2022a, Lemma 4.7) are (iv) and (v).

(iv). Given $a \in H$, observe that from $a \leq \Box a$ we have $\neg\Box a \leq \neg a$. As to the inequality $\neg a \leq \neg\Box a$, recalling that $\Box x \to \Box y = x \to \Box y$ and $\Box 0 = 0$, we have $\neg a \to \neg\Box a = \neg a \to (\Box a \to 0) = \neg a \to (\Box a \to \Box 0) = \neg a \to (a \to \Box 0) = \neg a \to (a \to 0) = \neg a \to \neg a = 1$.

Similarly to the preceding case, we have $\neg a \leq \Box\neg a$ simply because $x \leq \Box x$. As to the inequality $\Box\neg a \leq \neg a$, we have $\Box\neg a \to \neg a = \Box\neg a \to (a \to 0) = a \to (\Box\neg a \to 0) = a \to (\Box\neg a \to \Box 0) = a \to (\neg a \to \Box 0) = a \to (\neg a \to 0) = \neg a \to (a \to 0) = \neg a \to \neg a = 1$, as required.

(v). Let $a, b \in H$. To establish the inequality $\neg\neg(a \to b) \leq a \to \neg\neg b$, we compute:

$$\begin{aligned}
\neg\neg(a \to b) \to (a \to \neg\neg b) &= a \to (\neg\neg(a \to b) \to \neg\neg b) && x \to (y \to z) = y \to (x \to z)\\
&= a \to (\neg b \to \neg\neg\neg(a \to b)) && \text{item (iii)}\\
&= a \to (\neg b \to \neg(a \to b)) && \text{item (ii)}\\
&= a \to ((a \to b) \to \neg\neg b) && \text{item (iii)}\\
&= (a \to b) \to (a \to \neg\neg b) && x \to (y \to z) = y \to (x \to z)\\
&= (a \to (b \to \neg\neg b) && x \to (y \to z) = (x \to y) \to (x \to z)\\
&= a \to 1 && \text{item (i)}\\
&= 1.
\end{aligned}$$

To establish the inequality $a \to \neg\neg b \leq \neg\neg(a \to b)$, let us begin by observing that $\neg a = a \to 0 \leq a \to b$ entails $\neg(a \to b) \leq \neg\neg a$ and $b \leq a \to b$ entails $\neg(a \to b) \leq \neg b$. Thus, letting $c := \neg(a \to b)$, we have that $c \leq \neg\neg a$ and $c \leq \neg b$. We claim that these two inequalities entail $c \to \neg(a \to \neg\neg b) = 1$. Indeed, we have:

$$\begin{aligned}
& c \to \neg(a \to \neg\neg b)\\
&= c \to \neg(\neg b \to \neg a) && \text{item (iii)}\\
&= c \to ((\neg b \to \neg a) \to 0)\\
&= (c \to (\neg b \to \neg a)) \to (c \to 0) && x \to (y \to z) = (x \to y) \to (x \to z)\\
&= ((c \to \neg b) \to (c \to \neg a)) \to (c \to 0) && x \to (y \to z) = (x \to y) \to (x \to z)\\
&= (1 \to (c \to \neg a)) \to (c \to 0) && c \leq \neg b\\
&= (c \to \neg a) \to (c \to 0) && x = 1 \to x\\
&= c \to (\neg a \to 0) && x \to (y \to z) = (x \to y) \to (x \to z)\\
&= c \to \neg\neg a\\
&= 1 && c \leq \neg\neg a.
\end{aligned}$$

Thus we have:

$$(a \to \neg\neg b) \to \neg\neg(a \to b) = \neg(a \to b) \to \neg(a \to \neg\neg b) \qquad \text{item (iii)}$$

$$= c \to \neg(a \to \neg\neg b)$$
$$= 1$$

which gives us $a \to \neg\neg b \leq \neg\neg(a \to b)$, as desired. ■

Proof of Proposition 6.6: (i). Assuming $\sim\langle a_1, a_2\rangle \preceq \langle a_1, a_2\rangle$, we have $\langle 1, 0\rangle = \sim\langle a_1, a_2\rangle \to \langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle \to \langle a_1, a_2\rangle = \langle a_2 \to a_1, a_2 \odot a_2\rangle = \langle a_2 \to a_1, \Box a_2\rangle = \langle a_2 \to a_1, a_2\rangle$. Hence, $a_2 = 0$. For the converse, observe that $\sim\langle a, 0\rangle \to \langle a, 0\rangle = \langle 0, \Box a\rangle \to \langle a, 0\rangle = \langle 0 \to a, 0 \odot 0\rangle = \langle 1, 0\rangle$. Hence, $\sim\langle a_1, a_2\rangle \preceq \langle a_1, a_2\rangle$, as required.

(ii). Let $a \in S$ be such that $\neg a = 0$, Consider an element $b \in S$ such that $\langle a, b\rangle \in A$. Recalling that $a \odot b = 0$, we have:

$$\begin{aligned} b \to \neg a &= \Box b \to (a \to 0) && \Box b = b \\ &= \Box b \to (a \to \Box 0) && \Box 0 = 0 \\ &= \Box b \to (\Box a \to 0) && x \to \Box y = \Box x \to \Box y \\ &= (b \odot a) \to 0 && \text{Definition 4.4(v)} \\ &= 0 \to 0 && a \odot b = 0 \\ &= 1. \end{aligned}$$

Hence, $b \leq \neg a = 0$, entailing $b = 0$. Then $\langle a, b\rangle = \langle a, 0\rangle \in A^+$, which means that $a \in \nabla_{\mathbf{A}}$, as required.

(iii). Assume $a \in \nabla_{\mathbf{A}}$, i.e. $\langle a, 0\rangle \in A^+$. Let us compute $(\langle a, 0\rangle \to \langle 0, 1\rangle) \to \sim\langle a, 0\rangle = \langle \neg a, \Box a\rangle \to \langle 0, \Box a\rangle = \langle \neg\neg a, \Box\neg a \odot \Box a\rangle = \langle \neg\neg a, 0\rangle$. The latter equality holds because, by Rivieccio (2022a, Lemma 4.9 (i)) and items (iv) and (vi) of Definition 4.4, we have $\Box\neg a \odot \Box a = \neg a \odot a = (a \to 0) \odot a = a \odot 0 = 0$. Hence, $\neg\neg a \in \nabla_{\mathbf{A}}$, as claimed. Now, let $b \in S$. Then $\langle b, c\rangle \in A$ for some $c \in S$, and (using Definition 4.4(vi)) we can compute $\langle b, c\rangle \to \langle a, 0\rangle = \langle b \to a, b \odot 0\rangle = \langle b \to a, 0\rangle$, which gives us $b \to a \in \nabla_{\mathbf{A}}$, as claimed.

(iv). Assume $\langle a, 0\rangle, \langle b, 0\rangle \in A^+$. Then $\langle a, 0\rangle * (\langle a, 0\rangle \to \langle b, 0\rangle) = \langle a \wedge (a \to b), (a \to 0) \wedge \neg(a \to b)\rangle = \langle a \wedge b, 0\rangle \in A^+$, as required. Note that $(a \to 0) \wedge \neg(a \to b) = 0$ holds because it is equivalent, by the property of the pseudo-complement, to $\neg(a \to b) \leq \neg(a \to 0)$, which in turn follows from the pseudo-complement properties and the observation that $a \to 0 \leq a \to b$.

Now assume $\neg\neg a \in \nabla_{\mathbf{A}}$, so $\langle \neg\neg a, 0\rangle \in A^+$. Notice that $\neg\neg a \to a \in D(\mathbf{S}) \subseteq \nabla_{\mathbf{A}}$. Indeed, using items (ii) and (v) of Lemma 6.5, we have $\neg(\neg\neg a \to a) = \neg\neg\neg(\neg\neg a \to a) = \neg(\neg\neg a \to \neg\neg a) = \neg 1 = 0$. As we have seen in item (ii), this entails $\neg\neg a \to a \in \nabla_{\mathbf{A}}$, so $\langle \neg\neg a \to a, 0\rangle \in A^+$. Hence, $\langle \neg\neg a, 0\rangle * \langle \neg\neg a \to a, 0\rangle = \langle \neg\neg a \wedge (\neg\neg a \to a), \neg\neg\neg a \wedge \neg(\neg\neg a \to a)\rangle = \langle \neg\neg a \wedge a, \neg a \wedge \neg(\neg\neg a \to a)\rangle = \langle a, 0\rangle \in A^+$. Regarding the second component, the last equality is justified by the following reasoning. By the property of the pseudo-complement, we have $\neg a \wedge \neg(\neg\neg a \to a) = 0$ iff $\neg a \leq \neg\neg(\neg\neg a \to a)$. The latter holds because, by Lemma 6.5(v), we have $\neg\neg(\neg\neg a \to a) = \neg\neg a \to \neg\neg a = 1$.

(v). Taking the previous observations into account, it suffices to verify that $\nabla_{\mathbf{A}}$ is a $\leq$-increasing set. For this, assume $a \in \nabla_{\mathbf{A}}$ (so $\langle a, 0\rangle \in A^+$) and $a \leq b$ for some $b \in S$. Letting $c \in S$ be such that $\langle b, c\rangle \in A$, we have $\langle a, 0\rangle \vee \langle b, c\rangle = \langle a \vee b, 0 \wedge c\rangle = \langle b, 0\rangle$, which immediately implies the required result. ■

Proof of Proposition 6.7: (i). Observe that $\langle a, \neg a\rangle \in Tw(S, \nabla)$ for all $a \in S$. Indeed, we have $\Box\neg a = \neg a$ (Lemma 6.5(iv)) and, by items (iv) and (vi) of Definition 4.4, $a \odot \neg a = a \odot (a \to 0) = a \odot 0 = 0$ and (using Lemma 6.5(ii)) $\neg a \to \neg\neg\neg a = \neg a \to \neg a = 1 \in \nabla$. Hence $\pi_1[Tw(S, \nabla)] = S$, as required by Definition 4.6. For the rest of the proof, assume $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Tw(S, \nabla)$, so $a_1 \odot a_2 = b_1 \odot b_2 = 0$ and $\neg a_1 \to \neg\neg a_2, \neg b_1 \to \neg\neg b_2 \in \nabla$. Let us show that $Tw(S, \nabla)$ is closed under the QNI twist-algebra operations. The case of the constants is easy, for we have $\neg 0 \to \neg\neg 1 = 0 \to 1 = 1 = \neg 1 \to \neg\neg 0 = 0 \to 0$.

($\sim$). To show that $\sim\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle \in Tw(S, \nabla)$, recall that $\neg a_1 \to \neg\neg a_2 = \neg a_2 \to \neg\neg a_1$ (Lemma 6.5(iii)) and $\neg\Box a_1 = \neg a_1$ (Lemma 6.5(iv)). Then, $\neg a_2 \to \neg\neg\Box a_1 = \neg a_2 \to \neg\neg a_1 = \neg a_1 \to \neg\neg a_2 \in \nabla$, as required.

($\to$). We need to show that $\neg(a_1 \to b_1) \to \neg\neg(a_1 \odot b_2) \in \nabla$. Using the identities $x \to \neg y = y \to \neg x$, and $\neg\neg(x \to y) = x \to \neg\neg y$, we have $\neg(a_1 \to b_1) \to \neg\neg(a_1 \odot b_2) = \neg(a_1 \odot b_2) \to \neg\neg(a_1 \to b_1) = \neg(a_1 \odot b_2) \to (a_1 \to \neg\neg b_1) = (a_1 \to \neg b_2) \to (a_1 \to \neg\neg b_1)$. The last equality holds because, using Definition 4.4, we have $\neg(a_1 \odot b_2) = (a_1 \odot b_2) \to 0 = \Box a_1 \to (\Box b_2 \to 0) = \Box a_1 \to (b_2 \to 0) = \Box a_1 \to \neg b_2 = \Box a_1 \to \Box\neg b_2 = a_1 \to \Box\neg b_2 = a_1 \to \neg b_2$. Resuming our computation, we have $\neg(a_1 \to b_1) \to \neg\neg(a_1 \odot b_2) = (a_1 \to \neg b_2) \to (a_1 \to \neg\neg b_1) = a_1 \to (\neg b_2 \to \neg\neg b_1) = a_1 \to (\neg b_1 \to \neg\neg b_2)$. Since $\neg b_1 \to \neg\neg b_2 \in \nabla$ by assumption, the required result follows from item (iii) of Proposition 6.6.

(ii). We need to check that $\neg(a_1 \wedge b_1) \to \neg\neg((a_1 \to b_2) \wedge (b_1 \to a_2)) \in \nabla$. We have:

$$
\begin{aligned}
&\neg(a_1 \wedge b_1) \to \neg\neg((a_1 \to b_2) \wedge (b_1 \to a_2)) \\
&\quad = (a_1 \to \neg b_1) \to (\neg\neg(a_1 \to b_2) \wedge \neg\neg(b_1 \to a_2)) \\
&\quad = (a_1 \to \neg b_1) \to ((a_1 \to \neg\neg b_2) \wedge (b_1 \to \neg\neg a_2)) \\
&\quad = ((a_1 \to \neg b_1) \to (a_1 \to \neg\neg b_2)) \wedge ((a_1 \to \neg b_1) \to (b_1 \to \neg\neg a_2)) \\
&\quad = (a_1 \to (\neg b_1 \to \neg\neg b_2)) \wedge ((b_1 \to \neg a_1) \to (b_1 \to \neg\neg a_2)) \\
&\quad = (a_1 \to (\neg b_1 \to \neg\neg b_2)) \wedge (b_1 \to (\neg a_1 \to \neg\neg a_2)) \\
&\quad = ((a_1 \wedge \neg b_1) \to \neg\neg b_2) \wedge ((b_1 \wedge \neg a_1) \to \neg\neg a_2).
\end{aligned}
$$

At this point, taking $a := a_1$, $b := \neg\neg b_2$, $c := \neg\neg a_2$ and $d := b_1$, we can rewrite our initial assumptions as follows: $\neg a \to c, \neg d \to b \in \nabla$. Then Proposition 6.6(iii) gives us $d \to (\neg a \to c) = (\neg a \wedge d) \to c, a \to (\neg d \to b) = (a \wedge \neg d) \to b \in \nabla$, so we can apply Proposition 6.6(iv) to obtain the required result.

(iii). We need to show that $\neg(a_1 \wedge b_1) \to \neg\neg(a_2 \oplus b_2) \in \nabla$. Using Lemmas 6.5(iii) and 5.5(iv), we compute:

$$
\begin{aligned}
\neg(a_1 \wedge b_1) \to \neg\neg(a_2 \oplus b_2) &= \neg(a_2 \oplus b_2) \to \neg\neg(a_1 \wedge b_1) \\
&= (\neg a_2 \wedge \neg b_2) \to (\neg\neg a_1 \wedge \neg\neg b_1) \\
&= ((\neg a_2 \wedge \neg b_2) \to \neg\neg a_1) \wedge ((\neg a_2 \wedge \neg b_2) \to \neg\neg b_1).
\end{aligned}
$$

Since we have assumed ∇ to be closed under finite meets, it suffices to verify that $(\neg a_2 \wedge \neg b_2) \to \neg\neg a_1, (\neg a_2 \wedge \neg b_2) \to \neg\neg b_1 \in \nabla$. Since $(\neg a_2 \wedge \neg b_2) \to \neg\neg a_1 = \neg b_2 \to (\neg a_2 \to \neg\neg a_1) = \neg b_2 \to (\neg a_1 \to \neg\neg a_2)$, the result follows from the assumption $\neg a_1 \to \neg\neg a_2 \in \nabla$ together with item (iii) of Proposition 6.6. A similar reasoning shows that $(\neg a_2 \wedge \neg b_2) \to \neg\neg b_1 \in \nabla$.

(iv). Assuming ∇ is a lattice filter, we need to show that $\neg(a_1 \vee b_1) \to \neg\neg(a_2 \wedge b_2) \in \nabla$. We compute $\neg(a_1 \vee b_1) \to \neg\neg(a_2 \wedge b_2) = (\neg a_1 \wedge \neg b_1) \to (\neg\neg a_2 \wedge \neg\neg b_2) = ((\neg a_1 \wedge \neg b_1) \to \neg\neg a_2) \wedge ((\neg a_1 \wedge \neg b_1) \to \neg\neg b_2)$. From the assumption that $\neg a_1 \to \neg\neg a_2 \in \nabla$ and the inequality $\neg a_1 \to \neg\neg a_2 \leq (\neg a_1 \wedge \neg b_1) \to \neg\neg a_2$, we have $(\neg a_1 \wedge \neg b_1) \to \neg\neg a_2 \in \nabla$. Similarly, using the assumption that $\neg b_1 \to \neg\neg b_2 \in \nabla$, we conclude that $(\neg a_1 \wedge \neg b_1) \to \neg\neg b_2 \in \nabla$ as well. Hence, $((\neg a_1 \wedge \neg b_1) \to \neg\neg a_2) \wedge ((\neg a_1 \wedge \neg b_1) \to \neg\neg b_2) = \neg(a_1 \vee b_1) \to \neg\neg(a_2 \wedge b_2) \in \nabla$, as required. ■

Proof of Proposition 6.8: (i). Given $\langle a_1, a_2\rangle \in A$, we have:

$$
\begin{aligned}
(\langle a_1, a_2\rangle \to \langle 0, 1\rangle) \to \sim\langle a_1, a_2\rangle &= \langle \neg a_1, \Box a_1\rangle \to \langle a_2, \Box a_1\rangle \\
&= \langle \neg a_1 \to a_2, \Box\neg a_1 \wedge \Box a_1\rangle \\
&= \langle \neg a_1 \to a_2, \Box(\neg a_1 \wedge a_1)\rangle \\
&= \langle \neg a_1 \to a_2, \Box 0\rangle
\end{aligned}
$$

$$= \langle \neg a_1 \to a_2, 0 \rangle \in A.$$

Thus $\neg a_1 \to a_2 \in \nabla_{\mathbf{A}}$. Since $\neg\neg(\neg a_1 \to a_2) = \neg a_1 \to \neg\neg a_2$ (Lemma 6.5(vi)), we can use Proposition 6.6(iii) to obtain $\neg a_1 \to \neg\neg a_2 \in \nabla_{\mathbf{A}}$. Hence, $\langle a_1, a_2 \rangle \in Tw(S, \nabla)$.

(ii). Assume $\langle a_1, a_2 \rangle \in Tw(S, \nabla)$, i.e. $\langle a_1, a_2 \rangle \in S^{\bowtie}$ and $\neg a_1 \to \neg\neg a_2 \in \nabla_{\mathbf{A}}$. We claim that the latter entails $\neg a_2 \to a_1 \in \nabla_{\mathbf{A}}$. To see this, observe that, using items (iii) and (v) of Lemma 6.5, we have $\neg a_1 \to \neg\neg a_2 = \neg a_2 \to \neg\neg a_1 = \neg\neg(\neg a_2 \to a_1) \in \nabla_{\mathbf{A}}$. By Proposition 6.6(iv), the latter gives us $\neg a_2 \to a_1 \in \nabla_{\mathbf{A}}$. Hence, $\langle \neg a_2 \to a_1, 0 \rangle \in A^+ \subseteq A$. Further notice that, since $a_2 \in S$, there is $b \in S$ such that $\langle a_2, b \rangle \in A$. Then $\langle a_2, b \rangle \to \langle 0, 1 \rangle = \langle \neg a_2, \Box a_2 \rangle = \langle \neg a_2, a_2 \rangle \in A$ as well. We therefore have:

$$\begin{aligned} \langle \neg a_2 \to a_1, 0 \rangle * \langle \neg a_2, a_2 \rangle &= \langle (\neg a_2 \to a_1) \wedge \neg a_2, ((\neg a_2 \to a_1) \to a_2) \wedge (\neg a_2 \to 0) \rangle \\ &= \langle a_1 \wedge \neg a_2, a_2 \rangle \\ &= \langle a_1, a_2 \rangle \in A. \end{aligned}$$

The equality $a_1 \wedge \neg a_2 = a_1$ follows from the observation that, since $a_1 \wedge a_2 = 0$, by the property of the pseudo-complement we have $a_1 \leq \neg a_2$. To justify $((\neg a_2 \to a_1) \to a_2) \wedge (\neg a_2 \to 0) = a_2$, observe that the inequality $a_2 \leq ((\neg a_2 \to a_1) \to a_2) \wedge (\neg a_2 \to 0)$ follows from $a_2 \leq (\neg a_2 \to a_1) \to a_2$ and $a_2 \leq \neg\neg a_2$. On the other hand, by residuation, we have $((\neg a_2 \to a_1) \to a_2) \wedge (\neg a_2 \to 0) \leq a_2$ iff $(\neg a_2 \to a_1) \to a_2 \leq (\neg a_2 \to 0) \to a_2$, and the latter inequality holds because $\neg a_2 \to 0 \leq \neg a_2 \to a_1$. ∎

Proof of Proposition 8.3: As we have noted, $\mathrm{Con}(\langle S, \wedge, \to \rangle) = \mathrm{Con}(\langle S, \to \rangle)$. It will thus suffice to check that every congruence $\theta \in \mathrm{Con}(\langle S, \wedge, \to \rangle)$ is compatible with the $\oplus$ operation. Let us preliminary observe that, for all $a, b \in S$, we have that $\langle a \to b, 1 \rangle, \langle b \to a, 1 \rangle \in \theta$ entail $\langle a, b \rangle \in \theta$. Indeed, from $\langle a \to b, 1 \rangle, \langle b \to a, 1 \rangle \in \theta$ we have $\langle (a \to b) \wedge (b \to a), 1 \rangle \in \theta$ and $\langle ((a \to b) \wedge (b \to a)) \to a, 1 \to a \rangle = \langle ((a \to b) \wedge (b \to a)) \to a, a \rangle \in \theta$. Similarly we obtain $\langle ((a \to b) \wedge (b \to a)) \to b, 1 \to b \rangle = \langle ((a \to b) \wedge (b \to a)) \to b, b \rangle \in \theta$. But the equality $((a \to b) \wedge (b \to a)) \to a = ((a \to b) \wedge (b \to a)) \to b$ holds on any implicative semilattice, giving us $\langle a, b \rangle \in \theta$ by transitivity. Keeping this in mind, assume $\langle a, b \rangle \in \theta$. Then $\langle a \to b, 1 \rangle, \langle b \to a, 1 \rangle \in \theta$. By Lemma 5.5(iii), we have $\langle (a \oplus c) \to (b \oplus c), 1 \rangle, \langle (b \oplus c) \to (a \oplus c), 1 \rangle \in \theta$ for all $c \in S$. This, as observed, gives us $\langle a \oplus c, b \oplus c \rangle \in \theta$. Since $\oplus$ is commutative, this is sufficient to establish the claimed result. ∎

Proof of Proposition 8.7: The proof is analogous but even easier than that of Proposition 8.6. Let us show that $\eta^{\bowtie} \in \mathrm{Con}(\mathbf{A})$ for all $\eta \in \mathrm{Con}(\mathbf{A}_{\bowtie})$ and $\theta_{\bowtie} \in \mathrm{Con}(\mathbf{A}_{\bowtie})$ for all $\theta \in \mathrm{Con}(\mathbf{A})$. Regarding the latter, recall that, from our earlier observations and Proposition 8.3, we have $\mathrm{Con}(\langle \mathbf{A}/\equiv, \wedge, \oplus, \to \rangle) = \mathrm{Con}(\langle \mathbf{A}/\equiv, \to \rangle)$. Then $\theta_{\bowtie} \in \mathrm{Con}(\mathbf{A}_{\bowtie})$ simply holds by Proposition 8.4. To show $\eta^{\bowtie} \in \mathrm{Con}(\mathbf{A})$, it suffices to verify that $\eta^{\bowtie}$ is compatible with the meet.

Let then $a, b \in A$ be such that $\langle a, b \rangle \in \eta^{\bowtie}$. As before, we shall write $[a], [b]$ etc. for the elements of $\mathbf{A}_{\bowtie}$. By assumption, $\langle [a], [b] \rangle, \langle [\sim a], [\sim b] \rangle \in \eta$. Let $c \in A$. From $\langle [a], [b] \rangle \in \eta$ we have $\langle [a] \wedge [c], [b] \wedge [c] \rangle = \langle [a \wedge c], [b \wedge c] \rangle \in \eta$. From $\langle [\sim a], [\sim b] \rangle \in \eta$, using Lemma 5.2(ii) we obtain $\langle [\sim a] \oplus [\sim c], [\sim b] \oplus [\sim c] \rangle = \langle [\sim a \oplus \sim c], [\sim b \oplus \sim c] \rangle = \langle [\sim(a \wedge c)], [\sim(b \wedge c)] \rangle \in \eta$. Hence, $\langle [a \wedge c], [b \wedge c] \rangle, \langle [\sim(a \wedge c)], [\sim(b \wedge c)] \rangle \in \eta$, i.e. $\langle a \wedge c, b \wedge c \rangle \in \eta^{\bowtie}$. Since the meet is commutative, this entails $\eta^{\bowtie} \in \mathrm{Con}(\mathbf{A})$. ∎

Proof of Lemma 8.12: Taking into the account the equivalence of $r(x, y, z)$ and $q(x, y, z)$, one realises that items (i)–(iv) have been established in Rivieccio and Jansana (2021, Lemma 33). Let us verify (v). By Theorem 4.16, assume $\mathbf{A} \leq \mathbf{S}^{\bowtie}$, and let $a = \langle a_1, a_2 \rangle, b = \langle b_1, b_2 \rangle$ etc. To improve readability, we shall also abbreviate $\alpha := (a_1 \to b_1) \wedge (b_1 \to a_1) \wedge (a_2 \to b_2) \wedge (b_2 \to a_2)$. Then $r(a, b, c) * r(a, b, d)$ gives us the following:

$$\langle (\alpha \to c_1) \wedge (\alpha \to d_1), ((\alpha \to c_1) \to (\Box\alpha \wedge d_2)) \wedge ((\alpha \to d_1) \to (\Box\alpha \wedge c_2)) \rangle.$$

The first component of $r(a, b, r(a, b, c) * r(a, b, d))$ is thus:

$$\alpha \to ((\alpha \to c_1) \wedge (\alpha \to d_1))$$

and the second is:

$$\Box\alpha \wedge ((\alpha \to c_1) \to (\Box\alpha \wedge d_2)) \wedge ((\alpha \to d_1) \to (\Box\alpha \wedge c_2)).$$

On the other hand we have:

$$r(a, b, c * d) = \langle \alpha \to (c_1 \wedge d_1), \Box\alpha \wedge (c_1 \to d_2) \wedge (d_1 \to c_2)\rangle.$$

Equality of the first components is easily established: using the properties of the intuitionistic implication, we have $\alpha \to ((\alpha \to c_1) \wedge (\alpha \to d_1)) = \alpha \to (\alpha \to (c_1 \wedge d_1)) = \alpha \to (c_1 \wedge d_1)$.

Regarding the second component, we have: $\Box\alpha \wedge ((\alpha \to c_1) \to (\Box\alpha \wedge d_2)) \wedge ((\alpha \to d_1) \to (\Box\alpha \wedge c_2)) = \Box\alpha \wedge ((\alpha \to c_1) \to \Box\alpha) \wedge ((\alpha \to c_1) \to d_2) \wedge ((\alpha \to d_1) \to \Box\alpha) \wedge ((\alpha \to d_1) \to c_2) = \Box\alpha \wedge ((\alpha \to c_1) \to d_2) \wedge ((\alpha \to d_1) \to c_2)$. Observe that $\Box\alpha \wedge ((\alpha \to c_1) \to d_2) \wedge ((\alpha \to d_1) \to c_2) \leq \Box\alpha \wedge (c_1 \to d_2) \wedge (d_1 \to c_2)$ simply because the implication is order-reversing in the first argument. Thus, it suffices to show $\Box\alpha \wedge (c_1 \to d_2) \wedge (d_1 \to c_2) \leq \Box\alpha \wedge ((\alpha \to c_1) \to d_2) \wedge ((\alpha \to d_1) \to c_2)$ or, equivalently, $\Box\alpha \wedge (c_1 \to d_2) \wedge (d_1 \to c_2) \leq ((\alpha \to c_1) \to d_2) \wedge ((\alpha \to d_1) \to c_2)$. Let us show $\Box\alpha \wedge (c_1 \to d_2) \wedge (d_1 \to c_2) \leq (\alpha \to c_1) \to d_2$. By residuation, the latter is equivalent to $(\alpha \to c_1) \wedge \Box\alpha \wedge (c_1 \to d_2) \wedge (d_1 \to c_2) \leq d_2$. The result then is a consequence of the following (in)equalities:

$$\begin{aligned}
&(\alpha \to c_1) \wedge \Box\alpha \wedge (c_1 \to d_2) \wedge (d_1 \to c_2) && \\
&\leq \Box(\alpha \to c_1) \wedge \Box\alpha \wedge (c_1 \to d_2) && x \leq \Box x \\
&= \Box((\alpha \to c_1) \wedge \alpha) \wedge (c_1 \to d_2) && \Box(x \wedge y) = \Box x \wedge \Box y \\
&= \Box(c_1 \wedge \alpha) \wedge (c_1 \to d_2) && x \wedge y = x \wedge (x \to y) \\
&= \Box c_1 \wedge \Box\alpha \wedge (c_1 \to \Box d_2) && x \wedge y = x \wedge (x \to y) \\
&= \Box c_1 \wedge \Box\alpha \wedge (\Box c_1 \to \Box d_2) && x \to \Box y = \Box x \to \Box y \\
&= \Box c_1 \wedge \Box\alpha \wedge \Box d_2 = \Box c_1 \wedge \Box\alpha \wedge d_2 \leq d_2. &&
\end{aligned}$$

A similar reasoning allows us to establish $\Box\alpha \wedge (c_1 \to d_2) \wedge (d_1 \to c_2) \leq (\alpha \to d_1) \to c_2$, thus concluding our proof. ∎

Proof of Proposition 9.2: (i). It is easy to see that **M** satisfies $\sim\sim x = x$ if and only if the map $\Box$ is the identity on **S**. The second equivalence is an immediate consequence of Proposition 3.8.

(ii). Obviously, on every (pseudo-complemented) semilattice, it holds that $x = x * x$ and $\equiv$ is the identity relation (so $\langle S, \wedge, \neg, 0, 1\rangle \cong$ **M**). For the other implications, assume **M** satisfies $x = x * x$. Then $*$ is a semilattice operation and, by Definition 3.2(iii).6, the negation $\sim$ is the corresponding pseudo-complement. Furthermore, recall that $\langle a_1, a_2\rangle * \langle a_1, a_2\rangle = \langle a_1, \neg a_1\rangle$. Then every element of **M** is of the form $\langle a_1, \neg a_1\rangle$ for some $a_1 \in S$. This entails that the map $a \mapsto \langle a, \neg a\rangle$ is bijective, and it is easy to see that it preserves the algebraic operations of $\langle S, \wedge, \neg, 0, 1\rangle$.

(iii). Obviously every Boolean algebra satisfies $\sim\sim x = x * x$. Conversely assume **M** satisfies $\sim\sim x = x * x$. This means that, for all $\langle a_1, a_2\rangle \in M$, we have $\langle \Box a_1, a_2\rangle = \langle a_1, \neg a_1\rangle$. As we have seen in item (i) above, $a_1 = \Box a_1$ entails that **M** is involutive. Likewise, by item (ii), from $a_2 = \neg a_1$ we conclude that $\langle S, \wedge, \neg, 0, 1\rangle \cong$ **M** via the map $a \mapsto \langle a, \neg a\rangle$. So **M** is a pseudo-complemented semilattice and, as observed earlier, any involutive pseudo-complemented semilattice is a Boolean algebra.

(iv). It is easy to check that **M** satisfies $(x \rightharpoonup 0) \rightharpoonup 0 \preceq x$ if and only if **S** satisfies $\neg\neg x \leq x$. Then, by Corollary 3.13, $\langle S, \wedge, \rightharpoonup, 0, 1\rangle$ is a Boolean algebra, which (as observed in Subsection 2.2) entails that $\Box$ is the identity map. Thus, in particular, **M** satisfies item (i) as well (i.e. it is involutive). ∎

Proof of Lemma 9.4: It is clear that (i) entails (ii). Regarding the latter, we claim that (ii) is equivalent to the following property: $\neg a_1 \to a_2 = \neg a_2 \to \Box a_1$ for all $\langle a_1, a_2\rangle \in A$.

To see this, let us preliminarily compute $\langle a_1, a_2\rangle \Rightarrow \sim\langle a_1, a_2\rangle = \langle (a_1 \to a_2) \wedge (\Box a_1 \to a_2), \Box a_1 \wedge \Box a_1\rangle = \langle \neg a_1 \wedge (\Box a_1 \to \Box a_2), \Box a_1\rangle = \langle \neg a_1 \wedge (a_1 \to a_2), \Box a_1\rangle = \langle \neg a_1 \wedge \neg a_1, \Box a_1\rangle = \langle \neg a_1, \Box a_1\rangle$ and $\sim\langle a_1, a_2\rangle \Rightarrow \langle a_1, a_2\rangle = \langle (a_2 \to a_1) \wedge (a_2 \to \Box a_1), \Box a_2 \wedge a_2\rangle = \langle (a_2 \to a_1) \wedge (a_2 \to \Box a_1), a_2\rangle = \langle \neg a_2 \wedge \neg a_2, a_2\rangle = \langle \neg a_2, a_2\rangle$. At this point we can more easily compute $\langle a_1, a_2\rangle * (\langle a_1, a_2\rangle \Rightarrow \sim\langle a_1, a_2\rangle) = \langle a_1, a_2\rangle * \langle \neg a_1, \Box a_1\rangle = \langle a_1 \wedge \neg a_1, (a_1 \to \Box a_1) \wedge (\neg a_1 \to a_2)\rangle = \langle 0, 1 \wedge (\neg a_1 \to a_2)\rangle = \langle 0, \neg a_1 \to a_2\rangle$ and $\sim\langle a_1, a_2\rangle * (\sim\langle a_1, a_2\rangle \Rightarrow \langle a_1, a_2\rangle) = \sim\langle a_1, a_2\rangle * \langle \neg a_2, a_2\rangle = \langle a_2 \wedge \neg a_2, (a_2 \to a_2) \wedge (\neg a_2 \to \Box a_1)\rangle = \langle 0, 1 \wedge (\neg a_2 \to \Box a_1)\rangle = \langle 0, \neg a_2 \to \Box a_1\rangle$. Thus, $\mathbf{A} \vDash x * (x \Rightarrow \sim x) = \sim x * (\sim x \Rightarrow x)$ if and only if $\neg a_1 \to a_2 = \neg a_2 \to \Box a_1$ for all $\langle a_1, a_2\rangle \in A$, as claimed.

Now, assume (ii), and let $\langle a_1, a_2\rangle \in A$. Since $\langle a_1, a_2\rangle^2 = \langle a_1, \neg a_1\rangle$, we can instantiate the equality $\neg a_1 \to a_2 = \neg a_2 \to \Box a_1$ obtaining $\neg a_1 \to \neg a_1 = 1 = \neg\neg a_1 \to \Box a_1$. Thus, $\neg\neg a_1 \leq \Box a_1$. This gives us $\langle \neg\neg a_1 \to \Box a_1, \Box\neg\neg a_1 \wedge a_2\rangle = \neg\neg\langle a_1, a_2\rangle \to \sim\sim\langle a_1, a_2\rangle = \langle 1, 0\rangle$, obtaining (iii).

By the preceding computations, it is clear that (iii) entails $\mathbf{S} \vDash \neg\neg x \leq \Box x$. Since the inequality $\mathbf{S} \vDash \Box x \leq \neg\neg x$ holds generally, we have (iv).

To conclude the proof, let us show that (iv) entails (i).

Assuming (iv), let us compute: $\langle a_1, a_2\rangle * (\langle a_1, a_2\rangle \Rightarrow \langle b_1, b_2\rangle) = \langle a_1 \wedge (a_1 \to b_1) \wedge (b_2 \to a_2), (a_1 \to (\Box a_1 \wedge b_2)) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2)\rangle = \langle a_1 \wedge b_1 \wedge (b_2 \to a_2), (a_1 \to b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2)\rangle = \langle a_1 \wedge b_1, (a_1 \to b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2)\rangle$. In the last passage we have used the following observation: from the requirement $b_1 \wedge b_2 = 0 \leq a_2$, by residuation, we have $b_1 \leq b_2 \to a_2$. Thus, by symmetry, we have $\langle b_1, b_2\rangle * (\langle b_1, b_2\rangle \Rightarrow \langle a_1, a_2\rangle) = \langle b_1 \wedge a_1, (b_1 \to a_2) \wedge (((b_1 \to a_1) \wedge (a_2 \to b_2)) \to b_2)\rangle$. The first components are thus equal. Regarding the second ones, it suffices to show e.g. that $(a_1 \to b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) \leq (b_1 \to a_2) \wedge (((b_1 \to a_1) \wedge (a_2 \to b_2)) \to b_2)$. The latter, in turn, can be split into the two inequalities $(a_1 \to b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) \leq b_1 \to a_2$ and $(a_1 \to b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) \leq ((b_1 \to a_1) \wedge (a_2 \to b_2)) \to b_2$. The former holds generally. Indeed, by residuation, we have $(a_1 \to b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) \leq b_1 \to a_2$ iff $((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2 \leq ((a_1 \to b_2) \wedge b_1) \to a_2$. Since the implication is order-reversing in the first argument, it suffices to verify that $(a_1 \to b_2) \wedge b_1 \leq (a_1 \to b_1) \wedge (b_2 \to a_2)$, or equivalently the two inequalities $(a_1 \to b_2) \wedge b_1 \leq a_1 \to b_1$ and $(a_1 \to b_2) \wedge b_1 \leq b_2 \to a_2$. The former is clear. The latter is equivalent, by residuation, to $b_2 \wedge (a_1 \to b_2) \wedge b_1 \leq a_2$, which follows from the requirement $b_1 \wedge b_2 = 0$. It thus remains to verify:

$$(a_1 \to b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) \leq ((b_1 \to a_1) \wedge (a_2 \to b_2)) \to b_2$$

which is equivalent, by residuation, to:

$$(b_1 \to a_1) \wedge (a_2 \to b_2) \wedge (a_1 \to b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) \leq b_2.$$

By assumption, $b_2 = \Box b_2 = \neg\neg b_2$. Thus the preceding inequality can be rewritten as:

$$(b_1 \to a_1) \wedge (a_2 \to \neg\neg b_2) \wedge (a_1 \to \neg\neg b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) \leq \neg\neg b_2$$

which is equivalent, by residuation, to the following one:

$$\neg b_2 \wedge (b_1 \to a_1) \wedge (a_2 \to \neg\neg b_2) \wedge (a_1 \to \neg\neg b_2) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) = 0.$$

Using the identity $x \to \neg y = y \to \neg x$ (which is valid on all bounded implicative semilattices), we can further rewrite the left-hand side of the preceding equality as follows:

$$\neg b_2 \wedge (b_1 \to a_1) \wedge (\neg b_2 \to \neg a_2) \wedge (\neg b_2 \to \neg a_1) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2)$$

and, using the identity $x \wedge y = x \wedge (x \to y)$, we further obtain:

$$\neg b_2 \wedge \neg a_2 \wedge \neg a_1 \wedge (b_1 \to a_1) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2).$$

By residuation, we have

$$\neg b_2 \wedge \neg a_2 \wedge \neg a_1 \wedge (b_1 \to a_1) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) \leq 0$$

iff

$$\neg b_2 \wedge \neg a_1 \wedge (b_1 \to a_1) \wedge (((a_1 \to b_1) \wedge (b_2 \to a_2)) \to a_2) \leq \neg\neg a_2$$

iff (using also $a_2 = \Box a_2 = \neg\neg a_2$)

$$((a_1 \to b_1) \wedge (b_2 \to a_2)) \to \neg\neg a_2 \leq (\neg b_2 \wedge \neg a_1 \wedge (b_1 \to a_1)) \to \neg\neg a_2.$$

Thus, it suffices to verify the inequality $\neg b_2 \wedge \neg a_1 \wedge (b_1 \to a_1) \leq (a_1 \to b_1) \wedge (b_2 \to a_2)$ or, equivalently, the two inequalities $\neg b_2 \wedge \neg a_1 \wedge (b_1 \to a_1) \leq a_1 \to b_1$ and $\neg b_2 \wedge \neg a_1 \wedge (b_1 \to a_1) \leq b_2 \to a_2$. Both are easy: indeed, by the monotonicity of the implication (in the second argument), we have $\neg a_1 = a_1 \to 0 \leq a_1 \to b_1$ and $\neg b_2 = b_2 \to 0 \leq b_2 \to a_2$. This concludes our proof. ■