AHMAD KARIMI
Department of Mathematics
Tarbiat Modares University
P.O.Box 14115–134
Tehran, IRAN

Tel: +98 (0)919 510 2790
Fax: +98 (0)21 8288 3493
E-mail: a.karimi40@yahoo.com
Behbahan KA Univ. of Tech.
61635–151 Behbahan, IRAN

SAEED SALEHI
Department of Mathematics
University of Tabriz
P.O.Box 51666–17766
Tabriz, IRAN

Tel: +98 (0)411 339 2905
Fax: +98 (0)411 334 2102
E-mail: /root@SaeedSalehi.ir/
/SalehiPour@TabrizU.ac.ir/
Web: http://SaeedSalehi.ir/

$\oint^{\Sigma\alpha\epsilon\epsilon\partial}_{\Sigma\alpha\ell\epsilon\hbar i}$.ir

Theoremizing Yablo's Paradox

Abstract

To counter a general belief that all the paradoxes stem from a kind of circularity (or involve some self–reference, or use a diagonal argument) Stephen Yablo designed a paradox in 1993 that seemingly avoided self–reference. We turn Yablo's paradox, the most challenging paradox in the recent years, into a genuine mathematical theorem in Linear Temporal Logic (LTL). Indeed, Yablo's paradox comes in several varieties; and he showed in 2004 that there are other versions that are equally paradoxical. Formalizing these versions of Yablo's paradox, we prove some theorems in LTL. This is the first time that Yablo's paradox(es) become new(ly discovered) theorems in mathematics and logic.

2010 Mathematics Subject Classification: 03B44 ≀ 03A05.
Keywords: Yablo's Paradox ≀ Linear Temporal Logic.

Date: 01.06.14 (01 June 2014)

1 Introduction

Paradoxes are interesting puzzles in philosophy and mathematics. They can be more interesting when they turn into genuine theorems. For example, Russell's paradox which collapsed Frege's foundations of mathematics, is now a classical theorem in set theory, implying that no set of all sets can exist. Or, as another example, the Liar paradox has turned into Tarski's theorem on the undefinability of truth in sufficiently rich languages. This paradox also appears implicitly in the proof of Gödel's first incompleteness theorem. For this particular theorem, some other paradoxes such as Berry's ([2, 3]) or Yablo's ([12, 13]) have been used to give alternative proofs ([5, 9]). A more recent example is the surprise examination paradox [4] that has turned into a beautiful proof for Gödel's second incompleteness theorem ([8]).

In this paper we transform Yablo's paradox into a theorem in the Linear Temporal Logic. This paradox, which is the first one of its kind that supposedly avoids self–reference and circularity has been used for proving an old theorem ([5, 9]) but not a new theorem had been made out of it. In this paper, for the very first time, we use this paradox (actually its argument) for proving some genuine mathematical theorem in Linear Temporal Logic. Roughly speaking, we show that certain operators do not have fixed–points in this logic, where the proof is exactly Yablo's paradox (reaching to a contradiction by assuming the existence of certain fixed–point sentences). Let us note that very many other operations in the Linear Temporal Logic do have fixed–points, which constitute some other genuine mathematical theorems.

2 Yablo's Paradox

To counter a general belief that all the paradoxes stem from a kind of circularity (or involve some self–reference, or use a diagonal argument) Stephen Yablo designed a paradox in 1993 that seemingly avoided self–reference ([13, 12]). Let us fix our reading of Yablo's Paradox: Consider the sequence of sentences $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$: $\mathcal{Y}_n \Longleftrightarrow \forall k>n\ (\mathcal{Y}_k \text{ is not true})$.

The paradox follows from the following deductions. For each $n\in\mathbb{N}$,

$$\begin{aligned}
\mathcal{Y}_n \ &\Longrightarrow\ \forall k>n\ (\mathcal{Y}_k \text{ is not true})\\
&\Longrightarrow\ (\mathcal{Y}_{n+1} \text{ is not true}) \text{ and } \forall k>n+1\ (\mathcal{Y}_k \text{ is not true})\\
&\Longrightarrow\ (\mathcal{Y}_{n+1} \text{ is not true}) \text{ and } (\mathcal{Y}_{n+1} \text{ is true}),
\end{aligned}$$

thus $\mathcal{Y}_n$ is not true. So, $\forall k\ (\mathcal{Y}_k$ is not true), and in particular $\forall k>0\ (\mathcal{Y}_k$ is not true), and so $\mathcal{Y}_0$ must be true (and not true at the same time); contradiction!

Some paradoxes turn into mathematical–logical tautologies and so become (interesting) theorems. For example, Liar's paradox when translated into first–order logic is a sentence L such that $L\leftrightarrow\neg L$. The fact that this is contradictory is equivalent to the fact that the formula $\neg(\varphi\leftrightarrow\neg\varphi)$ is a tautology in propositional logic. As another less trivial paradox, take Russell's paradox: there can be no set S such that for every x we have $x\in S\leftrightarrow x\notin x$. Writing this in first–order logic (in the language $\{\in\}$) we have a logical theorem: $\neg\exists y\forall x(x\in y\leftrightarrow x\notin x)$. Indeed, this first–order logical tautology still holds when we replace the membership relation $\in$ with an arbitrary binary relation R: the sentence $\neg\exists y\forall x(xRy\leftrightarrow\neg xRx)$ is again a first–order logical tautology. On the other hand if xRy is interpreted as "y shaves x" then the above tautology is nothing but Barber's Paradox. As for Yablo's paradox, J. Ketland has translated it into first–order logic (called Uniform Homogeneous Yablo Scheme) in [7]:

$$(\mathsf{Y}):\ \forall x\big(\varphi(x)\leftrightarrow\forall y[xRy\rightarrow\neg\varphi(y)]\big),$$

where R is a binary formula (which could be a binary relation symbol, i.e. an atomic formula) with the auxiliary axioms stating that R is total and transitive:

$$(\mathsf{A}_1):\ \forall x\exists y(xRy) \quad\text{and}\quad (\mathsf{A}_2):\ \forall x,y,z(xRyRz\rightarrow xRz).$$

A Yablo-like argument can show that the formula $\neg(\mathsf{Y}\wedge\mathsf{A}_1\wedge\mathsf{A}_2)$ is a first–order tautology.

3 Linear Temporal Logic

Here, we show that there is another way to have a formal version of Yablo's paradox (different from the formalized version discussed above), and that is in Linear Temporal Logic. The (propositional) linear temporal logic (LTL) is a logical formalism that can refer to time; in LTL one can encode formulae about the future, e.g., a condition will eventually be true, a condition will be true until another fact becomes true, etc. LTL was first proposed for the formal verification of computer programs in 1977 by Amir Pnueli [10]. For a modern introduction to LTL and its syntax and semantics see e.g. [6]. Two modality operators in LTL that we will use are the "next" modality denoted by $\bigcirc$ and the "always" modality denoted as $\Box$.

3.1 Syntax and Semantics of LTL

We assume the reader is familiar with the general framework of LTL, but for the sake of accessibility, we list the main notations, definitions and theorems which will be referred to later on. For details we refer the reader to [6]. Let $\mathbf{V}$ be a set of *propositional constants*. The alphabet of a basic language $\mathcal{L}_{\rm LTL}(\mathbf{V})$ (also shortly: $\mathcal{L}_{\rm LTL}$) of propositional linear temporal logic LTL is given by

all propositional constants of $\mathbf{V}$ and the symbols $\{\mathbf{false}, \rightarrow, \bigcirc, \Box, (,)\}$.

The inductive definition of formulas (of $\mathcal{L}_{\rm LTL}(\mathbf{V})$) is as follows:

1—Every propositional constant of $\mathbf{V}$ and also the constant symbol **false** is a formula.

2—If φ and ψ are formulas then $(\varphi \rightarrow \psi)$ is a formula.

3—If φ is a formula then $\bigcirc\varphi$ and $\Box\varphi$ are formulas.

Further operators can be introduced as abbreviations:

$$\neg, \vee, \wedge, \leftrightarrow, \mathbf{true} \text{ as in classical logic, and } \Diamond\varphi \equiv \neg\Box\neg\varphi.$$

The temporal operators $\bigcirc, \Box$, and $\Diamond$ are called *next time*, *always (or henceforth)*, and *sometime (or eventuality)* operators, respectively. Formulas $\bigcirc\varphi$, $\Box\varphi$, and $\Diamond\varphi$ are typically read "next φ", "always φ", and "sometime φ".

Semantical interpretations in classical propositional logic are given by Boolean valuations. For LTL we have to extend this concept according to our informal idea that formulas are evaluated over sequences of states (time scales). Let $\mathbf{V}$ be a set of propositional constants. A *temporal (or Kripke) structure* for $\mathbf{V}$ is an infinite sequence $\mathcal{K} = (\eta_0, \eta_1, \eta_2, ...)$ of mappings $\eta_i : \mathbf{V} \rightarrow \{\mathfrak{ff}, \mathfrak{tt}\}$ called *states*, and η_0 is called the *initial state* of $\mathcal{K}$. Observe that states are just valuations in the classical logic sense. For $\mathcal{K}$ and $i \in \mathcal{K}$, we define $\mathcal{K}_i(F) \in \{\mathfrak{ff}, \mathfrak{tt}\}$ (informally meaning the "truth value of F in the $i^{\rm th}$ state of $\mathcal{K}$") for every formula F inductively as follows:

01. $\mathcal{K}_i(v) = \eta_i(v)$ for $v \in \mathbf{V}$.
02. $\mathcal{K}_i(\mathbf{false}) = \mathfrak{ff}$.
03. $\mathcal{K}_i(\varphi \rightarrow \psi) = \mathfrak{tt} \iff \mathcal{K}_i(\varphi) = \mathfrak{ff}$ or $\mathcal{K}_i(\psi) = \mathfrak{tt}$.
04. $\mathcal{K}_i(\bigcirc\varphi) = \mathcal{K}_{i+1}(\varphi)$.
05. $\mathcal{K}_i(\Box\varphi) = \mathfrak{tt} \iff \mathcal{K}_j(\varphi) = \mathfrak{tt}$ for every $j \geq i$.

Obviously, the formula **false** and the operator $\rightarrow$ behave classically in each state. The definitions for $\bigcirc$ and $\Box$ make these operators formalize the phrases *in the next state* and *from this step onward*. More precisely, the formula $\Box\varphi$ informally means "φ holds in all forthcoming states including the present one". The definitions induce the following truth values for the formula abbreviations:

06. $\mathcal{K}_i(\neg\varphi) = \mathfrak{tt} \iff \mathcal{K}_i(\varphi) = \mathfrak{ff}$.
07. $\mathcal{K}_i(\varphi \vee \psi) = \mathfrak{tt} \iff \mathcal{K}_i(\varphi) = \mathfrak{tt}$ or $\mathcal{K}_i(\psi) = \mathfrak{tt}$.
08. $\mathcal{K}_i(\varphi \wedge \psi) = \mathfrak{tt} \iff \mathcal{K}_i(\varphi) = \mathfrak{tt}$ and $\mathcal{K}_i(\psi) = \mathfrak{tt}$.
09. $\mathcal{K}_i(\varphi \leftrightarrow \psi) = \mathfrak{tt} \iff \mathcal{K}_i(\varphi) = \mathcal{K}_i(\psi)$.
10. $\mathcal{K}_i(\mathbf{true}) = \mathfrak{tt}$.
11. $\mathcal{K}_i(\Diamond\varphi) = \mathfrak{tt} \iff \mathcal{K}_j(\varphi) = \mathfrak{tt}$ for some $j \geq i$.

Definition 3.1 ([6]) *A formula φ of $\mathcal{L}_{LTL}(\boldsymbol{V})$ is called valid in the temporal structure $\mathcal{K}$ for $\boldsymbol{V}$ (or $\mathcal{K}$ satisfies φ), denoted by $\models_{\mathcal{K}} \varphi$, if $\mathcal{K}_i(\varphi) = \mathtt{tt}$ for every $i \in \mathbb{N}$. The formula φ is called a consequence of a set $\mathcal{F}$ of formulas ($\mathcal{F} \models \varphi$) if $\models_{\mathcal{K}} \varphi$ holds for every $\mathcal{K}$ such that $\models_{\mathcal{K}} \psi$ for all $\psi \in \mathcal{F}$. The formula φ is called (universally) valid ($\models \varphi$) if $\emptyset \models \varphi$. A formula φ is called (locally) satisfiable if there is a temporal structure $\mathcal{K}$ and $i \in \mathbb{N}$ such that $\mathcal{K}(\varphi) = \mathtt{tt}$.* ▲

The formula $\bigcirc\varphi$ holds (in the current moment) when φ is true in the "next step", and the formula $\Box\varphi$ is true (in the current moment) when φ is true "now and forever" ("always in the future"). In the other words, $\Box$ is the reflexive and transitive closure of $\bigcirc$. So the formula $\bigcirc\Box\psi$ is true when ψ is true from the next step onward, that is ψ holds in the next step, and the step after that, and the step after that, etc. The same holds for $\Box\bigcirc\psi$; indeed the formula $\bigcirc\Box\psi \longleftrightarrow \Box\bigcirc\psi$ is a law of *LTL* (T12 on page 28 of [6]). It can also be seen that the formula $\bigcirc\neg\varphi \longleftrightarrow \neg\bigcirc\varphi$ is always true (is a law of LTL, see T1 on page 27 of [6]), since φ is untrue in the next step if and only if it is not the case that "φ is true in the next step". Whence, we have the equivalences $\bigcirc\Box\neg\varphi \longleftrightarrow \Box\bigcirc\neg\varphi \longleftrightarrow \Box\neg\bigcirc\varphi$ in LTL. The following theorem will be used in our arguments.

Theorem 3.2 ([6]) LTL $\models \varphi$ *if and only if $\neg\varphi$ is not satisfiable.*

3.2 Paradoxical and Non–Paradoxical Fixed–Points

A version of Yablo's paradox is a sentence $\mathscr{Y}$ that satisfies the followng equivalences

$$\mathscr{Y} \longleftrightarrow \bigcirc\Box\neg\mathscr{Y} \quad \big(\longleftrightarrow \Box\bigcirc\neg\mathscr{Y} \longleftrightarrow \Box\neg\bigcirc\mathscr{Y}\big)$$

In the other words $\mathscr{Y}$ is a fixed–point of the operator $x \mapsto \bigcirc\Box\neg x$ $\big(\equiv \Box\bigcirc\neg x \equiv \Box\neg\bigcirc x\big)$. Yablo's argument in his paradox amounts to showing that this operator does not have any fixed–point in LTL. The semantic proof (i.e. non–existence of any such fixed–point in any Kripke model of LTL) is exactly the same as Yablo's argument. Now, Yablo's paradox becomes the following theorem.

Theorem 3.3 LTL $\models \neg\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$.

Proof. To show this formula is valid will exactly follow the line of Yablo's reasoning to obtain his paradox, this time in LTL. By Theorem 3.2, to prove the formula $\neg\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$ is valid in LTL, we need to show the formula $\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$ is not satisfiable. For a moment assume that there is a Kripke structure $\mathcal{K}$ and $n \in \mathbb{N}$ for which $\mathcal{K}_n\big(\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)\big) = \mathtt{tt}$. Then $\forall i \geq n\ \mathcal{K}_i(\varphi \leftrightarrow \bigcirc\Box\neg\varphi) = \mathtt{tt}$ which implies that $\forall i \geq n\ \mathcal{K}_i(\varphi) = \mathcal{K}_i(\bigcirc\Box\neg\varphi) = \mathcal{K}_{i+1}(\Box\neg\varphi)$. We distinguish two cases:

(1) For some $j \geq n$ we have $\mathcal{K}_j(\varphi) = \mathtt{tt}$. Then $\mathcal{K}_{j+1}(\Box\neg\varphi) = \mathtt{tt}$ so $\mathcal{K}_{j+l}(\varphi) = \mathtt{ff}$ for all $l \geq 1$. In particular $\mathcal{K}_{j+1}(\varphi) = \mathtt{ff}$ whence $\mathcal{K}_{j+2}(\Box\neg\varphi) = \mathtt{ff}$ which is in contradiction with $\mathcal{K}_{j+1}(\Box\neg\varphi) = \mathtt{tt}$.

(2) For all $j \geq n$ we have $\mathcal{K}_j(\varphi) = \mathtt{ff}$. So $\mathtt{ff} = \mathcal{K}_n(\varphi) = \mathcal{K}_{n+1}(\Box\neg\varphi)$ hence there must exist some $i > n$ with $\mathcal{K}_i(\varphi) = \mathtt{tt}$ which contradicts (1) above.

Thus, the formula $\Box(\varphi \leftrightarrow \bigcirc\Box\neg\varphi)$ cannot be satisfiable in LTL. ❏

Also, a Gödel–like argument can show that the operators $x \mapsto \neg\Box x$ and $x \mapsto \Box\neg x$ cannot have any fixed–points in LTL as well.

Proposition 3.4 *The operators $x \mapsto \neg\Box x$ and $x \mapsto \Box\neg x$ do not have any fixed–points in* LTL*; i.e. for any formula φ we have* LTL $\models \neg\Box(\varphi \leftrightarrow \neg\Box\varphi)$ *and* LTL $\models \neg\Box(\varphi \leftrightarrow \Box\neg\varphi)$.

Proof. We show that satisfiability of $\Box(\varphi \leftrightarrow \Box\neg\varphi)$ in LTL leads to a contradiction. For a moment let there exist some Kripke structure $\mathcal{K}$ and $n \in \mathbb{N}$ for which $\mathcal{K}_n(\Box(\varphi \leftrightarrow \Box\neg\varphi)) = \mathtt{tt}$. Then for any $i \geq n$ we have $\mathcal{K}_i(\varphi \leftrightarrow \Box\neg\varphi) = \mathtt{tt}$ whence $\forall i \geq n\ \mathcal{K}_i(\varphi) = \mathcal{K}_i(\Box\neg\varphi)$. This already implies that $\forall i \geq n\ \mathcal{K}_i(\varphi) = \mathtt{ff}$

(since $\models \Box\neg\varphi \to \neg\varphi$). Then, in particular, $\mathfrak{ff} = \mathcal{K}_n(\varphi) = \mathcal{K}_n(\Box\neg\varphi)$ and so there must exist some $m \geq n$ such that $\mathcal{K}_m(\neg\varphi) = \mathfrak{ff}$ contradiction! ❏

Some other operators like $x \mapsto \Box x$ or $x \mapsto \neg\bigcirc x$ do have fixed–points; **true** or **false** for the former and the sequences $\langle \mathfrak{ff}, \mathfrak{tt}, \mathfrak{ff}, \mathfrak{tt}, \mathfrak{ff}, \mathfrak{tt}, \cdots \rangle$ or $\langle \mathfrak{tt}, \mathfrak{ff}, \mathfrak{tt}, \mathfrak{ff}, \mathfrak{tt}, \mathfrak{ff}, \cdots \rangle$ for the latter (see [1]).

4 Other Versions of Yablo's Paradox

Yablo's paradox comes in several varieties [14]; here we show that other versions of Yablo's paradox become interesting theorems in LTL as well.

(always): $\mathcal{Y}_n \iff \forall i > n\,(\mathcal{Y}_i \text{ is not true })$.
(sometimes): $\mathcal{Y}_n \iff \exists i > n\,(\mathcal{Y}_i \text{ is not true })$.
(almost always): $\mathcal{Y}_n \iff \exists i > n\,\forall j \geq i\,(\mathcal{Y}_i \text{ is not true })$.
(infinitely often): $\mathcal{Y}_n \iff \forall i > n\,\exists j \geq i\,(\mathcal{Y}_i \text{ is not true })$.

It can be seen that all the sequences $\{\mathcal{Y}_n\}_{n\in\mathbb{N}}$ of sentences above are paradoxical. These sequences of sentences can be formalized in LTL as follows:

(always): $\mathscr{Y} \longleftrightarrow \bigcirc\Box\neg\mathscr{Y} \quad \big(\longleftrightarrow \Box\bigcirc\neg\mathscr{Y} \longleftrightarrow \Box\neg\bigcirc\mathscr{Y}\big)$.
(sometimes): $\mathscr{Y} \longleftrightarrow \bigcirc\Diamond\neg\mathscr{Y} \quad \big(\longleftrightarrow \Diamond\bigcirc\neg\mathscr{Y} \longleftrightarrow \Diamond\neg\bigcirc\mathscr{Y}\big)$.
(almost always): $\mathscr{Y} \longleftrightarrow \bigcirc\Diamond\Box\neg\mathscr{Y} \quad \big(\longleftrightarrow \Diamond\bigcirc\Box\neg\mathscr{Y} \longleftrightarrow \Diamond\Box\bigcirc\neg\mathscr{Y} \longleftrightarrow \Diamond\Box\neg\bigcirc\mathscr{Y}\big)$.
(infinitely often): $\mathscr{Y} \longleftrightarrow \bigcirc\Box\Diamond\neg\mathscr{Y} \quad \big(\longleftrightarrow \Box\bigcirc\Diamond\neg\mathscr{Y} \longleftrightarrow \Box\Diamond\bigcirc\neg\mathscr{Y} \longleftrightarrow \Box\Diamond\neg\bigcirc\mathscr{Y}\big)$.

The following (sometimes) counterpart of Theorem 3.3 directly follows.

Theorem 4.1 LTL $\models \neg\Box(\varphi \leftrightarrow \bigcirc\Diamond\neg\varphi)$.

Proof. By Theorem 3.3 we have LTL $\models \neg\Box(\psi \leftrightarrow \bigcirc\Box\neg\psi)$ for any arbitrary formula ψ. In particular for $\psi = \neg\varphi$ we have LTL $\models \neg\Box(\neg\varphi \leftrightarrow \bigcirc\Box\neg\neg\varphi \leftrightarrow \bigcirc\neg\Diamond\neg\varphi \leftrightarrow \neg\bigcirc\Diamond\neg\varphi)$, whence for any φ we conclude that LTL $\models \neg\Box(\varphi \leftrightarrow \bigcirc\Diamond\neg\varphi)$. ❏

Let us focus now on the "almost always" version of Yablo's paradox. Let $Y_0, Y_1, Y_2, \ldots$ be a sequence of sentences that each sentence, roughly speaking, says "all sentences, except finitely many, after this sentence are false". Mathematically, this sequence is as below:

$$\begin{array}{ll} Y_0 : & \exists i > 0\,\forall j \geq i\,(\mathcal{Y}_j \text{ is not true }). \\ Y_1 : & \exists i > 1\,\forall j \geq i\,(\mathcal{Y}_j \text{ is not true }). \\ Y_2 : & \exists i > 2\,\forall j \geq i\,(\mathcal{Y}_j \text{ is not true }). \\ \vdots & \vdots \end{array}$$

The paradox arises when we try to assign truth values in a consistent way to all Y_i's. Assume for a moment that there is a sentence (say) Y_n which is true; so there exists $i > n$ for which all Y_j with $j \geq i$ are untrue. In particular, Y_i is untrue. Since all the sentences $Y_{i+1}, Y_{i+2}, \ldots$ are untrue, so Y_i has to be true. Therefore, Y_i is true and false the same time, which is a contradiction. Whence, all Y_n's are untrue, so Y_0 is true, a contradiction again. Now we turn this version of Yablo's paradox to a theorem in LTL.

Theorem 4.2 LTL $\models \neg\Box(\varphi \leftrightarrow \bigcirc\Diamond\Box\neg\varphi)$.

Proof. We show that the formula $\Box(\varphi \leftrightarrow \bigcirc\Diamond\Box\neg\varphi)$ is not satisfiable in LTL. For a moment, assume that there is a Kripke structure $\mathcal{K}$ and a state $n \in \mathbb{N}$ for which $\mathcal{K}_n\big(\Box(\varphi \leftrightarrow \bigcirc\Diamond\Box\neg\varphi)\big) = \mathfrak{tt}$. So, we have $\forall i \geq n\ \mathcal{K}_i(\varphi \leftrightarrow \bigcirc\Diamond\Box\neg\varphi) = \mathfrak{tt}$ which implies $\forall i \geq n\ \mathcal{K}_i(\varphi) = \mathcal{K}_i(\bigcirc\Diamond\Box\neg\varphi)$ which is equivalent to $\forall i \geq n\ \exists j \geq 0\ \mathcal{K}_i(\varphi) = \mathcal{K}_{i+j+1}(\Box\neg\varphi)$.

(1) If there is some $l \geq n$ such that $\mathcal{K}_l(\varphi) = \mathfrak{tt}$, then $\mathcal{K}_{l+m+1}(\Box\neg\varphi) = \mathfrak{tt}$ for some m; so $\mathcal{K}_{l+m+1}(\varphi) = \mathfrak{ff}$ and also $\mathcal{K}_k(\varphi) = \mathfrak{ff}$ for all $k \geq l+m+1$. On the other hand there must exist some $p \geq 0$ such that $\mathcal{K}_{l+m+1+p+1}(\Box\neg\varphi) = \mathfrak{ff}$ which implies that $\mathcal{K}_{l+m+1+p+1+q}(\varphi) = \mathfrak{tt}$ for some $q \geq 0$. This is a contradiction since $l+m+1+p+1+q \geq l+m+1$.

(2) If $\mathcal{K}_l(\varphi) = \mathfrak{ff}$ holds for all $l \geq n$, then in particular $\mathcal{K}_n(\varphi) = \mathfrak{ff}$ and so there exists some $m \geq 0$ such that $\mathcal{K}_{n+m+1}(\Box\neg\varphi) = \mathfrak{ff}$; whence $\mathcal{K}_{n+m+1+p}(\varphi) = \mathfrak{tt}$ for some $p \geq 0$, which contradicts (1) above. ❑

Again by the technique of the proof of Theorem 4.1 we can deduce the following from Theorem 4.2.

Theorem 4.3 LTL $\models \neg\Box(\varphi \leftrightarrow \bigcirc\Box\Diamond\neg\varphi)$.

Proof. LTL $\models \neg\Box(\psi \leftrightarrow \bigcirc\Diamond\Box\neg\psi)$ holds for any formula ψ by Theorem 4.2. For $\psi = \neg\varphi$ we obtain the deduction LTL $\models \neg\Box(\neg\varphi \leftrightarrow \bigcirc\Diamond\Box\neg\neg\varphi \leftrightarrow \bigcirc\neg\Box\Diamond\neg\varphi \leftrightarrow \neg\bigcirc\Box\Diamond\neg\varphi)$ which completes the proof. ❑

References

[1] Bernardi, C.; A Topological Approach to Yablo's Paradox, *Notre Dame Journal of Formal Logic* 50 (2009) pp. 331–338. `http://dx.doi.org/10.1215/00294527-2009-014`

[2] Boolos, G.; A New Proof of the Gödel Incompleteness Theorem, *Notice of the American Mathematical Society* 36 (1989) pp. 388–390.

[3] Chaitin, G.J.; Information-Theoretic Limitations of Formal Systems, *Journal of Association for Computing Machinery* 21 (1974) pp. 403–424. `https://www.cs.auckland.ac.nz/~chaitin/acm74.pdf`

[4] Chow, T.Y.; The Surprise Examination or Unexpected Hanging Paradox, *The American Mathematical Monthly* 105 (1998) pp. 41–51. `http://www-math.mit.edu/~tchow/unexpected.pdf`

[5] Cieśliński, C., & Urbaniak, R.; Gödelizing the Yablo Sequence, *Journal of Philosophical Logic* 42 (2013) pp. 679–695. `http://dx.doi.org/10.1007/s10992-012-9244-4`

[6] Kröger, F. & Merz S.; *Temporal Logic and State Systems*, Springer (2008) ISBN 9783540674016. `http://dx.doi.org/10.1007/978-3-540-68635-4`

[7] Ketland, J.; Yablo's Paradox and $\omega-$Inconsistency, *Synthese* 145 (2005) pp. 295–302. `http://dx.doi.org/10.1007/s11229-005-6201-6`

[8] Kritchman, S., & Raz, R.; The Surprise Examination Paradox and the Second Incompleteness Theorem, *Notice of the American Mathematical Society* 57 (2010) pp. 1454–1458. `http://www.ams.org/notices/201011/rtx101101454p.pdf`

[9] Leach-Krouse, G.; Yablifying the Rosser Sentence, *Journal of Philosophical Logic* (to appear). `http://dx.doi.org/10.1007/s10992-013-9291-5`

[10] Pnueli, A.; The Temporal Logic of Programs, *Proceedings of the 18th Annual Symposium on Foundations of Computer Science (SFCS'77) IEEE Computer Society*, Washington DC, USA (1977) pp. 46–57. `http://dx.doi.org/10.1109/SFCS.1977.32`

[11] Priest, G.; Yablo's Paradox, *Analysis* 57 (1997) pp. 236–242. `http://dx.doi.org/10.1111/1467-8284.00081`

[12] Yablo, S.; Truth and Reflection, *Journal of Philosophical Logic* 14 (1985) pp. 297–349. `http://dx.doi.org/10.1007/BF00249368`

[13] Yablo, S.; Paradox without Self-Reference, *Analysis* 53 (1993) pp. 251–252. `http://dx.doi.org/10.2307/3328245`

[14] Yablo, S.; Circularity and Paradox, in: Thomas Bolander & Vincent F. Hendricks & Stig Andur Pedersen (eds.), *Self-Reference* (2004) CSLI Publications, pp. 139–157.

