Can Gravitons Be Detected?

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LaTeX-ed February 4, 2008

Abstract

Freeman Dyson has questioned whether any conceivable experiment in the real universe can detect a single graviton. If not, is it meaningful to talk about gravitons as physical entities? We attempt to answer Dyson's question and find it is possible concoct an idealized thought experiment capable of detecting one graviton; however, when anything remotely resembling realistic physics is taken into account, detection becomes impossible, indicating that Dyson's conjecture is very likely true. We also point out several mistakes in the literature dealing with graviton detection and production.

PACS: .03.65.Sq, .04.30.Db, .04.30.Nk, .04.80.Cc, .04.80.Nn, 95.30.Cq
Keywords: Gravitons, Gravitational Waves, Gravitational Bremsstrahlung, Graviton-electron Cross Sections.

1 Introduction

The search for gravitational waves, one of the central predictions of general relativity, has been ongoing for several decades. Strong indirect evidence for their existence comes from the timing of the orbital decay rate of the binary pulsar PSR1913+16, and with the completion of LIGO II and other gravity-wave observatories, researchers expect a direct detection. Gravitons themselves present a thornier issue. Although physicists routinely speak as if bosons mediating the gravitational force exist, the extraordinary weakness of the gravitational interaction makes the detection of one

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gravitational quantum a remote proposition. Recently, Dyson[1] has suggested that detection of a single graviton may in fact be ruled out in the real universe. If so, is it meaningful to talk about gravitons as physical, or do they become metaphysical entities?

In this note we attempt to answer Dyson's question "in principle." For both physical and philosophical reasons the matter turns out to be not entirely trivial, and both considerations require that the rules of the game be defined at the outset. We concede at once that there appear to be no fundamental laws disallowing the detection of a graviton, and so we take the approach of designing thought experiments that might be able to detect one. Such an "experimental" approach, however, also has ambiguities; allowing an infinite amount of time for an experiment in any imaginable universe would certainly allow the detection of anything. Thus we impose a few physical restrictions: We consider only experiments using standard physics in four dimensions that could be performed in something like the age of the real universe.

One must also agree on what sort of experimental result one would accept as establishing the existence of a quantum of gravity. When one recalls that to establish the photon picture of light took a decade, if not two, one sees that this "philosophical" issue requires perhaps even more negotiation than the physical one. We discuss the matter in greater detail in §3 and §7. Moreover, in addressing Dyson's conjecture, one is inevitably led to rather subtle physics, which apart from its own intrinsic interest, has caused some confusion and errors in the literature. For pedagogical reasons, we attempt to explain the basis for our calculations as well as the others, leaving however most of the fine details to the Appendix and a more technical companion paper[2], henceforth BR.

Should Dyson's question be answered in the affirmative, that is, if one decides that detection of a single graviton is physically impossible, this immediately raises the issue of whether it is necessary to quantize gravity. There are of course theoretical reasons for wanting to do so—after all, every other fundamental field is quantized—and there has been some discussion in the literature about the consistency of gravitational theories without quantization [3, 4, 5, 6]. These discussions appear to be inconclusive and we do not enter into them. Rather, we restrict ourselves to the more limited question we have already set forth: can one experimentally detect a graviton?

To begin, we assume the most general physics and units in which $c = 1$, but $G = G$ and $\hbar = \hbar$, and write down the criterion to detect a graviton.

2 Detection Criterion

Regardless of the exact nature of the detector or interaction, we demand that to detect a single graviton with high probability, its path-length λ in the detector should exceed a mean-free- path, or

$$n\sigma\lambda \geq 1 , \tag{2.1}$$

where σ is the interaction cross section and n is the density of detector particles. We wish to maximize the left hand side of this expression. For simplicity we consider a detector of atomic hydrogen, which implies $n = M_\mathrm{d}/(m_\mathrm{p} V)$, with M_d the detector mass, V its volume and and m_p the proton mass. Assume the detector is spherical with a diameter $\ell = \lambda_{\mathrm{max}}$. Then, if a total number of gravitons N_γ is incident over the lifetime of the experiment, the criterion for detecting a single particle becomes:

$$\frac{6\,\sigma}{\pi}\,\frac{M_\mathrm{d}}{m_\mathrm{p}\ell^2}\,N_\gamma \geq 1\ . \tag{2.2}$$

N_γ can be expressed in terms of the source's graviton luminosity L_γ and distance R as

$$N_\gamma = \frac{L_\gamma}{4\pi R^2}\frac{A_\mathrm{d}\tau_\mathrm{s}}{\epsilon_\gamma}. \tag{2.3}$$

Here, $A_\mathrm{d} = \pi\ell^2/6$ is the effective cross-sectional area of the detector (i.e., the cross section averaged over possible paths), τ_s is the time the source is on and ϵ_γ is the graviton energy. Thus,

$$\frac{\sigma}{4\pi}\,\frac{M_\mathrm{d}}{m_\mathrm{p}}\,\frac{L_\gamma}{R^2}\frac{\tau_\mathrm{s}}{\epsilon_\gamma} \geq 1\ . \tag{2.4}$$

Reasonably, $\tau_\mathrm{s} \leq M_\mathrm{s}/L$ by energy conservation, where M_s is the mass of the source and L its total luminosity. We also expect that any astrophysical source emits only a small fraction of its energy as gravitons. Letting $f_\gamma \equiv L_\gamma/L$ yields

$$\frac{f_\gamma\,\sigma}{4\pi}\,\frac{M_\mathrm{d}}{m_\mathrm{p}}\,\frac{M_\mathrm{s}}{R^2}\,\frac{1}{\epsilon_\gamma} \geq 1\ . \tag{2.5}$$

Thus far this is almost completely general. We now assume that the detector should not be so large as to undergo gravitational collapse to greater than atomic densities. It is well known[7] that the requirement that gravitational forces do not overwhelm the electrostatic forces supporting the object gives a maximum detector mass of about the mass of Jupiter: $M_\mathrm{d} \sim (\alpha/\alpha_\mathrm{g})^{3/2} m_\mathrm{p}$, where $\alpha = e^2/\hbar$ and $\alpha_\mathrm{g} \equiv G m_\mathrm{p}^2/\hbar$ are the fine structure and gravitational fine structure constants, respectively. And so, finally, the criterion for detecting a graviton becomes:

$$\frac{f_\gamma\,\sigma}{4\pi}\left(\frac{\alpha}{\alpha_\mathrm{g}}\right)^{3/2}\frac{M_\mathrm{s}}{R^2}\frac{1}{\epsilon_\gamma} \geq 1\ . \tag{2.6}$$

The crux of the problem is to determine σ for a given process and f_γ.

3 Cross sections: Heuristic Arguments

Whenever dealing with quantum gravity, it is fruitful to enlist analogies with ordinary quantum mechanics and electromagnetism, and we shall do so throughout. In choosing an experiment, and hence a cross section to calculate, we also follow this stratagem and attempt to learn from history. A century ago two experiments were crucial in establishing the particle nature of light: the photoelectric effect and Compton scattering of photons by electrons. The photoelectric effect of course presented severe difficulties for classical electromagnetic theory, which predicted that the energy of the photoelectrons should increase with the intensity of the incident light, but should be independent of the light frequency, neither of which proved true. Einstein explained everything by introducing light quanta and postulating the celebrated one-liner: $E = h\nu - W$, which gave the photoelectrons' energy in terms of the frequency of the quantum and the work function, the energy needed for the electron to escape the surface of the detector. The number of photoelectrons simply became equal to the number of incident quanta. Whether the formula actually held, though, was debated for ten years as physicists plotted E versus ν to get the slope h, but in 1916 Millikan was able to announce that "Planck's h has been photoelectrically determined with a precision of about 0.5% to have the value $h = 6.57 \times 10^{-27}$"[8].

With perfect hindsight, a devil's advocate would point out that the photoelectric effect is not so conclusive. In semi-classical derivations of the photoelectric cross section (§4), such as Schiff's[9], the electromagnetic field is never quantized. Thus, although the computed cross section gives good agreement with experiment, nothing in the mathematics implies a photon. There are two answers to this rather subtle point: one is that the photoelectric effect is a fact independent of the derivation of the cross section, and it still can't be explained by classical physics. The other is that we will accept the existence of a quantum of light (or gravity) if the detector can "fire" when there are fewer than one quantum in the detector at a given instant. We make this statement more precise in §7, after we have estimated expected fluxes, but will consider a "gravito-electric" experiment in which gravitons knock electrons from a detector to be a viable candidate.

Compton scattering might be considered a more conclusive demonstration of the existence of photons, because the change in wavelength under scattering is calculated by treating the X-rays as nothing more than billiard balls with quantum-mechanical energy $E = h\nu$ and momentum $p = h/\lambda$. Thus, we can also imagine a "gravito-Compton" experiment in which a graviton is scattered off an electron and one attempts to detect the electron's recoil energy. (Detecting the scattered graviton itself, in analogy to the usual detection of Compton-scattered photons, is not kosher because it only returns the original problem of detecting the graviton.)

The gravito-Compton effect presents its own subtleties. The ordinary Compton scattering cross section is $\sigma_{\rm C} \sim (e^4/m^2)$, where e and m are the electron's charge and mass. One might naively assume by the dimensional substitution $e \rightarrow \sqrt{G}m$ that gravito-Compton scattering should have cross section $\sigma \sim G^2m^2$. Indeed, numerous authors in the 1960s and 1970s considered scattering of gravitons off neutral scalar particles. Dewitt, for example[10], gives for the nonrelativistic limit the differential scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{G^2m^2}{sin^4(\frac{\theta}{2})}\left[cos^8(\frac{\theta}{2}) + sin^8(\frac{\theta}{2})\right], \tag{3.1}$$

with an analogous relativistic limit. Perhaps to one's surprise, although the G^2m^2 dutifully appears, (3.1) resembles the Rutherford scattering cross section with its $\theta = 0$ divergence rather than the Thomson cross section $\frac{8\pi}{3}(e^4/m^2)$, which is the nonrelativistic limit of the Compton cross section. This is a hint to beware analogies between electromagnetism and gravity. To the best of our knowledge the scattering of gravitons off true fermions has not been calculated, but conversion of gravitons to photons by scattering off charged scalar particles has[11],with the result that for unpolarized incident radiation:

$$\frac{d\sigma}{d\Omega} = \frac{e^2G}{4\pi}cot^2\left(\frac{\theta}{2}\right)\left[sin^2 2\phi + cos^2\theta cos^2 2\phi\right]. \tag{3.2}$$

This expression also diverges, but is larger than (3.1) by a factor of $\alpha/\alpha_{\rm g}(m_{\rm p}/m)^2 \approx 4 \times 10^{42}$. As to why these cross sections diverge, the usual statement is that it is due to the long-range nature of the gravitational and electromagnetic force, but the different forms of the divergence or lack thereof in ordinary Compton scattering show that more detailed considerations play a role.

Because the total cross sections of (3.1) and (3.2) are formally infinite, in practice one must put in a cutoff in the θ integration. This can be accomplished by relating θ to the initial graviton and final electron energies in the usual Compton scattering formula; however certainly for (3.1) any such detail is irrelevant because $G^2m^2 \sim 10^{-110}cm^2$. In other words, $G^2m^2 \sim 10^{-45}\ell_{\rm pl}^2$, where $\ell_{\rm pl} \sim 10^{-33}cm$ is the Planck length. Regardless of what other numbers one inserts into (2.6), the detection criterion cannot be satisfied. Gravitons cannot be detected by Compton scattering of gravitons off neutral particles.

All hope is not lost, however. One might reasonably assume the enormously larger (3.2) must be the nonrelativistic result for true electrons and contemplate using it as the basis for an experiment. In fact no speculation is required. For any "gravito-atomic process," the actual cross section turns out to be, within numerical factors, the square of the Planck length, in our units $G\hbar \sim 10^{-66}cm^2$, comparable in size to (3.2) but without divergences. To get some idea of how this comes about, consider an idealized gravitational wave detector, consisting of two balls on a spring that behave as damped harmonic oscillators under the passage of a gravitational wave:

$$\ddot{\xi} + \dot{\xi}/\tau_{\rm o} + \omega_{\rm o}^2\xi = acceleration, \tag{3.3}$$

where ξ is the displacement of the balls. The basic definition of differential cross section is:

$$d\sigma = \frac{P(ergs/s)}{I(ergs/s/cm^2)}. \tag{3.4}$$

By computing via (3.3) the power absorbed in the detector, P, and dividing it by the incoming flux $I \sim h^2\omega^2/G$ of the gravitational wave (see §5), is not difficult to show[12] that near resonance ($|\omega \pm \omega_o| << \omega_o$) the cross section is given by the Lorentzian

$$\sigma = \frac{\pi GML^2(\omega_o^2/\tau_o)sin^4\theta}{(|\omega| - \omega_o)^2 + (1/2\tau_o)^2} \,. \tag{3.5}$$

The mean cross section is just $\frac{1}{\omega_o}\int \sigma d\omega$, which gives $\sigma_{avg} \sim GML^2\omega_o$. If we now assume that $L = a = \hbar^2/me^2$, the Bohr radius, $M = m$, the electron mass, and impose the Nicholson-Bohr quantization condition $m\omega a^2 = \hbar$, then we immediately have $\sigma_{avg} \sim G\hbar$, the square of the Planck length. Notice that the "atomic process" enters only in determining the size of a; if the atom were bound by gravity, the Nicholson-Bohr condition would still apply, but but a would be about 10^{39} times larger and ω about 10^{78} times smaller.

Consequently, the "Planck-length squared" cross section stems directly from the imposition of angular momentum quantization onto the detector and nothing more. This result has the immediate consequence that it is physically impossible to detect a single given graviton with high probability. Detection criterion (2.1) can now be written approximately as

$$\frac{M}{m_p\ell^3}\,\ell_{pl}^2\,\ell = \frac{M}{\ell}\,\frac{\ell_{pl}^2}{m_p\ell} \geq 1$$

for detector mass M and size ℓ. However, $M/\ell < 1$ for any object that is not a black hole. If we take $\ell_{pl} = 1$, then $m_p = 10^{-19}$. The smallest conceivable detector is presumably about the size of a proton, or $\ell \sim 10^{20}$, making the inequality impossible to satisfy. Smolin [6] has argued that this conclusion holds for any physical detector and that therefore it is impossible for gravitational radiation to be in thermodynamic equilibrium with it surroundings.

However, this still leaves open the possibility that given a large flux of gravitons, a suitable detector might be able to detect one or more of them. Therefore, heartened by a non-divergent cross section that is forty some orders of magnitude larger than it could have been, we proceed to more detailed considerations.

4 Matrix Elements

The near universal prescription for the quantum mechanical calculation of cross sections is to select an interaction Hamiltonian between the particle and field, then employ perturbation theory to calculate the transition probability P_{ab} between individual states:

$$\begin{aligned} P_{ab} &\propto \frac{2\pi}{\hbar}|<b|H|a>|^2 \\ &\equiv \frac{2\pi}{\hbar}|\int \Psi_b^* H \Psi_a d^3x|^2, \end{aligned} \tag{4.1}$$

where Ψ_a and Ψ_b are the initial and final wavefunctions. Multiplying by the density of final states $\rho = dn/dE$ gives the famous golden rule and the total transition rate

$$\Gamma = \frac{2\pi}{\hbar}| < a|H|b > |^2 \rho \,. \tag{4.2}$$

The differential cross section $d\sigma$ for the process under consideration then follows immediately from the fundamental definition (3.4) as the ratio between $\hbar\omega\Gamma$ and the incoming flux, I. We will follow this prescription, but for gravity it turns out that the determination of the Hamiltonian and consequently Γ does involve subtleties, and we will try to clarify them. Doing so not only proves necessary to get the correct σ but presents a few surprises.

Consider first the calculation of the matrix element for the electromagnetic field via the standard semi-classical prescription. In this approach one begins by writing down the classical interaction Hamiltonian $H = -e/m\, \mathbf{A}\cdot\mathbf{p}$, where $\mathbf{A}$ is the vector potential and $\mathbf{p}$ is the electron momentum. To compute $| < b|H|a > |$ textbooks discuss replacing $\mathbf{p}$ by the momentum operator $-i\hbar\nabla$, but in fact what is usually done is to apply the correspondence principle in reverse: $-i\hbar\nabla \to \mathbf{p} \to md\mathbf{r}/dt$, by which point both $\hbar$ and quantization of the field have dropped out of the problem. Additionally, $\mathbf{r}$ in the "energy representation" is taken to have time dependence $e^{i\omega_{ab}t}$, where $\hbar\omega_{ab}$ is the energy difference between two levels a and b, implying that $\mathbf{p} = im\omega_{ab}\mathbf{r}$. One moreover assumes that the field itself is varying sinusoidally, i.e., $\mathbf{A} = \mathbf{A_o} e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}$. The time-independent part of the matrix element thus becomes $A_o e\omega_{ab}| < b|e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{r}|a > |$. Usually, the wavelength in the problem is much larger than atomic dimensions, allowing one to make the "dipole approximation," $e^{i\mathbf{k}\cdot\mathbf{r}} = 1$. And so, the matrix element one actually computes is simply $| < b|H|a > | = A_o e\omega_{ab}| < b|\mathbf{r}|a > |$.

For the gravitational case, we copy the procedure, considering the interaction of an electron with a gravitational wave. A plane gravitational wave can be viewed as a weak perturbation $h_{\mu\nu}$ traveling on a flat background $\eta_{\mu\nu}$, such that the full spacetime metric becomes:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,.$$

In analogy to the electromagnetic case, we take for a monochromatic gravitational plane wave $h_{\mu\nu} = he_{\mu\nu}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}$, where $e_{\mu\nu}$ is the polarization tensor. Because the perturbations are assumed weak, one takes the amplitude $h << 1$.

To illustrate how to get an interaction Hamiltonian, it is perhaps easiest to proceed as follows: Recall that in special relativity, the Minkowski metric can be written $d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu = dt^2(1-v^2)$. This leads to the free-particle Lagrangian $L = -m\sqrt{1-v^2}$. For an interaction with a gravitational wave we replace $\eta_{\mu\nu}$ by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ to get

$$L = -m\sqrt{1 - v_i v^i - h_{\mu\nu}v^\mu v^\nu} \,, \tag{4.3}$$

Here, $v^\mu \equiv dx^\mu/dt$; we use the summation convention throughout and Latin indices refer to spatial components. Although this Lagrangian is not manifestly covariant, it serves for our purposes.

Taking $\partial L/\partial v^\alpha$ yields by definition the canonical momentum π_α. In the nonrelativistic limit $v << 1$. To lowest order in v the canonical momentum becomes:

$$\pi_\alpha = p_i \delta^i_\alpha + h_{\alpha\beta} p^\beta,$$

where δ^i_α is the Kronecker delta and $p_\alpha = m v_\alpha$ is the ordinary momentum. By definition, the Hamiltonian is $H = \pi_\alpha v^\alpha - L$. Working again to lowest order gives:

$$H = \frac{p_i p^i}{2m} + \frac{1}{2} \frac{h_{\alpha\beta} p^\alpha p^\beta}{m}.$$

(Here we have ignored a physically irrelevant constant m.) If we define $T^{\mu\nu} \equiv m v^\mu v^\nu$ to be the classical stress-energy tensor for an electron (integrated over volume) the interaction Hamiltonian is then

$$H_{int} = \frac{1}{2} h_{\mu\nu} T^{\mu\nu}, \tag{4.4}$$

and we see that $T^{\mu\nu}$ plays the same role as $\mathbf{p}$ did in the in the electromagnetic case, while $h_{\mu\nu}$ corresponds to $\mathbf{A}$. This derivation agrees with the more general result found in [13, 14].

Now, the polarization tensor $e_{\mu\nu}$ has only two independent components. For the moment let us make the standard choice $e_{11} = -e_{22}$ and $e_{12} = e_{21}$. If we again make the dipole approximation, then sandwiching H_{int} between the initial and final state functions gives the required matrix element $1/2 < b|e^{ij} p_i p_j |a >$. To evaluate this expression exactly, one should let $p \to -ih\partial/\partial x$. However, if following the semi-classical procedure, we set $p = m\dot{x}$, then we have $m^2 < b|e^{ij} \dot{x}_i \dot{x}_j |a >$, which dimensionally is $m^2\omega^2 < b|e^{ij} x_i x_j |a >$. Indeed, for the harmonic oscillator wavefunction, the expectation value $m^2\omega^2 < a|x^2|a >$ is identically equal to $< a|p^2|a >$, and one can also show that the matrix element $m^2 < b|\dot{x}_i \dot{x}_j |a >= m^2\omega^2 < b|x_i x_j |a >$ for any case in which all the transition frequencies ω are the same. Thus we expect:

$$\begin{aligned} < b|H|a > &\approx \frac{m h \omega^2}{2} \int \Psi_b^* e^{ij} x_i x_j \Psi_a d^3x \\ &= \frac{m h \omega^2}{2} e^{ij} D_{ij}, \end{aligned} \tag{4.5}$$

where

$$D_{ij} \equiv \int \Psi_b x_i x_j \Psi_a d^3x. \tag{4.6}$$

In principle, all we need to do now to get Γ is insert the relevant wavefunctions into D_{ij} and calculate $| < b|H|a > |^2$. The matter is not so simple, however. First, for any absorption or scattering process the graviton can be incident from any direction.

Therefore, we need to square (4.5) for each polarization, average the results over the unit sphere and sum them. To do this, take $|e_{ij}| = 1/\sqrt{2}$. Then,

$$< |D_{ij}|^2 >\equiv \frac{1}{4\pi}\int d\Omega \left[2|\Psi_b xy\Psi_a|^2 + \frac{1}{2}|\Psi_b(x^2 - y^2)\Psi_a|^2 \right].$$

In BR, we give a quick way to evaluate this expression. The result is:

$$< |D_{ij}|^2 >= \frac{2}{5}(D_{ij}D_{ij} - \frac{1}{3}|TrD|^2). \qquad (4.7)$$

In what follows, we will thus assume that

$$| < b|H|a > |^2 = \frac{1}{4}\frac{m^2h^2\omega^4}{4} < |D_{ij}|^2 >, \qquad (4.8)$$

where $< |D_{ij}|^2 >$ is given by Eq. (4.7). The extra 1/4 in this equation gives an exact expression in a sense explained in the Appendix. It arises, in part, when demanding that detailed balance be satisfied among the absorption, spontaneous and stimulated emission rates. The appearance of this factor also helps resolve some discrepancies in the literature and leads to some fundamental issues regarding gauge transformations in general relativity. Because these considerations are somewhat more technical than the general discussion and do not seriously affect the numerical result, we leave greater detail to the Appendix and to BR. Here, we proceed to calculate a cross section.

5 Ionization cross section

Despite our devil's advocate insistence that nothing in the previous discussion implied the existence of quanta, we again defer comment until §7 and choose to calculate the gravito-electric (ionization) cross section of atomic hydrogen. Although some metals have work functions that are somewhat smaller than hydrogen's, hydrogen's is typical and the element does compose most of the universe; for this reason we chose a hydrogen detector in §2. It turns out that in the energy regime of greatest interest, the result is not sensitive to the work function.

As in the standard calculation of the photoelectric cross section[9], we take the initial state to be the ground (1s) state of hydrogen and for the final state a plane wave with momentum **k**:

$$\Psi_a = \frac{1}{\sqrt{\pi}a^{3/2}}e^{-r/a} \quad ; \quad \Psi_b = \frac{1}{L^{3/2}}e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad (5.1)$$

where L is the normalization constant. The use of the plane wave final state (the Born approximation) is reasonably good for large final momentum, or $a^2k^2 >> 1$, which is again the case of most interest.

Before one computes the matrix element with these wavefunctions, however, there is a complication. As in the ordinary photoelectric effect (see Schiff, p. 287), one should average the direction of the incident graviton over all angles. This average, though, is exactly the one we have already performed, leading to Eq. (4.7), and therefore, in evaluating the matrix elements expression (4.8) should be used for $|<b|H|a>|^2$.

This actually simplifies matters greatly, because with the gravitational field averaged over all directions, we may assume that the electron is ejected along the z-axis and that θ, the angle between $\mathbf{k}$ and $\mathbf{r}$, is the usual polar angle. Once again the integrals can all be carried out analytically in terms of elementary functions and the result is:

$$|<b|H|a>|^2 = \frac{3(32)^2\pi}{5}\frac{h^2\omega^4 m^2 a^7(a^4k^4)}{L^3(1+a^2k^2)^8} \tag{5.2}$$

The total absorption rate Γ follows directly from the golden rule (4.2). With the standard expression for the density of states $\rho = L^3/(2\pi\hbar)^3 km d\Omega$, this gives

$$\Gamma = \frac{3(32)^2}{4\cdot 5\pi}\frac{h^2\omega^4 m^3 a^{11}k^5}{\hbar^3(1+a^2k^2)^8}d\Omega \tag{5.3}$$

The differential cross section now follows from (3.4) as $d\sigma = \hbar\omega\Gamma/I$. For a gravitational plane wave, the equivalent of the Poynting flux is[12] $I = \omega^2h^2 e_{\rm ij}e^{\rm ij}/16\pi G = \omega^2h^2/8\pi G$, yielding

$$d\sigma = \frac{6(32)^2}{5}\frac{G\omega^3 m^3 a^{11}k^5}{\hbar^2(1+a^2k^2)^8}d\Omega \tag{5.4}$$

But the photoelectric equation requires $E = (\hbar k)^2/2m = \hbar\omega - e^2/2a$, where $e^2/2a$ is the hydrogen work function. Hence, $1 + k^2a^2 = 2m\omega a^2/\hbar$. Inserting this into the above and integrating yields, finally, the total cross section

$$\sigma = \frac{3072\,\pi}{5}\frac{\hbar G(ak)^5}{(1+a^2k^2)^5}\,. \tag{5.5}$$

As promised, $\sigma \sim \hbar G$ to "within factors of order unity," but is strongly dependent on ak. (Indeed, for $ak >> 1$, the dependence is the same as in the ordinary photoelectric effect.) We attempt to evaluate this factor in the following section.

6 Graviton Production

The second half of the task is to determine the graviton luminosity of astrophysical sources, that is estimate f_γ in Eq.(2.6).

There are only a few conceivable sources of gravitons: spontaneous emission of gravitons from neutral hydrogen, black hole decay, bremsstrahlung from electron-electron collisions in stellar interiors and conversion of photons to gravitons by interstellar magnetic fields. We examine each in turn.

Spontaneous emission of gravitons, such as that from the $3d \rightarrow 1s$ state of hydrogen discussed in the Appendix, will not produce gravitons energetic enough to ionize hydrogen. One can, alternatively, imagine a gravito-Compton experiment with the cross section $\sigma \sim e^2 G$ from §3. We leave it to the reader to show that, with the calculated decay rate $\Gamma \approx 5 \times 10^{-40} s^{-1}$, even if one assumes that the entire mass of the galaxy, $M_{\rm g} \approx 10^{50} M_{\rm pl}$ resides in hydrogen in the $3d$ state at an average distance of 30 kiloparsecs, $R \approx 10^{56} \ell_{\rm pl}$, the detection criterion falls orders of magnitude short of being satisfied. Detection of gravitons from the spontaneous emission of hydrogen is impossible.

Graviton production by black hole decay is in some sense more promising. The Hawking temperature of black holes is $T_{\rm bh} \sim \hbar/kGM$. To evaporate $10eV$ or higher energy gravitons requires black holes of $M \leq 10^{27} M_{\rm pl}$ or $M \leq 10^{22} g$. As this is far less than a stellar mass, such black holes would necessarily be primordial. Although theoretical prejudice may be aligned against primordial black holes, observational constraints on this mass range of PBHs are almost nonexistent[17] and in principle they could make up most of the universe's dark matter. On the other hand, PBHs with $M \leq 10^{15} g$, which would emit particles with energy $\geq 10^8 eV$, would have evaporated by the current age of the universe. Constraints on such PBHs are much tighter due to the distortion they produce on the X-ray background[17]. Let us then assume PBHs with $10^{15} g < M < 10^{22} g$ make up the entire mass of the galaxy.

Detection criterion (2.6) requires to an order of magnitude

$$10^{53} f_\gamma \, \sigma \frac{M_{\rm s}}{R^2} \frac{1}{\epsilon_\gamma} \geq 1. \tag{6.1}$$

To determine f_γ over the range of interest, note that the temperature of a $10^{22} g$ black hole is too low to evaporate anything but massless particles: neutrinos, photons and gravitons. By equipartition one expects approximately equal numbers of these species to be radiated, but the higher spin of the graviton suppresses its emission compared to the neutrino and photon. Page's detailed calculations[18] have shown in this case that $f_\gamma = .02$. For PBHs with temperature above the electron emission threshold, f_γ drops only to .01, so it can be considered a constant $\approx .01$ over the entire mass range.

To estimate $M\sigma/\epsilon_\gamma$, first note that from the photoelectric equation we have $a^2 k^2 = (\omega/\omega_{\rm Ry}) - 1$, where the Rydberg frequency $\omega_{\rm Ry} \equiv \hbar/(2ma^2)$. Thus $ak \approx (\epsilon_\gamma)^{1/2}$ when the graviton energy ϵ_γ is in Rydbergs. The mass spectrum is expected to be dominated by the smaller holes[17] of $M \sim 10^{15} g$, which emit gravitons at about $10^7 Ry$ or $ak \sim 10^{3.5}$. If we assume all the holes are in this mass range, the $(ak)^5$ in the denominator of the ionization cross section reduces it to only $\sigma \sim 10^{-14} \hbar G$. The high graviton energy $\epsilon_\gamma \sim 10^7 Ry \sim 10^{-20} \epsilon_{\rm pl}$ is similarly unfavorable in criterion (6.1). If we again take $M_{\rm g} \approx 10^{50} M_{\rm pl}$ and $R \approx 10^{56} \ell_{\rm pl}$, we find that the number of detections over the source lifetime for a Jupiter mass detector is

$$N_{\rm d} \sim 10^{-5}. \tag{6.2}$$

Properly one should average $(M\sigma/\epsilon_\gamma)$ over the mass spectrum of PBHs, or equivalently the lifetime of the source τ_s in Eq. (2.4), but the estimate we've just done should be adequate. Larger holes give a larger cross section, but their lifetime $\tau \sim M^3$ is so much enormously larger than the age of the universe, that the detections per age of universe is negligible. We should also mention that our ionization cross section is not valid in the relativistic regime, but a proper calculation may only make things worse; if so this result does indicate that even if primordial black holes exist, they do not produce a sufficient quantity of gravitons to be detected.

A more definite astrophysical source of gravitons is gravitational bremsstrahlung produced by electron-ion or electron-electron collisions in stellar interiors. Gravitational bremsstrahlung has been studied by numerous authors (see Weinberg[14] and Gould[19]). It is calculated by much the same procedure as in electrodynamics, however there the radiation is primarily dipolar, whereas as discussed in the Appendix, gravitational radiation is quadrupolar. In this case one calculates radiated gravitational energy in analogy to electromagnetic quadrupole radiation, by taking the the square of the third time derivative of the moment of inertia tensor, $D_{ij}(t) = \mu x_i(t) x_j(t)$, for reduced mass μ. In the zero-frequency limit, one finds that the energy radiated per unit frequency is:

$$\frac{dE}{d\omega} = \frac{8G}{5\pi}\mu^2 v^4 sin^2\theta, \tag{6.3}$$

where θ is the scattering angle of the electron.

If two scatterers with charges e_a and e_b undergo Coulomb scattering with impact parameter b, the deflection angle will be $\theta \approx e_a e_b/(b\mu v^2)$. The maximum frequency of emitted radiation will be $\omega_{max} \sim v/b \sim \mu v^3\theta/(e_a e_b)$. The important point is that the frequency of emitted radiation depends on θ. Integrating the above to ω_{max} yields for small angles

$$E \approx \frac{8G}{5\pi}\frac{\mu^3 v^7 sin^3\theta}{e_a e_b}. \tag{6.4}$$

The gravitational power radiated per unit volume is found simply by multiplying this expression by the number of collisions per unit time and volume, or

$$P = v\sum_{a,b} n_a n_b \int E\frac{d\sigma_r}{d\Omega}d\Omega,$$

where n_e and n_b are the number density of scattering species. In the nonrelativistic regime $d\sigma_r/d\Omega$ can be taken to be the differential Rutherford scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{e_a^2 e_b^2}{4v^4\mu^2 sin^4\theta/2}.$$

For a Maxwellian velocity distribution the power per unit volume works out to be

$$P \approx \frac{64\pi G}{5}\frac{15(kT)^2}{\mu}\sum_{a,b} n_a n_b e_a e_b, \tag{6.5}$$

Multiplying P by the volume of the star gives the power radiated in gravitational bremsstrahlung. We note that this formula differs somewhat from Weinberg's Eq. (10.4.31), which contains a lower-limit θ cutoff, $\ln\Lambda$, due to the divergence in the Rutherford cross section. His expression, however, neglects the fact that the upper limit of integration for ω depends on θ (and vice versa). When carried out as we have, the $\ln\Lambda$ does not appear.

In any case, we present this derivation primarily for illustrative purposes because Eq. (6.3) is only valid for low-energy electrons. Stellar interiors, where $T > 10^7 K$ ($kT > KeV$), require the opposite extreme; and it is in this limit where the Born approximation used in the calculation of the ionization cross section is valid. Because Gould has treated in some detail the astrophysical regime, we here merely quote his results. He finds that for the Sun, $L_{\gamma *} \sim 7.9 \times 10^{14}\, ergs\, s^{-1}$ implying that $f_{\gamma *} \sim 2 \times 10^{-19}$. For white dwarfs he finds $L_{\gamma \mathrm{wd}} \sim 10^4 L_{\gamma *}$, but for a typical white dwarf, $L_{\mathrm{wd}} \sim 10^{-2} L_*$, implying that $f_{\gamma \mathrm{wd}} \sim 10^{-13}$. For neutron stars he gets $L_{\gamma \mathrm{ns}} \sim 10^{25}\, ergs\, s^{-1}$. The total thermal luminosity of neutrons stars is less well determined, because much of the luminosity is due to rotational energy. Recent observations[20], however, give a thermal luminosity $L_{\mathrm{ns}} \sim 10^{31-34} ergs\, s^{-1}$, implying $f_{\gamma \mathrm{ns}} \sim 10^{-8}$, very roughly.

The number of graviton detections from bremsstrahlung is now easily estimated. If $T \sim 1KeV$, then $ak \sim 10$, which gives for the ionization cross section $\sigma \sim .01\hbar G$. With M and R as before, and making the assumption that all the mass of the galaxy resides in Main Sequence stars, Eq. (6.1) yields

$$N_\gamma \sim 10^{-5}, \tag{6.6}$$

ruling out detection of gravitons, but if a substantial fraction of the galaxy resides in white dwarfs or hot neutron stars (for the latter: $T \sim 10^9; ak \sim 100$),

$$N_{\gamma \mathrm{wd}} \sim 10; \;\; N_{\gamma \mathrm{ns}} \sim .1, \tag{6.7}$$

giving a faint hope. Unfortunately, the uncertainties in the various quantities preclude more precise answers. In this regard we point out that in this calculation and the previous, the numbers are not significantly changed by going to the mass and radius of the observable universe, due to the inverse square law; this is a manifestation of Olber's paradox. However, one can do better by merely parking the detector at 1 AU from the source, which yields

$$N_{\gamma *} \sim 10^3; \quad N_{\gamma \mathrm{wd}} \sim 10^9; \quad N_{\gamma \mathrm{ns}} \sim 10^7. \tag{6.8}$$

At the tidal disruption radius $r_{\mathrm{T}} \approx (M_*/M_{\mathrm{J}})^{1/3} R_{\mathrm{J}} \approx 30 R_{\mathrm{J}}$, these numbers can be pushed to

$$N_{\gamma *} \sim 10^7; \quad N_{\gamma \mathrm{wd}} \sim 10^{13}; \quad N_{\gamma \mathrm{ns}} \sim 10^{11} \tag{6.9}$$

For a neutron star thermal lifetime of roughly 10^5 years[20], the last figure gives potentially 10^6 graviton detections per year.

All these numbers are, however, somewhat optimistic. First of all, current estimates of the number of neutron stars in the galaxy is $\sim 10^9$, for a mass fraction $\sim 10^{-3}$. Furthermore, detection criterion (6.1) assumed that the upper limit for the lifetime of a source was $\tau = M/L$. In real life, for a main sequence star $\tau \sim 10^{-3}M/L$; for white dwarfs $\tau < 10^{-5}M/L$; and for neutron stars $\tau_{\rm th} < 10^{-8}M/L$. Thus, the white dwarf result in (6.7), as well as the MS and NS results of Eq. (6.8) are also ruled out absolutely. At this stage, the only possibility for detecting gravitational bremsstrahlung appears to be putting the Jupiter-mass detector in close orbit around a white dwarf or neutron star; the latter might result in as many as 10^{-2} detections per year.

A surprising, but perhaps significant, potential source for gravitons is photon-graviton oscillations due to passage of photons through the galactic magnetic field. This mechanism, analogous to neutrino oscillations, was first discussed by Gertsenshtein[21]. If $T_{\mu\nu}$ in interaction Hamiltonian (4.4) contains stress-energy due to a background magnetic field **B**, as well as to the magnetic field **b** of the photon, then the linearized wave equation for the gravitational wave $h_{\mu\nu}$ contains both these fields in the source term. Moreover, $h_{\mu\nu}$ must also satisfy Maxwell's equations in a gravitational field, which couples the photon magnetic field to the gravitational wave. Simultaneous solution of these equations leads to a wave-packet that oscillates between the photon and graviton states with mixing length

$$L = (2/G^{1/2}B), \tag{6.10}$$

independent of wavelength. (Here B is the component of the magnetic field perpendicular to the direction of propagation.) For B on the order of a Gauss, L comes out to be roughly a megaparsec. Furthermore, the oscillation period shows that if a single photon travels a distance D through the field B, it will emerge as a graviton with probability

$$P = \sin^2(D/L) = \sin^2(G^{1/2}BD/2). \tag{6.11}$$

Taking the magnetic field of the galaxy to be $B \approx 5\mu G$ with D the distance to the galactic center, yields $P \sim 10^{-15}$. This is equivalent to $f_\gamma \sim 10^{-15}$ in the previous calculations. For UV photons with $\epsilon_\gamma \approx 2\,Ry$ the detection criterion gives

$$N_\gamma \sim 10^5. \tag{6.12}$$

The situation improves if one places the detector 1 AU from a neutron star with a magnetic field $B \sim 10^{12}$. We take $D \sim 10km$. Then P $\sim 10^{-14}$ and the detection criterion gives

$$N_\gamma \sim 10^8, \tag{6.13}$$

assuming $\epsilon_\gamma \sim 1KeV$ (which may be somewhat high for an estimate of neutron star surface temperature). If one put the detector at the tidal disruption radius, the number of detections could be raised to $N_\gamma \sim 10^{12}$. Taking into account factor of 10^{-8} already mentioned, we might expect 10^{-1} detections per year for the Jupiter-mass detector.

In reality, however, the outlook for the Gertsenshtein mechanism is less positive. Large magnetic fields will result in electron-positron pair production, which produces an effective index of refraction that lowers the speed of light. This in turn introduces a coherence length above which the gravitons and photons will no longer be in phase; the Gertsenshtein process is quenched. Dyson has recently shown[22] that the quench condition is $DB^2\omega \leq 10^{43}$, implying for neutron stars that $\omega < 10^{13}$. This is below the ionization threshold for hydrogen; neither is it of great astrophysical interest.[1] Evidently, neutron star gravitons produced by the Gertsenshtein mechanism cannot be detected.

7 Discussion

We are now in a position to address the issue that we have so far avoided: given that the calculation of the cross section did not involve quantizing the field, what is our justification for claiming that a click in the above detector constitutes detection of a graviton? To see the answer, consider the upcoming generation of gravitational wave detectors, which are expected to be able to measure a wave amplitude $h \sim 10^{-20}$ at frequencies of approximately $10^3 Hz$. From the expression for the flux I above, this corresponds to roughly $10^7 eV\, cm^{-3}$ or about $10^{26} eV$ per cubic wavelength. At an energy $\hbar\omega$ per graviton this amounts to approximately 10^{38} gravitons per cubic wavelength. In other words, LIGO and its successors will never be said to have detected anything like a single graviton. The picture just described is one of a classical wave, which is what LIGO will have detected, as designed. On the other hand, the fluxes for the sources we have been considering amount to $\leq 10^{-18} eV$ per cubic wavelength, far smaller than the $10\, eV$ imparted to an electron in an ordinary photoelectric experiment. Such a situation cannot be reconciled with a classical picture of a wave, because there is insufficient energy at a given location to ever eject an electron. Hence, if the ejected electron is registered, we are entitled to claim we have detected a single graviton.

This last "if," however, is a large one. Throughout we have assumed an ideal detector, of one hundred-percent efficiency, the mass of Jupiter. This is not reasonable. One must therefore examine the physics of the detector itself. For an ejected electron to be registered by a sensor, the sensor should be located within a mean-free-path of the ejection point. If the mean-free-path λ for an electron in this case is determined by the ordinary Compton scattering cross section $\sim 10^{-24} cm^2$, the density is

[1] We have recently learned that Zel'dovich arrived at a similar conclusion regarding the quenching of the Gertsenshtein mechanism in 1973. See Ya. B. Zel'dovich, Sov. Phys. JETP **38**, 652-653 (1974).

$\sim 1gcm^{-3}$ and the temperature $T \sim 10K$, then $\lambda \sim 1m$, meaning that a fraction $\lambda/R_{\rm J} \sim 10^{-6}$ of all ejected electrons reach the surface. This essentially rules out any of the above scenarios that were not ruled out by other factors.

One does not wish to compress the detector, because the mean-free-path scales as ℓ^3, but it does suggest building detectors with $\ell < \lambda$. As one example, imagine a honeycomb structure made of silicon, whose interstices are filled with hydrogen. Both the silicon and hydrogen would become targets (with presumably comparable cross sections) and the question becomes, How large can one make such a detector before self gravity causes structural failure? A straightforward calculation shows that the maximum radius of such a detector is give by

$$R = \sqrt{\frac{3\epsilon {v_{\rm s}}^2}{2\pi\rho G}} \tag{7.1}$$

where ϵ is the structural strength as a fraction of the bulk modulus, $v_{\rm s}$ the velocity of sound, and ρ the density of the material. Suppose $\rho = 0.233\, g\, cm^{-3}$ (i.e., 0.1 the density of silicon), $\epsilon \sim 0.03$ and $v_{\rm s} = 4 \times 10^6 cm\, s^{-1}$. This yields $R \sim 4 \times 10^8\, cm$ with a mass of $M \sim 5 \times 10^{25} g$, about $.01 M_{\rm Earth}$. With such a device the detection criterion cannot be met for any of the sources discussed above and detection of gravitons is ruled out.

This result, however, does not *absolutely* exclude detection of gravitons; one can imagine filling the solar system and beyond with tiny detectors. At this point, though, the possibilities go out of sight.

Before that point, we must address two other issues. The first is noise. Any detector needs to be shielded against background noise. Two serious noise sources are neutrinos and cosmic rays. The cross section for the interaction of neutrinos with matter is about $10^{-45} cm^2$, or at least twenty orders of magnitude greater than the gravito-electric cross section. In a typical white dwarf, neutrino emission exceeds photon emission[23], meaning that 10^{13-14} neutrinos are emitted for every graviton. Therefore, without shielding, one would expect 10^{33-34} neutrino events for every graviton event. A shield should be thicker than the mean-free-path for neutrinos, which for materials of ordinary density amounts to light years. Such a shield would collapse into a black hole. Unless one can find another way to discriminate against neutrinos, this appears to make detection of thermal gravitons impossible. In light of this result, we do not pursue shielding against cosmic rays, which would activate the detector material, inundating it with secondary particles.

The second issue we have ignored brings us back to the philosophical side. We have assumed that a click in the detector amounts to a detection of a graviton. Historically, as mentioned in §3, Millikan's "proof" of the existence of photons rested on measuring the slope of the graph of the photoelectrons' energy versus frequency, i.e., the determination of h. If one insists that the existence of gravitons is not established until h is fished out of the data, then further obstacles immediately present themselves. At the very least, one requires many more detections, in order to plot gravito-electron energy vs. frequency. Unfortunately, in contrast to Millikan's situation, we here do not have a monochromatic graviton source. To deconvolve the signal from the source spectrum presents additional hurdles.

In sum, we can say that to detect a single graviton was *a priori* going to be a difficult proposition, but it was not obvious that it was fundamentally impossible. Although, as we stated at the outset, we have found no basic principle ruling out graviton detection, reasonable physics appears to do so. Perhaps the most interesting aspect of the investigation is that it leads to some fairly subtle physics, which, as the Appendix shows, has caused significant confusion in the literature. Certainly, if a "no graviton" law appears elusive, we do feel entitled to predict that no one will ever detect one in our universe.

Acknowledgements. We are grateful to Freeman Dyson, not only for asking the question that prompted this paper, but for numerous conversations, enthusiasm and for generously making available unpublished notes. He has effectively been an invisible third author, although he should not be held responsible for errors on our part. We also thank the physics department at Princeton University where much of this work was carried out.

Appendix: Spontaneous Emission and Hamiltonians

The extra factor of one-quarter in the squared matrix element (4.7) surfaces during the resolution of a matter concerning the spontaneous emission of gravitons by hydrogen. Specifically, Steven Weinberg in his standard text *Gravitation and Cosmology*[14] employs semi-classical methods to compute the spontaneous decay rate from the $3d2 \rightarrow 1s$ state of hydrogen by graviton emission and arrives at one answer, whereas the authors of the well-known *Problem Book in General Relativity and Gravitation*[15] use a field theoretic approach for the same problem and arrive at an answer 2.1×10^5 times larger. This is a significant discrepancy, particularly in that for the electromagnetic case, the field-theoretic and semi-classical approaches are known to give identical results. Although spontaneous emission did not directly figure in the body of the paper, it does provide an important check for our calculations because transition rates must necessarily satisfy detailed balance, i.e. conservation of energy. Resolving the discrepancy also leads to some fundamental issues about the Hamiltonian employed in computing the matrix element.

Consider, then, the spontaneous emission of a graviton by the decay of a hydrogen atom from the $3d$ to the $1s$ state (which one might conceivably contemplate as a graviton source). In the semi-classical approach to conventional quantum mechanics[9], one does not use a Hamiltonian to compute the spontaneous emission rate. (If the photon or graviton does not exist before emission, it is unclear what the interaction is.) Indeed, there is no classical analog to spontaneous photon emission, but in the semi-classical approach one nevertheless constructs one by imagining a current flowing from the upper state to the lower. One assumes the current density is $\mathbf{J} = \rho\mathbf{v} = \rho\mathbf{p}/m$. Next one identifies ρ with the probability density $e\Psi^*\Psi$ taken over the initial and final states, such that $\mathbf{J} = e/m\Psi_b^*\mathbf{p}\Psi_a$. As in §4, one assumes that $\mathbf{p} = m\mathbf{v} = im\omega\mathbf{r}$. Then $\mathbf{J} = ie\omega\Psi_b^*\mathbf{r}\Psi_a$, and quantization of the field is avoided. In classical electromagnetism, the total power radiated by a dipole is $\frac{4k^2}{3}|I_o|^2$, where I_o is the total current, so from the spontaneous transition we have for the total radiated power:

$$P = \frac{4k^2e^2\omega^2}{3}\left|\int \Psi_b^* x_i \Psi_a d^3x\right|^2. \tag{A.1}$$

If we now say that each transition emits a quantum with $E = \hbar\omega$, where ω is the transition frequency between the two states, we can reinterpret (A.1) as the transition rate

$$\Gamma = \frac{P}{\hbar\omega}. \tag{A.2}$$

There is no reason to believe this derivation, except that when the stimulated emission and absorption are independently computed, the imposition of detailed balance leads to the Planck formula and, miraculously, a proper field theoretic calculation also leads to the same result.

Thus as in §4 we again copy the procedure for gravity, with one important difference. The power emitted by a simple electric dipole is $P = (2/3)\ddot{d}^2$, where $d = ex$ is the dipole moment. By analogy, one would expect a gravitational dipole to emit $P \sim (m\ddot{\mathbf{x}})^2$. Due to conservation of momentum, however, $m\ddot{\mathbf{x}} = \dot{\mathbf{p}} \equiv 0$ for an isolated system and gravitational dipole radiation does not exist. The lowest radiating moment is the quadrupole, and like an electromagnetic quadrupole the radiated power goes as $\dddot{Q}^2$. With the substitution $Q \sim ex^2 \rightarrow \sqrt{G}mx^2$, for an x that varies harmonically in time we have $P \sim Gm^2\omega^6x^4$.

More precisely, when the source dimension is much smaller than a wavelength (the dipole approximation) one finds[14] that the emitted power is:

$$P = \frac{2G\omega^6m^2}{5}\left[D_{ij}^* D_{ij} - \frac{1}{3}|TrD|^2\right], \tag{A.3}$$

where the moments are exactly those defined by Eq. (4.6) and 2/5 times the bracketed quantity is exactly the average given by Eq. (4.7). Eq. (A.3) is essentially the square of the third time derivative of the moment of inertia; the spatial average results in the subtraction of $\frac{1}{3}|TrD|^2$. The important point is that all the matrix elements are now seen to be the same.

The spontaneous emission rate is now found by plugging (A.3) into (A.2). For the $3d2 \rightarrow 1s$ transition the normalized hydrogen wavefunctions are

$$\Psi_{1s} = \frac{1}{\sqrt{\pi} a^{3/2}} e^{-r/a} \quad ; \quad \Psi_{3d2} = \frac{1}{162\sqrt{\pi}} \frac{1}{a^{3/2}} \left(\frac{r^2}{a^2} \right) e^{-r/3a} sin^2\theta e^{2i\phi} \tag{A.4}$$

and the integrals can all be evaluated analytically in terms of elementary functions. Weinberg[14] apparently calculates this transition by such a direct substitution (see his Eqs. 10.8.5-10.8.7) and arrives at

$$\Gamma(3d2 \rightarrow 1s) = \frac{2^{23} G m^3}{3^7 5^{15} (137)^6 \hbar^2} = \frac{2^{13} 3^3}{5^{15}} \frac{G}{\hbar} m^2 a^4 \omega^5 = 2.5 \times 10^{-44} s^{-1}, \tag{A.5}$$

where to get the second equality we have used the exact transition frequency $\omega = 4\hbar/9ma^2$. This gives a lifetime $\Gamma^{-1} = 4.0 \times 10^{43} s$.

By direct substitution, however, we obtain

$$\Gamma(3d2 \rightarrow 1s) = \Gamma(3d0 \rightarrow 1s) = \frac{3^8}{5\, 2^{13}} \frac{G}{\hbar} m^2 a^4 \omega^5 = 4.9 \times 10^{-40} s^{-1}, \tag{A.6}$$

which is about 2.2×10^4 times larger than Weinberg's and roughly a factor of ten smaller than the Problem Book's. Apparently Weinberg's result merely contains a numerical error.

We can nevertheless convince ourselves that the general procedure is correct, in particular Eq. (A.3), by checking detailed balance. We assume the spontaneous emission rate given by Eqs. (A.2) and (A.3). The rate for stimulated and emission and absorption follow from (4.2) and (4.8). The crucial point is that detailed balance will only satisfied when the factor of 1/2 in the matrix element (4.5) is replaced by 1/4 (see Eq. (A.12) below). Then:

$$\Gamma_{abs} = \Gamma_{se} = \frac{2\pi}{16\hbar} m^2 h^2 \omega^4 < |D_{ij}|^2 > . \tag{A.7}$$

The intensity I for a gravitational wave is $I = \omega^2 h^2 / 8\pi G$. Demanding that the spontaneous plus stimulated emission rates equal the absorption rate requires

$$e^{-\hbar\omega/kT} \left(G\pi^2 \omega^2 I + G\omega^5 \right) < |D_{ij}|^2 >= G\pi^2 \omega^2 < |D_{ij}|^2 >,$$

or

$$I(\omega) = \frac{1}{\pi^2} \frac{\omega^3}{e^{\hbar\omega/kT} - 1}, \tag{A.8}$$

showing that detailed balance is indeed satisfied. Because the matrix elements cancel out from both sides of this equation, this procedure cannot provide a check that they have been correctly computed but it does show that if Eq. (A.3) is the correct expression for spontaneous emission, then Γ_{abs} in (A.7) and $| < a|H|b > |^2$ in (4.8) are the correct expression to be used in the calculation of cross sections.

The reason for the correction can be found by examining the Problem Book's field-theoretical calculation. A proper field theoretical derivation is the only utterly convincing way to obtain the spontaneous emission rate—and the only independent verification of the semi-classical results. Why then does the Problem Book fail to produce the semi-classical answer, which one would expect it to do? The crucial step in the authors' calculation is to write Hamiltonian (4.4) as

$$H = (8\pi G)^{1/2} \frac{p_i p_j \phi^{ij}}{m}. \tag{A.9}$$

Here, $\phi_{\mu\nu} \equiv h_{\mu\nu}/\sqrt{G}$. They have also chosen the same components of the polarization tensor as we did in §4, implying that $T_{\mu\nu} = p_i p_j /m$.

Next, in the standard field-theoretic manner, ϕ is decomposed into a series of harmonic oscillators by the use of creation and annihilation operators. (The $8\pi G$ is introduced to normalize the number operator to the energy density; see BR.)

To get Γ, they compute $< 3d|H|1s >$ for the spontaneous emission of a graviton. This is carried out in by the standard field-theoretic prescription with the result that in the dipole approximation

$$\frac{d\Gamma}{d\Omega} = \frac{G\omega}{\pi\hbar m^2} | < 1s|p_i p_j\, e^{ij}_{k\lambda}|3d > |^2. \tag{A.10}$$

(The conceptual difficulty about using an interaction Hamiltonian for spontaneous emission is obviated in field theory by saying that the interaction is with vacuum fluctuations.)

The remainder of the solution consists of a sophisticated averaging procedure over all spin states via Clebsch-Gordon coefficients and the Wigner-Eckart theorem. For a final result the Problem Book gets $\Gamma = 5.3 \times 10^{-39} s^{-1}$, which as mentioned is about 10 times larger than our result. Nevertheless, a simple field-theoretic calculation performed with Eq. (4.8), does yield exactly the semi-classical answer, which shows that, not only Weinberg's result, but the Problem Book's must be incorrect. There are several minor mistakes involved, which we discuss in BR; here, however, we do point out a fundamental error that has been made:

In general relativity, coordinate transformations and gauge transformations are the same thing, and it turns out that the Hamilonian (4.4) is not gauge invariant. In setting $e_{11} = -e_{22}$ and $e_{12} = e_{21}$, the Problem Book authors have chosen to work in the so-called transverse-traceless, or TT, gauge, in which the polarization directions are purely spatial and orthogonal to the direction of the wave's propagation. It is a textbook exercise to show[16] that to first-order in the TT gauge, two particles initially at rest remain at rest under the passage of a gravitational wave. In other words, nothing happens, which suggests that the TT gauge cannot easily be used for this problem. Properly, physics should be calculated in a locally inertial frame (LIF), in which the effects of gravity are absent. Otherwise, in the TT gauge for example, not only are "non-inertial" forces present, but the laws of electromagnetism are significantly modified as well. In a LIF, such effects are second-order, typically smaller than in the TT gauge by a factor of v^2, where v is the velocity of particles in the system.

In an LIF, the energy density ρ dominates over the momentum and pressure, and so the only non-negligible component of the stress-energy tensor is $T_{00} = \rho$. A straightforward coordinate transformation from the TT gauge to the LIF (see BR) yields for the Lagrangian density

$$\mathcal{L}_{\mathrm{LIF}} = -\frac{1}{4}\omega^2 \rho h e_{\mathrm{ij}}^{\mathrm{TT}} x^{\mathrm{i}} x^{\mathrm{j}} \,, \tag{A.11}$$

Thus, the LIF interaction Hamiltonian is

$$H_{\mathrm{LIF}} = -\int \mathcal{L} d^3x = \frac{1}{4}\omega^2 m h e_{\mathrm{ij}}^{\mathrm{TT}} x^{\mathrm{i}} x^{\mathrm{j}} \,. \tag{A.12}$$

We see though, in the case $p_{\mathrm{i}} = im\omega x_{\mathrm{i}}$, the locally inertial Hamiltonian itself has exactly the same form as the TT-gauge Hamiltonian in (A.9), but that each term is smaller by a factor of ≈ 2. Thus, it is not entirely surprising that a TT-gauge calculation leads to almost the locally inertial result. Nevertheless, the LIF Hamiltonian (A.12) is the easiest one to use in the computation of the cross section. One can carry out the calculation in the TT gauge (see BR) but in that case one needs to include the electromagnetic stresses in the stress-energy tensor.

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