Errata for Nathan Salmón,“Modal Logic Kalish-and-Montague Style (1994*u*)”

Proof page no.:	**Line: (fb = from bottom)**	**Current text:**	**Should instead be:**
113	18 fb	premise	premises
114	4 fb	‘Show’	‘~~*Show*~~’
115	9 fb	*Show*	~~*Show*~~
115	7 fb	*Show*	~~*Show*~~
115	5 fb	*Show* □→□P	~~*Show*~~ □~□P
116	1	*Show* ◇□P→P	~~*Show*~~ □P→□□P
116	4	*Show*	~~*Show*~~
116	5	*Show*	~~*Show*~~
116	5	Assertion (SD)	Assertion (ID)
116	8	*Show*	~~*Show*~~
117	13	$\alpha = \beta$	□□...□$\alpha = \beta$
117	16	$\alpha = \beta$	□□...□$\alpha = \beta$
117	19	$\alpha = \beta$	□□...□$\alpha = \beta$
117	22	where ϕ_β	where ‘□□. . . □’ represents a string of occurrences of ‘□’ (to be explained below), ϕ_β

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Modal Logic Kalish-and-Montague Style (1994*u*)

A natural-deduction apparatus for the propositional modal system *S*5, based on the nonmodal deductive apparatus of D. Kalish, R. Montague, and G. Mar, *Logic: Techniques of Formal Reasoning*, Second Edition (Harcourt Brace Jovanovich, 1964, 1980), chapter 2, was given in Jordan Howard Sobel, 'A Natural Deduction System for Sentential Modal Logic,' *Philosophy Research Archives* (1979).[1] The Sobel system can be significantly improved, and made sufficiently flexible to accommodate other well-known modal systems, by utilizing additional natural-deduction techniques. Besides its more extensive reliance on the general approach of Kalish, Montague, and Mar (hereafter 'K&M'), the apparatus proposed below provides genuinely natural-deduction derivations not only for *T, S*4, and *S*5, but also for the unduly neglected modal system *B* (the *Brouwersche* or *Brouwerian* system), which I have argued is less vulnerable than *S*5 and *S*4 to counter-example.[2]

Specifically, we augment and modify the deductive apparatus of K&M as follows. The following clause is added to the characterization of the class of *symbolic formulas* given on p. 309.

(5′) If ϕ is a symbolic formula, then so are

$$\Box\phi$$
$$\Diamond\phi.$$

The following primitive inference rules for the new sentential connectives are added to the rules given in K&M, pp. 60–61.

Necessity instantiation (NI): $\dfrac{\Box\phi}{\phi}$

Modal negation (MN), in four forms:

$$\frac{\sim\Box\phi}{\Diamond\sim\phi}\quad\frac{\Diamond\sim\phi}{\sim\Box\phi}\quad\frac{\sim\Diamond\phi}{\Box\sim\phi}\quad\frac{\Box\sim\phi}{\sim\Diamond\phi}$$

I thank Allen Hazen, Ilhan Inan, Andrrzej Indrzejcza k, and Gary Mar for their comments and suggestions.

[1] Sobel's original system was unsound, and was later corrected in his 'Names and Indefinite Descriptions in Ontological Arguments,' *Dialogue*, 22 (1983), pp. 195–201, at 199–200.

[2] See my 'The Logic of What Might Have Been,' *The Philosophical Review*, 98, 1 (January 1989), pp. 3–34, concerning the philosophical superiority of *T* and *B* over *S*4 and *S*5.

The rule of necessity instantiation and the third form of modal negation taken together yield the following derived modal rule.

Possibility generalization (PG): $\dfrac{\phi}{\Diamond\phi}$

These are known as the *modal inference rules*. We also introduce a new form of derivation, known as *strict derivation* (SD), which provides both a $\Box$-introduction rule and a sort of combination elimination-introduction rule for $\Diamond$, and which is subject to restrictions roughly analogous to those for universal derivation (UD in K&M, p. 143). Whereas all of the inference rules are available in each of the propositional modal systems *T, B, S*4, and *S*5, in each of these systems one rule is designated as that system's characteristic *strict importation rule*. A strict importation rule enables one to enter a necessary truth into a subsidiary strict derivation. The *T*-importation rule is NI. The *B*-importation rule is PG. The *S*4-importation rule, $\Box$R, is simply repetition (R in K&M, p. 15) applied to a symbolic formula of the form

$$\Box\phi.$$

And the *S*5-importation rule, $\Diamond$R, is repetition applied to a symbolic formula of the form

$$\Diamond\phi.$$

Each of the modal systems *B, S*4, and *S*5 also admits NI as a primitive strict importation rule together with its own characteristic strict importation rule, thus admitting two primitive strict importation rules apiece.[3]

An *antecedent line* in an incomplete derivation is defined in K&M, p. 24, as a preceding line that is neither boxed nor contains an uncancelled occurrence of '*Show*'. Strict derivations are explained in terms of a distinction between two kinds

[3] The strict importation rules are usually called '(strict) reiteration rules'—a term that fits $\Box$R and $\Diamond$R better than the other two. The characteristic strict importation rules for *T* and *S*4 were first given in Frederic Brenton Fitch, *Symbolic Logic: An Introduction* (New York: Ronald Press, 1952), chapter 3, pp. 64–80 (referring to the former system as 'almost the same as the system Lewis calls *S*2'). The *S*5-importation rule was first given in William A. Wisdom, 'Possibility-Elimination in Natural Deduction,' *Notre Dame Journal of Formal Logic*, 5, 4 (October 1964), pp. 295–298, at 298*n*2, wherein the rule is credited to Robert Price. The *B*-importation rule proposed here (which was discovered independently by the author) is a variation of the *B*-importation rule given by Fitch in 'Natural Deduction Rules for Obligation,' *American Philosophical Quarterly*, 3, 1 (January 1966), pp. 27–38, at 32. (I thank Max J. Cresswell and Allen Hazen for this bibliographical information.) Although PG is a derived modal rule, its role in *B* as a strict importation rule is not derived but primitive.

The admission of NI as a primitive strict importation rule is redundant in *S*4 (as the reader will easily verify) and in *B* (though this is less easily verified and has not been noted before now). As C. Anthony Anderson pointed out to me, it is also redundant in *S*5. Indeed, the natural-deduction apparatus for *S*5 given in G. E. Hughes and Cresswell, *An Introduction to Modal Logic* (London: Methuen and Co., 1986), at pp. 331–334, employs $\Diamond$R as the only primitive strict importation rule. However, the argument (pp. 333–334) for the apparatus's being at least as strong as *S*5 (and hence for its completeness) fallaciously assumes that the axiomatic system whose basis is $\Box\phi \rightarrow \phi$ plus $\Diamond\phi \rightarrow \Box\Diamond\phi$ together with the classical rule of necessitation is sufficient without the *K* axiom $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$ for *S*5. (Strict derivation with NI as a strict importation rule does the work of necessitation and the *K* axiom simultaneously.)

of antecedent lines. Let us say that an antecedent line is *accessible* if there is at most one line of the form

$$\textit{Show}\ \Box\phi$$

or of the form

$$\textit{Show}\ \Diamond\phi$$

where ϕ is a symbolic formula, and containing uncancelled '*Show*', subsequent to that antecedent line, and that it is *inaccessible* otherwise. As a derivation proceeds in stages, with the writing of new lines containing '*Show*' and the cancelling of previous occurrences of '*Show*', a single antecedent line that is accessible at one stage may become inaccessible at a later stage, and then become accessible again at a still later stage. There are two forms of strict derivation. The initial line of a strict derivation (which may be a subsidiary derivation wholly contained within a larger derivation) is either of the form

$$\textit{Show}\ \Box\phi$$

or of the form

$$\textit{Show}\ \Diamond\phi.$$

If the initial line is of the first form, no special assumption is made. If the initial line is of the second form, and a symbolic formula of the form

$\Diamond\psi$

occurs as an accessible line antecedent to the initial line, on the next line one may write the symbolic formula ψ as an assumption. In either case, one then proceeds by inference rules, subsidiary derivations, and citing of premise until the symbolic formula ϕ is secured. One may then cancel the occurrence of '*Show*' in the initial line and box all subsequent lines provided that there is no uncancelled occurrence of '*Show*' among those lines, and provided further that none of those lines (inclusive of boxed lines) was entered as a premise, by an application of an inference rule to an inaccessible line (inaccessible at the current stage, immediately prior to boxing and cancelling), or by an application of an inference rule other than an admissible strict importation rule to an accessible line (accessible at the current stage) antecedent to the initial line. In a strict derivation, any inference rule may be applied to any accessible lines subsequent to the initial line. The first form of strict derivation is known as *necessity derivation* (ND). The second form, invoking a special assumption, is known as *possibility derivation* (PD).

More accurately, besides the addition of the modal inference rules, the following new clause is added to the directions for constructing a *derivation* from given symbolic premise, as it appears in K&M, pp. 24–25:

(4′) If ϕ is a symbolic formula such that

$$\textit{Show}\ \Diamond\phi$$

occurs as a line, then any symbolic formula

$$\psi$$

may occur as the next line, provided that the symbolic formula

$$\Diamond\psi,$$

occurs as a preceding accessible line. [The annotation should refer to the number of the preceding line involved, followed by 'Assumption for possibility derivation' or simply 'Assumption (PD)'.]

In addition, clause (6) (the 'box and cancel' clause), as it appears in K&M, pp. 24–25, is replaced with the following:

(6′) When the following arrangement of lines has appeared:

$$\begin{array}{l} \textit{Show } \phi \\ \chi^{1} \\ \vdots \\ \chi m, \end{array}$$

where none of $\chi 1$ through χm contains uncancelled 'Show' and either

(i′) ϕ occurs unboxed among $\chi 1$ through χm, and $\chi 1$ does not occur as an assumption for possibility derivation,

(ii′) ϕ is of the form

$$(\psi 1 \rightarrow \psi 2)$$

and $\psi 2$ occurs unboxed among $\chi 1$ through χm,

(iii′) for some symbolic formula χ, both χ and its negation occur unboxed among $\chi 1$ through χm, and $\chi 1$ does not occur as an assumption for possibility derivation,

or

(iv′) ϕ is either of the form

$$\Box\psi,$$

or of the form

$$\Diamond\psi,$$

ψ occurs unboxed among $\chi 1$ through χm, and none of $\chi 1$ through χm occurs as a premise, by an application of an inference rule to an inaccessible line, or by an application of an inference rule other than an admissible strict importation rule to an accessible line antecedent to the displayed occurrence of

$$\textit{Show } \phi,$$

then one may simultaneously cancel the displayed occurrence of 'Show' and box all subsequent lines. [When we say that a symbolic formula ϕ occurs among certain lines, we mean that one of those lines is either ϕ or ϕ preceded by 'Show'. Further, annotations for clause (6′), parts (i′), (ii), (iii′), and (iv′) are '*DD*', 'CD', 'ID', and 'SD', respectively, to be entered parenthetically after the annotation for the line in which '*Show*' is cancelled.]

Some of the virtues of this deductive apparatus become more evident upon performing the following.

EXERCISES

1. Construct a T-proof of K: $\Box(\mathrm{P} \to \mathrm{Q}) \to (\Box\mathrm{P} \to \Box\mathrm{Q})$.
2. Prove that PG is derivable in each of the modal systems T, B, $S4$, and $S5$.
3. Prove that the following result obtains in each of the modal systems.

$$N: \qquad \text{If} \vdash \phi, \text{ then} \vdash \Box\phi.$$

4. Prove by induction that interchange of equivalents (IE in K&M, pp. 362–363) is derivable in each of the modal systems.
5. Prove that the T-importation rule of NI is redundant in $S4$ (i.e. that it is a derivable strict importation rule in the system that results by declassifying it as a primitive strict importation rule of $S4$).
6. Construct a T-derivation for the following argument: $\Box\Box(\mathrm{P} \to \mathrm{Q}).\Box\Diamond\mathrm{P}.\ \Box(\Diamond\mathrm{Q} \to \mathrm{R} \vee \mathrm{S}).\ {\sim}\Box\mathrm{R} \quad \therefore \Diamond\mathrm{S}$
7. Construct an $S5$-derivation for the following argument: $\Box(\mathrm{P} \to \Box Q).\Diamond\mathrm{P} \therefore \Box\mathrm{Q}$
8. Construct a B-derivation for the following argument: $\Box(\mathrm{P} \to \Box\mathrm{P}).\Diamond\mathrm{P} \therefore \Box\mathrm{P}$
9. Construct a B-proof of '$\Diamond\Box\mathrm{P} \to \mathrm{P}$'.
10. Construct an $S4$-proof of '$\Diamond\Diamond\mathrm{P} \to \Diamond\mathrm{P}$'.
11. Construct S5-proofs of the following.

B:	$\mathrm{P} \to \Box\Diamond\mathrm{P}$
E:	$\Diamond\Box\mathrm{P} \to \Box\mathrm{P}$
4:	$\Box\mathrm{P} \to \Box\Box\mathrm{P}$

12. Prove that the characteristic strict importation rules PG and $\Box$R of B and $S4$, respectively, are derivable strict importation rules in $S5$.
13. Prove that the T-importation rule of NI is redundant in B.
14. Prove that the T-importation rule of NI is redundant in $S5$.

Exercises 9 and 11, part 3, are solved here for illustration.

1.	*Show* $\Diamond\Box\mathrm{P} \to \mathrm{P}$	Assertion (CD)
2.	$\Diamond\Box\mathrm{P}$	Assumption CD
3.	*Show* P	Assertion (ID)
4.	${\sim}\mathrm{P}$	Assumption ID
5.	*Show* $\Box \to \Box\mathrm{P}$	Assertion (SD)
6.	$\Diamond{\sim}\mathrm{P}$	4, PG
7.	${\sim}\Box\mathrm{P}$	6. MN
8.	${\sim}\Diamond\Box\mathrm{P}$	5,MN
9.	$\Diamond\Box\mathrm{P}$	2,R

1.	*Show* $\Diamond\Box P \rightarrow P$	Assertion (CD)
2.	$\Box P$	Assumption CD
3.	$\Diamond\Box P$	2, PG
4.	*Show* $\Box\Box P$	Assertion (SD)
5.	*Show* $\Box P$	Assertion (SD)
6.	$\sim\Box P$	Assumption ID
7.	$\Diamond\sim P$	6, MN
8.	*Show* $\Box\sim\Box P$	Assertion (SD)
9.	$\Diamond\sim P$	7, $\Diamond$R
10.	$\sim\Box P$	9, MN
11.	$\sim\Diamond\Box P$	8, MN
12.	$\Diamond\Box P$	3, $\Diamond$R

These proofs also yield solutions to exercises 12–14. The second proof illustrates several features of the deductive apparatus. Obtaining line 10 was sufficient for cancelling the occurrence of '*Show*' at line 8 and boxing lines 9 and 10, since those subsequent lines comply with the restriction of importing only from accessible lines in accordance with admissible importation rules. (They do not comply with the restrictions for strict derivation in any of the modal systems other than *S*5, since line 9 was imported by the *S*5-importation rule.) At the stage at which line 12 is imported into the subsidiary derivation beginning at line 5, line 3 (from which 12 is imported) is in fact inaccessible, since at that stage both of the lines 4 and 5 are of the form

$$\textit{Show } \Box\phi$$

with uncancelled '*Show*'. The subsidiary derivation beginning at line 5, however, is a (uniform) indirect derivation (K&M, pp. 20–21, 32) rather than a strict derivation, and is therefore not required to comply with the restrictions for strict derivation. When the occurrence of '*Show*' is cancelled in line 5, line 3 becomes accessible once again, so that, by that stage, the newly boxed line 12 now occurs by an application of $\Diamond$R to an accessible line. The boxed line 12 thus complies with the restrictions for the strict derivation beginning at line 4. It is important to notice also that it is permissible simply to repeat line 2 in place of the subsidiary derivation at lines 5–12 (or alternatively to repeat line 2 in place of the entire sequence of boxed lines 6–12). But had we done so, we would have been prevented from cancelling the occurrence of '*Show*' in line 4 and boxing the subsequent lines as we did, since application subsequent to line 4 of an inference rule other than an admissible importation rule to any line antecedent to line 4 disqualifies the subsidiary derivation beginning at line 4 from the boxing-and-cancelling privileges of a strict derivation.

In light of exercise 12, PG and $\square$R may be admitted as derived strict importation rules of *S*5, thereby significantly increasing the ease of abbreviated *S*5-derivations (K&M, pp. 71–73).

To extend the deductive apparatus to quantified modal logic with identity, the inference rules of universal instantiation, existential generalization, and existential instantiation (UI, EG, and EI in K&M, pp. 140–141) should be replaced with the following free-logical forms.

Free universal instantiation (FUI)
$$\frac{\begin{array}{c}\wedge\alpha\phi_\alpha\\ \vee\alpha\ \alpha=\beta\end{array}}{\phi_\beta}$$

Free existential generalization (FEG)
$$\frac{\begin{array}{c}\phi_\beta\\ \vee\alpha\ \alpha=\beta\end{array}}{\vee\alpha\phi_\alpha}$$

Free existential instantiation (FEI)
$$\frac{\vee\alpha\phi_\alpha}{\phi_\beta\wedge\ \vee\alpha\ \alpha=\beta}$$

where ϕ_β comes from ϕ_α by *proper substitution* (K&M, p. 219) of the singular term β for the individual variable α, i.e. where ϕ_β is the same symbolic formula as ϕ_α except for having free occurrences of β wherever ϕ_α has free occurrences of α.[4] (In the case of FEI, the instantial term β must be a variable new to the derivation.) In addition, universal derivation (K&M, p. 143) should be replaced with a free-logical form. More precisely, besides the substitution of the free-logical quantifier inference rules, the following clause should be added to the old directions for constructing a *derivation* from given symbolic premises, as it appears in K&M, pp. 144–145, 199–200.

(4″) If ϕ is a symbolic formula such that

$$\textit{Show}\ \wedge\alpha_1\ldots\wedge\alpha_k\psi,$$

occurs as a line, then

$$\vee\beta_1\beta_1=\alpha_1\wedge\ldots\wedge\vee\beta_k\beta_k=\alpha_k,$$

may occur as the next line, where for each i, the variable β_i does not occur free in the term α_r [Annotation: 'Assumption for universal derivation' or simply 'Assumption (UD)'.]

This extension of the apparatus excludes the Barcan Formula '$\wedge x\square Fx\rightarrow\square\wedge xFx$' and its converse, as well as '$\vee x\square Fx\rightarrow\square\vee xFx$', sometimes called *the Buridan Formula*, and its converse as theorems, replacing the converse Barcan Formula with the

[4] Compare UID and EGD in K&M, pp. 399–400; and my 'Existence,' in J. Tomberlin, ed., *Philosophical Perspectives, 1: Metaphysics* (Atascadero, Ca.: Ridgeview, 1987), pp. 49–108, at 92–93.

weakened version '$\Box \wedge xFx \to \wedge x\Box(\vee yx = y \to Fx)$' and the Buridan Formula with the weakened version '$\vee x\Box(\vee yx = y \wedge Fx) \to \Box \vee x\ Fx$'. In addition, the inference rule of Leibniz' law (LL in K&M, p. 270) is replaced with the following:

$$\begin{array}{l} \Box\Box\ldots\Box\alpha = \beta \\ \underline{\phi_\alpha \qquad\qquad} \\ \phi_\beta \end{array}$$

where '$\Box\Box\ldots\Box$' represents a string of occurrences of $\Box$, and ϕ_β is the same symbolic formula as ϕ_α except for having free occurrences of the term β where ϕ_α has free occurrences of the term α. When each of the terms α and β is either an individual variable or an individual constant (i.e. a 0-place operation letter, or 'name letter' in K&M, pp. 119, 202), the string of occurrences of $\Box$ may be of any length, including 0. Otherwise the length of the string is subject to a lower-bound restriction: It must be at least as great as the largest number of occurrences in ϕ_α of symbolic formulas of the form

$$\Box\psi$$

or of the form

$$\Diamond\psi$$

where ψ is a symbolic formula, such that there is a single free occurrence of α standing within each (i.e. the largest number of modal-operator occurrences having the same free occurrence of α in their scope). Other modifications are possible.[5] For example, the string of occurrences of $\Box$ in Leibniz' law might be taken to be subject to the same lower-bound restriction with regard to length if α or β is an individual constant.[6]

It is possible also to extend the natural-deduction apparatus to the separate modal systems obtained by adding a modal operator for 'actually' (in the indexical sense) to the logical vocabulary.[7]

[5] In 'Gödel's Ontological Proof,' in J. J. Thompson, ed., *On Being and Saying* (Cambridge, Mass.: MIT Press, 1987), pp. 241–261, at 259–260*n*, Sobel extends his K&M-based modal natural-deduction apparatus (see note 1 above) to QML in a significantly different manner from that proposed here. (In particular, Sobel rightly objects to the inclusion of the Barcan and Buridan Formulas and their converses as theorems, but adds modal inference rules that yield equally objectionable versions of the Buridan and Converse Barcan Formulas.) For alternative extensions to QML, see Andrzej Indrzejcz ak, 'Natural Deduction in Quantifier Modal Logic,' *Bulletin of the Section of Logic*, 23, 1 (March 1994), pp. 30–40.

[6] See my 'How to Become a Millian Heir,' *Noûs*, 23, 2 (April 1989), pp. 211–220, at 212–215, for an argument against so extending the lower-bound restriction.

[7] I provide some details in 'A Natural-Deduction Apparatus for Modal Logics with "Actually",' unpublished notes, University of California, Santa Barbara.