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Representation of interlaced trilattices

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ABSTRACT

Trilattices are algebraic structures introduced ten years ago into logic with the aim to provide a uniform framework for the notions of constructive truth and constructive falsity. In more recent years, trilattices have been used to introduce a number of many-valued systems that generalize the Belnap–Dunn logic of first-degree entailment, proposed as logics of how several computers connected together in a network should think in order to deal with incomplete and possibly contradictory information. The aim of the present work is to develop a first purely algebraic study of trilattices, focusing in particular on the problem of representing certain subclasses of trilattices as special products of bilattices. This approach allows to extend the known representation results for interlaced bilattices to the setting of trilattices and to reduce many algebraic problems concerning these new structures to the better-known framework of lattice theory.

1. Introduction

Trilattices were first introduced into logic by Y. Shramko, J.M. Dunn and T. Takenaka [21] with the aim to provide a uniform framework for the notions of constructive truth and constructive falsity. These algebraic structures were used to define some interesting many-valued logics that Shramko and his collaborators proposed as generalizations of the systems introduced by A. Heyting as a formal counterpart of constructive (intuitionistic) logic and by D. Nelson [15] as a logic for constructive falsity.

Logics based on trilattices are also closely related to other well-known formal systems such as bilattice and relevance logics. This relationship has been stressed and investigated in several works by Y. Shramko and H. Wansing [19,20,22], who presented their trilattice logics as a generalization of the "useful four-valued logic" introduced by N. Belnap and J.M. Dunn [3,1]. While the Belnap–Dunn system was originally proposed as a logic of how a computer should think in order to handle information coming from different and possibly conflicting sources, Shramko and Wansing proposed trilattice-based systems as logics meant to model how several computers connected together in a network should think in order to deal with incomplete and possibly contradictory information.

The aim of the present work is to provide a first algebraic approach to the study of trilattices, focusing in particular on the relationship between trilattices and bilattices, in order to extend some of the representation results obtained in [4] for bilattices to the setting of trilattices. The main appeal of this approach, that proved to be useful in the case of bilattices, is that it allows to reduce many algebraic problems concerning these new structures to the better-known framework of lattices, in which they can be solved using powerful tools and results of lattice theory.

The paper is organized as follows. The next section contains the main definitions and fixes the terminology that we are going to use; it presents as well some basic results on bilattices and trilattices that we shall need in the subsequent sections. Section 3 contains some of the main results of this paper, namely representation theorems stating that various kinds of trilattices can be constructed as special products of two bilattices. At the end of Section 3.5 we briefly compare

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our approach to a previous work by S. Odintsov on the representation of a particular example of trilattice. In Section 4 we use the representation results of Section 3 to obtain characterizations of the congruences of trilattices in terms of those of their bilattice factors. These results are then used in Section 5 in order to identify the generators of minimal varieties of trilattices (i.e., the distributive ones). Finally, Section 6 mentions some open problems and lines for future research.

2. Definitions and basic results

In this section we introduce the main definitions, terminology and notation that we are going to use throughout the present work.

2.1. Bilattices

A *pre-bilattice* [8] is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ such that $\langle B, \leqslant, \wedge, \vee \rangle$ and $\langle B, \sqsubseteq, \sqcap, \sqcup \rangle$ are both lattices. For notational convenience, we shall sometimes indicate the pre-bilattice $\langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ just as $\langle B, \leqslant, \sqsubseteq \rangle$, but let us stress that we always treat these structures as algebras (rather than as doubly partially ordered sets).

In the literature on bilattices it is usually required that both lattices be complete or at least bounded, but here none of these assumptions is made. The minimum and maximum element of the lattice $\langle B, \wedge, \vee \rangle$, in case they exist, will be denoted, respectively, by f and t. Similarly, $\bot$ and $\top$ will refer to the minimum and maximum of $\langle B, \sqcap, \sqcup \rangle$, when they exist.

In logical contexts, where the underlying set of a pre-bilattice is understood as a space of truth values, the two lattice orders are usually thought of as representing the degree of truth ($\leqslant$) and the degree of information ($\sqsubseteq$) associated with a given sentence; accordingly, they are called respectively the *truth order* (or "logical order") and the *information order* (or "knowledge order"). This accounts for the use of f (for *false*) and t for (*true*) to denote the least and greatest elements w.r.t. the truth order, while $\bot$ should represent a complete absence of information and $\top$ an excess of it (a contradiction).

We reserve the term *bilattice* [12] for what is sometimes called a "bilattice with negation", i.e., an algebra $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ such that $\langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ is a pre-bilattice and the negation $\neg : B \to B$ is an operation satisfying that, for all $a, b \in B$:

$$\text{if } a \leqslant b, \quad \text{then } \neg b \leqslant \neg a$$
$$\text{if } a \sqsubseteq b, \quad \text{then } \neg a \sqsubseteq \neg b$$
$$a = \neg\neg a.$$

Negation is thus anti-monotonic with respect to the truth order and monotonic with respect to the information order; it is not difficult to convince oneself that these requirements constitute a plausible generalization of the behavior of negation within classical logic. The following identities (that we will call *De Morgan laws*) hold in any bilattice:

$$\neg(a \wedge b) = \neg a \vee \neg b \qquad \neg(a \vee b) = \neg a \wedge \neg b$$
$$\neg(a \sqcap b) = \neg a \sqcap \neg b \qquad \neg(a \sqcup b) = \neg a \sqcup \neg b.$$

Moreover, if the bilattice is bounded, then $\neg\top = \top$, $\neg\bot = \bot$, $\neg\mathsf{t} = \mathsf{f}$ and $\neg\mathsf{f} = \mathsf{t}$. So, if a bilattice $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ is distributive, or at least the reduct $\langle B, \wedge, \vee \rangle$ is distributive, then $\langle B, \wedge, \vee, \neg \rangle$ is a De Morgan lattice.

The most interesting algebraic results known on (pre-)bilattices, in particular the representation theorems that we are going to state below, do not apply to all bilattices, but only to the subclass of the interlaced ones (most of these results may be found in [4,5], to which we refer for more details and the proofs that we are going to omit).

A pre-bilattice is called *interlaced* [7] when all four lattice operations are monotone w.r.t. to both lattice orders. It is called *distributive* [12] when all possible distributive laws concerning the four lattice operations, i.e., any identity of the following form, hold:

$$a \circ (b \bullet c) = (a \circ b) \bullet (a \circ c) \quad \text{for every } \circ, \bullet \in \{\wedge, \vee, \sqcap, \sqcup\}.$$

We say that a bilattice is interlaced (or distributive) when its pre-bilattice reduct is.

Fig. 1 shows the double Hasse diagram of some of the best-known (pre-)bilattices: the four- and nine-element ones are distributive, while the seven-element one is not (in fact, it is not even interlaced). The diagrams should be read as follows: $a \leqslant b$ if there is a path from a to b which goes uniformly from left to right, while $a \sqsubseteq b$ if there is a path from a to b which goes uniformly from the bottom to the top. The four lattice operations are thus uniquely determined by the diagram, while negation, if there is one, corresponds to reflection along the vertical axis joining $\bot$ and $\top$. It is then clear that all the pre-bilattices shown in Fig. 1 can be endowed with a negation in a unique way and turned in this way into bilattices. When no confusion is likely to arise, we will use the same name to denote a particular pre-bilattice and its associated bilattice. The names used in the diagrams are by now more or less standard in the literature, except for the subscripts, that we use to indicate that we are now considering structures endowed with two lattice orders (whereas further below we shall consider three orders).

The smallest non-trivial bilattice, $\mathcal{FOUR}_2$, has a fundamental role among bilattices, both from an algebraic and a logical point of view. $\mathcal{FOUR}_2$ is distributive and, as a bilattice, it is a simple algebra. It is in fact, up to isomorphism, the only

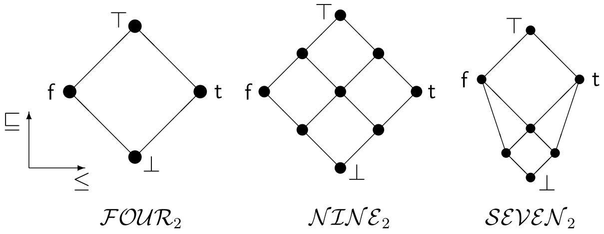

Fig. 1. Some examples of (pre-)bilattices.

subdirectly irreducible distributive bilattice (this was proved for the bounded case in [13], then generalized in [4] to the unbounded one).

A natural expansion of the bilattice language considered above is obtained by adding a unary operator that behaves as a dual of the bilattice negation. Such an operator has been introduced by Fitting [9] who called it "conflation". A *bilattice with conflation* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg, - \rangle$ such that $\langle B, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ is a bilattice and the conflation $- : B \to B$ is an operation satisfying that, for all $a, b \in B$:

$$\text{if } a \leqslant b, \quad \text{then } -a \leqslant -b$$

$$\text{if } a \sqsubseteq b, \quad \text{then } -b \sqsubseteq -a$$

$$a = --a.$$

More briefly, one could say that a bilattice with conflation is a structure $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg, - \rangle$ such that both $\langle B, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ and $\langle B, \sqcap, \sqcup, \wedge, \vee, - \rangle$ are bilattices, and we could call the two operations simply t-negation and i-negation.

We say that $\mathbf{B}$ is *commutative* when negation and conflation commute, i.e., when, for all $a \in B$,

$$\neg - a = - \neg a.$$

Notice that $\mathcal{FOUR}_2$ and $\mathcal{NINE}_2$ can be endowed with a conflation (that in fact commutes with negation), which corresponds in Fig. 1 to reflection along the horizontal axis joining f and t.

All the classes of (pre-)bilattices introduced above are varieties, i.e., definable by means of equations only. Pre-bilattices, for instance, are axiomatized by the lattice identities for the two lattices, while for bilattices we have to add the involutive identity $x = \neg\neg x$ plus the following (De Morgan laws):

$$\neg(x \wedge y) = \neg x \vee \neg y \qquad \neg(x \vee y) = \neg x \wedge \neg y$$
$$\neg(x \sqcap y) = \neg x \sqcap \neg y \qquad \neg(x \sqcup y) = \neg x \sqcup \neg y.$$

For bilattices with conflation we also have to add $x = --x$, plus the following:

$$-(x \sqcap y) = -x \sqcup -y \qquad -(x \sqcup y) = -x \sqcap -y$$
$$-(x \wedge y) = -x \wedge -y \qquad -(x \vee y) = -x \vee -y.$$

The classes of interlaced and distributive (pre-)bilattices (with or without conflation) are also varieties. Moreover, the class of distributive (pre-)bilattices (with conflation) is a proper subvariety of the interlaced, which is a proper subvariety of the class of all (pre-)bilattices (with conflation).

2.2. Product (pre-)bilattices

A fundamental result in bilattice theory is a representation theorem stating that any interlaced (pre-)bilattice is isomorphic to a special product of two lattices. We describe the constructions involved as they will have a key role in our approach to the representation of trilattices. The following definitions were first introduced by Fitting [7,9].

Let $\mathbf{L_1} = \langle L_1, \wedge_1, \vee_1 \rangle$ and $\mathbf{L_2} = \langle L_2, \wedge_2, \vee_2 \rangle$ be lattices with associated orders $\leqslant_1$ and $\leqslant_2$. The *product pre-bilattice* $\mathbf{L_1} \odot \mathbf{L_2} = \langle L_1 \times L_2, \wedge, \vee, \sqcap, \sqcup \rangle$ is defined as follows. For all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L_1 \times L_2$,

$$\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle := \langle a_1 \wedge_1 b_1, a_2 \vee_2 b_2 \rangle$$

$$\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle := \langle a_1 \vee_1 b_1, a_2 \wedge_2 b_2 \rangle$$

$$\langle a_1, a_2 \rangle \sqcap \langle b_1, b_2 \rangle := \langle a_1 \wedge_1 b_1, a_2 \wedge_2 b_2 \rangle$$

$$\langle a_1, a_2 \rangle \sqcup \langle b_1, b_2 \rangle := \langle a_1 \vee_1 b_1, a_2 \vee_2 b_2 \rangle.$$

$\mathbf{L_1} \odot \mathbf{L_2}$ is always an interlaced pre-bilattice, and it is distributive if and only if both $\mathbf{L_1}$ and $\mathbf{L_2}$ are distributive lattices. From the definition it follows immediately that

$$\langle a_1, a_2\rangle \sqsubseteq \langle b_1, b_1\rangle \quad \text{iff} \quad a_1 \leqslant_1 b_1 \text{ and } a_2 \leqslant_2 b_2$$
$$\langle a_1, a_2\rangle \leqslant \langle b_1, b_1\rangle \quad \text{iff} \quad a_1 \leqslant_1 b_1 \text{ and } a_2 \geqslant_2 b_2.$$

If $\mathbf{L_1}$ and $\mathbf{L_2}$ are isomorphic, then it is possible to define a negation in $\mathbf{L_1} \odot \mathbf{L_2}$, and we speak of *product bilattice* instead of product pre-bilattice. If h: $\mathbf{L_1} \cong \mathbf{L_2}$ is an isomorphism, then the negation is defined as

$$\neg\langle a_1, a_2\rangle := \big\langle h^{-1}(a_2), h(a_1)\big\rangle.$$

In particular, if $\mathbf{L_1} = \mathbf{L_2}$, the definition gives $\neg\langle a_1, a_2\rangle := \langle a_2, a_1\rangle$.

The representation theorem for interlaced (pre-)bilattices states then that *any interlaced pre-bilattice* $\mathbf{B}$ *is isomorphic to a product* $\mathbf{L_1} \odot \mathbf{L_2}$. *Moreover, if* $\mathbf{B}$ *is an interlaced bilattice, then* $\mathbf{L_1} \cong \mathbf{L_2}$.

This result was obtained in [12,7] for bounded distributive (pre-)bilattices. It was later on generalized in [2] to bounded interlaced (pre-)bilattices and in [14,4] to the unbounded case.

If $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \mathsf{f}, \mathsf{t}, \perp, \top\rangle$ is a bounded interlaced (pre-)bilattice, then $\mathbf{L_1}$ and $\mathbf{L_2}$ can be obtained as sublattices of $\mathbf{B}$ as follows: defining $L_1 := \{a \vee \perp\colon\ a \in B\}$ and $L_2 := \{a \wedge \perp\colon\ a \in B\}$, we have that

$$\mathbf{B} \cong \langle L_1, \sqcap, \sqcup\rangle \odot \langle L_2, \sqcap, \sqcup\rangle.$$

In the unbounded case $\mathbf{L_1}$ and $\mathbf{L_2}$ can instead be obtained as quotients of $\mathbf{B}$ (see Section 3.1 for the details of the construction).

In order to construct a bilattice with conflation we need an *involutive lattice*, i.e., an algebra $\mathbf{L} = \langle L, \wedge, \vee, '\rangle$ such that the reduct $\langle L, \wedge, \vee\rangle$ is a lattice and the operation $' : L \to L$ satisfies that, for all $a, b \in L$:

$$\text{if } a \leqslant b, \quad \text{then } b' \leqslant a'$$
$$a = a''.$$

Given an involutive lattice $\mathbf{L} = \langle L, \wedge, \vee, '\rangle$, we denote by $\mathbf{L} \odot \mathbf{L}$ the bilattice with conflation whose bilattice reduct is the product bilattice $\langle L, \wedge, \vee\rangle \odot \langle L, \wedge, \vee\rangle$ defined as above and where the conflation is defined, for all $a, b \in L$, as

$$-\langle a, b\rangle = \big\langle b', a'\big\rangle.$$

It can be easily checked that $\mathbf{L} \odot \mathbf{L}$ is always an interlaced bilattice with conflation; in addition, it is commutative. Conversely, a representation theorem analogous to the one mentioned above states that *any commutative bilattice with conflation can be represented as a product of this kind* [18,5].

2.3. Trilattices

The terminology used so far in the literature on trilattices is neither uniform nor quite precise as far as the signature is concerned. The one we are going to adopt here is meant to be precise enough for our algebraic approach and as consistent as possible with the established notation on trilattices.

By a *trilattice* we mean an algebra

$$\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i\rangle$$

such that the reducts $\langle A, \wedge_t, \vee_t\rangle$, $\langle A, \wedge_f, \vee_f\rangle$ and $\langle A, \wedge_i, \vee_i\rangle$ are lattices.[1] For brevity, we sometimes indicate a trilattice just as $\langle A, \leqslant_t, \leqslant_f, \leqslant_i\rangle$, but we always view it as an algebra rather than a relational structure.

The lattice orders of a trilattice may be interpreted in various ways (see for instance [19]). Let us just recall one of the interpretations that can be seen as a generalization of the one introduced above for bilattices: we still have one information order ($\leqslant_i$), but we also have two independent orders, one for the degree of truth ($\leqslant_t$) and the other for the degree of falsity ($\leqslant_f$) associated with a sentence. We are thus adopting a paraconsistent view, in that we do not require that an increase in truth should necessarily imply a decrease in falsity and vice versa.

We say that a trilattice $\mathbf{A}$ has a *t-involution* (respectively, an *f-involution* or an *i-involution*) when there is a unary operation which is involutive, anti-monotone w.r.t. $\leqslant_t$ (respectively, w.r.t. $\leqslant_f$ or $\leqslant_i$) and monotone w.r.t. to the other two lattice orders. That is (in the case of the t-involution), when there is an operation $-_t : A \to A$ such that, for all $a, b \in A$:

[1] In order to be more general, one could simply define an *n-lattice* to be a set endowed with n lattice orders: according to this definition pre-bilattices are just 2-lattices, while trilattices correspond to 3-lattices [19, Definition 3.1].

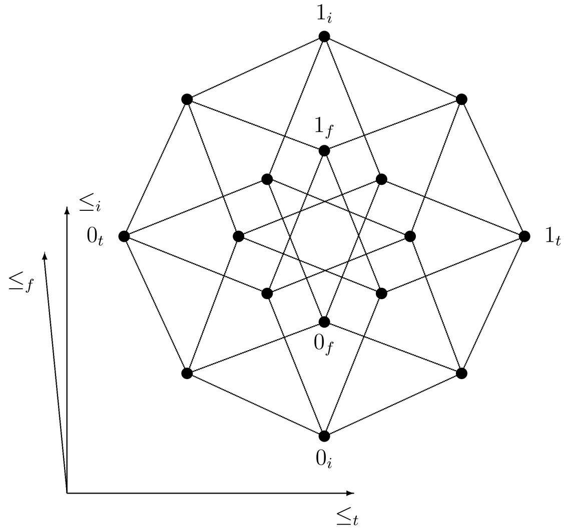

Fig. 2. The trilattice $\mathcal{SIXTEEN}_3$.

$$\text{if } a \leqslant_t b, \quad \text{then } -_t b \leqslant_t -_t a$$
$$\text{if } a \leqslant_f b, \quad \text{then } -_t a \leqslant_f -_t b$$
$$\text{if } a \leqslant_i b, \quad \text{then } -_t a \leqslant_i -_t b$$
$$a = -_t -_t a.$$

Fig. 2 shows the trilattice $\mathcal{SIXTEEN}_3$, which has a fundamental role among trilattices, analogous to the one played by $\mathcal{FOUR}_2$ among bilattices. The diagram can be read like the ones introduced above to represent bilattices, but notice that it is only possible to represent two orders at a time in a perspicuous way ($\leqslant_t$ and $\leqslant_i$ in our diagram), while the third one ($\leqslant_f$) should be visualized as a third dimension in perspective. We have put names just for the top and bottom elements of each of the three orders in order to give a rough idea of the three dimensions of the trilattice.

In analogy with bilattices, we define a trilattice (possibly enriched with involutions) to be *interlaced* when all six lattice operations are monotone w.r.t. to all three lattice orders, and *distributive* when all possible distributive laws concerning all lattice operations hold.

An obvious fact, but important to our approach, is that any trilattice

$$\langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle$$

has three pre-bilattice reducts, namely $\langle A, \wedge_t, \vee_t, \wedge_f, \vee_f \rangle$, $\langle A, \wedge_t, \vee_t, \wedge_i, \vee_i \rangle$ and $\langle A, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle$, all of which inherit the property of being interlaced (distributive).

It follows from the results on bilattices mentioned above that distributivity implies the interlacing conditions, therefore distributive trilattices are a subclass of the interlaced ones. Another result that can be straightforwardly transferred from the theory of bilattices is that this inclusion is strict, i.e., that there are trilattices which are interlaced but non-distributive (this is an easy consequence of a more general result that we are going to prove in Proposition 2.2).

We say that two involution operations (for instance $-_t$ and $-_f$) commute when, for all $a \in A$,

$$-_t -_f a = -_f -_t a.$$

It is easy to see that, as happens with (pre-)bilattices, all the conditions involved in the various definitions of trilattices (with or without involutions) can be expressed by equations, for instance through De Morgan laws of the following form:

$$-_t(x \wedge_t y) = -_t x \vee_t -_t y \qquad -_t(x \vee_t y) = -_t x \wedge_t -_t y$$
$$-_t(x \wedge_f y) = -_t x \wedge_f -_t y \qquad -_t(x \vee_f y) = -_t x \vee_f -_t y.$$

Hence, all the classes of trilattices introduced above are varieties.

2.4. Product trilattices

We are now going to introduce constructions that allow to build trilattices (with involutions) as special products of two (pre-)bilattices. Our ultimate aim will be to show that all the trilattices satisfying certain conditions can be represented as products of this kind.

We first consider the case of trilattices without involution operations. Let $\mathbf{B_1} = \langle B_1, \leqslant_1, \sqsubseteq_1 \rangle$ and $\mathbf{B_2} = \langle B_2, \leqslant_2, \sqsubseteq_2 \rangle$ be pre-bilattices, and define the *product trilattice*

$$\mathbf{B_1} \odot \mathbf{B_2} = \langle B_1 \times B_2, \leqslant_t, \leqslant_f, \leqslant_i \rangle$$

as follows. For all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in B_1 \times B_2$:

$$\begin{aligned}
\langle a_1, a_2 \rangle \leqslant_t \langle b_1, b_2 \rangle \quad &\text{iff} \quad a_1 \leqslant_1 b_1 \text{ and } a_2 \leqslant_2 b_2 \\
\langle a_1, a_2 \rangle \leqslant_f \langle b_1, b_2 \rangle \quad &\text{iff} \quad b_1 \sqsubseteq_1 a_1 \text{ and } a_2 \sqsubseteq_2 b_2 \\
\langle a_1, a_2 \rangle \leqslant_i \langle b_1, b_2 \rangle \quad &\text{iff} \quad a_1 \sqsubseteq_1 b_1 \text{ and } a_2 \sqsubseteq_2 b_2.
\end{aligned}$$

The six lattice operations of $\mathbf{B_1} \odot \mathbf{B_2}$ are thus determined, and let us stress that they behave as in a direct product, except for those relative to $\leqslant_f$ that are given by

$$\langle a_1, a_2 \rangle \wedge_f \langle b_1, b_2 \rangle = \langle a_1 \sqcup_1 b_1, a_2 \sqcap_2 b_2 \rangle$$

$$\langle a_1, a_2 \rangle \vee_f \langle b_1, b_2 \rangle = \langle a_1 \sqcap_1 b_1, a_2 \sqcup_2 b_2 \rangle.$$

It can be easily checked that $\mathbf{B_1} \odot \mathbf{B_2}$ is indeed a trilattice. Let us also observe that:

- The reduct $\langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_i, \vee_i \rangle$ of $\mathbf{B_1} \odot \mathbf{B_2}$ is a pre-bilattice that coincides with the usual direct product $\mathbf{B_1} \times \mathbf{B_2}$, where $\leqslant_t$ corresponds to $\leqslant_1 \times \leqslant_2$ and $\leqslant_i$ to $\sqsubseteq_1 \times \sqsubseteq_2$.
- The reduct $\langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_f, \vee_f \rangle$ of $\mathbf{B_1} \odot \mathbf{B_2}$ is also a pre-bilattice that coincides with the usual direct product $\langle B_1, \leqslant_1, \sqsupseteq_1 \rangle \times \langle B_2, \leqslant_2, \sqsubseteq_2 \rangle$. Note that in the first factor the $\sqsubseteq_1$ order of $\mathbf{B_1}$ is reversed.
- The algebra $\langle B_1 \times B_2, \vee_f, \wedge_f, \wedge_i, \vee_i \rangle$, where the $\leqslant_f$ order of $\mathbf{B_1} \odot \mathbf{B_2}$ is reversed, is a pre-bilattice that is isomorphic to the product pre-bilattice $\langle B_1, \sqsubseteq_1 \rangle \odot \langle B_2, \sqsubseteq_2 \rangle$.

The above facts will be used to simplify the proofs of the next statements, starting from the following:

Proposition 2.1. *The trilattice $\mathbf{B_1} \odot \mathbf{B_2}$ is interlaced if and only if both $\mathbf{B_1}$ and $\mathbf{B_2}$ are interlaced pre-bilattices.*

Proof. Assume $\mathbf{B_1} \odot \mathbf{B_2}$ is an interlaced trilattice. Then all its pre-bilattice reducts are interlaced, in particular $\langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_i, \vee_i \rangle$ is. As observed above, this reduct coincides with the direct product $\mathbf{B_1} \times \mathbf{B_2}$, therefore we know that $\mathbf{B_1}$ and $\mathbf{B_2}$ are homomorphic images of $\langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_i, \vee_i \rangle$. Since the class of interlaced pre-bilattices is a variety (so closed under homomorphic images), we conclude that $\mathbf{B_1}$ and $\mathbf{B_2}$ are also interlaced pre-bilattices.

Conversely, assume the pre-bilattices $\mathbf{B_1}$ and $\mathbf{B_2}$ are interlaced. The class of interlaced pre-bilattices is obviously also closed under direct products and this implies that $\mathbf{B_1} \times \mathbf{B_2}$ is an interlaced pre-bilattice. In the light of the above observations, this means that the t-lattice operations are monotonic w.r.t. $\leqslant_i$ and the i-lattice operations are monotonic w.r.t. $\leqslant_t$. Notice also that the class of interlaced pre-bilattices is closed under dual algebras, in the sense that if $\langle B_1, \leqslant_1, \sqsubseteq_1 \rangle$ is an interlaced pre-bilattice, then so is for example $\langle B_1, \leqslant_1, \sqsupseteq_1 \rangle$. It follows then that $\langle B_1, \leqslant_1, \sqsupseteq_1 \rangle \times \langle B_2, \leqslant_2, \sqsubseteq_2 \rangle$ is an interlaced pre-bilattice, therefore the t-lattice connectives are monotonic w.r.t. $\leqslant_f$ and, conversely, the f-lattice connectives are monotonic w.r.t. $\leqslant_t$. Finally, as we have observed, the algebra $\langle B_1 \times B_2, \vee_f, \wedge_f, \wedge_i, \vee_i \rangle$ is isomorphic to a product pre-bilattice, hence it is interlaced. It follows that the algebra $\langle B_1 \times B_2, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle$ obtained by reversing the $\leqslant_f$ order is also interlaced, and this allows us to conclude that the f-lattice connectives are monotonic w.r.t. $\leqslant_i$ and the i-lattice operations are monotonic w.r.t. $\leqslant_f$. Thus the trilattice $\mathbf{B_1} \odot \mathbf{B_2}$ is interlaced (and notice that for the last step of the proof we do not even need to assume that $\mathbf{B_1}$ and $\mathbf{B_2}$ be interlaced). $\square$

By examining the proof of the previous proposition it is not difficult to see that the same reasoning may be employed to prove the following:

Proposition 2.2. *The trilattice $\mathbf{B_1} \odot \mathbf{B_2}$ is distributive if and only if both $\mathbf{B_1}$ and $\mathbf{B_2}$ are distributive pre-bilattices.*

As anticipated, the latter results easily allow to build an example of non-interlaced trilattice as well as an interlaced but non-distributive trilattice, thus showing that the inclusions between the above-mentioned varieties of algebras are all strict.

We are now going to see how to extend the product trilattice construction introduced above in order to define involution operators. Due to the dualities implicit in the definition of trilattices, it is obvious that there are only three basic cases to consider, namely:

- trilattices with just one (say, the t-involution) operation
- trilattices with just two (say, t- and f-involution) operations
- trilattices with three (t-, f- and i-involution) operations.

t-involution. Let $\mathbf{B_1} = \langle B_1, \leqslant_1, \sqsubseteq_1, \neg_1\rangle$ and $\mathbf{B_2} = \langle B_2, \leqslant_2, \sqsubseteq_2, \neg_2\rangle$ be bilattices. We define the product trilattice with t-involution

$$\mathbf{B_1} \odot \mathbf{B_2} := \langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t\rangle$$

as follows. The reduct $\langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i\rangle$ is defined as before and the t-involution operation is given, for all $\langle a, b\rangle \in B_1 \times B_2$, by

$$-_t\langle a, b\rangle := \langle \neg_1 a, \neg_2 b\rangle.$$

It is easy to check that the operation $-_t$ satisfies the axioms for being a t-involution, i.e., is involutive, anti-monotone w.r.t. $\leqslant_t$ and monotone w.r.t. $\leqslant_f$ and $\leqslant_i$. Notice also that, for any product trilattice with t-involution $\mathbf{B_1} \odot \mathbf{B_2}$, the reduct $\langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_i, \vee_i, -_t\rangle$ is a bilattice that coincides with the direct product $\mathbf{B_1} \times \mathbf{B_2}$.

{t, f}-involutions. Let $\mathbf{B_1} = \langle B_1, \leqslant_1, \sqsubseteq_1, \neg_1\rangle$ and $\mathbf{B_2} = \langle B_2, \leqslant_2, \sqsubseteq_2, \neg_2\rangle$ be bilattices such that there is an isomorphism h: $\mathbf{B_1} \cong \mathbf{B_2}$. Then we define the product trilattice with t- and f-involutions

$$\mathbf{B_1} \odot \mathbf{B_2} := \langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t, -_f\rangle$$

as before, with the f-involution operation given, for all $\langle a, b\rangle \in B_1 \times B_2$, by

$$-_f\langle a, b\rangle := \big\langle h^{-1}(b), h(a)\big\rangle.$$

In particular, if $\mathbf{B_1} = \mathbf{B_2}$, we have

$$-_f\langle a, b\rangle := \langle b, a\rangle.$$

It easy to check that the operation thus defined satisfies the conditions for being an f-involution. Moreover, notice that the two involutions always commute.

{t, f, i}-involutions. Let $\mathbf{B_1} = \langle B_1, \leqslant_1, \sqsubseteq_1, \neg_1, -_1\rangle$ and $\mathbf{B_2} = \langle B_2, \leqslant_2, \sqsubseteq_2, \neg_2, -_2\rangle$ be bilattices with conflation such that there is an isomorphism $h : \mathbf{B_1} \cong \mathbf{B_2}$. Then we define the product trilattice with t-, f- and i-involutions

$$\mathbf{B_1} \odot \mathbf{B_2} := \langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t, -_f, -_i\rangle$$

as before, with the i-involution operation given, for all $\langle a, b\rangle \in B_1 \times B_2$, by

$$-_i\langle a, b\rangle := \big\langle -_1 h^{-1}(b), -_2 h(a)\big\rangle.$$

In particular, if $\mathbf{B_1} = \mathbf{B_2} = \langle B, \leqslant, \sqsubseteq, \neg, -\rangle$, we have

$$-_i\langle a, b\rangle := \langle -b, -a\rangle.$$

It is easy to check that the above-defined operation is actually an i-involution, and that the commutative laws

$$-_t -_f = -_f -_t \quad \text{and} \quad -_f -_i = -_i -_f$$

always hold, while $-_t$ and $-_i$ commute if and only if in $\mathbf{B_1}$ and $\mathbf{B_2}$ negation and conflation commute.

Let us also observe that, for any interlaced trilattice $\mathbf{B} \odot \mathbf{B}$, where $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg, -\rangle$, the algebra

$$\langle B \times B, \vee_f, \wedge_f, \wedge_i, \vee_i, -_f, -_i\rangle$$

where the $\leqslant_f$ order of $\mathbf{B} \odot \mathbf{B}$ is reversed, is a commutative interlaced bilattice with conflation that is isomorphic to the product bilattice $\langle B, \sqcap, \sqcup, -\rangle \odot \langle B, \sqcap, \sqcup, -\rangle$. In fact, the f-involution $-_f$ is an isomorphism between $\langle B \times B, \vee_f, \wedge_f, \wedge_i, \vee_i, -_f, -_i\rangle$ and $\langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f, -_i\rangle$, so we may conclude that

$$\langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f, -_i\rangle \cong \langle B, \sqcap, \sqcup, -\rangle \odot \langle B, \sqcap, \sqcup, -\rangle.$$

3. Representation of interlaced trilattices

In this section we are going to prove representation theorems that establish which classes of trilattices can be represented through the product constructions defined in the previous section. We start with the case of trilattices without involutions.

3.1. Trilattices without involutions

Let $\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle$ be an interlaced trilattice. Let us focus on its pre-bilattice reduct $\langle A, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle$ and consider the relations $\sim_1$ and $\sim_2$ defined as follows:

$$\sim_1 := \big\{ \langle a, b \rangle \in A \times A \colon a \wedge_i b = a \vee_f b \big\}$$
$$\sim_2 := \big\{ \langle a, b \rangle \in A \times A \colon a \wedge_i b = a \wedge_f b \big\}.$$

It is easy to show that

$$\sim_1 = \big\{ \langle a, b \rangle \in A \times A \colon a \vee_i b = a \wedge_f b \big\}$$
$$\sim_2 = \big\{ \langle a, b \rangle \in A \times A \colon a \vee_i b = a \vee_f b \big\}.$$

We will use the following result [4, Proposition 3.8]:

Proposition 3.1. *The relations $\sim_1$ and $\sim_2$ defined above are factor congruences of any interlaced pre-bilattice $\langle A, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle$.*

Let us remind the reader that two congruences θ_1, θ_2 of an algebra $\mathbf{A}$ are called *factor congruences* of $\mathbf{A}$ when the following conditions are satisfied [6, Definition II.7.4]:

(i) $\theta_1 \cap \theta_2 = Id_{\mathbf{A}}$
(ii) $\theta_1 \vee \theta_2 = A \times A$
(iii) θ_1 and θ_2 permute.

This implies that $\mathbf{A}$ is isomorphic to the direct product $\mathbf{A}/\theta_1 \times \mathbf{A}/\theta_2$. In our case we have then that $\langle A, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle$ is isomorphic to the direct product

$$\langle A, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle / {\sim_1} \times \langle A, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle / {\sim_2}.$$

This is also true for the trilattice as a whole:

Proposition 3.2. *The relations $\sim_1$ and $\sim_2$ are factor congruences of any interlaced trilattice $\mathbf{A}$.*

Proof. Examining the definition of factor congruence, one sees that the only part that needs to be checked is that $\sim_1$ and $\sim_2$ are indeed congruences of $\mathbf{A}$, i.e., that they are compatible with $\{\wedge_t, \vee_t\}$. To see this recall that, by [4, Definition 3.7], $a \sim_1 b$ is equivalent to the condition that there be $c, d \in A$ such that $a \leqslant_f c \leqslant_i b$ and $b \leqslant_f d \leqslant_i a$. Now consider an arbitrary element $e \in A$. Applying the interlacing conditions to the above inequalities, we obtain $a \wedge_t e \leqslant_f c \wedge_t e \leqslant_i b \wedge_t e$ and $b \wedge_t e \leqslant_f d \wedge_t e \leqslant_i a \wedge_t e$, which means $(a \wedge_t e) \sim_1 (b \wedge_t e)$. Since we are in a lattice, this is enough to conclude that $\sim_1$ is compatible with $\wedge_t$. A similar reasoning may be applied to establish the remaining cases. □

The previous result immediately yields the following:

Theorem 3.3. *For any interlaced trilattice $\mathbf{A}$, it holds that $\mathbf{A} \cong \mathbf{A}/{\sim_1} \times \mathbf{A}/{\sim_2}$.*

Let us note that $\mathbf{A}/{\sim_1}$ and $\mathbf{A}/{\sim_2}$ are degenerated trilattices, in the sense that in $\mathbf{A}/{\sim_1}$ the $\leqslant_f$ order is the dual of $\leqslant_i$, while in $\mathbf{A}/{\sim_2}$ we have $\leqslant_f = \leqslant_i$. It is then easy to check that the direct product $\mathbf{A}/{\sim_1} \times \mathbf{A}/{\sim_2}$ coincides with the product trilattice $\langle A, \leqslant_t, \leqslant_i \rangle / {\sim_1} \odot \langle A, \leqslant_t, \leqslant_i \rangle / {\sim_2}$. Therefore we obtain the following:

Theorem 3.4. *Any interlaced trilattice $\mathbf{A}$ is isomorphic to a product trilattice*

$$\mathbf{A} \cong \langle A, \leqslant_t, \leqslant_i \rangle / {\sim_1} \odot \langle A, \leqslant_t, \leqslant_i \rangle / {\sim_2}$$

where $\langle A, \leqslant_t, \leqslant_i \rangle / {\sim_1}$ and $\langle A, \leqslant_t, \leqslant_i \rangle / {\sim_2}$ are interlaced pre-bilattices.

3.2. Trilattices with t-involution

Let $\mathbf{A} = \langle A, \leqslant_t, \leqslant_f, \leqslant_i, -_t \rangle$ be an interlaced trilattice with t-involution. Using De Morgan laws, it is not difficult to prove the following:

Proposition 3.5. *The relations $\sim_1$ and $\sim_2$ defined in the previous section are congruences of any interlaced trilattice with t-involution.*

Proof. Obviously we only need to check that $\sim_1$ and $\sim_2$ are compatible with the t-involution, and this is easily proved. In fact $a \sim_1 b$ means $a \wedge_i b = a \vee_f b$ and, using De Morgan laws, we have that the latter equality implies $-_t a \wedge_i -_t b = -_t(a \wedge_i b) = -_t(a \vee_f b) = -_t a \vee_f -_t b$. So $-_t a \sim_1 -_t b$, and the same reasoning applies to $\sim_2$. □

The quotients $\langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_1}$ and $\langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_2}$ are thus bilattices, and it is not difficult to obtain the following:

Theorem 3.6. *Any interlaced trilattice with t-involution* **A** *is isomorphic to the product trilattice*:

$$\mathbf{A} \cong \langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_1} \odot \langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_2}$$

where $\langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_1}$ *and* $\langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_2}$ *are interlaced bilattices.*

Proof. The isomorphism is defined as for trilattices without involutions, i.e., is given by the map

$$\iota : a \longmapsto \big\langle [a]_1, [a]_2\big\rangle$$

that to any $a \in A$ assigns the ordered pair formed by its equivalence class $[a]_1$ modulo $\sim_1$ and its equivalence class $[a]_2$ modulo $\sim_2$. We have to show that this map preserves the t-involution, and this is easy because we have

$$-_t\iota(a) = -_t\big\langle [a]_1, [a]_2\big\rangle = \big\langle -_t[a]_1, -_t[a]_2\big\rangle = \big\langle [-_t a]_1, [-_t a]_2\big\rangle = \iota(-_t a). \qquad \square$$

Using the above representation, it is easy to see that the smallest non-trivial trilattice with t-involution has four elements[2] and can be represented as a product $\mathbf{B_1} \odot \mathbf{B_2}$ where $\mathbf{B_1}$ (or, equivalently, $\mathbf{B_2}$) is trivial and $\mathbf{B_2}$ (or $\mathbf{B_1}$) is the four-element Belnap bilattice $\mathcal{FOUR}_2$. This implies that either $\leqslant_f = \leqslant_i$ or $\leqslant_f = \geqslant_i$. Another easy consequence is then that the smallest non-degenerated interlaced trilattice with t-involution (non-degenerated meaning that, for any two orders of the trilattice $\leqslant, \leqslant'$, neither $\leqslant = \leqslant'$ nor $\leqslant = \geqslant'$) must have sixteen elements, being isomorphic to $\mathcal{FOUR}_2 \odot \mathcal{FOUR}_2$.

3.3. Trilattices with {t, f}-involutions

If **A** is an interlaced trilattice with t- and f-involutions, then the reduct $\langle A, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f\rangle$ is an interlaced bilattice. Thus, we know [4, Proposition 3.8] that

$$\langle A, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f\rangle \cong \langle A, \wedge_i, \vee_i\rangle/{\sim_1} \odot \langle A, \wedge_i, \vee_i\rangle/{\sim_2}.$$

Moreover, there is a lattice isomorphism

$$h\colon \langle A, \wedge_i, \vee_i\rangle/{\sim_1} \cong \langle A, \wedge_i, \vee_i\rangle/{\sim_2}$$

defined, for all $a \in A$, as

$$h : [a]_1 \longmapsto [-_f a]_2.$$

The following result shows that under an additional assumption this map is also a bilattice isomorphism between the bilattices $\langle A, \wedge_t, \vee_t, \wedge_i, \vee_i, -_t\rangle/{\sim_1}$ and $\langle A, \wedge_t, \vee_t, \wedge_i, \vee_i, -_t\rangle/{\sim_2}$.

Theorem 3.7. *Let* **A** *be an interlaced trilattice with t- and f-involutions such that the two involutions commute, i.e.,* $-_t -_f = -_f -_t$. *Then*:

(i) $h\colon \langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_1} \cong \langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_2}$
(ii) $\mathbf{A} \cong \langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_1} \odot \langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_1}$, *where* $\langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_1}$ *is an interlaced bilattice.*

Proof. (i) It is easy to check that h is a homomorphism w.r.t. the operations $\{\wedge_t, \vee_t\}$, for using De Morgan laws we have

$$\begin{aligned} h\big([a]_1 \wedge_t [b]_1\big) &= h\big([a \wedge_t b]_1\big) \\ &= \big[-_f(a \wedge_t b)\big]_2 \\ &= [-_f a \wedge_t -_f b]_2 \\ &= [-_f a]_2 \wedge_t [-_f b]_2 \\ &= h\big([a]_1\big) \wedge_t h\big([b]_1\big) \end{aligned}$$

[2] Notice that any non-trivial trilattice with t-involution $\langle A, \leqslant_t, \leqslant_f, \leqslant_i, -_t\rangle$, independently on whether it is interlaced or not, must have at least four elements, because for instance the reduct $\langle A, \leqslant_t, \leqslant_f, -_t\rangle$ is a bilattice and we know that the smallest non-trivial bilattice has four elements.

and similarly for the operation $\vee_t$. Notice that this step does not depend on the assumption that the two involutions commute. Adding such a requirement, we may show that h also preserves the t-involution, for we have

$$h\big([-_t\, a]_1\big) = [-_f\, -_t\, a]_2 = [-_t\, -_f\, a]_2 = -_t[-_f a]_2 = -_t h\,\big([a]_1\big).$$

Therefore we conclude that h is an isomorphism between the bilattices $\langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_1}$ and $\langle A, \leqslant_t, \leqslant_i, -_t\rangle/{\sim_2}$.

(ii) Follows immediately from the previous item. □

As in the previous section, we may use the result obtained above to conclude that the smallest non-trivial trilattice with t- and f-involution operations (notice that we do not need to assume that they commute) has sixteen elements, as it is again isomorphic to the product trilattice $\mathcal{FOUR}_2 \odot \mathcal{FOUR}_2$.

3.4. Trilattices with {t, f, i}-involutions

Suppose $\mathbf{A} = \langle A, \leqslant_t, \leqslant_f, \leqslant_i, -_t, -_f, -_i\rangle$ is an interlaced trilattice with t-, f- and i-involutions. Notice that neither $-_f$ nor $-_i$ is compatible with $\sim_1$ (nor with $\sim_2$), but it is easy to prove that the composition of the two operation is.

Proposition 3.8. *Let* $\mathbf{A}$ *be an interlaced trilattice with t-, f- and i-involutions and* $a, b \in A$. *Then*:

(i) $a \sim_1 b$ *implies* $(-_f\, -_i\, a) \sim_1 (-_f\, -_i\, b)$ *and* $(-_i\, -_f\, a) \sim_1 (-_i\, -_f\, b)$
(ii) $a \sim_2 b$ *implies* $(-_f\, -_i\, a) \sim_2 (-_f\, -_i\, b)$ *and* $(-_i\, -_f\, a) \sim_2 (-_i\, -_f\, b)$.

Proof. (i) Assume $a \sim_1 b$, that is $a \wedge_i b = a \vee_f b$. Using De Morgan laws, we have

$$\begin{aligned}
-_f\, -_i\, a \vee_i -_f\, -_i\, b &= -_f(-_i a \vee_i -_i b)\\
&= -_f\, -_i\, (a \wedge_i b)\\
&= -_f\, -_i\, (a \vee_f b)\\
&= -_f(-_i a \vee_f -_i b)\\
&= -_f\, -_i\, a \wedge_f -_f\, -_i\, b
\end{aligned}$$

which implies $(-_f\, -_i\, a) \sim_1 (-_f\, -_i\, b)$. The remaining cases can be proved by the same reasoning. □

Let us introduce the following abbreviations: $-_{fi} := -_f -_i$ and $-_{if} := -_i -_f$. Taking into account the above proposition, one sees that it makes sense to consider the quotients

$$\langle A, \leqslant_t, \leqslant_i, -_t -_{fi}\rangle/{\sim_1}$$

and

$$\langle A, \leqslant_t, \leqslant_i, -_t, -_{if}\rangle/{\sim_1}.$$

If the two involution operations commute, i.e., if $-_f -_i = -_i -_f$, then obviously

$$\langle A, \leqslant_t, \leqslant_i, -_t, -_{fi}\rangle/{\sim_1} = \langle A, \leqslant_t, \leqslant_i, -_t, -_{if}\rangle/{\sim_1}.$$

In such a case it is easy to check that $\langle A, \leqslant_t, \leqslant_i, -_t, -_{fi}\rangle/{\sim_1}$ is a bilattice with conflation, thus obtaining the following:

Theorem 3.9. *Let* $\mathbf{A}$ *be an interlaced trilattice with t-, f- and i-involutions such that* $-_f -_i = -_i -_f$. *Then*

$$\mathbf{A} \cong \langle A, \leqslant_t, \leqslant_i, -_t, -_{fi}\rangle/{\sim_1} \odot \langle A, \leqslant_t, \leqslant_i, -_t, -_{fi}\rangle/{\sim_1}$$

where $\langle A, \leqslant_t, \leqslant_i, -_t, -_{fi}\rangle/{\sim_1}$ *is an interlaced bilattice with conflation.*

Using the above result we may check that the smallest non-trivial trilattice with t-, f- and i-involutions is exactly the canonical trilattice $\mathcal{SIXTEEN}_3$, which is isomorphic to the product trilattice of the bilattice with conflation $\mathcal{FOUR}_2$ with itself. Fig. 3 shows a diagram of this trilattice represented as a product. Notice that, as happened in the case of the bilattice negation, the t-involution corresponds in the diagram to reflection along the vertical axis, but none of the other two involution operations has now a simple graphical characterization.

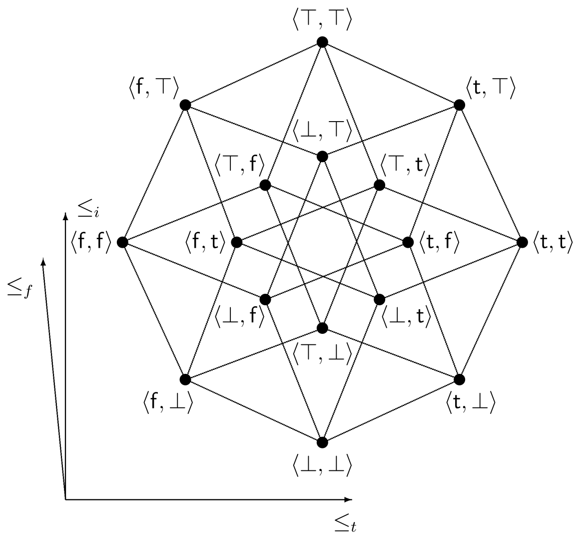

Fig. 3. The trilattice $\mathcal{SIXTEEN}_3$ represented as $\mathcal{FOUR}_2 \odot \mathcal{FOUR}_2$.

3.5. *Odintsov's construction*

In this section we compare the representation results obtained above with a construction introduced by S. Odintsov [16] that provides a representation of the trilattice $\mathcal{SIXTEEN}_3$ as a special kind of power of the two-element Boolean algebra.

Let us denote by $\mathbf{2} = \langle \{0,1\}, \wedge, \vee, ' \rangle$ the two-element Boolean algebra and by $\leqslant$ its lattice order. According to Odintsov's construction, an element of $\mathcal{SIXTEEN}_3$ is represented as a matrix of the form

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix}$$

where $n, f, t, b \in \{0,1\}$. The t- and f-orders on $\mathcal{SIXTEEN}_3$ are then defined as follows:

$$\begin{vmatrix} n_1 & f_1 \\ t_1 & b_1 \end{vmatrix} \leqslant_t \begin{vmatrix} n_2 & f_2 \\ t_2 & b_2 \end{vmatrix} \quad \text{iff} \quad \begin{matrix} n_2 \leqslant n_1 & f_2 \leqslant f_1 \\ t_1 \leqslant t_2 & b_1 \leqslant b_2 \end{matrix}$$

$$\begin{vmatrix} n_1 & f_1 \\ t_1 & b_1 \end{vmatrix} \leqslant_f \begin{vmatrix} n_2 & f_2 \\ t_2 & b_2 \end{vmatrix} \quad \text{iff} \quad \begin{matrix} n_1 \leqslant n_2 & f_2 \leqslant f_1 \\ t_1 \leqslant t_2 & b_2 \leqslant b_1 \end{matrix}$$

and the involution operations are given by:

$$-_t \begin{vmatrix} n & f \\ t & b \end{vmatrix} = \begin{vmatrix} t & b \\ n & f \end{vmatrix} \qquad -_f \begin{vmatrix} n & f \\ t & b \end{vmatrix} = \begin{vmatrix} f & n \\ b & t \end{vmatrix}$$

It is easy to check that the i-order, although not considered in [16], is given by:

$$\begin{vmatrix} n_1 & f_1 \\ t_1 & b_1 \end{vmatrix} \leqslant_i \begin{vmatrix} n_2 & f_2 \\ t_2 & b_2 \end{vmatrix} \quad \text{iff} \quad \begin{matrix} n_1 \leqslant n_2 & f_1 \leqslant f_2 \\ t_1 \leqslant t_2 & b_1 \leqslant b_2 \end{matrix}$$

while the i-involution is given by

$$-_i \begin{vmatrix} n & f \\ t & b \end{vmatrix} = \begin{vmatrix} b' & t' \\ f' & n' \end{vmatrix}$$

where b', t', f', n' denote the Boolean complements of, respectively, b, t, f, n. All the algebraic operations of $\mathcal{SIXTEEN}_3$ are thus determined. Let us now see how this trilattice can be represented through the construction introduced in Section 2.4.

Recall that we may assume without loss of generality that any interlaced (pre-)bilattice is of the form $\mathbf{L_1} \odot \mathbf{L_2}$, where $\mathbf{L_1}$ and $\mathbf{L_2}$ are lattices. Moreover, we proved that any interlaced trilattice has the form $\mathbf{B_1} \odot \mathbf{B_2}$, where $\mathbf{B_1}$ and $\mathbf{B_2}$ are interlaced (pre-)bilattices. Putting these results together we may conclude that any interlaced trilattice $\mathbf{A} = \langle A, \leqslant_t, \leqslant_f, \leqslant_i \rangle$ can be seen as a product

$$(\mathbf{L_1} \odot \mathbf{L_2}) \odot (\mathbf{L_3} \odot \mathbf{L_4})$$

where each $\mathbf{L_n} = \langle L_n, \leqslant_n \rangle$ with $1 \leqslant n \leqslant 4$ is a lattice. In this way, any element $a \in A$ is represented as a 4-tuple $\langle a_1, a_2, a_3, a_4 \rangle \in L_1 \times L_2 \times L_3 \times L_4$. Using this notation, it is easy to see that the three lattice orders on $\mathbf{A}$ are given by:

$$\begin{array}{lll} \langle a_1, a_2, a_3, a_4 \rangle \leqslant_t \langle b_1, b_2, b_3, b_4 \rangle & \text{iff} & a_1 \leqslant_1 b_1 \quad b_2 \leqslant_2 a_2 \\ & & a_3 \leqslant_3 b_3 \quad b_4 \leqslant_4 a_4 \\ \langle a_1, a_2, a_3, a_4 \rangle \leqslant_f \langle b_1, b_2, b_3, b_4 \rangle & \text{iff} & b_1 \leqslant_1 a_1 \quad b_2 \leqslant_2 a_2 \\ & & a_3 \leqslant_3 b_3 \quad a_4 \leqslant_4 b_4 \\ \langle a_1, a_2, a_3, a_4 \rangle \leqslant_i \langle b_1, b_2, b_3, b_4 \rangle & \text{iff} & a_1 \leqslant_1 b_1 \quad a_2 \leqslant_2 b_2 \\ & & a_3 \leqslant_3 b_3 \quad a_4 \leqslant_4 b_4. \end{array}$$

As we have seen in the previous sections, if $\mathbf{A}$ has a t-involution, then $\mathbf{L_1} \cong \mathbf{L_2}$ and $\mathbf{L_3} \cong \mathbf{L_4}$. Similarly, the existence of an f-involution entails that $\mathbf{L_1} \cong \mathbf{L_3}$ and $\mathbf{L_2} \cong \mathbf{L_4}$. Finally, if all three involution operations exist, then $\mathbf{L_1} \cong \mathbf{L_2} \cong \mathbf{L_3} \cong \mathbf{L_4}$ and in addition each $\mathbf{L_n}$ has an involution operation, which we denote by $'$. The involution operations in $\mathbf{A}$, in case they exist, are defined by

$$-_t \langle a_1, a_2, a_3, a_4 \rangle = \langle a_2, a_1, a_4, a_3 \rangle$$

$$-_f \langle a_1, a_2, a_3, a_4 \rangle = \langle a_3, a_4, a_1, a_2 \rangle$$

$$-_i \langle a_1, a_2, a_3, a_4 \rangle = \langle a_4', a_3', a_2', a_1' \rangle.$$

Notice that, in order to simplify the notation, we are assuming that $\mathbf{L_m} = \mathbf{L_n}$ whenever $\mathbf{L_m} \cong \mathbf{L_n}$ for $1 \leqslant m, n \leqslant 4$.

It is then easy to see that, taking $\mathbf{L_n} = \mathbf{2}$ for all $1 \leqslant n \leqslant 4$, where $\mathbf{2}$ denotes the two-element Boolean algebra, we obtain the trilattice $\mathcal{SIXTEEN}_3$ as a special case of our construction. Using Odintsov's matrix notation, our 4-tuple

$$\langle a_1, a_2, a_3, a_4 \rangle$$

would be rewritten as

$$\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}$$

and, conversely, the matrix

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix}$$

corresponds to the 4-tuple $\langle b, f, t, n \rangle$. It is easy to check that, using this notation, our definitions of the trilattice operations coincide with Odintsov's.

4. Congruences of interlaced trilattices

We now turn to the study of congruences of interlaced trilattices, with the aim to obtain more information on these algebras from a universal algebraic point of view.

4.1. Trilattices without involutions

Let us start with trilattices without any involution, and let us keep in mind that, as observed above, all the classes of trilattices we deal with are varieties.

We know from the theory of pre-bilattices [4, Proposition 3.8] that the congruences of any interlaced pre-bilattice $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ coincide with those of either of its lattice reducts $\langle B, \wedge, \vee \rangle$ and $\langle B, \sqcap, \sqcup \rangle$. This observation immediately implies the following:

Proposition 4.1. *The congruences of any interlaced trilattice $\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle$ coincide with those of any of its lattice reducts, i.e.,*

$$\mathrm{Con}(\mathbf{A}) = \mathrm{Con}(\langle A, \wedge_t, \vee_t \rangle) = \mathrm{Con}(\langle A, \wedge_f, \vee_f \rangle) = \mathrm{Con}(\langle A, \wedge_i, \vee_i \rangle).$$

It is then obvious that $\mathrm{Con}(\mathbf{A})$ also coincides with the congruences of any of the pre-bilattice reducts of $\mathbf{A}$. This observation can be used to prove the following:

Proposition 4.2. *Let $\mathbf{B_1} \odot \mathbf{B_2}$ be an interlaced trilattice without any involution, $\mathbf{B_1}$ and $\mathbf{B_2}$ being interlaced pre-bilattices. Then*

$$\mathrm{Con}(\mathbf{B_1} \odot \mathbf{B_2}) \cong \mathrm{Con}(\mathbf{B_1}) \times \mathrm{Con}(\mathbf{B_2}).$$

Proof. We have observed that $\mathrm{Con}(\mathbf{B_1} \odot \mathbf{B_2})$ coincides with the congruences of any of its pre-bilattice reducts, for instance those of $\langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_i, \vee_i \rangle$. As noted in Section 2.4, we have $\langle B_1 \times B_2, \wedge_t, \vee_t, \wedge_i, \vee_i \rangle = \mathbf{B_1} \times \mathbf{B_2}$. Since we are in a congruence-distributive variety, we may invoke the Fraser–Horn–Hu property [11, Corollary 1] to conclude that $\mathrm{Con}(\mathbf{B_1} \times \mathbf{B_2}) \cong \mathrm{Con}(\mathbf{B_1}) \times \mathrm{Con}(\mathbf{B_2})$, which completes our proof. □

Recall that, by [4, Proposition 3.8], for any pre-bilattice $\mathbf{B}$ there are lattices $\mathbf{L_1}$ and $\mathbf{L_2}$ such that $\mathbf{B} \cong \mathbf{L_1} \odot \mathbf{L_2}$ and $\mathrm{Con}(\mathbf{B}) \cong \mathrm{Con}(\mathbf{L_1}) \times \mathrm{Con}(\mathbf{L_2})$. In the light of the previous proposition we may then observe that if $\mathbf{A}$ is an interlaced trilattice, then there are lattices $\mathbf{L_1}, \ldots, \mathbf{L_4}$ such that

$$\mathbf{A} \cong (\mathbf{L_1} \odot \mathbf{L_2}) \odot (\mathbf{L_3} \odot \mathbf{L_4})$$

and

$$\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{L_1}) \times \mathrm{Con}(\mathbf{L_2}) \times \mathrm{Con}(\mathbf{L_3}) \times \mathrm{Con}(\mathbf{L_4}).$$

This means then that any question concerning the lattice of congruences of trilattices can be reduced to a question concerning congruences of lattices, which is of course a fairly well-known topic. We are going to see that, adding sometimes a few restrictions, it will be possible to obtain an analogous reduction also in the case of trilattices with involution operations.

4.2. Trilattices with t-involution

In case $\mathbf{A}$ has just one involution, we may reason as in the previous case to obtain the following:

Proposition 4.3. *Let $\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t \rangle$ be an interlaced trilattice with t-involution. Then*

$$\mathrm{Con}(\mathbf{A}) = \mathrm{Con}\big(\langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, -_t \rangle\big) = \mathrm{Con}\big(\langle A, \wedge_t, \vee_t, -_t \rangle\big).$$

The reduct $\langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, -_t \rangle$ is an interlaced bilattice, while the reduct $\langle A, \wedge_t, \vee_t, -_t \rangle$ is an involutive lattice. Thanks to this result we then have a characterization of the congruences of interlaced trilattices with t-involution in terms of the congruences of either of these two classes of algebras. Moreover, we know [4, Proposition 3.13] that any interlaced bilattice $\mathbf{B}$ is isomorphic to a product $\mathbf{L} \odot \mathbf{L}$ for some lattice $\mathbf{L}$ such that $\mathrm{Con}(\mathbf{B}) \cong \mathrm{Con}(\mathbf{L})$. Thus, letting $\langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, -_t \rangle = \mathbf{L} \odot \mathbf{L}$, we may conclude that

$$\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{L}).$$

4.3. Trilattices with {t, f}-involutions

If $\mathbf{A}$ has two involution operations, we may repeat the previous reasoning to obtain the following:

Proposition 4.4. *Let $\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t, -_f \rangle$ be an interlaced trilattice with t- and f-involutions. Then*

$$\mathrm{Con}(\mathbf{A}) = \mathrm{Con}\big(\langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, -_t, -_f \rangle\big)$$

where the reduct $\langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, -_t, -_f \rangle$ is an interlaced bilattice with conflation.

In general we do not know of a nice characterization of the congruences of interlaced bilattices with conflation, since the only known representation theorem for this class of algebras [5, Theorem 4.2] holds just for the commutative case. However, if the two involutions commute, then the above-mentioned reduct is a commutative bilattice with conflation, so it can be represented as a product $\mathbf{L} \odot \mathbf{L}$, where $\mathbf{L}$ is an involutive lattice. In this case [5, Theorem 4.3] we have that

$$\mathrm{Con}(\mathbf{L}) \cong \mathrm{Con}\big(\langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, -_t, -_f \rangle\big) = \mathrm{Con}(\mathbf{A}).$$

Notice that this lattice $\mathbf{L}$ does not coincide with any of the $\mathbf{L_n}$ obtained from the representation of trilattices through the construction described in Section 3.5.

4.4. Trilattices with {t, f, i}-involutions

The case where $\mathbf{A}$ has three involution operations, i.e.,

$$\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t, -_f, -_i \rangle$$

is somewhat more involved. We will assume that in $\mathbf{A}$ all three involution operations commute with each other. By the representation result proved in Section 3.4, this implies that $\mathbf{A}$ has the form $\mathbf{B} \odot \mathbf{B}$ for some interlaced bilattice with conflation $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg, - \rangle$. It is then our aim to prove the following statement:

Proposition 4.5. *If* **A** *is an interlaced trilattice with t, f- and i-involutions such that* $\mathbf{A} \cong \mathbf{B} \odot \mathbf{B}$ *and all three involutions commute with each other, then* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{B})$.

Proof. Reasoning as in the previous cases, we start by noting that

$$\mathrm{Con}(\mathbf{B} \odot \mathbf{B}) = \mathrm{Con}\big(\langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t, -_f, -_i\rangle\big).$$

As observed at the end of Section 2.4, the reduct $\langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f, -_i\rangle$ is a commutative interlaced bilattice with conflation and

$$\langle B, \sqcap, \sqcup, -\rangle \odot \langle B, \sqcap, \sqcup, -\rangle \cong \langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f, -_i\rangle.$$

By [5, Theorem 4.3], we then know that there is an isomorphism

$$H\colon \mathrm{Con}\big(\langle B, \sqcap, \sqcup, -\rangle\big) \cong \mathrm{Con}\big(\langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f, -_i\rangle\big)$$

which can be defined, for all $\theta \in \mathrm{Con}(\langle B, \sqcap, \sqcup, -\rangle)$ and all $a_1, a_2, b_1, b_2 \in B$, as follows:

$$\big\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\big\rangle \in H(\theta) \quad \text{iff} \quad \langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in \theta.$$

The inverse H^{-1} may be defined, for all $\eta \in \mathrm{Con}(\langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f, -_i\rangle)$ and all $a, b \in B$, as follows:

$$\langle a, b\rangle \in H^{-1}(\eta) \quad \text{iff} \quad \big\langle\langle a, a\rangle, \langle b, b\rangle\big\rangle \in \eta.$$

Let us check that the maps H and H^{-1} are actually mutually inverse. Let us introduce terms $p(x, y)$ and $q(x)$ defined as follows:

$$p(x, y) := \big(x \wedge_i (x \wedge_f y)\big) \vee_i \big(y \wedge_i (x \vee_f y)\big)$$
$$q(x) := \big(x \vee_f (x \vee_i -_f x)\big) \wedge_i -_f\big(x \vee_f (x \vee_i -_f x)\big).$$

Using the product representation it is easy to check that, for all $\langle a_1, a_2\rangle \in B \times B$, it holds that $\langle a_1, a_2\rangle = p(\langle a_1, a_1\rangle, \langle a_2, a_2\rangle)$ and $\langle a_1, a_1\rangle = q(\langle a_1, a_2\rangle)$. This clearly implies (cf. [5, Proposition 3.3]) that, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in B \times B$ and for all $\eta \in \mathrm{Con}(\langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f, -_i\rangle)$,

$$\big\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\big\rangle \in \eta \quad \text{iff} \quad \big\langle\langle a_1, a_1\rangle, \langle b_1, b_1\rangle\big\rangle, \big\langle\langle a_2, a_2\rangle, \langle b_2, b_2\rangle\big\rangle \in \eta.$$

Now for all $\eta \in \mathrm{Con}(\langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f, -_i\rangle)$ we have that, by definition, $\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle \in H(H^{-1}(\eta))$ means that $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in H^{-1}(\eta)$. This means that

$$\big\langle\langle a_1, a_1\rangle, \langle b_1, b_1\rangle\big\rangle, \big\langle\langle a_2, a_2\rangle, \langle b_2, b_2\rangle\big\rangle \in \eta$$

which is equivalent, as we have seen, to $\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle \in \eta$. Hence, $H(H^{-1}(\eta)) = \eta$.

Conversely, for all $\theta \in \mathrm{Con}(\langle B, \sqcap, \sqcup, -\rangle)$, we have $\langle a, b\rangle \in H^{-1}(H(\theta))$ if and only if $\langle\langle a, a\rangle, \langle b, b\rangle\rangle \in H(\theta)$ if and only if $\langle a, b\rangle \in \theta$. Hence, $\theta = H^{-1}(H(\theta))$.

As observed above, we also know that

$$\mathrm{Con}\big(\langle B, \sqcap, \sqcup, -\rangle\big) = \mathrm{Con}\big(\langle B, \wedge, \vee, \sqcap, \sqcup, -\rangle\big).$$

Therefore,

$$H\colon \mathrm{Con}\big(\langle B, \wedge, \vee, \sqcap, \sqcup, -\rangle\big) \cong \mathrm{Con}\big(\langle B \times B, \wedge_f, \vee_f, \wedge_i, \vee_i, -_f, -_i\rangle\big).$$

In order to prove that $H\colon \mathrm{Con}(\mathbf{B}) \cong \mathrm{Con}(\mathbf{B} \odot \mathbf{B})$, it will then be sufficient to show that any congruence $\theta \in \mathrm{Con}(\langle B, \wedge, \vee, \sqcap, \sqcup, -\rangle)$ is compatible with the operation $\neg$ (i.e., is indeed a congruence of **B**) if and only if $H(\theta)$ is compatible with $-_t$ (i.e., is a congruence of $\mathbf{B} \odot \mathbf{B}$). Assume then $\theta \in \mathrm{Con}(\mathbf{B})$ and $\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle \in H(\theta)$. By the definition of H, this means $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in \theta$. Since θ is a congruence of **B**, this implies $\langle \neg a_1, \neg b_1\rangle, \langle \neg a_2, \neg b_2\rangle \in \theta$, which means that $\langle\langle \neg a_1, \neg a_2\rangle, \langle \neg b_1, \neg b_2\rangle\rangle \in H(\theta)$. By definition $-_t\langle a_1, a_2\rangle = \langle \neg a_1, \neg a_2\rangle$, so we are allowed to conclude that $\langle -_t\langle a_1, a_2\rangle, -_t\langle b_1, b_2\rangle\rangle \in H(\theta)$. Conversely, suppose $\eta \in \mathrm{Con}(\mathbf{B} \odot \mathbf{B})$ and $\langle a, b\rangle \in H^{-1}(\eta)$. By the definition of H, this means that $\langle\langle a, a\rangle, \langle b, b\rangle\rangle \in \eta$. Since η is a congruence of $\mathbf{B} \odot \mathbf{B}$, this implies $\langle -_t\langle a, a\rangle, -_t\langle b, b\rangle\rangle \in \eta$, i.e., $\langle\langle \neg a, \neg a\rangle, \langle \neg b, \neg b\rangle\rangle \in \eta$ and this means that $\langle \neg a, \neg b\rangle \in H^{-1}(\eta)$. □

An easy consequence of the above result is the following. Since we assumed that $-_t$ and $-_i$ commute in $\mathbf{B} \odot \mathbf{B}$, we know that in **B** negation and conflation commute. Hence we may apply the representation theorem for commutative bilattices with conflation to conclude that

$$\mathrm{Con}(\mathbf{B} \odot \mathbf{B}) \cong \mathrm{Con}(\mathbf{B}) \cong \mathrm{Con}(\mathbf{L})$$

where **L** is an involutive lattice such that $\mathbf{B} \cong \mathbf{L} \odot \mathbf{L}$.

5. Distributive trilattices

The characterization of congruences obtained in the previous section allows one to transfer some results that are known for distributive (bi)lattices to the context of distributive trilattices.

In the first place we are now able to individuate the subdirectly irreducible distributive trilattices, i.e., the algebras that are generators of the corresponding varieties.

We know, for instance, that any subdirectly irreducible distributive trilattice without any involution $\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i \rangle$ must have two elements, and it is easy to check that there are only two non-isomorphic algebras of this kind: one is such that $\leqslant_t = \leqslant_f = \leqslant_i$ and the other such that $\leqslant_t = \leqslant_f = \geqslant_i$. Therefore *the variety of distributive trilattices is generated by its two two-element members*.

Let us now consider distributive trilattices with involutions.

t-involution. For any subdirectly irreducible distributive trilattice with t-involution $\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t \rangle$, the reducts $\langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, -_t \rangle$ and $\langle A, \wedge_t, \vee_t, \wedge_i, \vee_i, -_t \rangle$ must be isomorphic to the bilattice $\mathcal{FOUR}_2$ (which is, as mentioned before, the only subdirectly irreducible distributive bilattice). It is then easy to see that there are only two algebras of this kind, namely the one in which $\leqslant_f = \leqslant_i$ and the one in which $\leqslant_f = \geqslant_i$. Therefore *the variety of distributive trilattices with t-involution is generated by its two four-element members*.

{t, f}-involutions. Now suppose $\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t, -_f \rangle$ is a subdirectly irreducible distributive trilattice with t- and f-involutions such that the two involutions commute. By the previous results we may assume that $A = L \times L$, where L is the universe of a subdirectly irreducible distributive involutive lattice $\mathbf{L}$. In other words, $\mathbf{L}$ is a subdirectly irreducible De Morgan lattice, which implies (see [10]) that L may only have either two, three or four elements. As observed at the end of Section 3.3, the smallest non-trivial trilattice with t- and f-involution operations has sixteen elements. We may then conclude that $|A| = 16$. We then have that *the variety of distributive trilattices with commuting t- and f-involutions is generated by its sixteen-element member*. Moreover, by Theorem 3.7, we know that this algebra $\mathbf{A}$ is such that $\mathbf{A} \cong \mathcal{FOUR}_2 \odot \mathcal{FOUR}_2$.

{t, f, i}-involutions. Finally, let $\mathbf{A} = \langle A, \wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i, -_t, -_f, -_i \rangle$ be a subdirectly irreducible distributive trilattice (with t-, f- and i-involutions) such that all involutions commute. By Theorem 3.9 we may assume that $A = B \times B$, where B is the universe of a commutative distributive bilattice with conflation. Then we also know that $B = L \times L$, where L is the universe of a subdirectly irreducible De Morgan lattice. Reasoning as in the previous case, we may conclude that $2 \leqslant |L| \leqslant 4$, so $|A| \in \{2^4, 3^4, 4^4\}$. Therefore *the variety of distributive trilattices with commuting t-, f- and i-involutions is generated by its three members* $\mathbf{A_1}, \mathbf{A_2}$ *and* $\mathbf{A_3}$ *such that* $|A_1| = 2^4$, $|A_2| = 3^4$ *and* $|A_3| = 4^4$.

6. Future work

As mentioned above, the present work is the first purely algebraic study devoted to trilattices and the results presented here are to be considered but preliminary. We mention some lines of research that, in our opinion, deserve further investigation:

- the formulation of the representation results stated above in terms of category theory, along the same lines of the work done in [5] for bilattices, with the aim to obtain categorical equivalences between different classes of trilattices and of lattices;
- the study of trilattices satisfying weaker interlacing conditions, for instance monotonicity of the lattice operations of just one of the three orders with respect to the other two (an analogous study has been developed, for bilattices, by Pynko [17]);
- the study of bounded interlaced trilattices, in which one could hope to obtain results similar to the ones proved by Avron [2] on bounded interlaced bilattices (for instance, that bounded interlaced trilattices are equivalent, up to algebraic signature, to bounded interlaced bilattices with some extra constants satisfying certain properties);
- the generalization of the results obtained in the previous sections to n-lattices, i.e., structures on which an arbitrary number n of lattice orders is simultaneously defined;
- the expansion of the trilattice language considered in the previous sections through the introduction of implication operations, with the possibility to extend the representation results to the new classes of algebras thus obtained, along the line of the study developed in [18,5] for bilattices with implication;
- the application of algebraic techniques to the study of trilattice logics, which appear to have strong similarities with the bilattice logics studied from an algebraic logic point of view in [18,4].

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